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We study the subalgebra of the lattice vertex operator algebra $V_{\sqrt{2}A_2}$ consisting of the fixed points of an automorphism which is induced from an order-three isometry of the root lattice A_2. We classify the simple modules for the subalgebra. The rationality and the C_2-cofiniteness are also established.

1. Introduction

The space of fixed points of an automorphism group of finite order in a vertex operator algebra is a vertex operator subalgebra. The study of such subalgebras and their modules is called orbifold theory. It is a rich field both in conformal field theory and in the theory of vertex operator algebras. However, orbifold theory is difficult to study in general. One reason is that the subalgebra of fixed points usually has more complicated structure than the original vertex operator algebra.

The first example of orbifold theory in vertex operator algebras is the moonshine module $V^\natural$ by Frenkel, Lepowsky, and Meurman [Frenkel et al. 1988], constructed as an extension of V_Λ^+ by its simple module $V_\Lambda^{T,+}$, where V_Λ^+ is the space of fixed points of an automorphism θ of order two in the Leech lattice vertex operator algebra V_Λ. This construction is called a $2B$-orbifold construction because θ corresponds to a $2B$ involution of the monster simple group. More generally, Frenkel et al. defined a vertex operator algebra V_L associated with an arbitrary positive definite even lattice L. These lattice vertex operator algebras provide a large family of vertex operator algebras. Such a lattice vertex operator algebra admits an automorphism θ of order two, which is a lift of the isometry $\alpha \mapsto -\alpha$ of the underlying lattice L. Orbifold theory for the fixed point subalgebra V_L^+ of θ has been developed extensively. The simple V_L^+-modules have been classified

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[Abe and Dong 2004], the fusion rules have been determined [Abe et al. 2005], and it has been established that V_L^+ is C_2-cofinite [Abe et al. 2004; Yamskulna 2004].

Here we study the fixed point subalgebra by an automorphism of order three for a certain lattice vertex operator algebra. Namely, let $L = \sqrt{2}A_2$ be $\sqrt{2}$ times an ordinary root lattice of type A_2 and let τ be an isometry of the root lattice of type A_2 induced from an order-three permutation on the set of positive roots. We classify the simple modules for the subalgebra V_L^τ of fixed points by τ. Moreover, we show that V_L^τ is rational and C_2-cofinite.

In [Dong et al. 2004; Kitazume et al. 2003] we have already discussed the vertex operator algebra V_L^τ. It was shown that $V_L^\tau = M^0 \oplus W^0$ is a direct sum of a subalgebra M^0 and its simple highest-weight module W^0. Actually, M^0 is a tensor product of a W_3 algebra of central charge $6/5$ and a W_3 algebra of central charge $4/5$. The property of a W_3 algebra of central charge $6/5$ as the first component of the tensor product M^0 was investigated in [Dong et al. 2004]. It is generated by the Virasoro element $\tilde{\omega}^1$ and a weight-three vector J. The second component of M^0, a W_3 algebra of central charge $4/5$, was studied in [Kitazume et al. 2000b]. It is generated by the Virasoro element $\tilde{\omega}^2$ and a weight-three vector K. Each of these W_3 algebras possesses a symmetry of order three. The order-three symmetry of the second W_3 algebra is related to the $\mathbb{Z}_3$ part of $L^\perp/L \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$, where $L^\perp$ denotes the dual lattice of L. As an M^0-module, W^0 is generated by a highest-weight vector P of weight 2. Thus the vertex operator algebra V_L^τ is generated by the five elements $\tilde{\omega}^1$, $\tilde{\omega}^2$, J, K, and P.

There are 12 inequivalent simple V_L-modules, which correspond to the cosets of L in its dual lattice $L^\perp$ [Dong 1993]. Let (U, Y_U) be a simple V_L-module. One can define a new simple V_L-module $(U \circ \tau, Y_{U\circ\tau})$ by $U \circ \tau = U$ as vector spaces and $Y_{U\circ\tau}(v, z) = Y_U(\tau v, z)$ for $v \in V_L$. Then $U \mapsto U \circ \tau$ is a permutation on the set of simple V_L-modules. In the case where U and $U \circ \tau$ are equivalent V_L-modules, U is said to be τ-stable. If U is τ-stable, the eigenspace $U(\varepsilon)$ of τ with eigenvalue ξ^ε, where $\xi = \exp(2\pi\sqrt{-1}/3)$, $\varepsilon = 0, 1, 2$, is a simple V_L^τ-module, while if U belongs to a τ-orbit of length three, U itself is a simple V_L^τ-module and the three members in the τ-orbit are equivalent [Dong and Yamskulna 2002, Theorem 6.14]. Among the 12 inequivalent simple V_L-modules, three are τ-stable and the remaining nine are divided into three τ-orbits. In this way we obtain 12 simple V_L^τ-modules. It is known that there are three inequivalent simple τ-twisted V_L-modules and three inequivalent simple τ^2-twisted V_L-modules. We denote them respectively by

$$V_L^j(\tau) := V_L^{T_{\chi_j}}(\tau), \quad V_L^j(\tau^2) := V_L^{T_{\chi'_j}}(\tau^2), \quad j = 0, 1, 2. \tag{1-1}$$

The automorphism τ acts on these τ-twisted or τ^2-twisted V_L-modules and each eigenspace of τ is a simple V_L^τ-module [Miyamoto and Tanabe 2004, Theorem 2].

There are 18 such simple V_L^τ-modules, all of them inequivalent. Hence there are at least 30 inequivalent simple V_L^τ-modules.

The main part of our argument is to show that every simple V_L^τ-module is isomorphic to one of these 30 simple V_L^τ-modules. Recall that $V_L^\tau = M^0 \oplus W^0$ and that M^0 is a tensor product of two W_3 algebras. The W_3 algebra of central charge 6/5 (resp. 4/5) possesses 20 (resp. 6) inequivalent simple modules. Thus there are 120 inequivalent simple M^0-modules. It turns out that among these simple M^0-modules, 60 of them cannot appear as an M^0-submodule in any simple V_L^τ-module and that each simple V_L^τ-module is a direct sum of two of the remaining 60 simple M^0-modules. We note that W^0 is not a simple current M^0-module. Thus V_L^τ is a nonsimple current extension of M^0. A discussion on simple modules for another nonsimple current extension of a certain vertex operator algebra can be found in [Lam et al. 2005, Appendix C].

The organization of this paper is as follows. In Section 2 we review various notions about untwisted or twisted modules for vertex operator algebras, together with some basic tools which will be used in later sections. In Section 3 we fix notation for the vertex operator algebra V_L^τ and collect its properties. We clarify an argument on the simplicity of $M_T^0(\tau^i)$ and $W_T^0(\tau^i)$, $i = 1, 2$, in [Kitazume et al. 2003, Proposition 6.8]. Furthermore, we correct some misprints in [Kitazume et al. 2003, (6.46)] and in an equation of [Dong et al. 2004, page 265] concerning a decomposition of a simple τ-twisted V_L-module $V_L^j(\tau)$, $j = 1, 2$ as a τ-twisted $M_k^0 \otimes M_t^0$-module (see Remark 3.5). In Section 4 we discuss the structure of the 30 known simple V_L^τ-modules. In particular, we calculate the action of $o(\tilde{\omega}^1)$, $o(\tilde{\omega}^2)$, $o(J)$, $o(K)$, and $o(P)$ on the top level of these simple modules. Finally, in Section 5 we complete the classification of simple V_L^τ-modules. We also show the rationality of V_L^τ.

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2. Preliminaries

We recall some notation for untwisted or twisted modules for a vertex operator algebra. We also review the twisted version of Zhu's theory. A basic reference to twisted modules is [Dong et al. 1998a]. For untwisted modules, see also [Lepowsky and Li 2004]. Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra and g be an automorphism of V of finite order T. Set $V^r = \{v \in V \mid gv = e^{2\pi\sqrt{-1}r/T} v\}$, so that $V = \bigoplus_{r \in \mathbb{Z}/T\mathbb{Z}} V^r$.

Definition 2.1. A weak g-twisted V-module M is a vector space equipped with a linear map

$$Y_M(\,\cdot\,,z) : v \in V \mapsto Y_M(v,z) = \sum_{n\in\mathbb{Q}} v_n z^{-n-1} \in (\operatorname{End} M)\{z\}$$

satisfying the following conditions.

(1) $Y_M(v,z) = \sum_{n\in r/T+\mathbb{Z}} v_n z^{-n-1}$ for $v \in V^r$.

(2) $v_n w = 0$ if $n \gg 0$, where $v \in V$ and $w \in M$.

(3) $Y_M(\mathbf{1},z) = \mathrm{id}_M$.

(4) For $u \in V^r$ and $v \in V$, the g-twisted Jacobi identity holds:

$$z_0^{-1}\delta\Big(\frac{z_1-z_2}{z_0}\Big)Y_M(u,z_1)Y_M(v,z_2) - z_0^{-1}\delta\Big(\frac{z_2-z_1}{-z_0}\Big)Y_M(v,z_2)Y_M(u,z_1) = z_2^{-1}\Big(\frac{z_1-z_0}{z_2}\Big)^{-r/T}\delta\Big(\frac{z_1-z_0}{z_2}\Big)Y_M(Y(u,z_0)v,z_2). \tag{2-1}$$

Compare the coefficients of $z_0^{-l-1}z_1^{-m-1}z_2^{-n-1}$ in both sides of (2-1) for $u \in V^r$, $v \in V^s$, $l \in \mathbb{Z}$, $m \in \frac{r}{T}+\mathbb{Z}$, and $n \in \frac{s}{T}+\mathbb{Z}$. Then we obtain

$$\sum_{i=0}^{\infty}\binom{m}{i}(u_{l+i}v)_{m+n-i} = \sum_{i=0}^{\infty}(-1)^i\binom{l}{i}\big(u_{l+m-i}v_{n+i} - (-1)^l v_{l+n-i}u_{m+i}\big). \tag{2-2}$$

In the case $l = 0$, this reduces to

$$[u_m, v_n] = \sum_{i=0}^{\infty}\binom{m}{i}(u_i v)_{m+n-i}. \tag{2-3}$$

The Virasoro element ω is contained in V^0. Let $L(n) = \omega_{n+1}$ for $n \in \mathbb{Z}$. Then

$$[L(m), L(n)] = (m-n)L(m+n) + \tfrac{1}{12}(m^3-m)\delta_{m+n,0}(\operatorname{rank} V),$$
$$\frac{d}{dz}Y_M(v,z) = Y_M(L(-1)v,z)$$

for $v \in V$; see [Dong et al. 1998a, (3.8), (3.9)].

An important consequence of (2-1) is the associativity formula

$$(z_0+z_2)^{k+r/T}Y_M(u,z_0+z_2)Y_M(v,z_2)w = (z_2+z_0)^{k+r/T}Y_M(Y(u,z_0)v,z_2)w \tag{2-4}$$

(see [Dong et al. 1998a, (3.5)]), where $u \in V^r$, $v \in V$, $w \in M$, and k is a nonnegative integer such that $z^{k+r/T}Y_M(u,z)w \in M[[z]]$.

Let (M, Y_M) and (N, Y_N) be weak g-twisted V-modules. A homomorphism of M to N is a linear map $f : M \to N$ such that $fY_M(v,z) = Y_N(v,z)f$ for all $v \in V$.

Let $\mathbb{N}$ be the set of nonnegative integers.

Definition 2.2. A $\frac{1}{T}\mathbb{N}$-graded weak g-twisted V-module M is a weak g-twisted V-module with a $\frac{1}{T}\mathbb{N}$-grading $M = \bigoplus_{n\in\frac{1}{T}\mathbb{N}} M_{(n)}$ such that

$$v_m M_{(n)} \subset M_{(n+\mathrm{wt}(v)-m-1)} \tag{2-5}$$

for any homogeneous vectors $v \in V$.

A $\frac{1}{T}\mathbb{N}$-graded weak g-twisted V-module here is called an admissible g-twisted V-module in [Dong et al. 1998a]. Without loss we can shift the grading of a $\frac{1}{T}\mathbb{N}$-graded weak g-twisted V-module M so that $M_{(0)} \neq 0$ if $M \neq 0$. We call such an $M_{(0)}$ the top level of M.

Definition 2.3. A g-twisted V-module M is a weak g-twisted V-module with a $\mathbb{C}$-grading $M = \bigoplus_{\lambda\in\mathbb{C}} M_\lambda$, where $M_\lambda = \{w \in M \mid L(0)w = \lambda w\}$. Moreover, each M_λ is a finite dimensional space and for any fixed λ, $M_{\lambda+n/T} = 0$ for all sufficiently small integers n.

A g-twisted V-module is sometimes called an ordinary g-twisted V-module. By [Dong et al. 1998a, Lemma 3.4], any g-twisted V-module is a $\frac{1}{T}\mathbb{N}$-graded weak g-twisted V-module. Indeed, assume that M is a g-twisted V-module. For each $\lambda \in \mathbb{C}$ with $M_\lambda \neq 0$, let $\lambda_0 = \lambda + m/T$ be such that $m \in \mathbb{Z}$ is minimal subject to $M_{\lambda_0} \neq 0$. Let Λ be the set of all such λ_0 and let $M_{(n)} = \bigoplus_{\lambda\in\Lambda} M_{n+\lambda}$. Then $M_{(n)}$ satisfies the condition in Definition 2.2. Thus we have the following inclusions.

$$\begin{aligned}\{g\text{-twisted } V\text{-modules}\} &\subset \{\tfrac{1}{T}\mathbb{N}\text{-graded weak } g\text{-twisted } V\text{-modules}\}\\ &\subset \{\text{weak } g\text{-twisted } V\text{-modules}\}\end{aligned}$$

Definition 2.4. A vertex operator algebra V is said to be g-rational if every $\frac{1}{T}\mathbb{N}$-graded weak g-twisted V-module is semisimple, that is, a direct sum of simple $\frac{1}{T}\mathbb{N}$-graded weak g-twisted V-modules.

Let M be a weak g-twisted V-module. The next lemma is a twisted version of [Li 2001, Lemma 3.12]. In fact, using the associativity formula (2-4) we can prove it by essentially the same argument as in [Li 2001].

Lemma 2.5. *Let $u \in V^r$, $v \in V^s$, $w \in M$, and k be a nonnegative integer such that $z^{k+r/T} Y_M(u, z)w \in M[[z]]$. Let $p \in \frac{r}{T} + \mathbb{Z}$, $q \in \frac{s}{T} + \mathbb{Z}$, and N be a nonnegative integer such that $z^{N+1+q} Y_M(v, z)w \in M[[z]]$. Then*

$$u_p v_q w = \sum_{i=0}^{N}\sum_{j=0}^{\infty} \binom{p-k-r/T}{i}\binom{k+r/T}{j} (u_{p-k-r/T-i+j}v)_{q+k+r/T+i-j}w. \tag{2-6}$$

Conversely, $(u_p v)_q w$ can be written as a linear combination of some vectors of the form $u_i v_j w$.

Lemma 2.6. *Let $u \in V^r$, $v \in V^s$, $w \in M$. Then for $p \in \mathbb{Z}$ and $q \in \frac{r+s}{T} + \mathbb{Z}$, the vector $(u_p v)_q w$ is a linear combination of $u_i v_j w$ with $i \in \frac{r}{T} + \mathbb{Z}$ and $j \in \frac{s}{T} + \mathbb{Z}$.*

Proof. Let $X = \operatorname{span}\{u_i v_j w \mid i \in \frac{r}{T} + \mathbb{Z},\ j \in \frac{s}{T} + \mathbb{Z}\}$. We use (2-2). Take $m \in \frac{r}{T} + \mathbb{Z}$ such that $u_{m+i} w = 0$ for $i \geq 0$. Let $N \in \mathbb{Z}$ be such that $u_{N+i} v = 0$ for $i > 0$. If $p > N$, then $u_p v = 0$ and the assertion is trivial. Assume that $p \leq N$. For $j = 0, 1, \ldots, N - p$, let $l = p + j$ and $n = q - m - j$ in (2-2). Then

$$\sum_{i=0}^{\infty} \binom{m}{i} (u_{p+j+i} v)_{q-j-i} w = \sum_{i=0}^{\infty} (-1)^i \binom{p+j}{i} u_{p+m+j-i} v_{q-m-j+i} w.$$

The right hand side of this equation is contained in X. Consider the left hand side for each of $j = N - p, N - p - 1, \ldots, 1, 0$. Then we see that $(u_N v)_{q-N+p} w \in X$, $(u_{N-1} v)_{q-N+p+1} w \in X, \ldots,$ and $(u_p v)_q w \in X$. □

For subsets A, B of V and a subset X of M, set $A \cdot X = \operatorname{span}\{u_n w \mid u \in A, w \in X, n \in \frac{1}{T}\mathbb{Z}\}$ and $A \cdot B = \operatorname{span}\{u_n v \mid u \in A, v \in B, n \in \mathbb{Z}\}$. Then it follows from (2-6) that $A \cdot (B \cdot X) \subset (A \cdot B) \cdot X$ (see also [Yamauchi 2004, (2.2)]). For a vector $w \in M$, this in particular implies that $V \cdot w$ is a weak g-twisted V-submodule of M. If w is an eigenvector for $L(0)$, then $V \cdot w$ is a direct sum of eigenspaces for $L(0)$. Each eigenspace is not necessarily of finite dimension. Thus $V \cdot w$ is not a g-twisted module in general. This subject was discussed in [Abe et al. 2004; Buhl 2002; Yamauchi 2004]. We will review it later in this section.

Zhu [1996] introduced an associative algebra $A(V)$ called the Zhu algebra for a vertex operator algebra V, which plays a crucial role in the study of representations for V. Later, Dong, Li and Mason [Dong et al. 1998a] constructed an associative algebra $A_g(V)$ called the g-twisted Zhu algebra in order to generalize Zhu's theory to g-twisted representations for V. The definition of $A_g(V)$ is similar to that of $A(V)$. Let V, g, T, and V^r be as before. Roughly speaking, $A_g(V) = V/O_g(V)$ is a quotient space of V with a binary operation $*_g$. It is in fact an associative algebra with respect to $*_g$. If $r \neq 0$, then $V^r \subset O_g(V)$. Thus $A_g(V) = (V^0 + O_g(V))/O_g(V)$. For the case $g = 1$, see (5-1) in Section 5.

A certain Lie algebra $V[g]$ was considered in [Dong et al. 1998a]. Any weak g-twisted V-module is a module for the Lie algebra $V[g]$ (see Lemma 5.1 of that reference). Moreover, for a $V[g]$-module M, the space $\Omega(M)$ of lowest-weight vectors with respect to $V[g]$ was defined. If M is a weak g-twisted V-modules, then $\Omega(M)$ is the set of $w \in M$ such that $v_{\mathrm{wt}(v)-1+n} w = 0$ for all homogeneous vectors $v \in V$ and $0 < n \in \frac{1}{T}\mathbb{Z}$. The map $v \mapsto o(v)$ for homogeneous vectors $v \in V^0$ induces a representation of the associative algebra $A_g(V)$ on $\Omega(M)$, where $o(v) = v_{\mathrm{wt}(v)-1}$. If M is a $\frac{1}{T}\mathbb{N}$-graded weak g-twisted V-module, then the top level

$M_{(0)}$ is contained in $\Omega(M)$. In the case where M is a simple $\frac{1}{T}\mathbb{N}$-graded weak g-twisted V-module, $M_{(0)} = \Omega(M)$ and $M_{(0)}$ is a simple $A_g(V)$-module (see [Dong et al. 1998a, Proposition 5.4]).

For any $A_g(V)$-module U, a certain $\frac{1}{T}\mathbb{N}$-graded $V[g]$-module $M(U)$ such that $M(U)_{(0)} = U$ was defined (see [Dong et al. 1998a, (6.1)]). Let W be the subspace of $M(U)$ spanned by the coefficients of

$$(z_0+z_2)^{\mathrm{wt}(u)-1+\delta_r+r/T} Y_M(u, z_0+z_2) Y_M(v, z_2) w \\ - (z_2+z_0)^{\mathrm{wt}(u)-1+\delta_r+r/T} Y_M(Y(u, z_0)v, z_2) w$$

for all homogeneous $u \in V^r$, $v \in V$, $w \in U$ (see [Dong et al. 1998a, (6.3)]). Set $\bar{M}(U) = M(U)/U(V[g])W$, which is a quotient module of $M(U)$ by the $V[g]$-submodule generated by W.

The following results will be necessary in Sections 3 and 5.

Theorem 2.7 [Dong et al. 1998a, Theorem 6.2]. *$\bar{M}(U)$ is a $\frac{1}{T}\mathbb{N}$-graded weak g-twisted V-module such that its top level $\bar{M}(U)_{(0)}$ is equal to U and such that it has the following universal property: for any weak g-twisted V-module M and any homomorphism $\varphi : U \to \Omega(M)$ of $A_g(V)$-modules, there is a unique homomorphism $\bar{\varphi} : \bar{M}(U) \to M$ of weak g-twisted V-modules which is an extension of φ.*

Let J be the sum of all $\frac{1}{T}\mathbb{N}$-graded $V[g]$-submodules of $M(U)$ which intersect trivially with U. Since $M(U)_{(0)} = U$, it is a unique $\frac{1}{T}\mathbb{N}$-graded $V[g]$-submodule of $M(U)$ being maximal subject to $J \cap U = 0$. The principal point is that $U(V[g])W \subset J$. Set $L(U) = M(U)/J$.

Theorem 2.8 [Dong et al. 1998a, Theorem 6.3]. *$L(U)$ is a $\frac{1}{T}\mathbb{N}$-graded weak g-twisted V-module such that $\Omega(L(U)) \cong U$ as $A_g(V)$-modules.*

Remark 2.9. If M is a $\frac{1}{T}\mathbb{N}$-graded weak g-twisted V-module and $\varphi : U \to M_{(0)}$ is a homomorphism of $A_g(V)$-modules, then the homomorphism $\bar{\varphi} : \bar{M}(U) \to M$ of weak g-twisted V-modules in Theorem 2.7 preserves the $\frac{1}{T}\mathbb{N}$-grading. Indeed, $\bar{M}(U) = \mathrm{span}\{v_n U \mid v \in V, n \in \frac{1}{T}\mathbb{Z}\}$ by (2-6), since $\bar{M}(U)$ is generated by U as a $\frac{1}{T}\mathbb{N}$-graded weak g-twisted V-module. By (2-5), $v_{\mathrm{wt}(v)-1-n}\bar{M}(U)_{(0)} \subset \bar{M}(U)_{(n)}$ for any homogeneous $v \in V$ and $n \in \frac{1}{T}\mathbb{Z}$. Since $\bar{M}(U)_{(0)} = U$, it follows that $\bar{M}(U)_{(n)}$ is spanned by $v_{\mathrm{wt}(v)-1-n}U$ for all homogeneous $v \in V$. Now, $\bar{\varphi}(v_{\mathrm{wt}(v)-1-n}U) = v_{\mathrm{wt}(v)-1-n}\bar{\varphi}(U)$ is contained in $v_{\mathrm{wt}(v)-1-n}M_{(0)} \subset M_{(n)}$. Hence $\bar{\varphi}(\bar{M}(U)_{(n)}) \subset M_{(n)}$ as required. In the case where both of $\bar{M}(U)$ and M are ordinary g-twisted V-modules, $\bar{\varphi}$ becomes a homomorphism of ordinary g-twisted V-modules since $\bar{\varphi}$ commutes with $L(0)$.

Lemma 2.10. *Let U be an $A_g(V)$-module. Let S be a $\frac{1}{T}\mathbb{N}$-graded weak g-twisted V-module such that it is generated by its top level $S_{(0)}$ and such that $S_{(0)}$ is isomorphic to U as an $A_g(V)$-module. Then there is a surjective homomorphism $S \to L(U)$ of weak g-twisted V-modules which preserves the $\frac{1}{T}\mathbb{N}$-grading.*

Proof. By Theorem 2.7 and Remark 2.9, an isomorphism $\varphi : U \to S_{(0)}$ of $A_g(U)$-modules can be extended to a surjective homomorphism $\bar{\varphi} : \bar{M}(U) \to S$ of weak g-twisted V-modules which preserves the $\frac{1}{T}\mathbb{N}$-grading. The kernel $\operatorname{Ker}\bar{\varphi}$ of $\bar{\varphi}$ intersects trivially with $\bar{M}(U)_{(0)}$ and so is contained in $\bigoplus_{0<n\in\frac{1}{T}\mathbb{N}}\bar{M}(U)_{(n)}$. Let I be a $\frac{1}{T}\mathbb{N}$-graded $V[g]$-submodule of $M(U)$ such that $\operatorname{Ker}\bar{\varphi} = I/U(V[g])W$. Then $I \cap U = 0$. This implies that $I \subset J$. Hence $L(U) = M(U)/J$ is a homomorphic image of $M(U)/I \cong S$. □

Theorem 2.11 [Dong et al. 1998a, Theorem 7.2]. *L is a functor from the category of simple $A_g(V)$-modules to the category of simple $\frac{1}{T}\mathbb{N}$-graded weak g-twisted V-modules such that $\Omega \circ L = \mathrm{id}$ and $L \circ \Omega = \mathrm{id}$.*

Theorem 2.12 [Dong et al. 1998a, Theorem 8.1]. *Let V be a g-rational vertex operator algebra.*

(1) *$A_g(V)$ is a finite dimensional semisimple associative algebra.*

(2) *V has only finitely many isomorphism classes of simple $\frac{1}{T}\mathbb{N}$-graded weak g-twisted V-modules.*

(3) *Every simple $\frac{1}{T}\mathbb{N}$-graded weak g-twisted V-module is an ordinary g-twisted V-modules.*

In case of $g = 1$, the above argument reduces to the untwisted case. In particular, $A_g(V)$ is identical with the original Zhu algebra $A(V)$ if $g = 1$.

There is an important intrinsic property of a vertex operator algebra, namely, the C_2-cofiniteness. Let $C_2(V) = \operatorname{span}\{u_{-2}v \mid u, v \in V\}$. More generally, we set $C_2(M) = \operatorname{span}\{u_{-2}w \mid u \in V, w \in M\}$ for a weak V-module M. If the dimension of the quotient space $V/C_2(V)$ is finite, V is said to be C_2-cofinite. Similarly, a weak V-module M is said to be C_2-cofinite if $M/C_2(M)$ is of finite dimension. The notion of C_2-cofiniteness of a vertex operator algebra was first introduced by Zhu [1996]. The subspace $C_2(M)$ of a weak V-module M was studied in [Li 1999b]. We refer the reader to [Nagatomo and Tsuchiya 2005] also.

Theorem 2.13 [Dong et al. 2000, Proposition 3.6]. *If V is C_2-cofinite, then $A_g(V)$ is of finite dimension.*

If $V = \bigoplus_{n=0}^{\infty} V_n$ and $V_0 = \mathbb{C}\mathbf{1}$, then V is said to be of CFT type. Here V_n denotes the homogeneous subspace of weight n, that is, the eigenspace of $L(0) = \omega_1$ with eigenvalue n.

Theorem 2.14 [Yamauchi 2004, Lemma 3.3]. *Suppose V is C_2-cofinite and of CFT type. Choose a finite dimensional $L(0)$-invariant and g-invariant subspace U of V such that $V = U + C_2(V)$. Let W be a weak g-twisted V-module generated by a vector w. Then W is spanned by the vectors of the form $u^1_{-n_1}u^2_{-n_2}\cdots u^k_{-n_k}w$ with $n_1 > n_2 > \cdots > n_k > -N$ and $u^i \in U$, $i = 1, 2, \ldots, k$, where $N \in \frac{1}{T}\mathbb{Z}$ is a constant such that $u_m w = 0$ for all $u \in U$ and $m \geq N$.*

Theorem 2.15 [Yamauchi 2004, Corollaries 3.8 and 3.9]. *Suppose V is C_2-cofinite and of CFT type. Then the following assertions hold.*

(1) *Every weak g-twisted V-module is a $\frac{1}{T}\mathbb{N}$-graded weak g-twisted V-module.*

(2) *Every simple weak g-twisted V-module is a simple ordinary g-twisted V-module.*

Remark 2.16. Suppose V is C_2-cofinite and of CFT type. Let M be a weak g-twisted V-module and $w^1, \ldots, w^k$ be eigenvectors of $L(0)$ in M. Then the weak g-twisted V-submodule W generated by $w^1, \ldots, w^k$ is an ordinary g-twisted V-module. Indeed, W is a direct sum of eigenspaces for $L(0)$ and each homogeneous subspace is of finite dimension by Theorem 2.14.

For the untwisted case, that is, the case $g = 1$, we refer the reader to [Abe et al. 2004; Buhl 2002; Dong et al. 1997; Li 1999b]. A spanning set for a vertex operator algebra was first studied in [Gaberdiel and Neitzke 2003, Proposition 8].

3. The fixed point subalgebra $(V_{\sqrt{2}A_2})^\tau$

In this section we fix notation. We tend to follow the notation in [Dong et al. 2004; Kitazume et al. 2000a; Kitazume et al. 2003] unless otherwise specified. We also recall certain properties of the lattice vertex operator algebra $V_{\sqrt{2}A_2}$ associated with $\sqrt{2}$ times an ordinary root lattice of type A_2 and its subalgebras (see [Dong et al. 2004; Kitazume et al. 2000a; Kitazume et al. 2003; Kitazume et al. 2000b]).

Let α_1, α_2 be the simple roots of type A_2 and set $\alpha_0 = -(\alpha_1+\alpha_2)$. Thus $\langle\alpha_i, \alpha_i\rangle = 2$ and $\langle\alpha_i, \alpha_j\rangle = -1$ if $i \neq j$. Set $\beta_i = \sqrt{2}\alpha_i$ and let $L = \mathbb{Z}\beta_1 + \mathbb{Z}\beta_2$ be the lattice spanned by β_1 and β_2. We denote the cosets of L in its dual lattice $L^\perp = \{\alpha \in \mathbb{Q}\otimes_{\mathbb{Z}} L \mid \langle\alpha, L\rangle \subset \mathbb{Z}\}$ as follows.

$$L^0 = L, \quad L^1 = \frac{-\beta_1+\beta_2}{3} + L, \quad L^2 = \frac{\beta_1-\beta_2}{3} + L,$$

$$L_0 = L, \quad L_a = \frac{\beta_2}{2} + L, \quad L_b = \frac{\beta_0}{2} + L, \quad L_c = \frac{\beta_1}{2} + L,$$
$$L^{(i,j)} = L_i + L^j$$

for $i = 0, a, b, c$ and $j = 0, 1, 2$, where $\{0, a, b, c\} \cong \mathbb{Z}_2\times\mathbb{Z}_2$ is Klein's four-group. Note that $L^{(i,j)}$, $i \in \{0, a, b, c\}$, $j \in \{0, 1, 2\}$ are all the cosets of L in $L^\perp$ and $L^\perp/L \cong \mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3$.

We adopt the standard notation for the vertex operator algebra $(V_L, Y(\cdot, z))$ associated with the lattice L (see [Frenkel et al. 1988]). In particular, $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$ is an abelian Lie algebra, $\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$ is the corresponding affine Lie algebra, $M(1) = \mathbb{C}[\alpha(n)\,;\, \alpha \in \mathfrak{h}, n < 0]$, where $\alpha(n) = \alpha \otimes t^n$, is the unique simple $\hat{\mathfrak{h}}$-module such that $\alpha(n)1 = 0$ for all $\alpha \in \mathfrak{h}$ and $n > 0$ and $c = 1$. As a vector space $V_L = M(1) \otimes \mathbb{C}[L]$ and for each $v \in V_L$, a vertex operator $Y(v, z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1} \in \mathrm{End}(V_L)[[z, z^{-1}]]$ is defined. The vector $\mathbf{1} = 1 \otimes 1$ is called the vacuum vector. In our case $\langle \alpha, \beta \rangle \in 2\mathbb{Z}$ for any $\alpha, \beta \in L$. Thus the twisted group algebra $\mathbb{C}\{L\}$ of [Frenkel et al. 1988] is naturally isomorphic to the ordinary group algebra $\mathbb{C}[L]$.

There are exactly 12 inequivalent simple V_L-modules, which are represented by $V_{L^{(i,j)}}$, $i = 0, a, b, c$ and $j = 0, 1, 2$ (see [Dong 1993]). We use the symbol e^α, $\alpha \in L^\perp$ to denote a basis of $\mathbb{C}\{L^\perp\}$.

We consider the following three isometries of $(L, \langle \cdot, \cdot \rangle)$.

$$\begin{aligned} &\tau : \beta_1 \to \beta_2 \to \beta_0 \to \beta_1, \\ &\sigma : \beta_1 \to \beta_2, \qquad \beta_2 \to \beta_1, \\ &\theta : \beta_i \to -\beta_i, \quad i = 1, 2. \end{aligned} \tag{3-1}$$

Note that τ is fixed-point-free and of order 3. The isometries τ, σ, and θ of L can be extended linearly to isometries of $L^\perp$. Moreover, the isometry τ lifts naturally to an automorphism of V_L:

$$\alpha^1(-n_1) \cdots \alpha^k(-n_k) e^\beta \longmapsto (\tau\alpha^1)(-n_1) \cdots (\tau\alpha^k)(-n_k) e^{\tau\beta}.$$

By abuse of notation, we denote it by τ also. We can consider the action of τ on $V_{L^{(i,j)}}$ in a similar way. We apply the same argument to σ and θ. Our purpose is the classification of simple modules for the fixed point subalgebra $V_L^\tau = \{v \in V_L \mid \tau v = v\}$ of V_L by the automorphism τ.

For a simple V_L-module (U, Y_U), let $(U \circ \tau, Y_{U\circ\tau})$ be a new V_L-module such that $U \circ \tau = U$ as vector spaces and $Y_{U\circ\tau}(v, z) = Y_U(\tau v, z)$ for $v \in V_L$ [Dong et al. 2000]. Then $U \mapsto U \circ \tau$ induces a permutation on the set of simple V_L-modules. If U and $U \circ \tau$ are equivalent V_L-modules, U is said to be τ-stable. The following lemma is a straightforward consequence of the definition of $V_{L^{(i,j)}}$.

Lemma 3.1. (1) $V_{L^{(0,j)}}$, $j = 0, 1, 2$ *are* τ*-stable.*

(2) $V_{L^{(a,j)}} \circ \tau = V_{L^{(c,j)}}$, $V_{L^{(c,j)}} \circ \tau = V_{L^{(b,j)}}$, *and* $V_{L^{(b,j)}} \circ \tau = V_{L^{(a,j)}}$, $j = 0, 1, 2$.

A family of simple twisted modules for lattice vertex operator algebras was constructed in [Dong and Lepowsky 1996; Lepowsky 1985]. Following Dong and Lepowsky, three inequivalent simple τ-twisted V_L-modules $(V_L^j(\tau), Y^\tau(\cdot, z))$, $j = 0, 1, 2$ were studied in [Dong et al. 2004; Kitazume et al. 2003]. By the preceding lemma and [Dong et al. 2000, Theorem 10.2], we know that $(V_L^j(\tau), Y^\tau(\cdot, z))$,

$j = 0, 1, 2$, are all the inequivalent simple τ-twisted V_L-modules. Similarly, there are exactly three inequivalent simple τ^2-twisted V_L-modules $(V_L^j(\tau^2), Y^{\tau^2}(\,\cdot\,, z))$, $j = 0, 1, 2$.

We use the same notation for $(V_L^j(\tau), Y^{\tau}(\,\cdot\,, z))$ and $(V_L^j(\tau^2), Y^{\tau^2}(\,\cdot\,, z))$ as in [Dong et al. 2004, Section 4]. Thus

$$V_L^j(\tau) = S[\tau] \otimes T_{\chi_j},$$

where T_{χ_j}, $j = 0, 1, 2$, is the one-dimensional representation of a certain central extension of L affording the character χ_j. Let

$$h_1 = \tfrac{1}{3}(\beta_1 + \xi^2\beta_2 + \xi\beta_0), \qquad h_2 = \tfrac{1}{3}(\beta_1 + \xi\beta_2 + \xi^2\beta_0).$$

Then $\tau h_i = \xi^i h_i$, $\langle h_1, h_1\rangle = \langle h_2, h_2\rangle = 0$, and $\langle h_1, h_2\rangle = 2$. Moreover, $\beta_i = \xi^{i-1}h_1 + \xi^{2(i-1)}h_2$, $i = 0, 1, 2$. As a vector space, $S[\tau]$ is isomorphic to a polynomial algebra with variables $h_1(1/3+n)$, $h_2(2/3+n)$, $n \in \mathbb{Z}_{<0}$. The isometry τ acts on $S[\tau]$ by $\tau h_j = \xi^j h_j$. We define the action of τ on T_{χ_j} to be the identity. The weight in $S[\tau]$ is given by $\operatorname{wt} h_i(i/3+n) = -i/3 - n$, $i = 1, 2$ and $\operatorname{wt} 1 = 1/9$. The weight of any element of T_{χ_j} is defined to be 0. Note that the weight in $V_L^j(\tau)$ is identical with the eigenvalue for the action of the coefficient of z^{-2} in the τ-twisted vertex operator $Y^{\tau}(\omega, z)$, where ω denotes the Virasoro element of V_L.

The simple τ^2-twisted V_L-modules $(V_L^j(\tau^2), Y^{\tau^2}(\,\cdot\,, z))$, $j = 0, 1, 2$ are

$$V_L^j(\tau^2) = S[\tau^2] \otimes T_{\chi'_j},$$

where $T_{\chi'_j}$, $j = 0, 1, 2$, are the one-dimensional representations of a certain central extension of L affording the character χ'_j. Moreover, $S[\tau^2]$ is isomorphic to a polynomial algebra with variables $h'_1(1/3+n)$, $h'_2(2/3+n)$, $n \in \mathbb{Z}_{<0}$ as a vector space, where $h'_1 = h_2$ and $h'_2 = h_1$. Thus $\tau^2 h'_i = \xi^i h'_i$, $i = 1, 2$. The action of τ on $S[\tau^2]$ is given by $\tau h'_i = \xi^{2i} h'_i$, $i = 1, 2$. The action of τ on $T_{\chi'_j}$ is defined to be the identity. The weight in $S[\tau^2]$ is given by $\operatorname{wt} h'_i(i/3+n) = -i/3 - n$, $i = 1, 2$ and $\operatorname{wt} 1 = 1/9$. The weight of any element of $T_{\chi'_j}$ is defined to be 0. The weight in $V_L^j(\tau^2)$ is identical with the eigenvalue for the action of the coefficient of z^{-2} in the τ^2-twisted vertex operator $Y^{\tau^2}(\omega, z)$.

By Lemma 3.1, [Dong and Mason 1997, Theorem 4.4], and [Dong and Yamskulna 2002, Theorem 6.14],

$$V_{L^{(0,j)}}(\varepsilon) = \{v \in V_{L^{(0,j)}} \mid \tau v = \xi^{\varepsilon} v\}, \quad j, \varepsilon = 0, 1, 2$$

are inequivalent simple V_L^{τ}-modules. For each of $j = 0, 1, 2$, we have that $V_{L^{(i,j)}}$, $i = a, b, c$ are equivalent simple V_L^{τ}-modules. Moreover, $V_{L^{(c,j)}}$, $j = 0, 1, 2$ are inequivalent simple V_L^{τ}-modules. From [Miyamoto and Tanabe 2004, Theorem 2], it follows that

$$V_L^j(\tau)(\varepsilon) = \{v \in V_L^j(\tau) \mid \tau v = \xi^\varepsilon v\}, \quad j, \varepsilon = 0, 1, 2$$

are inequivalent simple V_L^τ-modules. Similar assertions hold for simple τ^2-twisted modules, namely,

$$V_L^j(\tau^2)(\varepsilon) = \{v \in V_L^j(\tau^2) \mid \tau^2 v = \xi^\varepsilon v\}, \quad j, \varepsilon = 0, 1, 2$$

are inequivalent simple V_L^τ-modules. In this way we obtain 30 simple V_L^τ-modules. These 30 simple V_L^τ-modules are inequivalent by [Miyamoto and Tanabe 2004, Theorem 2]. We summarize the result as follows.

Lemma 3.2. *The following* 30 *simple V_L^τ-modules are inequivalent.*

(1) $V_{L^{(0,j)}}(\varepsilon)$, $j, \varepsilon = 0, 1, 2$,

(2) $V_{L^{(c,j)}}$, $j = 0, 1, 2$,

(3) $V_L^j(\tau)(\varepsilon)$, $j, \varepsilon = 0, 1, 2$,

(4) $V_L^j(\tau^2)(\varepsilon)$, $j, \varepsilon = 0, 1, 2$.

We consider the structure of V_L^τ in detail. Set

$$x(\alpha) = e^{\sqrt{2}\alpha} + e^{-\sqrt{2}\alpha}, \quad y(\alpha) = e^{\sqrt{2}\alpha} - e^{-\sqrt{2}\alpha}, \quad w(\alpha) = \tfrac{1}{2}\alpha(-1)^2 - x(\alpha)$$

for $\alpha \in \{\pm\alpha_0, \pm\alpha_1, \pm\alpha_2\}$ and let

$$\begin{gathered}\omega = \tfrac{1}{6}\big(\alpha_1(-1)^2 + \alpha_2(-1)^2 + \alpha_0(-1)^2\big),\\ \tilde{\omega}^1 = \tfrac{1}{5}\big(w(\alpha_1) + w(\alpha_2) + w(\alpha_0)\big), \qquad \tilde{\omega}^2 = \omega - \tilde{\omega}^1,\\ \omega^1 = \tfrac{1}{4}w(\alpha_1), \qquad \omega^2 = \tilde{\omega}^1 - \omega^1.\end{gathered}$$

Then ω is the Virasoro element of V_L and $\tilde{\omega}^1$, $\tilde{\omega}^2$ are mutually orthogonal conformal vectors of central charge $6/5, 4/5$ respectively. The subalgebra $\mathrm{Vir}(\tilde{\omega}^i)$ generated by $\tilde{\omega}^i$ is isomorphic to the Virasoro vertex operator algebra of given central charge, namely, $\mathrm{Vir}(\tilde{\omega}^1) \cong L(6/5, 0)$ and $\mathrm{Vir}(\tilde{\omega}^2) \cong L(4/5, 0)$. Moreover, $\tilde{\omega}^1$ is a sum of two conformal vectors ω^1 and ω^2 of central charge $1/2$ and $7/10$ respectively and ω^1, ω^2 and $\tilde{\omega}^2$ are mutually orthogonal. Note that $\tilde{\omega}^2$ was denoted by ω^3 in [Dong et al. 2004; Kitazume et al. 2000a; Kitazume et al. 2003; Kitazume et al. 2000b]. Such a decomposition of the Virasoro element of a lattice vertex operator algebra into a sum of mutually orthogonal conformal vectors was first studied in [Dong et al. 1998b].

Set

$$M_k^i = \{v \in V_{L_i} \mid (\tilde{\omega}^2)_1 v = 0\}, \quad W_k^i = \{v \in V_{L_i} \mid (\tilde{\omega}^2)_1 v = \tfrac{2}{5}v\}, \quad i = 0, a, b, c,$$

$$M_t^j = \{v \in V_{L^j} \mid (\omega^1)_1 v = (\omega^2)_1 v = 0\},$$
$$W_t^j = \{v \in V_{L^j} \mid (\omega^1)_1 v = 0,\ (\omega^2)_1 v = \tfrac{3}{5} v\}, \quad j = 0, 1, 2.$$

Then M_k^0 and M_t^0 are simple vertex operator algebras. Moreover, $\{M_k^i,\ W_k^i;\ i = 0, a, b, c\}$ and $\{M_t^j,\ W_t^j;\ j = 0, 1, 2\}$ are complete sets of representatives of isomorphism classes of simple modules for M_k^0 and M_t^0, respectively (see [Kitazume et al. 2000a; Kitazume et al. 2000b; Lam and Yamada 2000]). As $\mathrm{Vir}(\omega^1) \otimes \mathrm{Vir}(\omega^2)$-modules,

$$\begin{aligned}
M_k^0 &\cong \big(L(\tfrac{1}{2}, 0) \otimes L(\tfrac{7}{10}, 0)\big) \oplus \big(L(\tfrac{1}{2}, \tfrac{1}{2}) \otimes L\tfrac{7}{10}, \tfrac{3}{2})\big),\\
M_k^a &\cong M_k^b \cong L(\tfrac{1}{2}, \tfrac{1}{16}) \otimes L(\tfrac{7}{10}, \tfrac{7}{16}),\\
M_k^c &\cong \big(L(\tfrac{1}{2}, \tfrac{1}{2}) \otimes L(\tfrac{7}{10}, 0)\big) \oplus \big(L(\tfrac{1}{2}, 0) \otimes L(\tfrac{7}{10}, \tfrac{3}{2})\big),\\
W_k^0 &\cong \big(L(\tfrac{1}{2}, 0) \otimes L(\tfrac{7}{10}, \tfrac{3}{5})\big) \oplus \big(L(\tfrac{1}{2}, \tfrac{1}{2}) \otimes L(\tfrac{7}{10}, \tfrac{1}{10})\big),\\
W_k^a &\cong W_k^b \cong L(\tfrac{1}{2}, \tfrac{1}{16}) \otimes L(\tfrac{7}{10}, \tfrac{3}{80}),\\
W_k^c &\cong \big(L(\tfrac{1}{2}, \tfrac{1}{2}) \otimes L(\tfrac{7}{10}, \tfrac{3}{5})\big) \oplus \big(L(\tfrac{1}{2}, 0) \otimes L(\tfrac{7}{10}, \tfrac{1}{10})\big),
\end{aligned} \tag{3-2}$$

and as $\mathrm{Vir}(\tilde{\omega}^2)$-modules,

$$\begin{aligned}
M_t^0 &\cong L(\tfrac{4}{5}, 0) \oplus L(\tfrac{4}{5}, 3), &\qquad M_t^1 &\cong M_t^2 \cong L(\tfrac{4}{5}, \tfrac{2}{3}),\\
W_t^0 &\cong L(\tfrac{4}{5}, \tfrac{2}{5}) \oplus L(\tfrac{4}{5}, \tfrac{7}{5}), &\qquad W_t^1 &\cong W_t^2 \cong L(\tfrac{4}{5}, \tfrac{1}{15}).
\end{aligned} \tag{3-3}$$

Furthermore,

$$V_{L^{(i,j)}} \cong (M_k^i \otimes M_t^j) \oplus (W_k^i \otimes W_t^j) \tag{3-4}$$

as $M_k^0 \otimes M_t^0$-modules. In particular,

$$V_L \cong (M_k^0 \otimes M_t^0) \oplus (W_k^0 \otimes W_t^0). \tag{3-5}$$

Note that $M_t^j = \{v \in V_{L^j} \mid (\tilde{\omega}^1)_1 v = 0\}$ and that M_k^0, W_k^0 and M_t^j, $j = 0, 1, 2$ are τ-invariant. However, W_t^j, $j = 0, 1, 2$ are not τ-invariant.

The fusion rules for M_k^0 and M_t^0 were determined in [Lam and Yamada 2000] and [Miyamoto 2001], respectively. They are

$$M_k^i \times M_k^j = M_k^{i+j}, \quad M_k^i \times W_k^j = W_k^{i+j}, \quad W_k^i \times W_k^j = M_k^{i+j} + W_k^{i+j} \tag{3-6}$$

for $i, j = 0, a, b, c$ and

$$M_t^i \times M_t^j = M_t^{i+j}, \quad M_t^i \times W_t^j = W_t^{i+j}, \quad W_t^i \times W_t^j = M_t^{i+j} + W_t^{i+j} \tag{3-7}$$

for $i, j = 0, 1, 2$.

The following two weight-three vectors are important.

$$\begin{aligned}
J &= w(\alpha_1)_0 w(\alpha_2) - w(\alpha_2)_0 w(\alpha_1) \\
&= -\frac{1}{6}\Big(\beta_1(-2)(\beta_2-\beta_0)(-1) + \beta_2(-2)(\beta_0-\beta_1)(-1) + \beta_0(-2)(\beta_1-\beta_2)(-1)\Big) \\
&\qquad - (\beta_2-\beta_0)(-1)y(\alpha_1) - (\beta_0-\beta_1)(-1)y(\alpha_2) - (\beta_1-\beta_2)(-1)y(\alpha_0), \\
K &= -\frac{1}{9}(\beta_1-\beta_2)(-1)(\beta_2-\beta_0)(-1)(\beta_0-\beta_1)(-1) \\
&\qquad + (\beta_2-\beta_0)(-1)x(\alpha_1) + (\beta_0-\beta_1)(-1)x(\alpha_2) + (\beta_1-\beta_2)(-1)x(\alpha_0).
\end{aligned}$$

Let $M(0) = (M_k^0)^\tau = \{u \in M_k^0 \mid \tau u = u\}$. The vertex operator algebra $M(0)$ was studied in [Dong et al. 2004]. Among other things, the classification of simple modules, the rationality and the C_2-cofiniteness for $M(0)$ were established. It is known that $M(0)$ is a W_3 algebra of central charge $6/5$ with the Virasoro element $\tilde{\omega}^1$. In fact, $M(0)$ is generated by $\tilde{\omega}^1$ and J. The following equations hold [Dong et al. 2004, (3.1)].

$$\begin{aligned}
J_5 J &= -84 \cdot \mathbf{1}, \\
J_4 J &= 0, \\
J_3 J &= -420\tilde{\omega}^1, \\
J_2 J &= -210(\tilde{\omega}^1)_0\tilde{\omega}^1, \\
J_1 J &= 9(\tilde{\omega}^1)_0(\tilde{\omega}^1)_0\tilde{\omega}^1 - 240(\tilde{\omega}^1)_{-1}\tilde{\omega}^1, \\
J_0 J &= 22(\tilde{\omega}^1)_0(\tilde{\omega}^1)_0(\tilde{\omega}^1)_0\tilde{\omega}^1 - 120(\tilde{\omega}^1)_0(\tilde{\omega}^1)_{-1}\tilde{\omega}^1.
\end{aligned} \tag{3-8}$$

Let $L^1(n) = (\tilde{\omega}^1)_{n+1}$ and $J(n) = J_{n+2}$ for $n \in \mathbb{Z}$, so that the weight of these operators is $\operatorname{wt} L^1(n) = \operatorname{wt} J(n) = -n$. Then

$$[L^1(m),\ L^1(n)] = (m-n)L^1(m+n) + \frac{m^3-m}{12}\cdot\frac{6}{5}\cdot\delta_{m+n,0}, \tag{3-9}$$

$$[L^1(m),\ J(n)] = (2m-n)J(m+n), \tag{3-10}$$

$$\begin{aligned}
&[J(m),\ J(n)] \\
&\quad = (m-n)\big(22(m+n+2)(m+n+3) + 35(m+2)(n+2)\big)L^1(m+n) \\
&\qquad -120(m-n)\Big(\sum_{k\le -2} L^1(k)L^1(m+n-k) + \sum_{k\ge -1} L^1(m+n-k)L^1(k)\Big) \\
&\qquad -\tfrac{7}{10}m(m^2-1)(m^2-4)\delta_{m+n,0}.
\end{aligned} \tag{3-11}$$

The vertex operator algebra M_t^0 is known as a 3-State Potts model. It is a W_3 algebra of central charge $4/5$ with the Virasoro element $\tilde{\omega}^2$ and is generated by

$\tilde{\omega}^2$ and K. Both of $\tilde{\omega}^2$ and K are fixed by τ, so that τ is the identity on M_t^0. The rationality of M_t^0 was established in [Kitazume et al. 2000b] and the C_2-cofiniteness of M_t^0 follows from [Buhl 2002]. By a direct calculation, we can verify that

$$\begin{aligned}
K_5K &= 104\cdot\mathbf{1},\\
K_4K &= 0,\\
K_3K &= 780\tilde{\omega}^2,\\
K_2K &= 390(\tilde{\omega}^2)_0\tilde{\omega}^2,\\
K_1K &= -27(\tilde{\omega}^2)_0(\tilde{\omega}^2)_0\tilde{\omega}^2+480(\tilde{\omega}^2)_{-1}\tilde{\omega}^2,\\
K_0K &= -46(\tilde{\omega}^2)_0(\tilde{\omega}^2)_0(\tilde{\omega}^2)_0\tilde{\omega}^2+240(\tilde{\omega}^2)_0(\tilde{\omega}^2)_{-1}\tilde{\omega}^2.
\end{aligned}\tag{3-12}$$

Let $L^2(n)=(\tilde{\omega}^2)_{n+1}$ and $K(n)=K_{n+2}$ for $n\in\mathbb{Z}$. Then

$$[L^2(m),\ L^2(n)]=(m-n)L^2(m+n)+\frac{m^3-m}{12}\cdot\frac{4}{5}\cdot\delta_{m+n,0},\tag{3-13}$$

$$[L^2(m),\ K(n)]=(2m-n)K(m+n),\tag{3-14}$$

$$\begin{aligned}
&[K(m), K(n)]\\
&\quad=-(m-n)\big(46(m+n+2)(m+n+3)+65(m+2)(n+2)\big)L^2(m+n)\\
&\qquad+240(m-n)\Bigg(\sum_{k\le -2}L^2(k)L^2(m+n-k)+\sum_{k\ge -1}L^2(m+n-k)L^2(k)\Bigg)\\
&\qquad\qquad+\tfrac{13}{15}m(m^2-1)(m^2-4)\delta_{m+n,0}.
\end{aligned}\tag{3-15}$$

Remark 3.3. Let $L_n=L^1(n)$, $W_n=\sqrt{-1/210}J(n)$, and $c=6/5$. Then the commutation relations above coincide with (2.1) and (2.2) of [Bouwknegt et al. 1996]. The same commutation relations also hold if we set $L_n=L^2(n)$, $W_n=K(n)/\sqrt{390}$, and $c=4/5$.

Let us review the 20 inequivalent simple $M(0)$-modules studied in [Dong et al. 2004]. Among those simple $M(0)$-modules, eight of them appear in simple M_k^0-modules, namely,

$$M(\varepsilon)=\{u\in M_k^0\mid \tau u=\xi^\varepsilon u\},\qquad W(\varepsilon)=\{u\in W_k^0\mid \tau u=\xi^\varepsilon u\}$$

for $\varepsilon=0,1,2$, M_k^c and W_k^c. The remaining 12 simple $M(0)$-modules appear in simple τ-twisted or τ^2-twisted V_L-modules. Let

$$\begin{aligned}
M_T(\tau)(\varepsilon)&=\{u\in V_L^0(\tau)\mid (\tilde{\omega}^2)_1u=0,\ \tau u=\xi^\varepsilon u\},\\
W_T(\tau)(\varepsilon)&=\{u\in V_L^0(\tau)\mid (\tilde{\omega}^2)_1u=\tfrac{2}{5}u,\ \tau u=\xi^\varepsilon u\}.
\end{aligned}$$

Then $M_T(\tau)(\varepsilon)$, $W_T(\tau)(\varepsilon)$, $\varepsilon = 0, 1, 2$ are inequivalent simple $M(0)$-modules. Similarly,

$$M_T(\tau^2)(\varepsilon) = \{u \in V_L^0(\tau^2) \mid (\tilde{\omega}^2)_1 u = 0,\ \tau^2 u = \xi^\varepsilon u\},$$
$$W_T(\tau^2)(\varepsilon) = \{u \in V_L^0(\tau^2) \mid (\tilde{\omega}^2)_1 u = \tfrac{2}{5} u,\ \tau^2 u = \xi^\varepsilon u\}$$

for $\varepsilon = 0, 1, 2$ are inequivalent simple $M(0)$-modules. In [Dong et al. 2004], it was shown that $M(\varepsilon)$, $W(\varepsilon)$, M_k^c, W_k^c, $M_T(\tau)(\varepsilon)$, $W_T(\tau)(\varepsilon)$, $M_T(\tau^2)(\varepsilon)$, and $W_T(\tau^2)(\varepsilon)$, $\varepsilon = 0, 1, 2$ form a complete set of representatives of isomorphism classes of simple $M(0)$-modules.

Let us describe the structure of the fixed point subalgebra V_L^τ. By the definition of $M(0)$ and M_t^0, we see that $V_L^\tau \supset M(0) \otimes M_t^0$. Since both of $M(0)$ and M_t^0 are rational, $M(0) \otimes M_t^0$ is also rational. Thus $V_L(\varepsilon) = \{u \in V_L \mid \tau u = \xi^\varepsilon u\}$, $\varepsilon = 0, 1, 2$ can be decomposed into a direct sum of simple modules for $M(0) \otimes M_t^0$. Any simple module for $M(0) \otimes M_t^0$ is of the form $A \otimes B$, where A and B are simple modules for $M(0)$ and M_t^0, respectively. By (3-5), it follows that $B \cong M_t^0$ or W_t^0. Moreover, $V_L(\varepsilon)$ contains the simple $M(0)$-modules $M(\varepsilon)$ and $W(\varepsilon)$. The eigenvalues of $(\tilde{\omega}^1)_1$ in $M(\varepsilon)$ (resp. $W(\varepsilon)$) are integers (resp. of the form $3/5 + n$, $n \in \mathbb{Z}$), while the eigenvalues of $(\tilde{\omega}^2)_1$ in M_t^0 (resp. W_t^0) are integers (resp. of the form $2/5 + n$, $n \in \mathbb{Z}$). Since the eigenvalues of $\omega_1 = (\tilde{\omega}^1)_1 + (\tilde{\omega}^2)_1$ in V_L are integers, we conclude that

$$V_L(\varepsilon) \cong (M(\varepsilon) \otimes M_t^0) \oplus (W(\varepsilon) \otimes W_t^0) \tag{3-16}$$

as $M(0) \otimes M_t^0$-modules, $\varepsilon = 0, 1, 2$. In particular,

$$V_L^\tau \cong (M(0) \otimes M_t^0) \oplus (W(0) \otimes W_t^0). \tag{3-17}$$

From now on we set $M^0 = M(0) \otimes M_t^0$ and $W^0 = W(0) \otimes W_t^0$. Thus $V_L^\tau = V_L(0) \cong M^0 \oplus W^0$. Let

$$P = y(\alpha_1) + y(\alpha_2) + y(\alpha_0).$$

Then we can verify that $(\tilde{\omega}^1)_n P = (\tilde{\omega}^2)_n P = 0$ for $n \geq 2$, $(\tilde{\omega}^1)_1 P = (8/5)P$, and $(\tilde{\omega}^2)_1 P = (2/5)P$. Moreover, $J_n P = K_n P = 0$ for $n \geq 2$. Thus W^0 is a simple M^0-module with P a highest-weight vector of weight $(8/5, 2/5)$. The vertex operator algebra V_L^τ is generated by $\tilde{\omega}^1$, $\tilde{\omega}^2$, J, K and P.

Theorem 3.4. *V_L^τ is a simple C_2-cofinite vertex operator algebra.*

Proof. We know that $M(0)$ and M_t^0 are C_2-cofinite. Thus M^0 is also C_2-cofinite. Since W^0 is generated by P as an M^0-module, it follows from [Buhl 2002] that V_L^τ is C_2-cofinite. By [Dong and Mason 1997, Theorem 4.4], V_L^τ is simple. □

Following the outline of the argument in [Dong et al. 2004; Kitazume et al. 2003], we discuss the structure of the simple τ-twisted V_L-modules $V_L^j(\tau)$, $j = 0, 1, 2$ as τ-twisted $M_k^0 \otimes M_t^0$-modules. Furthermore, we correct an error in [Dong et al. 2004; Kitazume et al. 2003] concerning a decomposition of $V_L^j(\tau)$ for $j = 1, 2$. We first consider $V_L^0(\tau)$. Let $0 \neq v \in T_{\chi_0}$ and 1 be the identity of $S[\tau]$. Then $1 \otimes v \in S[\tau] \otimes T_{\chi_0} = V_L^0(\tau)$. Since $M_t^0 \subset V_L^\tau$, we can decompose $V_L^0(\tau)$ into a direct sum of simple M_t^0-modules. By a direct calculation, we can verify that

$$(\tilde{\omega}^2)_1(1 \otimes v) = 0, \qquad (\tilde{\omega}^2)_1(h_2(-\tfrac{1}{3}) \otimes v) = \tfrac{2}{5} h_2(-\tfrac{1}{3}) \otimes v.$$

Thus we see that M_t^0 and W_t^0 appear as direct summands. Since $V_L^0(\tau)$ is simple as a τ-twisted V_L-module, (3-5) and the fusion rule $W_t^0 \times W_t^0 = M_t^0 + W_t^0$ (see (3-7)) imply that any simple M_t^0-submodule of $V_L^0(\tau)$ is isomorphic to M_t^0 or W_t^0. Hence

$$V_L^0(\tau) \cong (M_T^0(\tau) \otimes M_t^0) \oplus (W_T^0(\tau) \otimes W_t^0) \tag{3-18}$$

as τ-twisted $M_k^0 \otimes M_t^0$-modules, where

$$M_T^0(\tau) = \{u \in V_L^0(\tau) \mid (\tilde{\omega}^2)_1 u = 0\},$$
$$W_T^0(\tau) = \{u \in V_L^0(\tau) \mid (\tilde{\omega}^2)_1 u = \tfrac{2}{5} u\}.$$

The τ-twisted M_k^0-modules $M_T^0(\tau)$ and $W_T^0(\tau)$ are simple. Indeed, if N is a τ-twisted M_k^0-submodule of $M_T^0(\tau)$, then $N \otimes M_t^0$ is a τ-twisted $M_k^0 \otimes M_t^0$-submodule of $M_T^0(\tau) \otimes M_t^0$. By (2-6), $V_L \cdot (N \otimes M_t^0) = \operatorname{span}\{a_n(N \otimes M_t^0) \mid a \in V_L, n \in \mathbb{Q}\}$ is a τ-twisted V_L-submodule of $V_L^0(\tau)$. The fusion rule $W_t^0 \times M_t^0 = W_t^0$ and (3-5) imply that $V_L \cdot (N \otimes M_t^0)$ is contained in $(N \otimes M_t^0) \oplus (W_T^0(\tau) \otimes W_t^0)$. Since $V_L^0(\tau)$ is a simple τ-twisted V_L-module, we conclude that $M_T^0(\tau)$ is a simple τ-twisted M_k^0-module.

Because of the fusion rule $W_t^0 \times W_t^0 = M_t^0 + W_t^0$, we can not apply a similar argument to $W_T^0(\tau)$. Note that there are at most two inequivalent simple τ-twisted M_k^0-modules by [Dong et al. 2004, Lemma 4.1] and [Dong et al. 2000, Theorem 10.2]. Note also that a weight in $M_T^0(\tau)$ or in $W_T^0(\tau)$ means an eigenvalue of $(\tilde{\omega}^1)_1$. First several terms of the characters of $M_T^0(\tau)$ and $W_T^0(\tau)$ can be calculated easily from (3-18) (see [Dong et al. 2004]).

$$\operatorname{ch} M_T^0(\tau) = q^{1/9} + q^{1/9+2/3} + q^{1/9+1} + q^{1/9+4/3} + \cdots,$$
$$\operatorname{ch} W_T^0(\tau) = q^{2/45} + q^{2/45+1/3} + q^{2/45+2/3} + q^{2/45+1} + \cdots.$$

Suppose $W_T^0(\tau)$ is not a simple τ-twisted M_k^0-module. Let N be the τ-twisted M_k^0-submodule of $W_T^0(\tau)$ generated by the top level of $W_T^0(\tau)$. Then the top level of N is a one dimensional space of weight $2/45$. If N is not a simple τ-twisted M_k^0-module, then the sum U of all proper τ-twisted M_k^0-submodules of N is a unique maximal τ-twisted M_k^0-submodule of N. The quotient N/U is a simple τ-twisted M_k^0-module whose top level is of weight $2/45$. Denote the top level of U by U_λ, where the weight λ is $2/45+n/3$ for some $1 \le n \in \mathbb{Z}$. Consider the τ-twisted Zhu algebra $A_\tau(M_k^0)$ of M_k^0. Since U_λ is a finite dimensional $A_\tau(M_k^0)$-module, we can choose a simple $A_\tau(M_k^0)$-submodule S of U_λ. By [Dong et al. 1998a, Proposition 5.4 and Theorem 7.2], there is a simple $\frac{1}{3}\mathbb{N}$-graded weak τ-twisted M_k^0-module R with top level R_λ being isomorphic to S as an $A_\tau(M_k^0)$-module. It follows from [Yamauchi 2004, Corollary 3.8] that R is in fact a simple τ-twisted M_k^0-module. Here we note that M_k^0 is C_2-cofinite and of CFT type by its structure (3-2). Since the top levels of $M_T^0(\tau)$, N/U, and R have different weight, they are inequivalent simple τ-twisted M_k^0-modules. If N is a simple τ-twisted M_k^0-module, then it is not equal to $W_T^0(\tau)$ by our assumption. The quotient $W_T^0(\tau)/N$ is a τ-twisted M_k^0-module and the weight of its top level, say μ is $2/45+m/3$ for some $1 \le m \in \mathbb{Z}$. By a similar argument as above, we see that there is a simple τ-twisted M_k^0-module whose top level is of weight μ. Hence we have three inequivalent simple τ-twisted M_k^0-modules in both cases. This contradicts the fact that there are at most two inequivalent simple τ-twisted M_k^0-modules. Thus $W_T^0(\tau)$ is a simple τ-twisted M_k^0-module.

Next, let $0 \neq v \in T_{\chi_j}$, $j=1,2$. From the definition of $V_L^j(\tau)$ in [Dong et al. 2004; Kitazume et al. 2003], we can calculate that

$$(\tilde{\omega}^2)_1(1 \otimes v) = \tfrac{1}{15}(1 \otimes v), \qquad (\tilde{\omega}^2)_1 u^j = \tfrac{2}{3}u^j,$$

where $u^j = h_1(-\frac{2}{3}) \otimes v - (-1)^j \sqrt{-3}\, h_2(-\frac{1}{3})^2 \otimes v$. Thus M_t^1 or M_t^2 and W_t^1 or W_t^2 appear as M_t^0-submodules of $V_L^j(\tau)$. In order to distinguish M_t^1 and M_t^2 (resp. W_t^1 and W_t^2), we need to know the action of K_2 on these vectors (see [Kitazume et al. 2000b]). By a direct calculation, we can verify that

$$K_2(1 \otimes v) = -(-1)^j \tfrac{2}{9}(1 \otimes v), \qquad K_2 u^j = (-1)^j \tfrac{52}{9} u^j.$$

Hence M_t^{3-j} and W_t^{3-j} appear in $V_L^j(\tau)$ for $j=1,2$. Let

$$M_T^j(\tau) = \{u \in V_L^j(\tau) \mid (\tilde{\omega}^2)_1 u = \tfrac{2}{3}u\},$$
$$W_T^j(\tau) = \{u \in V_L^j(\tau) \mid (\tilde{\omega}^2)_1 u = \tfrac{1}{15}u\}, \quad j=1,2.$$

Then, $V_L^j(\tau) \cong (M_T^j(\tau) \otimes M_t^{3-j}) \oplus (W_T^j(\tau) \otimes W_t^{3-j})$ as τ-twisted $M_k^0 \otimes M_t^0$-modules for $j=1,2$. Moreover, $M_T^j(\tau)$ and $W_T^j(\tau)$, $j=1,2$ are simple τ-twisted M_k^0-modules.

Recall that there are at most two inequivalent simple τ-twisted M_k^0-modules. Looking at the smallest weight of $M_T^j(\tau)$ and $W_T^j(\tau)$, we see that the $M_T^j(\tau)$, $j=0,1,2$ are equivalent, and the $W_T^j(\tau)$, $j=0,1,2$ are equivalent, but $M_T^0(\tau)$ and

$W_T^0(\tau)$ are not equivalent. For simplicity, set $M_T(\tau) = M_T^0(\tau)$ and $W_T(\tau) = W_T^0(\tau)$. Then

$$V_L^0(\tau) \cong (M_T(\tau) \otimes M_t^0) \oplus (W_T(\tau) \otimes W_t^0),$$
$$V_L^j(\tau) \cong (M_T(\tau) \otimes M_t^{3-j}) \oplus (W_T(\tau) \otimes W_t^{3-j}), \quad j = 1, 2, \tag{3-19}$$

as τ-twisted $M_k^0 \otimes M_t^0$-modules.

The structure of the simple τ^2-twisted V_L-module $V_L^j(\tau^2)$, $j = 0, 1, 2$ as a τ^2-twisted $M_k^0 \otimes M_t^0$-module is similar to that of the case for $V_L^j(\tau)$. Let $0 \neq v \in T_{\chi_0'}$ and let 1 be the identity of $S[\tau^2]$. Then

$$(\tilde{\omega}^2)_1(1 \otimes v) = 0, \qquad (\tilde{\omega}^2)_1(h_2'(-\tfrac{1}{3}) \otimes v) = \tfrac{2}{5} h_2'(-\tfrac{1}{3}) \otimes v$$

and so

$$V_L^0(\tau^2) \cong (M_T^0(\tau^2) \otimes M_t^0) \oplus (W_T^0(\tau^2) \otimes W_t^0)$$

as τ^2-twisted $M_k^0 \otimes M_t^0$-modules, where

$$M_T^0(\tau^2) = \{u \in V_L^0(\tau^2) \mid (\tilde{\omega}^2)_1 u = 0\},$$
$$W_T^0(\tau^2) = \{u \in V_L^0(\tau^2) \mid (\tilde{\omega}^2)_1 u = \tfrac{2}{5} u\}.$$

By a similar argument as in the τ-twisted case, we can show that $M_T^0(\tau^2)$ and $W_T^0(\tau^2)$ are inequivalent simple τ^2-twisted M_k^0-modules.

Take a nonzero v in $T_{\chi_j'}$, $j = 1, 2$. Then

$$(\tilde{\omega}^2)_1(1 \otimes v) = \tfrac{1}{15}(1 \otimes v), \qquad (\tilde{\omega}^2)_1 v^j = \tfrac{2}{3} v^j,$$

where $v^j = h_1'(-\tfrac{2}{3}) \otimes v - (-1)^j \sqrt{-3}\, h_2'(-\tfrac{1}{3})^2 \otimes v$. Furthermore,

$$K_2(1 \otimes v) = (-1)^j \tfrac{2}{9}(1 \otimes v), \qquad K_2 v^j = -(-1)^j \tfrac{52}{9} v^j.$$

Hence $V_L^j(\tau^2) \cong (M_T^j(\tau^2) \otimes M_t^j) \oplus (W_T^j(\tau^2) \otimes W_t^j)$ as τ^2-twisted $M_k^0 \otimes M_t^0$-modules for $j = 1, 2$, where

$$M_T^j(\tau^2) = \{u \in V_L^j(\tau^2) \mid (\tilde{\omega}^2)_1 u = \tfrac{2}{3} u\},$$
$$W_T^j(\tau^2) = \{u \in V_L^j(\tau^2) \mid (\tilde{\omega}^2)_1 u = \tfrac{1}{15} u\}, \quad j = 1, 2.$$

As in the τ-twisted case, the $M_T^j(\tau^2)$, $j = 0, 1, 2$ are equivalent and the $W_T^j(\tau^2)$, $j = 0, 1, 2$ are equivalent. Set $M_T(\tau^2) = M_T^0(\tau^2)$ and $W_T(\tau^2) = W_T^0(\tau^2)$. Then

$$V_L^j(\tau^2) \cong (M_T(\tau^2) \otimes M_t^j) \oplus (W_T(\tau^2) \otimes W_t^j), \quad j = 0, 1, 2, \tag{3-20}$$

as τ^2-twisted $M_k^0 \otimes M_t^0$-modules.

Remark 3.5. The weight-three vector K was denoted by different symbols in previous papers, namely, v_t, v^3, and q were used in [Dong et al. 2004], [Kitazume et al. 2003], and [Kitazume et al. 2000b], respectively. They are related as follows: $K = -2\sqrt{2}v_t = -2\sqrt{2}v^3 = 2\sqrt{2}q$. Thus, in the proof of [Kitazume et al. 2003, Proposition 6.8] $(v^3)_2$ should act on the top level of $V_L^j(\tau)$ as a scalar multiple of $(-1)^j/9\sqrt{2}$ for $j = 1, 2$. Moreover, (6.46) of [Kitazume et al. 2003] and the equation for $V_L^j(\tau)$ on page 265 of [Dong et al. 2004] should be replaced with Equation (3-19). This correction does not affect the results in the latter paper. However, certain changes are necessary in [Kitazume et al. 2003] along with the correction.

Note that

$$M_T(\tau^i)(\varepsilon) = \{u \in M_T(\tau^i) \mid \tau^i u = \xi^\varepsilon u\},$$
$$W_T(\tau^i)(\varepsilon) = \{u \in W_T(\tau^i) \mid \tau^i u = \xi^\varepsilon u\}$$

for $i = 1, 2$, $\varepsilon = 0, 1, 2$. Another notation was used in [Kitazume et al. 2003], namely,

$$M_T(\tau^i)^\varepsilon = \bigoplus_{n\in 1/9+\varepsilon/3+\mathbb{Z}} (M_T(\tau^i))_n, \quad W_T(\tau^i)^\varepsilon = \bigoplus_{n\in 2/45+\varepsilon/3+\mathbb{Z}} (W_T(\tau^i))_n,$$

where U_n denotes the eigenspace of U with eigenvalue n for $(\tilde{\omega}^1)_1$. The two sets of notation are related by

$$M_T(\tau^i)^\varepsilon = M_T(\tau^i)(2\varepsilon), \quad W_T(\tau^i)^\varepsilon = W_T(\tau^i)(2\varepsilon - 1). \tag{3-21}$$

Likewise,

$$\big(V_L^{T_{\chi_j}}(\tau)\big)^\varepsilon = \bigoplus_{n\in 1/9+\varepsilon/3+\mathbb{Z}} \big(V_L^{T_{\chi_j}}(\tau)\big)_n, \qquad \big(V_L^{T_{\chi_j}}(\tau^2)\big)^\varepsilon = \bigoplus_{n\in 1/9+\varepsilon/3+\mathbb{Z}} \big(V_L^{T_{\chi_j}}(\tau^2)\big)_n$$

of [Kitazume et al. 2003, (7.16)] are denoted here by

$$(V_L^j(\tau))^\varepsilon = V_L^j(\tau)(2\varepsilon), \qquad (V_L^j(\tau^2))^\varepsilon = V_L^j(\tau^2)(2\varepsilon) \tag{3-22}$$

for $j = 0, 1, 2$ and $\varepsilon = 0, 1, 2$, where U_n is the eigenspace of U with eigenvalue n for ω_1.

By (3-3), the minimal eigenvalues of $(\tilde{\omega}^2)_1$ on M_t^0 and W_t^0 are 0 and $2/5$, respectively, while those on M_t^j and W_t^j, $j = 1, 2$, are $2/3$ and $1/15$, respectively. Hence it follows from (3-19) that

$$\begin{aligned} &(V_L^0(\tau))^\varepsilon \cong (M_T(\tau)^\varepsilon \otimes M_t^0) \oplus (W_T(\tau)^{\varepsilon-1} \otimes W_t^0), \\ &(V_L^j(\tau))^\varepsilon \cong (M_T(\tau)^{\varepsilon+1} \otimes M_t^{3-j}) \oplus (W_T(\tau)^\varepsilon \otimes W_t^{3-j}), \quad j = 1, 2, \end{aligned} \tag{3-23}$$

as M^0-modules for $\varepsilon = 0, 1, 2$, where $M^0 = M(0) \otimes M_t^0$. Similarly,

$$
\begin{aligned}
(V_L^0(\tau^2))^\varepsilon &\cong (M_T(\tau^2)^\varepsilon \otimes M_t^0) \oplus (W_T(\tau^2)^{\varepsilon-1} \otimes W_t^0), \\
(V_L^j(\tau^2))^\varepsilon &\cong (M_T(\tau^2)^{\varepsilon+1} \otimes M_t^j) \oplus (W_T(\tau^2)^\varepsilon \otimes W_t^j), \quad j = 1, 2,
\end{aligned}
\tag{3-24}
$$

as M^0-modules for $\varepsilon = 0, 1, 2$ (see [Kitazume et al. 2003, (7.17)]).

The following fusion rules of simple $M(0)$-modules will be necessary for the study of simple V_L^τ-modules.

$$
\begin{aligned}
W(0) \times M_k^c &= W_k^c, \\
W(0) \times W_k^c &= M_k^c + W_k^c, \\
W(0) \times M(\varepsilon) &= W(\varepsilon), \\
W(0) \times W(\varepsilon) &= M(\varepsilon) + W(\varepsilon), \\
W(0) \times M_T(\tau^i)(\varepsilon) &= W_T(\tau^i)(\varepsilon), \\
W(0) \times W_T(\tau^i)(\varepsilon) &= M_T(\tau^i)(\varepsilon) + W_T(\tau^i)(\varepsilon)
\end{aligned}
\tag{3-25}
$$

for $i = 1, 2$ and $\varepsilon = 0, 1, 2$. In fact, the first four fusion rules, that is, the fusion rules among simple $M(0)$-modules appearing in untwisted simple V_L-modules, can be found in [Tanabe 2005]. The last two fusion rules involve simple $M(0)$-modules that appear in τ^i-twisted simple V_L-modules. Their proofs can be found in the Appendix.

Fusion rules possess certain symmetries. Let M^i, $i = 1, 2, 3$ be modules for a vertex operator algebra V. Then by [Frenkel et al. 1993, Propositions 5.4.7 and 5.5.2]

$$
\dim I_V\begin{pmatrix} M^3 \\ M^1 \; M^2 \end{pmatrix} = \dim I_V\begin{pmatrix} M^3 \\ M^2 \; M^1 \end{pmatrix} = \dim I_V\begin{pmatrix} (M^2)' \\ M^1 \; (M^3)' \end{pmatrix},
$$

where $(M^i)'$ is the contragredient module of M^i. Recall that the contragredient module $(U', Y_{U'})$ of a V-module (U, Y_U) is defined as follows. As a vector space $U' = \bigoplus_n (U_n)^*$ is the restricted dual of U and $Y_{U'}(\,\cdot\,, z)$ is determined by

$$
\langle Y_{U'}(a, z)v, u\rangle = \langle v, Y_U(e^{zL(1)}(-z^{-2})^{L(0)}a, z^{-1})u\rangle
$$

for $a \in V$, $u \in U$, and $v \in U'$.

In our case $M(0)$ is generated by the Virasoro element $\tilde{\omega}^1$ and the weight-three vector J. Moreover, $\langle L^1(0)v, u\rangle = \langle v, L^1(0)u\rangle$ and $\langle J(0)v, u\rangle = -\langle v, J(0)u\rangle$. Since the 20 simple $M(0)$-modules are distinguished by the action of $L^1(0)$ and $J(0)$ on their top levels, we know from [Dong et al. 2004, Tables 1, 3, and 4] that the contragredient modules of the simple $M(0)$-modules are as follows.

$$M(\varepsilon)' \cong M(2\varepsilon), \quad W(\varepsilon)' \cong W(2\varepsilon), \quad \varepsilon = 0, 1, 2,$$
$$(M_k^c)' \cong M_k^c, \quad (W_k^c)' \cong W_k^c,$$
$$M_T(\tau)(\varepsilon)' \cong M_T(\tau^2)(\varepsilon), \quad W_T(\tau)(\varepsilon)' \cong W_T(\tau^2)(\varepsilon), \quad \varepsilon = 0, 1, 2$$

(see also [Dong et al. 1998a, Lemma 3.7] and [Tanabe 2005, Section 4.2]).

4. Structure of simple modules

Recall that $V_L^\tau = V_L(0) = M^0 \oplus W^0$ with $M^0 = M(0) \otimes M_t^0$ and $W^0 = W(0) \otimes W_t^0$. In this section we study the structure of the 30 known simple V_L^τ-modules listed in Lemma 3.2. We discuss decompositions of these simple modules as modules for M^0. Those decompositions have been obtained in [Kitazume et al. 2003]. We review them briefly. (Some corrections are needed in that paper; see Remark 3.5.)

A vector in a V_L^τ-module is said to be of weight h if it is an eigenvector for $L(0) = \omega_1$ with eigenvalue h. We calculate the action of $(\tilde{\omega}^1)_1$, $(\tilde{\omega}^2)_1$, J_2, K_2, P_1, $(J_1 P)_2$, and $(K_1 P)_2$ on the top levels of the 30 known simple V_L^τ-modules. Recall that the top level of a module means the homogeneous subspace of the module of smallest weight. The calculation is accomplished directly from the definition of untwisted or twisted vertex operators associated with the lattice L and the automorphisms τ and τ^2 (see [Dong and Lepowsky 1996; Frenkel et al. 1988; Lepowsky and Li 2004]). The results in this section will be used to determine the Zhu algebra $A(V_L^\tau)$ of V_L^τ in Section 5.

The vectors $J_1 P$ and $K_1 P$ are of weight 3. Their precise form in terms of the lattice vertex operator algebra V_L is as follows.

$$\begin{aligned}
J_1 P &= 2\beta_1(-1)^3 + 3\beta_1(-1)^2\beta_2(-1) - 3\beta_1(-1)\beta_2(-1)^2 - 2\beta_2(-1)^3 \\
&\qquad - 4\big((\beta_2 - \beta_0)(-1)x(\alpha_1) + (\beta_0 - \beta_1)(-1)x(\alpha_2) + (\beta_1 - \beta_2)(-1)x(\alpha_0)\big) \\
&= \tfrac{13}{9}\big(2\beta_1(-1)^3 + 3\beta_1(-1)^2\beta_2(-1) - 3\beta_1(-1)\beta_2(-1)^2 - 2\beta_2(-1)^3\big) - 4K, \\
K_1 P &= 3\big(\beta_1(-2)\beta_2(-1) - \beta_2(-2)\beta_1(-1)\big) \\
&\qquad - \big((\beta_2 - \beta_0)(-1)y(\alpha_1) + (\beta_0 - \beta_1)(-1)y(\alpha_2) + (\beta_1 - \beta_2)(-1)y(\alpha_0)\big) \\
&= \tfrac{7}{2}\big(\beta_1(-2)\beta_2(-1) - \beta_2(-2)\beta_1(-1)\big) + J.
\end{aligned}$$

The simple module $V_L(0)$. $V_L(0) = M^0 \oplus W^0$ as M^0-modules. The top level of $V_L(0)$ is $\mathbb{C}\mathbf{1}$. By a property of the vacuum vector, all of $(\tilde{\omega}^1)_1$, $(\tilde{\omega}^2)_1$, J_2, K_2, P_1, $(J_1 P)_2$, and $(K_1 P)_2$ act as 0 on $\mathbb{C}\mathbf{1}$.

The simple module $V_L(\varepsilon)$, $\varepsilon = 1, 2$. By (3-16), we have

$$V_L(\varepsilon) \cong (M(\varepsilon) \otimes M_t^0) \oplus (W(\varepsilon) \otimes W_t^0)$$

as M^0-modules for $\varepsilon=1,2$. The top level of $V_L(\varepsilon)$ is $\mathbb{C}\mathbf{v}^{2,\varepsilon}$, where $\mathbf{v}^{2,\varepsilon}=\alpha_1(-1)-\xi^\varepsilon\alpha_2(-1)\in W(\varepsilon)\otimes W_t^0$. We have

$$(\tilde{\omega}^1)_1\mathbf{v}^{2,\varepsilon}=\tfrac{3}{5}\mathbf{v}^{2,\varepsilon},\quad (\tilde{\omega}^2)_1\mathbf{v}^{2,\varepsilon}=\tfrac{2}{5}\mathbf{v}^{2,\varepsilon},\quad J_2\mathbf{v}^{2,\varepsilon}=-(-1)^\varepsilon 2\sqrt{-3}\,\mathbf{v}^{2,\varepsilon},$$
$$K_2\mathbf{v}^{2,\varepsilon}=0,\quad P_1\mathbf{v}^{2,\varepsilon}=0,\quad (J_1P)_2\mathbf{v}^{2,\varepsilon}=0,\quad (K_1P)_2\mathbf{v}^{2,\varepsilon}=(-1)^\varepsilon 12\sqrt{-3}\,\mathbf{v}^{2,\varepsilon}.$$

The simple module $V_{L^{(0,j)}}(0)$, $j=1,2$. For $j=1,2$, (3-4) implies that $V_{L^{(0,j)}}$ is a direct sum of simple M^0-modules of the form $A\otimes B$, where A is a simple $M(0)$-module and B is a simple M_t^0-module isomorphic to M_t^j or W_t^j. For convenience, set $U^j(\varepsilon)=V_{L^{(0,j)}}(\varepsilon)$, $j=1,2$, $\varepsilon=0,1,2$. Let

$$\mathbf{v}^{3,j}=e^{(-1)^j(\beta_1-\beta_2)/3}+e^{(-1)^j(\beta_2-\beta_0)/3}+e^{(-1)^j(\beta_0-\beta_1)/3}.$$

Then $\mathbf{v}^{3,j}\in U^j(0)$. Moreover, $(\omega^1)_1\mathbf{v}^{3,j}=(\omega^2)_1\mathbf{v}^{3,j}=0$ and $(\tilde{\omega}^2)_1\mathbf{v}^{3,j}=(2/3)\mathbf{v}^{3,j}$. Hence $\mathbf{v}^{3,j}\in M_t^j$ and $U^j(0)$ contains an M_t^0-submodule isomorphic to M_t^j. By the fusion rule $M_t^j\times W_t^0=W_t^j$ of M_t^0-modules and [Dong and Lepowsky 1996, Proposition 11.9], $U^j(0)$ contains an M_t^0-submodule isomorphic to W_t^j also. Thus $U^j(0)$ contains simple M^0-submodules of the form $A\otimes M_t^j$ and $A'\otimes W_t^j$ for some simple $M(0)$-modules A and A'.

The minimal weight of $V_{L^{(0,j)}}$ is $2/3$. Its weight subspace is of dimension 3 and spanned by $e^{(-1)^j(\beta_1-\beta_2)/3}$, $e^{(-1)^j(\beta_2-\beta_0)/3}$, and $e^{(-1)^j(\beta_0-\beta_1)/3}$. Thus the weight-$2/3$ subspace of $U^j(0)$ is $\mathbb{C}\mathbf{v}^{3,j}$. Since $(\tilde{\omega}^1)_1\mathbf{v}^{3,j}=0$ and since only $M(0)$ is the simple $M(0)$-module whose minimal weight (= eigenvalue of $(\tilde{\omega}^1)_1$) is 0 by [Dong et al. 2004], we conclude that $U^j(0)$ contains a simple M^0-submodule isomorphic to $M(0)\otimes M_t^j$.

The minimal eigenvalue of $(\tilde{\omega}^2)_1$ in W_t^j is $1/15$. Thus the eigenvalues of $(\tilde{\omega}^1)_1$ on A' must be of the form $3/5+n$, $n\in\mathbb{Z}$. By [Dong et al. 2004], only $W(0)$, $W(1)$, $W(2)$ are the simple $M(0)$-modules whose weights are of this form. The minimal weight of these simple modules are $8/5$, $3/5$ and $3/5$, respectively. Since the weight-$2/3$ subspace of $U^j(0)$ is one dimensional, we see that $U^j(0)$ contains a simple M^0-submodule isomorphic to $W(0)\otimes W_t^j$.

From the fusion rules for M_t^0-modules, we obtain the fusion rules

$$(M(\varepsilon)\otimes M_t^0)\times(M(0)\otimes M_t^j)=M(\varepsilon)\otimes M_t^j,$$
$$(W(\varepsilon)\otimes W_t^0)\times(M(0)\otimes M_t^j)=W(\varepsilon)\otimes W_t^j$$

for M^0-modules. Hence $U^j(\varepsilon)\cong(M(\varepsilon)\otimes M_t^j)\oplus(W(\varepsilon)\otimes W_t^j)$ for $j=1,2$ and $\varepsilon=0,1,2$ by (3-4) and (3-16). In particular, $V_{L^{(0,j)}}(0)\cong(M(0)\otimes M_t^j)\oplus(W(0)\otimes W_t^j)$ as M^0-modules, $j=1,2$. The top level of $V_{L^{(0,j)}}(0)$ is $\mathbb{C}\mathbf{v}^{3,j}\subset M(0)\otimes M_t^j$. We have

$$(\tilde{\omega}^1)_1\mathbf{v}^{3,j}=0,\quad (\tilde{\omega}^2)_1\mathbf{v}^{3,j}=\tfrac{2}{3}\mathbf{v}^{3,j},\quad J_2\mathbf{v}^{3,j}=0,\quad K_2\mathbf{v}^{3,j}=-(-1)^j\tfrac{52}{9}\mathbf{v}^{3,j},$$
$$P_1\mathbf{v}^{3,j}=0,\quad (J_1P)_2\mathbf{v}^{3,j}=0,\quad (K_1P)_2\mathbf{v}^{3,j}=0.$$

The simple module $V_{L^{(0,j)}}(\varepsilon)$, $j=1,2$, $\varepsilon=1,2$. We have shown above that $V_{L^{(0,j)}}(\varepsilon)\cong(M(\varepsilon)\otimes M_t^j)\oplus(W(\varepsilon)\otimes W_t^j)$ as M^0-modules, $j=1,2$, $\varepsilon=1,2$. The top level of $V_{L^{(0,j)}}(\varepsilon)$ is $\mathbb{C}\mathbf{v}^{4,j,\varepsilon}$, where

$$\mathbf{v}^{4,j,\varepsilon}=e^{(-1)^j(\beta_1-\beta_2)/3}+\xi^{2\varepsilon}e^{(-1)^j(\beta_2-\beta_0)/3}+\xi^{\varepsilon}e^{(-1)^j(\beta_0-\beta_1)/3}\in W(\varepsilon)\otimes W_t^j.$$

We have

$$(\tilde{\omega}^1)_1\mathbf{v}^{4,j,\varepsilon}=\tfrac{3}{5}\mathbf{v}^{4,j,\varepsilon},\quad (\tilde{\omega}^2)_1\mathbf{v}^{4,j,\varepsilon}=\tfrac{1}{15}\mathbf{v}^{4,j,\varepsilon},\quad J_2\mathbf{v}^{4,j,\varepsilon}=-(-1)^{\varepsilon}2\sqrt{-3}\,\mathbf{v}^{4,j,\varepsilon},$$
$$K_2\mathbf{v}^{4,j,\varepsilon}=(-1)^j\tfrac{2}{9}\mathbf{v}^{4,j,\varepsilon},\quad P_1\mathbf{v}^{4,j,\varepsilon}=-(-1)^{j+\varepsilon}\sqrt{-3}\,\mathbf{v}^{4,j,\varepsilon},$$
$$(J_1P)_2\mathbf{v}^{4,j,\varepsilon}=-(-1)^j24\mathbf{v}^{4,j,\varepsilon},\quad (K_1P)_2\mathbf{v}^{4,j,\varepsilon}=-(-1)^{\varepsilon}2\sqrt{-3}\,\mathbf{v}^{4,j,\varepsilon}.$$

The simple module $V_{L^{(c,0)}}$. By (3-4), $V_{L^{(c,0)}}\cong(M_k^c\otimes M_t^0)\oplus(W_k^c\otimes W_t^0)$ as M^0-modules. The top level of $V_{L^{(c,0)}}$ is of dimension 2 with basis $\{\mathbf{v}^{5,1},\mathbf{v}^{5,2}\}$, where $\mathbf{v}^{5,1}=e^{\beta_1/2}-e^{-\beta_1/2}\in M_k^c\otimes M_t^0$, $\mathbf{v}^{5,2}=e^{\beta_1/2}+e^{-\beta_1/2}\in W_k^c\otimes W_t^0$. We have

$$(\tilde{\omega}^1)_1\mathbf{v}^{5,1}=\tfrac{1}{2}\mathbf{v}^{5,1},\quad (\tilde{\omega}^1)_1\mathbf{v}^{5,2}=\tfrac{1}{10}\mathbf{v}^{5,2},\quad (\tilde{\omega}^2)_1\mathbf{v}^{5,1}=0,\quad (\tilde{\omega}^2)_1\mathbf{v}^{5,2}=\tfrac{2}{5}\mathbf{v}^{5,2},$$
$$J_2\mathbf{v}^{5,j}=0,\quad K_2\mathbf{v}^{5,j}=0,\ j=1,2,\quad P_1\mathbf{v}^{5,1}=-\mathbf{v}^{5,2},\quad P_1\mathbf{v}^{5,2}=\mathbf{v}^{5,1},$$
$$(J_1P)_2\mathbf{v}^{5,j}=0,\quad (K_1P)_2\mathbf{v}^{5,j}=0,\ j=1,2.$$

The simple module $V_{L^{(c,j)}}$, $j=1,2$. By (3-4), we have the isomorphism $V_{L^{(c,j)}}\cong(M_k^c\otimes M_t^j)\oplus(W_k^c\otimes W_t^j)$ as M^0-modules, $j=1,2$. The top level of $V_{L^{(c,j)}}$ is $\mathbb{C}\mathbf{v}^{6,j}$, where $\mathbf{v}^{6,j}=e^{-(-1)^j(\beta_2-\beta_0)/6}\in W_k^c\otimes W_t^j$. We have

$$(\tilde{\omega}^1)_1\mathbf{v}^{6,j}=\tfrac{1}{10}\mathbf{v}^{6,j},\quad (\tilde{\omega}^2)_1\mathbf{v}^{6,j}=\tfrac{1}{15}\mathbf{v}^{6,j},\quad J_2\mathbf{v}^{6,j}=0,\quad K_2\mathbf{v}^{6,j}=(-1)^j\tfrac{2}{9}\mathbf{v}^{6,j},$$
$$P_1\mathbf{v}^{6,j}=0,\quad (J_1P)_2\mathbf{v}^{6,j}=(-1)^j2\mathbf{v}^{6,j},\quad (K_1P)_2\mathbf{v}^{6,j}=0.$$

The simple module $V_L^0(\tau)(0)$. By (3-23), we have the isomorphism $V_L^0(\tau)(0)\cong(M_T(\tau)(0)\otimes M_t^0)\oplus(W_T(\tau)(0)\otimes W_t^0)$ as M^0-modules. The top level of $V_L^0(\tau)(0)$ is $\mathbb{C}\mathbf{v}^7$, where $\mathbf{v}^7=1\otimes v\in M_T(\tau)(0)\otimes M_t^0$ and $0\neq v\in T_{\chi_0}$. We have

$$(\tilde{\omega}^1)_1\mathbf{v}^7=\tfrac{1}{9}\mathbf{v}^7,\quad (\tilde{\omega}^2)_1\mathbf{v}^7=0,\quad J_2\mathbf{v}^7=\tfrac{14}{81}\sqrt{-3}\,\mathbf{v}^7,\quad K_2\mathbf{v}^7=0,$$
$$P_1\mathbf{v}^7=0,\quad (J_1P)_2\mathbf{v}^7=0,\quad (K_1P)_2\mathbf{v}^7=0.$$

The simple module $V_L^0(\tau)(1)$. By (3-23), we have the isomorphism $V_L^0(\tau)(1)\cong(M_T(\tau)(1)\otimes M_t^0)\oplus(W_T(\tau)(1)\otimes W_t^0)$ as M^0-modules. The top level of $V_L^0(\tau)(1)$ is of dimension 2 with basis $\{\mathbf{v}^{8,1},\mathbf{v}^{8,2}\}$, where

$$\mathbf{v}^{8,1}=h_2(-1/3)^2\otimes v\in M_T(\tau)(1)\otimes M_t^0,$$
$$\mathbf{v}^{8,2}=h_1(-2/3)\otimes v\in W_T(\tau)(1)\otimes W_t^0$$

and $0 \neq v \in T_{\chi_0}$. We have

$$(\tilde{\omega}^1)_1\mathbf{v}^{8,1} = \left(\tfrac{1}{9}+\tfrac{2}{3}\right)\mathbf{v}^{8,1}, \quad (\tilde{\omega}^2)_1\mathbf{v}^{8,1} = 0, \quad J_2\mathbf{v}^{8,1} = -\tfrac{238}{81}\sqrt{-3}\,\mathbf{v}^{8,1}, \quad K_2\mathbf{v}^{8,1} = 0,$$
$$P_1\mathbf{v}^{8,1} = -\tfrac{4}{3}\mathbf{v}^{8,2}, \quad (J_1P)_2\mathbf{v}^{8,1} = \tfrac{104}{9}\sqrt{-3}\,\mathbf{v}^{8,2}, \quad (K_1P)_2\mathbf{v}^{8,1} = 0,$$

$$(\tilde{\omega}^1)_1\mathbf{v}^{8,2} = \left(\tfrac{2}{45}+\tfrac{1}{3}\right)\mathbf{v}^{8,2}, \ (\tilde{\omega}^2)_1\mathbf{v}^{8,2} = \tfrac{2}{5}\mathbf{v}^{8,2}, \ J_2\mathbf{v}^{8,2} = -\tfrac{22}{81}\sqrt{-3}\,\mathbf{v}^{8,2}, \ K_2\mathbf{v}^{8,2} = 0,$$
$$P_1\mathbf{v}^{8,2} = 2\mathbf{v}^{8,1}, \ (J_1P)_2\mathbf{v}^{8,2} = -\tfrac{52}{3}\sqrt{-3}\,\mathbf{v}^{8,1}, \ (K_1P)_2\mathbf{v}^{8,2} = -\tfrac{20}{3}\sqrt{-3}\,\mathbf{v}^{8,2}.$$

The simple module $V_L^0(\tau)(2)$. By (3-23), we have the isomorphism $V_L^0(\tau)(2) \cong (M_T(\tau)(2)\otimes M_t^0)\oplus(W_T(\tau)(2)\otimes W_t^0)$ as M^0-modules. The top level of $V_L^0(\tau)(2)$ is $\mathbb{C}\mathbf{v}^9$, where $\mathbf{v}^9 = h_2(-1/3)\otimes v \in W_T(\tau)(2)\otimes W_t^0$ and $0 \neq v \in T_{\chi_0}$. We have

$$(\tilde{\omega}^1)_1\mathbf{v}^9 = \tfrac{2}{45}\mathbf{v}^9, \quad (\tilde{\omega}^2)_1\mathbf{v}^9 = \tfrac{2}{5}\mathbf{v}^9, \quad J_2\mathbf{v}^9 = -\tfrac{4}{81}\sqrt{-3}\,\mathbf{v}^9, \quad K_2\mathbf{v}^9 = 0,$$
$$P_1\mathbf{v}^9 = 0, \quad (J_1P)_2\mathbf{v}^9 = 0, \quad (K_1P)_2\mathbf{v}^9 = \tfrac{4}{3}\sqrt{-3}\,\mathbf{v}^9.$$

The simple module $V_L^j(\tau)(0)$, $j = 1, 2$. By (3-23), we have the isomorphism $V_L^j(\tau)(0) \cong (M_T(\tau)(2)\otimes M_t^{3-j})\oplus(W_T(\tau)(2)\otimes W_t^{3-j})$ as M^0-modules for $j = 1, 2$. The top level of $V_L^j(\tau)(0)$ is $\mathbb{C}\mathbf{v}^{10,j}$, where $\mathbf{v}^{10,j} = 1\otimes v \in W_T(\tau)(2)\otimes W_t^{3-j}$ and $0 \neq v \in T_{\chi_j}$. We have

$$(\tilde{\omega}^1)_1\mathbf{v}^{10,j} = \tfrac{2}{45}\mathbf{v}^{10,j}, \quad (\tilde{\omega}^2)_1\mathbf{v}^{10,j} = \tfrac{1}{15}\mathbf{v}^{10,j}, \quad J_2\mathbf{v}^{10,j} = -\tfrac{4}{81}\sqrt{-3}\,\mathbf{v}^{10,j},$$
$$K_2\mathbf{v}^{10,j} = -(-1)^j\tfrac{2}{9}\mathbf{v}^{10,j}, \quad P_1\mathbf{v}^{10,j} = (-1)^j\tfrac{1}{9}\sqrt{-3}\,\mathbf{v}^{10,j},$$
$$(J_1P)_2\mathbf{v}^{10,j} = (-1)^j\tfrac{8}{9}\mathbf{v}^{10,j}, \quad (K_1P)_2\mathbf{v}^{10,j} = -\tfrac{2}{9}\sqrt{-3}\,\mathbf{v}^{10,j}.$$

The simple module $V_L^j(\tau)(1)$, $j = 1, 2$. By (3-23), we have the isomorphism $V_L^j(\tau)(1) \cong (M_T(\tau)(0)\otimes M_t^{3-j})\oplus(W_T(\tau)(0)\otimes W_t^{3-j})$ as M^0-modules for $j = 1, 2$. The top level of $V_L^j(\tau)(1)$ is of dimension 2 with basis $\{\mathbf{v}^{11,j,1}, \mathbf{v}^{11,j,2}\}$, where

$$\mathbf{v}^{11,j,1} = h_1(-2/3)\otimes v - (-1)^j\sqrt{-3}\,h_2(-1/3)^2\otimes v \in M_T(\tau)(0)\otimes M_t^{3-j},$$
$$\mathbf{v}^{11,j,2} = 2h_1(-2/3)\otimes v + (-1)^j\sqrt{-3}\,h_2(-1/3)^2\otimes v \in W_T(\tau)(0)\otimes W_t^{3-j}$$

and $0 \neq v \in T_{\chi_j}$. We have

$$(\tilde{\omega}^1)_1\mathbf{v}^{11,j,1} = \tfrac{1}{9}\mathbf{v}^{11,j,1}, \quad (\tilde{\omega}^2)_1\mathbf{v}^{11,j,1} = \tfrac{2}{3}\mathbf{v}^{11,j,1}, \quad J_2\mathbf{v}^{11,j,1} = \tfrac{14}{81}\sqrt{-3}\,\mathbf{v}^{11,j,1},$$
$$K_2\mathbf{v}^{11,j,1} = (-1)^j\tfrac{52}{9}\mathbf{v}^{11,j,1}, \quad P_1\mathbf{v}^{11,j,1} = -(-1)^j\tfrac{4}{9}\sqrt{-3}\,\mathbf{v}^{11,j,2},$$
$$(J_1P)_2\mathbf{v}^{11,j,1} = (-1)^j\tfrac{52}{9}\mathbf{v}^{11,j,2}, \quad (K_1P)_2\mathbf{v}^{11,j,1} = -\tfrac{28}{9}\sqrt{-3}\,\mathbf{v}^{11,j,2},$$
$$(\tilde{\omega}^1)_1\mathbf{v}^{11,j,2} = \left(\tfrac{2}{45}+\tfrac{2}{3}\right)\mathbf{v}^{11,j,2}, \quad (\tilde{\omega}^2)_1\mathbf{v}^{11,j,2} = \tfrac{1}{15}\mathbf{v}^{11,j,2},$$

$$J_2\mathbf{v}^{11,j,2} = \tfrac{176}{81}\sqrt{-3}\,\mathbf{v}^{11,j,2}, \quad K_2\mathbf{v}^{11,j,2} = -(-1)^j\tfrac{2}{9}\mathbf{v}^{11,j,2},$$
$$P_1\mathbf{v}^{11,j,2} = -(-1)^j\tfrac{8}{9}\sqrt{-3}\,\mathbf{v}^{11,j,1} + (-1)^j\tfrac{5}{9}\sqrt{-3}\,\mathbf{v}^{11,j,2},$$
$$(J_1P)_2\mathbf{v}^{11,j,2} = (-1)^j\tfrac{104}{9}\mathbf{v}^{11,j,1} - (-1)^j\tfrac{200}{9}\mathbf{v}^{11,j,2},$$
$$(K_1P)_2\mathbf{v}^{11,j,2} = -\tfrac{56}{9}\sqrt{-3}\,\mathbf{v}^{11,j,1} - \tfrac{10}{9}\sqrt{-3}\,\mathbf{v}^{11,j,2}.$$

The simple module $V_L^j(\tau)(2)$, $j = 1, 2$. By (3-23), we have the isomorphism $V_L^j(\tau)(2) \cong (M_T(\tau)(1)\otimes M_t^{3-j})\oplus(W_T(\tau)(1)\otimes W_t^{3-j})$ as M^0-modules for $j=1,2$. The top level of $V_L^j(\tau)(2)$ is $\mathbb{C}\mathbf{v}^{12,j}$, where $\mathbf{v}^{12,j} = h_2(-1/3)\otimes v \in W_T(\tau)(1)\otimes W_t^{3-j}$ and $0 \neq v \in T_{\chi_j}$. We have

$$(\tilde\omega^1)_1\mathbf{v}^{12,j} = \left(\tfrac{2}{45}+\tfrac{1}{3}\right)\mathbf{v}^{12,j}, \quad (\tilde\omega^2)_1\mathbf{v}^{12,j} = \tfrac{1}{15}\mathbf{v}^{12,j}, \quad J_2\mathbf{v}^{12,j} = -\tfrac{22}{81}\sqrt{-3}\,\mathbf{v}^{12,j},$$
$$K_2\mathbf{v}^{12,j} = -(-1)^j\tfrac{2}{9}\mathbf{v}^{12,j}, \quad P_1\mathbf{v}^{12,j} = -(-1)^j\tfrac{5}{9}\sqrt{-3}\,\mathbf{v}^{12,j},$$
$$(J_1P)_2\mathbf{v}^{12,j} = (-1)^j\tfrac{8}{9}\mathbf{v}^{12,j}, \quad (K_1P)_2\mathbf{v}^{12,j} = \tfrac{10}{9}\sqrt{-3}\,\mathbf{v}^{12,j}.$$

The simple module $V_L^0(\tau^2)(0)$. By (3-24), we have the isomorphism $V_L^0(\tau^2)(0) \cong (M_T(\tau^2)(0)\otimes M_t^0)\oplus(W_T(\tau^2)(0)\otimes W_t^0)$ as M^0-modules. The top level of $V_L^0(\tau^2)(0)$ is $\mathbb{C}\mathbf{v}^{13}$, where $\mathbf{v}^{13} = 1\otimes v \in M_T(\tau^2)(0)\otimes M_t^0$ and $0 \neq v \in T_{\chi_0'}$. We have

$$(\tilde\omega^1)_1\mathbf{v}^{13} = \tfrac{1}{9}\mathbf{v}^{13}, \quad (\tilde\omega^2)_1\mathbf{v}^{13} = 0, \quad J_2\mathbf{v}^{13} = -\tfrac{14}{81}\sqrt{-3}\,\mathbf{v}^{13}, \quad K_2\mathbf{v}^{13} = 0,$$
$$P_1\mathbf{v}^{13} = 0, \quad (J_1P)_2\mathbf{v}^{13} = 0, \quad (K_1P)_2\mathbf{v}^{13} = 0.$$

The simple module $V_L^0(\tau^2)(1)$. By (3-24), we have the isomorphism $V_L^0(\tau^2)(1) \cong (M_T(\tau^2)(1)\otimes M_t^0)\oplus(W_T(\tau^2)(1)\otimes W_t^0)$ as M^0-modules. The top level of $V_L^0(\tau^2)(1)$ is of dimension 2 with basis $\{\mathbf{v}^{14,1}, \mathbf{v}^{14,2}\}$, where

$$\mathbf{v}^{14,1} = h_2'(-1/3)^2\otimes v \in M_T(\tau^2)(1)\otimes M_t^0,$$
$$\mathbf{v}^{14,2} = h_1'(-2/3)\otimes v \in W_T(\tau^2)(1)\otimes W_t^0,$$

and $0 \neq v \in T_{\chi_0'}$. We have

$$(\tilde\omega^1)_1\mathbf{v}^{14,1} = \left(\tfrac{1}{9}+\tfrac{2}{3}\right)\mathbf{v}^{14,1}, \quad (\tilde\omega^2)_1\mathbf{v}^{14,1} = 0,$$
$$J_2\mathbf{v}^{14,1} = \tfrac{238}{81}\sqrt{-3}\,\mathbf{v}^{14,1}, \quad K_2\mathbf{v}^{14,1} = 0,$$
$$P_1\mathbf{v}^{14,1} = -\tfrac{4}{3}\mathbf{v}^{14,2}, \quad (J_1P)_2\mathbf{v}^{14,1} = -\tfrac{104}{9}\sqrt{-3}\,\mathbf{v}^{14,2}, \quad (K_1P)_2\mathbf{v}^{14,1} = 0,$$
$$(\tilde\omega^1)_1\mathbf{v}^{14,2} = \left(\tfrac{2}{45}+\tfrac{1}{3}\right)\mathbf{v}^{14,2}, \quad (\tilde\omega^2)_1\mathbf{v}^{14,2} = \tfrac{2}{5}\mathbf{v}^{14,2},$$
$$J_2\mathbf{v}^{14,2} = \tfrac{22}{81}\sqrt{-3}\,\mathbf{v}^{14,2}, \quad K_2\mathbf{v}^{14,2} = 0,$$
$$P_1\mathbf{v}^{14,2} = 2\mathbf{v}^{14,1}, \quad (J_1P)_2\mathbf{v}^{14,2} = \tfrac{52}{3}\sqrt{-3}\,\mathbf{v}^{14,1}, \quad (K_1P)_2\mathbf{v}^{14,2} = \tfrac{20}{3}\sqrt{-3}\,\mathbf{v}^{14,2}.$$

The simple module $V_L^0(\tau^2)(2)$. By Equation (3-24), we have the isomorphism $V_L^0(\tau^2)(2) \cong (M_T(\tau^2)(2) \otimes M_t^0) \oplus (W_T(\tau^2)(2) \otimes W_t^0)$ as M^0-modules. The top level of $V_L^0(\tau^2)(2)$ is $\mathbb{C}\mathbf{v}^{15}$, where $\mathbf{v}^{15} = h_2'(-1/3) \otimes v \in W_T(\tau^2)(2) \otimes W_t^0$ and $0 \neq v \in T_{\chi_0'}$. We have

$$(\tilde{\omega}^1)_1\mathbf{v}^{15} = \tfrac{2}{45}\mathbf{v}^{15}, \quad (\tilde{\omega}^2)_1\mathbf{v}^{15} = \tfrac{2}{5}\mathbf{v}^{15}, \quad J_2\mathbf{v}^{15} = \tfrac{4}{81}\sqrt{-3}\,\mathbf{v}^{15}, \quad K_2\mathbf{v}^{15} = 0,$$
$$P_1\mathbf{v}^{15} = 0, \quad (J_1P)_2\mathbf{v}^{15} = 0, \quad (K_1P)_2\mathbf{v}^{15} = -\tfrac{4}{3}\sqrt{-3}\,\mathbf{v}^{15}.$$

The simple module $V_L^j(\tau^2)(0)$, $j = 1, 2$. By (3-24), we have the isomorphism $V_L^j(\tau^2)(0) \cong (M_T(\tau^2)(2) \otimes M_t^j) \oplus (W_T(\tau^2)(2) \otimes W_t^j)$ as M^0-modules for $j = 1, 2$. The top level of $V_L^j(\tau^2)(0)$ is $\mathbb{C}\mathbf{v}^{16,j}$, where $\mathbf{v}^{16,j} = 1 \otimes v \in W_T(\tau^2)(2) \otimes W_t^j$ and $0 \neq v \in T_{\chi_j'}$. We have

$$(\tilde{\omega}^1)_1\mathbf{v}^{16,j} = \tfrac{2}{45}\mathbf{v}^{16,j}, \quad (\tilde{\omega}^2)_1\mathbf{v}^{16,j} = \tfrac{1}{15}\mathbf{v}^{16,j}, \quad J_2\mathbf{v}^{16,j} = \tfrac{4}{81}\sqrt{-3}\,\mathbf{v}^{16,j},$$
$$K_2\mathbf{v}^{16,j} = (-1)^j\tfrac{2}{9}\mathbf{v}^{16,j}, \quad P_1\mathbf{v}^{16,j} = (-1)^j\tfrac{1}{9}\sqrt{-3}\,\mathbf{v}^{16,j},$$
$$(J_1P)_2\mathbf{v}^{16,j} = -(-1)^j\tfrac{8}{9}\mathbf{v}^{16,j}, \quad (K_1P)_2\mathbf{v}^{16,j} = \tfrac{2}{9}\sqrt{-3}\,\mathbf{v}^{16,j}.$$

The simple module $V_L^j(\tau^2)(1)$, $j = 1, 2$. By (3-24), we have the isomorphism $V_L^j(\tau^2)(1) \cong (M_T(\tau^2)(0) \otimes M_t^j) \oplus (W_T(\tau^2)(0) \otimes W_t^j)$ as M^0-modules for $j = 1, 2$. The top level of $V_L^j(\tau^2)(1)$ is of dimension 2 with basis $\{\mathbf{v}^{17,j,1}, \mathbf{v}^{17,j,2}\}$, where

$$\mathbf{v}^{17,j,1} = h_1'(-2/3) \otimes v - (-1)^j\sqrt{-3}\,h_2'(-1/3)^2 \otimes v \in M_T(\tau^2)(0) \otimes M_t^j,$$
$$\mathbf{v}^{17,j,2} = 2h_1'(-2/3) \otimes v + (-1)^j\sqrt{-3}\,h_2'(-1/3)^2 \otimes v \in W_T(\tau^2)(0) \otimes W_t^j$$

and $0 \neq v \in T_{\chi_j'}$. We have

$$(\tilde{\omega}^1)_1\mathbf{v}^{17,j,1} = \tfrac{1}{9}\mathbf{v}^{17,j,1}, \quad (\tilde{\omega}^2)_1\mathbf{v}^{17,j,1} = \tfrac{2}{3}\mathbf{v}^{17,j,1}, \quad J_2\mathbf{v}^{17,j,1} = -\tfrac{14}{81}\sqrt{-3}\,\mathbf{v}^{17,j,1},$$
$$K_2\mathbf{v}^{17,j,1} = -(-1)^j\tfrac{52}{9}\mathbf{v}^{17,j,1}, \quad P_1\mathbf{v}^{17,j,1} = -(-1)^j\tfrac{4}{9}\sqrt{-3}\,\mathbf{v}^{17,j,2},$$
$$(J_1P)_2\mathbf{v}^{17,j,1} = -(-1)^j\tfrac{52}{9}\mathbf{v}^{17,j,2}, \quad (K_1P)_2\mathbf{v}^{17,j,1} = \tfrac{28}{9}\sqrt{-3}\,\mathbf{v}^{17,j,2},$$
$$(\tilde{\omega}^1)_1\mathbf{v}^{17,j,2} = \left(\tfrac{2}{45} + \tfrac{2}{3}\right)\mathbf{v}^{17,j,2}, \quad (\tilde{\omega}^2)_1\mathbf{v}^{17,j,2} = \tfrac{1}{15}\mathbf{v}^{17,j,2},$$
$$J_2\mathbf{v}^{17,j,2} = -\tfrac{176}{81}\sqrt{-3}\,\mathbf{v}^{17,j,2}, \quad K_2\mathbf{v}^{17,j,2} = (-1)^j\tfrac{2}{9}\mathbf{v}^{17,j,2},$$
$$P_1\mathbf{v}^{17,j,2} = -(-1)^j\tfrac{8}{9}\sqrt{-3}\,\mathbf{v}^{17,j,1} + (-1)^j\tfrac{5}{9}\sqrt{-3}\,\mathbf{v}^{17,j,2},$$
$$(J_1P)_2\mathbf{v}^{17,j,2} = -(-1)^j\tfrac{104}{9}\mathbf{v}^{17,j,1} + (-1)^j\tfrac{200}{9}\mathbf{v}^{17,j,2},$$
$$(K_1P)_2\mathbf{v}^{17,j,2} = \tfrac{56}{9}\sqrt{-3}\,\mathbf{v}^{17,j,1} + \tfrac{10}{9}\sqrt{-3}\,\mathbf{v}^{17,j,2}.$$

The simple module $V_L^j(\tau^2)(2)$, $j = 1, 2$. By (3-24), we have the isomorphism $V_L^j(\tau^2)(2) \cong (M_T(\tau^2)(1) \otimes M_t^j) \oplus (W_T(\tau^2)(1) \otimes W_t^j)$ as M^0-modules for $j = 1, 2$.

The top level of $V_L^j(\tau^2)(2)$ is $\mathbb{C}\mathbf{v}^{18,j}$, where $\mathbf{v}^{18,j} = h_2'(-1/3)\otimes v \in W_T(\tau^2)(1)\otimes W_t^j$ and $0 \neq v \in T_{\chi_j'}$. We have

$$(\tilde{\omega}^1)_1\mathbf{v}^{18,j} = \left(\tfrac{2}{45}+\tfrac{1}{3}\right)\mathbf{v}^{18,j}, \quad (\tilde{\omega}^2)_1\mathbf{v}^{18,j} = \tfrac{1}{15}\mathbf{v}^{18,j}, \quad J_2\mathbf{v}^{18,j} = \tfrac{22}{81}\sqrt{-3}\,\mathbf{v}^{18,j},$$
$$K_2\mathbf{v}^{18,j} = (-1)^j\tfrac{2}{9}\mathbf{v}^{18,j}, \quad P_1\mathbf{v}^{18,j} = -(-1)^j\tfrac{5}{9}\sqrt{-3}\,\mathbf{v}^{18,j},$$
$$(J_1P)_2\mathbf{v}^{18,j} = -(-1)^j\tfrac{8}{9}\mathbf{v}^{18,j}, \quad (K_1P)_2\mathbf{v}^{18,j} = -\tfrac{10}{9}\sqrt{-3}\,\mathbf{v}^{18,j}.$$

Symmetries by σ. We now consider the automorphisms σ and θ of V_L that are lifts of the isometries σ and θ of the lattice L defined by (3-1). Clearly, $\sigma\tau\sigma = \tau^2$, $\sigma\theta = \theta\sigma$, and $\tau\theta = \theta\tau$. Thus σ and θ induce automorphisms of V_L^τ of order 2. We have $\sigma J = -J$, $\sigma K = -K$, $\sigma P = P$, $\theta J = J$, $\theta K = -K$, and $\theta P = -P$. Hence σ and θ induce the same automorphism of M_t^0 and θ is the identity on $M(0)$. Note also that $\sigma(J_1P) = -J_1P$ and $\sigma(K_1P) = -K_1P$.

From the action of σ on the top level of the 30 known simple V_L^τ-modules or the action of J_2, K_2, $(J_1P)_2$, and $(K_1P)_2$, we know how σ permutes those simple V_L^τ-modules. In fact, σ transforms $V_{L^{(c,0)}}$ into an equivalent simple V_L^τ-module and interchanges the remaining simple V_L^τ-modules as follows.

$$V_L(1) \leftrightarrow V_L(2), \qquad V_{L^{(0,1)}}(\varepsilon) \leftrightarrow V_{L^{(0,2)}}(2\varepsilon), \qquad \varepsilon = 0, 1, 2,$$
$$V_{L^{(c,1)}} \leftrightarrow V_{L^{(c,2)}}, \qquad V_L^j(\tau)(\varepsilon) \leftrightarrow V_L^j(\tau^2)(\varepsilon), \qquad j, \varepsilon = 0, 1, 2.$$

Note that $\sigma h_i = \xi^{3-i}h_i'$, $i = 1, 2$. The top level of $V_L^j(\tau^2)(\varepsilon)$ can be obtained by replacing $h_i(i/3+n)$ with $h_i'(i/3+n)$ in the top level of $V_L^j(\tau)(\varepsilon)$ for $j, \varepsilon = 0, 1, 2$. The corresponding action of σ on the simple $M(0)$-modules was discussed in [Dong et al. 2004, Section 4.4].

5. Classification of simple modules

We keep the notation in the preceding section. Thus $V_L^\tau = M^0 \oplus W^0$ with $M^0 = M(0)\otimes M_t^0$ and $W^0 = W(0)\otimes W_t^0$. In this section we show that any simple V_L^τ-module is equivalent to one of the 30 simple V_L^τ-modules listed in Lemma 3.2. The result will be established by considering the Zhu algebra $A(V_L^\tau)$ of V_L^τ.

First, we review some notation and basic formulas for the Zhu algebra $A(V)$ of a vertex operator algebra $(V, Y, \mathbf{1}, \omega)$. Define two binary operations

$$u * v = \sum_{i=0}^{\infty}\binom{\mathrm{wt}\,u}{i}u_{i-1}v, \quad u \circ v = \sum_{i=0}^{\infty}\binom{\mathrm{wt}\,u}{i}u_{i-2}v \tag{5-1}$$

for $u, v \in V$ with u being homogeneous and extend $*$ and $\circ$ for arbitrary $u \in V$ by linearity. Let $O(V)$ be the subspace of V spanned by all $u \circ v$ for $u, v \in V$. Set $A(V) = V/O(V)$. By [Zhu 1996, Theorem 2.1.1], $O(V)$ is a two-sided ideal with respect to the operation $*$. Thus $*$ induces an operation in $A(V)$. Denote by $[v]$ the image of $v \in V$ in $A(V)$. Then $[u]*[v]=[u*v]$ and $A(V)$ is an associative algebra by this operation. Moreover, $[\mathbf{1}]$ is the identity and $[\omega]$ is in the center of $A(V)$. For $u, v \in V$, we write $u \sim v$ if $[u]=[v]$. For $\varphi, \psi \in \operatorname{End} V$, we write $\varphi \sim \psi$ if $\varphi v \sim \psi v$ for all $v \in V$. We need some basic formulas from [Zhu 1996].

$$v * u \sim \sum_{i=0}^{\infty} \binom{\mathrm{wt}(u)-1}{i} u_{i-1} v, \tag{5-2}$$

$$\sum_{i=0}^{\infty} \binom{\mathrm{wt}(u)+m}{i} u_{i-n-2} v \in O(V), \quad n \geq m \geq 0. \tag{5-3}$$

Moreover (see [Wang 1993]),

$$L(-n) \sim (-1)^n\bigl\{(n-1)\bigl(L(-2)+L(-1)\bigr)+L(0)\bigr\}, \quad n \geq 1, \tag{5-4}$$

$$[\omega]*[u]=[(L(-2)+L(-1))u], \tag{5-5}$$

where $L(n)=\omega_{n+1}$. From (5-4) and (5-5) we have

$$[L(-n)u]=(-1)^n(n-1)[\omega]*[u]+(-1)^n[L(0)u], \quad n \geq 1. \tag{5-6}$$

If $u \in V$ is of weight 2, then $u(-n-3)+2u(-n-2)+u(-n-1) \sim 0$ by (5-3), where $u(n)=u_{n+1}$. Hence

$$u(-n) \sim (-1)^n\bigl((n-1)u(-2)+(n-2)u(-1)\bigr) \tag{5-7}$$

for $n \geq 1$. Then it follows from (5-1) and (5-2) that

$$u(-n)w \sim (-1)^n\bigl(-u*w+nw*u+u(0)w\bigr) \tag{5-8}$$

for $n \geq 1$, $w \in V$. Likewise, if u is of weight 3 and $u(n)=u_{n+2}$, then

$$u(-n) \sim (-1)^{n+1} \cdot\bigl(\tfrac{1}{2}(n-1)(n-2)u(-3)+(n-1)(n-3)u(-2)+\tfrac{1}{2}(n-2)(n-3)u(-1)\bigr), \tag{5-9}$$

$$u(-n)w \sim (-1)^{n+1}\bigl(nu(-1)w+(n-1)u(0)w-(n-1)u*w+\tfrac{1}{2}n(n-1)w*u\bigr), \tag{5-10}$$

for $n \geq 1$, $w \in V$.

For a homogeneous vector $u \in V$, $o(u)=u_{\mathrm{wt}(u)-1}$ is the weight zero component operator of $Y(u,z)$. Extend $o(u)$ for arbitrary $u \in V$ by linearity. Note that we call a module in the sense of [Zhu 1996] an $\mathbb{N}$-graded weak module here. If $M=\oplus_{n=0}^{\infty} M_{(n)}$ is an $\mathbb{N}$-graded weak V-module with $M_{(0)} \neq 0$, then $o(u)$ acts on its top level $M_{(0)}$. Zhu's theory [1996] says: (1) $o(u)o(v)=o(u*v)$ as operators on the top level $M_{(0)}$ and $o(u)$ acts as 0 on $M_{(0)}$ if $u \in O(V)$. Thus $M_{(0)}$ is an $A(V)$-module, where $[u]$ acts on $M_{(0)}$ as $o(u)$. (2) The map $M \mapsto M_{(0)}$ is a bijection between the set of isomorphism classes of simple $\mathbb{N}$-graded weak V-modules and the set of isomorphism classes of simple $A(V)$-modules.

We return to V_L^τ. As in Section 3, we write $L^i(n) = (\tilde{\omega}^i)_{n+1}$, $i = 1, 2$, $J(n) = J_{n+2}$, and $K(n) = K_{n+2}$. The Zhu algebras $A(M(0))$ and $A(M_t^0)$ were determined in [Dong et al. 2004] and [Kitazume et al. 2000b], respectively. Since $O(M^0) \subset O(V_L^\tau)$, the image of $M(0)$ (resp. M_t^0) in $A(V_L^\tau)$ is a homomorphic image of $A(M(0))$ (resp. $A(M_t^0)$). It is generated by $[\tilde{\omega}^1]$, $[J]$ (resp. $[\tilde{\omega}^2]$, $[K]$).

By a direct calculation, we have

$$
\begin{aligned}
P_1 P &= -16\tilde{\omega}^1 - 6\tilde{\omega}^2,\\
P_0 P &= -8(\tilde{\omega}^1)_{-2}\mathbf{1} - 3(\tilde{\omega}^2)_{-2}\mathbf{1},\\
P_{-1} P &= \tfrac{5}{273} J_1 K_1 P - \tfrac{12}{7}(\tilde{\omega}^1)_{-3}\mathbf{1} - \tfrac{18}{13}(\tilde{\omega}^2)_{-3}\mathbf{1}\\
&\qquad - \tfrac{36}{7}(\tilde{\omega}^1)_{-1}(\tilde{\omega}^1)_{-1}\mathbf{1} - \tfrac{9}{13}(\tilde{\omega}^2)_{-1}(\tilde{\omega}^2)_{-1}\mathbf{1} - 16(\tilde{\omega}^1)_{-1}(\tilde{\omega}^2)_{-1}\mathbf{1},\\
P_{-2} P &= \tfrac{1}{84} J_0 K_1 P + \tfrac{1}{156} J_1 K_0 P - \tfrac{8}{7}(\tilde{\omega}^1)_{-4}\mathbf{1} - \tfrac{12}{13}(\tilde{\omega}^2)_{-4}\mathbf{1} - \tfrac{36}{7}(\tilde{\omega}^1)_{-2}(\tilde{\omega}^1)_{-1}\mathbf{1}\\
&\qquad - \tfrac{9}{13}(\tilde{\omega}^2)_{-2}(\tilde{\omega}^2)_{-1}\mathbf{1} - 8(\tilde{\omega}^1)_{-2}(\tilde{\omega}^2)_{-1}\mathbf{1} - 8(\tilde{\omega}^1)_{-1}(\tilde{\omega}^2)_{-2}\mathbf{1}.
\end{aligned}
\tag{5-11}
$$

Moreover, $J_2 P = K_2 P = 0$. Then, using formulas (5-4)–(5-10), we obtain

$$
\begin{aligned}
[P] * [P] = \tfrac{5}{273}[J_1 K_1 P] - \tfrac{36}{7}[\tilde{\omega}^1] * [\tilde{\omega}^1] - \tfrac{9}{13}[\tilde{\omega}^2] * [\tilde{\omega}^2]&\\
- 16[\tilde{\omega}^1] * [\tilde{\omega}^2] + \tfrac{4}{7}[\tilde{\omega}^1] + \tfrac{6}{13}[\tilde{\omega}^2]&,
\end{aligned}
\tag{5-12}
$$

$$
\begin{aligned}
[P \circ P] &= \tfrac{1}{84}[J] * [K_1 P] - \tfrac{1}{84}[K_1 P] * [J] + \tfrac{1}{156}[K] * [J_1 P] - \tfrac{1}{156}[J_1 P] * [K]\\
&= 0.
\end{aligned}
\tag{5-13}
$$

It turns out that $A(V_L^\tau)$ is generated by $[\tilde{\omega}^1]$, $[\tilde{\omega}^2]$, $[J]$, $[K]$, and $[P]$ (Corollary 5.11). However, we first prove the following intermediate assertion.

Proposition 5.1. *The Zhu algebra $A(V_L^\tau)$ is generated by $[\tilde{\omega}^1]$, $[\tilde{\omega}^2]$, $[J]$, $[K]$, $[P]$, $[J_1 P]$, and $[K_1 P]$.*

Proof. Recall that $L^i(n)P = 0$ for $i = 1, 2$, $n \geq 1$, $L^1(0)P = \frac{8}{5}P$, $L^2(0)P = \frac{2}{5}P$, and $J(n)P = K(n)P = 0$ for $n \geq 0$. Thus from the commutation relations (3-9)–(3-11) and (3-13)–(3-15) we see that W^0 is spanned by the vectors of the form

$$
\begin{aligned}
L^1(-j_1) \cdots L^1(-j_r) L^2(-k_1) \cdots L^2(-k_s)&\\
\cdot J(-m_1) \cdots J(-m_p) K(-n_1) \cdots K(-n_q) P&
\end{aligned}
\tag{5-14}
$$

with $j_1 \geq \cdots \geq j_r \geq 1$, $k_1 \geq \cdots \geq k_s \geq 1$, $m_1 \geq \cdots \geq m_p \geq 1$, $n_1 \geq \cdots \geq n_q \geq 1$. Let v be a vector of this form. Its weight is

$$j_1 + \cdots + j_r + k_1 + \cdots + k_s + m_1 + \cdots + m_p + n_1 + \cdots + n_q + 2.$$

Since $V_L^\tau = M^0 \oplus W^0$ and since the image of M^0 in $A(V_L^\tau)$ is generated by $[\tilde{\omega}^1]$, $[\tilde{\omega}^2]$, $[J]$, and $[K]$, it suffices to show that the image $[v]$ of v in $A(V_L^\tau)$ is contained in the subalgebra generated by $[\tilde{\omega}^1]$, $[\tilde{\omega}^2]$, $[J]$, $[K]$, $[P]$, $[J_1 P]$, and $[K_1 P]$. We proceed by induction on the weight of v. By formula (5-8) with $u = \tilde{\omega}^i$, $i = 1, 2$ and the induction on the weight, we may assume that $r = s = 0$, that is,

$$v = J(-m_1)\cdots J(-m_p)K(-n_1)\cdots K(-n_q)P.$$

Moreover, by formula (5-10) with $u = J$, we may assume that $m_1 = \cdots = m_p = 1$. Since $J(m)$ and $K(n)$ commute, we may also assume that $n_1 = \cdots = n_q = 1$ by a similar argument. Then $v = J(-1)^p K(-1)^q P$.

Next, we reduce v to the case $p \le 1$. For this purpose, we use a singular vector

$$5J(-1)^2 P + 2496 L^1(-2)P - 195 L^1(-1)^2 P = 0. \tag{5-15}$$

in $W(0)$. Suppose $p \ge 2$. Then, since $K(-1)$ commutes with $J(m)$ and $L^1(n)$, (5-15) implies that $v = J(-1)^p K(-1)^q P$ is a linear combination of

$$J(-1)^{p-2} L^1(-2) K(-1)^q P \quad \text{and} \quad J(-1)^{p-2} L^1(-1)^2 K(-1)^q P.$$

By (3-10), these two vectors can be written in the form $L^1(-2) H K(-1)^q P$ and $L^1(-1)^2 H' K(-1)^q P$, where H (resp. H') is a polynomial in $J(-1)$ and $J(-3)$ (resp. $J(-1)$, $J(-2)$, and $J(-3)$). Then by (5-8) with $u = \tilde{\omega}^1$ and the induction on the weight, the assertion holds for v. Hence we may assume that $p \le 1$.

There is a singular vector

$$K(-1)^2 P - 210 L^2(-2) P = 0 \tag{5-16}$$

in W_t^0. Thus, by a similar argument as above, we may assume that $q \le 1$. Finally, it follows from (5-12) that $[J(-1)K(-1)P]$ can be written by $[\tilde{\omega}^1]$, $[\tilde{\omega}^2]$, and $[P]$ in $A(V_L^\tau)$. The proof is complete. $\square$

We will classify the simple V_L^τ-modules using our knowledge of simple modules for $M(0)$ and M_t^0 together with fusion rules (3-25) and (3-7). Set

$$\mathcal{M}_1 = \{M(\varepsilon), M_k^c, M_T(\tau^i)(\varepsilon) \mid i = 1, 2,\ \varepsilon = 0, 1, 2\},$$
$$\mathcal{W}_1 = \{W(\varepsilon), W_k^c, W_T(\tau^i)(\varepsilon) \mid i = 1, 2,\ \varepsilon = 0, 1, 2\},$$
$$\mathcal{M}_2 = \{M_t^j \mid j = 0, 1, 2\}, \quad \mathcal{W}_2 = \{W_t^j \mid j = 0, 1, 2\}.$$

Then $\mathcal{M}_1 \cup \mathcal{W}_1$ (resp. $\mathcal{M}_2 \cup \mathcal{W}_2$) is a complete set of representatives of isomorphism classes of simple $M(0)$-modules (resp. simple M_t^0-modules). A main point is that the fusion rules of the following form hold.

$$\begin{aligned} W(0) \times M^1 &= W^1, & W(0) \times W^1 &= M^1 + W^1, \\ W_t^0 \times M^2 &= W^2, & W_t^0 \times W^2 &= M^2 + W^2, \end{aligned} \tag{5-17}$$

where $M^i \in \mathcal{M}_i$, $i = 1, 2$, and $W^i \in \mathcal{W}_i$ is determined by M^i through the fusion rule $W(0) \times M^1 = W^1$ or $W_t^0 \times M^2 = W^2$.

Recall that M^0 is rational, C_2-cofinite, and of CFT type. Thus every $\mathbb{N}$-graded weak M^0-module is a direct sum of simple M^0-modules. As a result, every $\mathbb{N}$-graded weak V_L^τ-module is decomposed into a direct sum of simple M^0-modules, and in particular $L(0) = \omega_1$ acts semisimply on it. Each weight subspace, that is, each eigenspace for $L(0)$ is not necessarily a finite dimensional space. However, any simple weak V_L^τ-module is a simple ordinary V_L^τ-module by [Abe et al. 2004, Corollary 5.8], since V_L^τ is C_2-cofinite and of CFT type.

We note that

$$W^0 \cdot W^0 = V_L^\tau. \tag{5-18}$$

Indeed, $W^0 \cdot W^0 = \operatorname{span}\{a_n b \mid a, b \in W^0, n \in \mathbb{Z}\}$ is an M^0-submodule of V_L^τ by (2-6). Since P, $J_1 K_1 P \in W^0$ and $\tilde{\omega}^1, \tilde{\omega}^2 \in M^0$, (5-11) implies that $W^0 \cdot W^0 = M^0 \oplus W^0$.

Each simple M^0-module is isomorphic to a tensor product $A \otimes B$ of a simple $M(0)$-module A and a simple M_t^0-module B. We show that only restricted simple M^0-modules can appear in $\mathbb{N}$-graded weak V_L^τ-modules.

Lemma 5.2. *Let U be an $\mathbb{N}$-graded weak V_L^τ-module. Then any simple M^0-submodule of U is isomorphic to $M^1 \otimes M^2$ or $W^1 \otimes W^2$ for some $M^i \in \mathcal{M}_i$ and $W^i \in \mathcal{W}_i$, $i = 1, 2$.*

Proof. Suppose U contains a simple M^0-submodule $S^0 \cong M^1 \otimes W^2$ with $M^1 \in \mathcal{M}_1$ and $W^2 \in \mathcal{W}_2$. Let $S = V_L^\tau \cdot S^0 = \operatorname{span}\{a_n w \mid a \in V_L^\tau, w \in S^0, n \in \mathbb{Z}\}$. Then (2-6) implies that S is the $\mathbb{N}$-graded weak V_L^τ-submodule of U generated by S^0. By the construction of S, the difference of any two eigenvalues of $L(0)$ in S is an integer. In fact, S is an ordinary V_L^τ-module by Remark 2.16.

If v is a nonzero vector in V_L^τ, then $v_n S^0 \neq 0$ for some $n \in \mathbb{Z}$. Indeed, Lemma 2.6 implies that the set $\{v \in V_L^\tau \mid v_n S^0 = 0 \text{ for all } n \in \mathbb{Z}\}$ is an ideal of V_L^τ. It is in fact 0, since V_L^τ is a simple vertex operator algebra and S^0 is a simple M^0-module. Then by the fusion rules (5-17), a simple M^0-module isomorphic to $W^1 \otimes M^2$ or $W^1 \otimes W^2$ must appear in S. However, the difference of the minimal eigenvalues of $L(0)$ in $M^1 \otimes W^2$ and $W^1 \otimes M^2$, or in $M^1 \otimes W^2$ and $W^1 \otimes W^2$ is not an integer. This is a contradiction. Thus U does not contain a simple M^0-submodule isomorphic to $M^1 \otimes W^2$. By a similar argument, we can also show that there is no simple M^0-submodule isomorphic to $W^1 \otimes M^2$ in U. Hence the assertion holds. □

Set $\mathcal{M} = \{M^1 \otimes M^2 \mid M^i \in \mathcal{M}_i, i = 1, 2\}$ and $\mathcal{W} = \{W^1 \otimes W^2 \mid W^i \in \mathcal{W}_i, i = 1, 2\}$. Then each of $\mathcal{M}$ and $\mathcal{W}$ consists of 30 inequivalent simple M^0-modules. The top level of every simple M^0-module is of dimension one.

Lemma 5.3. *If U is a simple $\mathbb{N}$-graded weak V_L^τ-module whose top level is of dimension one, then U is isomorphic to one of the 23 known simple V_L^τ-modules* *with one dimensional top level, namely*, $V_{L^{(0,j)}}(\varepsilon)$, $j = 0, 1, 2$, $\varepsilon = 0, 1, 2$, $V_{L^{(c,j)}}$, $j = 1, 2$, $V_L^j(\tau)(\varepsilon)$, $j = 0, 1, 2$, $\varepsilon = 0, 2$, *and* $V_L^j(\tau^2)(\varepsilon)$, $j = 0, 1, 2$, $\varepsilon = 0, 2$.

Proof. Since U is a direct sum of simple M^0-modules and since the top level, say U_λ of U is assumed to be of dimension one, it follows from Lemma 5.2 that U_λ is isomorphic to the top level of $M^1 \otimes M^2$ or the top level of $W^1 \otimes W^2$ as an $A(M^0)$-module for some $M^i \in \mathcal{M}_i$, $W^i \in \mathcal{W}_i$, $i = 1, 2$. The Zhu algebra

$$A(M^0) \cong A(M(0)) \times A(M_t^0)$$

is commutative and the action of $A(M^0)$ on the top level of $M^1 \otimes M^2$ and the top level of $W^1 \otimes W^2$ are known. Indeed, we know all possible action of the elements $[\tilde{\omega}^1]$, $[\tilde{\omega}^2]$, $[J]$, and $[K]$ of $A(V_L^\tau)$ on U_λ. Let $[\tilde{\omega}^1]$, $[\tilde{\omega}^2]$, $[J]$, and $[K]$ act on U_λ as scalars a_1, a_2, b_1, and b_2, respectively. There are 60 possible such quadruplets (a_1, a_2, b_1, b_2).

Let $[P]$, $[J_1 P]$, and $[K_1 P]$ act on U_λ as scalars x_1, x_2, and x_3, respectively. Then it follows from (5-12) that $[J_1 K_1 P]$ acts on U_λ as a scalar

$$\tfrac{273}{5}x_1^2 + \tfrac{1404}{5}a_1^2 + \tfrac{189}{5}a_2^2 + \tfrac{4368}{5}a_1a_2 - \tfrac{156}{5}a_1 - \tfrac{126}{5}a_2. \tag{5-19}$$

From computer calculations, whose results are presented in an online supplement to this paper,[1] and from formulas (5-4)–(5-10), we conclude that the vanishing of $[P \circ (J_1 P)]$ and $[P \circ (K_1 P)]$ imply, respectively,

$$15b_2x_1 + 5a_2x_3 - 2x_3 = 0, \qquad (15a_2 - 1)x_2 = 0. \tag{5-20}$$

Using (5-19), we can calculate

$$[(J_1 P) * (J_1 P)], \quad [(K_1 P) * (K_1 P)], \quad [(J_1 P) * (K_1 P)]$$

in a similar way and verify that the following equations hold.

$$\begin{aligned} x_2^2 = \big(\tfrac{229164}{575}a_1 - \tfrac{37856}{425}a_2 + \tfrac{1669382}{48875}\big)x_1^2 - \tfrac{56}{85}b_2x_2 - \tfrac{4056}{115}b_1x_3 + \tfrac{348994464}{107525}a_1^3 \\ + \tfrac{137149584}{9775}a_1^2a_2 - \tfrac{1030224}{1375}a_1^2 + \tfrac{7064876}{9775}a_1a_2^2 - \tfrac{40788488}{48875}a_1a_2 \\ + \tfrac{16160456}{537625}a_1 - \tfrac{419184}{9775}a_2^3 - \tfrac{200994}{48875}a_2^2 + \tfrac{1065516}{48875}a_2 - \tfrac{3042}{187}b_1^2, \end{aligned} \tag{5-21}$$

$$\begin{aligned} x_3^2 = &\big(-\tfrac{37044}{575}a_1 - \tfrac{5684}{85}a_2 + \tfrac{741713}{97750}\big)x_1^2 \\ &+ \tfrac{28}{221}b_2x_2 + \tfrac{216}{115}b_1x_3 - \tfrac{54559344}{107525}a_1^3 - \tfrac{28217448}{9775}a_1^2a_2 + \tfrac{254982}{1375}a_1^2 - \tfrac{25042724}{9775}a_1a_2^2 \\ &+ \tfrac{26308184}{48875}a_1a_2 - \tfrac{8127098}{537625}a_1 - \tfrac{4775148}{25415}a_2^3 + \tfrac{188338017}{1270750}a_2^2 - \tfrac{9722139}{635375}a_2 - \tfrac{180}{187}b_1^2, \end{aligned} \tag{5-22}$$

(5-23)

$$x_2x_3 = \big(-\tfrac{864}{5}a_1^2 + \tfrac{1248}{25}a_1a_2 + \tfrac{1152}{5}a_2^2 + \tfrac{5904}{125}a_1 + \tfrac{184176}{125}a_2 - \tfrac{62112}{625}\big)x_1 - 36b_1b_2.$$

[1] The authors can supply these expressions in machine readable form upon request.

We have obtained a system of equations (5-20)–(5-23) for x_1, x_2, x_3. We can solve this system of equations with respect to the 60 possible quadruplets (a_1, a_2, b_1, b_2). Actually, there is no solution for 37 quadruplets of (a_1, a_2, b_1, b_2). For each of the remaining 23 quadruplets (a_1, a_2, b_1, b_2), the system of equations possesses a unique solution (x_1, x_2, x_3). Furthermore, the 23 sets $(a_1, a_2, b_1, b_2, x_1, x_2, x_3)$ of values determined in this way coincide with the action of $[\tilde{\omega}^1]$, $[\tilde{\omega}^2]$, $[J]$, $[K]$, $[P]$, $[J_1 P]$, and $[K_1 P]$ on the top level of the 23 known simple V_L^τ-modules with one dimensional top level described in Section 4. Since $A(V_L^\tau)$ is generated by these seven elements, this implies that U_λ is isomorphic to the top level of one of the 23 simple V_L^τ-modules listed in the assertion as an $A(V_L^\tau)$-module. Thus the lemma holds by Zhu's theorem. □

Remark 5.4. We also obtain some equations for $x_1 x_2$ and $x_1 x_3$ from $[P * (J_1 P)]$ and $[P * (K_1 P)]$. However, they are not sufficient to determine x_1, x_2, and x_3.

Lemma 5.5. *Every $\mathbb{N}$-graded weak V_L^τ-module contains a simple M^0-submodule isomorphic to a member of $\mathcal{M}$.*

Proof. Suppose false and let U be an $\mathbb{N}$-graded weak V_L^τ-module which contains no simple M^0-submodule isomorphic to a member of $\mathcal{M}$. Then by Lemma 5.2, there is a simple M^0-submodule W in U such that $W \cong W^1 \otimes W^2$ for some $W^i \in \mathcal{W}_i$, $i = 1, 2$. The top level of W, say W_λ for some $\lambda \in \mathbb{Q}$, is a one dimensional space. Take $0 \neq w \in W_\lambda$ and let $S = V_L^\tau \cdot w = \operatorname{span}\{a_n w \mid a \in V_L^\tau, n \in \mathbb{Z}\}$, which is an ordinary V_L^τ-module by (2-6) and Remark 2.16. Since $V_L^\tau = M^0 \oplus W^0$, it follows from our assumption and the fusion rules (5-17) that S is isomorphic to a direct sum of finite number of copies of W as an M^0-module. Thus $[\tilde{\omega}^1]$, $[\tilde{\omega}^2]$, $[J]$, and $[K]$ act on the top level S_λ of S as scalars, say a_1, a_2, b_1, and b_2, respectively. Then by a similar calculation as in the proof of Lemma 5.3, we see that $[P \circ (K_1 P)] = 0$ implies

$$(15a_2 - 1)\, o(J_1 P) = 0 \tag{5-24}$$

as an operator on the top level S_λ. Recall that $[u] \in A(V_L^\tau)$ acts on S_λ as $o(u) = u_{\mathrm{wt}(u)-1}$ for a homogeneous vector u of V_L^τ. Furthermore, we can calculate that

$$\begin{aligned} o(J_1 P)o(P) - o(P)o(J_1 P) &= 0,\\ o(K_1 P)o(P) - o(P)o(K_1 P) &= \tfrac{2}{13}(15a_2 - 1)\, o(J_1 P),\\ o(J_1 P)o(K_1 P) - o(K_1 P)o(J_1 P) &= \tfrac{96}{125}(15a_2 - 1)(65a_1 + 100a_2 + 441)\, o(P) \end{aligned} \tag{5-25}$$

as operators on S_λ.

By (5-24), $15a_2 - 1 = 0$ or $o(J_1 P) = 0$ and so $o(P)$, $o(J_1 P)$, and $o(K_1 P)$ commute each other. Thus the action of $A(V_L^\tau)$ on S_λ is commutative. Hence we can choose a one dimensional $A(V_L^\tau)$-submodule T of S_λ. Zhu's theory tells us that there is a simple $\mathbb{N}$-graded weak V_L^τ-module R whose top level R_λ is isomorphic to T as an $A(V_L^\tau)$-module. Since $\dim R_\lambda = 1$, R is isomorphic to one of the 23 simple V_L^τ-modules listed in Lemma 5.3. In particular, R contains a simple M^0-submodule M isomorphic to a member of $\mathcal{M}$. Now, consider the V_L^τ-submodule $V_L^\tau \cdot T$ of S generated by T. By Lemma 2.10, there is a surjective homomorphism of V_L^τ-modules from $V_L^\tau \cdot T$ onto R. Then $V_L^\tau \cdot T$ must contain a simple M^0-submodule isomorphic to M. This contradicts our assumption. The proof is complete. □

Lemma 5.6. *Let U be an $\mathbb{N}$-graded weak V_L^τ-module and M be a simple M^0-submodule of U such that $M \cong M^1 \otimes M^2$ as M^0-modules for some $M^i \in \mathcal{M}_i$, $i = 1, 2$. Then $V_L^\tau \cdot M = \operatorname{span}\{a_n u \mid a \in V_L^\tau, u \in M, n \in \mathbb{Z}\}$ is a simple V_L^τ-module. Moreover, $V_L^\tau \cdot M = M \oplus W$, where W is a simple M^0-module isomorphic to $W^1 \otimes W^2$ and W^i, $i = 1, 2$ are determined from M^i by the fusion rules $W(0) \times M^1 = W^1$ and $W_t^0 \times M^2 = W^2$ of (5-17).*

Proof. By Remark 2.16, $V_L^\tau \cdot M$ is an ordinary V_L^τ-module. Note that $V_L^\tau \cdot M = (M^0 + W^0) \cdot M = M + W^0 \cdot M$. We see that $W^0 \cdot M \neq 0$ by a similar argument as in the proof of Lemma 5.2. Actually, $W^0 \cdot (W^0 \cdot M) \supset (W^0 \cdot W^0) \cdot M = V_L^\tau \cdot M$ (see Lemma 2.6 and (5-18)) implies $W^0 \cdot M \neq 0$ also. Moreover, $W^0 \cdot M$ is an M^0-module by (2-6). Since M^0 is rational, $W^0 \cdot M$ is decomposed into a direct sum of simple M^0-modules, say $W^0 \cdot M = \bigoplus_{\gamma \in \Gamma} S^\gamma$. Let $W = W^1 \otimes W^2$, where $W^i \in \mathcal{W}_i$, $i = 1, 2$ are determined by the fusion rules $W(0) \times M^1 = W^1$ and $W_t^0 \times M^2 = W^2$. The space $I_{M^0}\binom{W}{W^0\ M}$ of intertwining operators of type $\binom{W}{W^0\ M}$ is of dimension one and each S^γ is isomorphic to W.

We want to show that $|\Gamma| = 1$. Suppose Γ contains at least two elements and take $\gamma_1, \gamma_2 \in \Gamma$, $\gamma_1 \neq \gamma_2$. Let $\psi : S^{\gamma_2} \to S^{\gamma_1}$ be an isomorphism of M^0-modules and $p_\gamma : W^0 \cdot M \to S^\gamma$ be a projection. For $a \in W^0$ and $u \in M$, set

$$\mathcal{G}_{\gamma_1}(a, z)u = p_{\gamma_1} Y_U(a, z)u, \qquad \mathcal{G}_{\gamma_2}(a, z)u = \psi p_{\gamma_2} Y_U(a, z)u,$$

where $Y_U(a, z)$ is the vertex operator of the $\mathbb{N}$-graded weak V_L^τ-module U. Then $\mathcal{G}_{\gamma_i}(\,\cdot\,, z)$, $i = 1, 2$ are nonzero members in the one dimensional space $I_{M^0}\binom{W}{W^0\ M}$, so that $\mu \mathcal{G}_{\gamma_1}(\,\cdot\,, z) = \mathcal{G}_{\gamma_2}(\,\cdot\,, z)$ for some $0 \neq \mu \in \mathbb{C}$. Let $0 \neq v \in S^{\gamma_1}$. Then $v \in W^0 \cdot M$ and so $v = \sum_j (a^j)_{n_j} u^j$ for some $a^j \in W^0$, $u^j \in M$, $n_j \in \mathbb{Z}$. Take the coefficients of z^{-n_j-1} in both sides of $\mu \mathcal{G}_{\gamma_1}(a^j, z)u^j = \mathcal{G}_{\gamma_2}(a^j, z)u^j$. Then $\mu p_{\gamma_1}((a^j)_{n_j} u^j) = \psi p_{\gamma_2}((a^j)_{n_j} u^j)$. Summing up both sides of the equation with respect to j, we have $\mu p_{\gamma_1} v = \psi p_{\gamma_2} v$. However, $v \in S^{\gamma_1}$ implies that $p_{\gamma_1} v = v$ and $p_{\gamma_2} v = 0$. This is a contradiction since $\mu \neq 0$ and $v \neq 0$. Thus $|\Gamma| = 1$ and $W^0 \cdot M \cong W$ as required.

If $V_L^\tau \cdot M$ is not a simple V_L^τ-module, then there is a proper V_L^τ-submodule N of $V_L^\tau \cdot M$. Since M and W are simple M^0-modules, N must be isomorphic to M or W as an M^0-module. Then the top level of N is of dimension one. The simple

V_L^τ-modules with one dimensional top level are classified in Lemma 5.3. Each of them is a direct sum of two simple M^0-modules. However, N is not of such a form. Thus $V_L^\tau \cdot M$ is a simple V_L^τ-module. □

Lemma 5.7. *Let $U = M \oplus W$ be an M^0-module such that $M \cong M^1 \otimes M^2$ and $W \cong W^1 \otimes W^2$ for some $M^i \in \mathcal{M}_i$ and $W^i \in \mathcal{W}_i$, $i = 1, 2$. Then U admits at most one simple V_L^τ-module structure.*

Proof. Assume that (U, Y_1) and (U, Y_2) are simple V_L^τ-modules such that $Y_i(a, z) = Y(a, z)$ for all $a \in M^0$, $i = 1, 2$, where (U, Y) is the given M^0-module structure. We denote the vertex operator of V_L^τ by $\tilde{Y}(v, z)$ for $v \in V_L^\tau$. Let $p_{M^0} : V_L^\tau \to M^0$ and $p_{W^0} : V_L^\tau \to W^0$ be projections and define $\mathcal{I}(\,\cdot\,, z)$ and $\mathcal{J}(\,\cdot\,, z)$ by

$$\mathcal{I}(a, z)b = p_{M^0}\tilde{Y}(a, z)b, \qquad \mathcal{J}(a, z)b = p_{W^0}\tilde{Y}(a, z)b$$

for $a, b \in W^0$. Then by (5-18), $\mathcal{I}(\,\cdot\,, z)$ and $\mathcal{J}(\,\cdot\,, z)$ are nonzero intertwining operators of respective types $\binom{M^0}{W^0\ W^0}$ and $\binom{W^0}{W^0\ W^0}$. By the fusion rules (5-17), the space $I_{M^0}\binom{M^0}{W^0\ W^0}$ of M^0-intertwining operators of type $\binom{M^0}{W^0\ W^0}$ is of dimension one. Likewise, $\dim I_{M^0}\binom{W^0}{W^0\ W^0} = 1$. Note that $W^0 \cdot M^0 \subset W^0$ and that $\mathcal{I}(a, z)b + \mathcal{J}(a, z)b = \tilde{Y}(a, z)b$.

Let $p_M : U \to M$ and $p_W : U \to W$ be projections. Define $\mathcal{F}_i^M(\,\cdot\,, z)$ and $\mathcal{F}_i^W(\,\cdot\,, z)$, $i = 1, 2$ by

$$\mathcal{F}_i^M(a, z)w = p_M Y_i(a, z)w, \qquad \mathcal{F}_i^W(a, z)w = p_W Y_i(a, z)w$$

for $a \in W^0$ and $w \in W$. Then $\mathcal{F}_i^M(\,\cdot\,, z)$ and $\mathcal{F}_i^W(\,\cdot\,, z)$ are intertwining operators of type $\binom{M}{W^0\ W}$ and $\binom{W}{W^0\ W}$, respectively. Clearly, $\mathcal{F}_i^M(a, z)w + \mathcal{F}_i^W(a, z)w = Y_i(a, z)w$. If $\mathcal{F}_i^M(\,\cdot\,, z) = 0$, then $W^0 \cdot W \subset W$ and so $V_L^\tau \cdot W = M^0 \cdot W + W^0 \cdot W \subset W$. This is a contradiction, since U is a simple V_L^τ-module. Hence $\mathcal{F}_i^M(\,\cdot\,, z) \neq 0$. Let

$$\mathcal{G}_i^W(a, z)v = Y_i(a, z)v$$

for $a \in W^0$, $v \in M$. Then $\mathcal{G}_i^W(\,\cdot\,, z)$ is a nonzero intertwining operator of type $\binom{W}{W^0\ M}$ by (5-17). The space of M^0-intertwining operators $I_{M^0}\binom{W}{W^0\ M}$ of type $\binom{W}{W^0\ M}$ is of dimension one by (5-17). Similarly, $\dim I_{M^0}\binom{M}{W^0\ W} = \dim I_{M^0}\binom{W}{W^0\ W} = 1$. Therefore, $\mathcal{F}_2^M(\,\cdot\,, z) = \lambda\mathcal{F}_1^M(\,\cdot\,, z)$, $\mathcal{F}_2^W(\,\cdot\,, z) = \mu\mathcal{F}_1^W(\,\cdot\,, z)$, and $\mathcal{G}_2^W(\,\cdot\,, z) = \gamma\mathcal{G}_1^W(\,\cdot\,, z)$ for some $\lambda, \mu, \gamma \in \mathbb{C}$ with $\lambda \neq 0$ and $\gamma \neq 0$.

Now,

$$Y_i(a, z_1)Y_i(b, z_2)v = \big(\mathcal{F}_i^M(a, z_1) + \mathcal{F}_i^W(a, z_1)\big)\mathcal{G}_i^W(b, z_2)v,$$
$$Y_i(b, z_2)Y_i(a, z_1)v = \big(\mathcal{F}_i^M(b, z_2) + \mathcal{F}_i^W(b, z_2)\big)\mathcal{G}_i^W(a, z_1)v,$$
$$Y_i(\tilde{Y}(a, z_0)b, z_2)v = Y_i(\mathcal{I}(a, z_0)b, z_2)v + \mathcal{G}_i^W(\mathcal{J}(a, z_0)b, z_2)v$$

for $a, b \in W^0$ and $v \in M$. Taking the image of both sides of the Jacobi identity

$$z_0^{-1}\delta\Big(\frac{z_1-z_2}{z_0}\Big)Y_i(a,z_1)Y_i(b,z_2)v - z_0^{-1}\delta\Big(\frac{z_2-z_1}{-z_0}\Big)Y_i(b,z_2)Y_i(a,z_1)v = z_2^{-1}\delta\Big(\frac{z_1-z_0}{z_2}\Big)Y_i(\tilde{Y}(a,z_0)b,z_2)v \tag{5-26}$$

under the projection p_M, we obtain

$$z_0^{-1}\delta\Big(\frac{z_1-z_2}{z_0}\Big)\mathscr{F}_i^M(a,z_1)\mathscr{G}_i^W(b,z_2)v - z_0^{-1}\delta\Big(\frac{z_2-z_1}{-z_0}\Big)\mathscr{F}_i^M(b,z_2)\mathscr{G}_i^W(a,z_1)v = z_2^{-1}\delta\Big(\frac{z_1-z_0}{z_2}\Big)Y_i(\mathscr{I}(a,z_0)b,z_2)v. \tag{5-27}$$

Likewise, if we take the image of both sides of (5-26) under the projection p_W, then

$$z_0^{-1}\delta\Big(\frac{z_1-z_2}{z_0}\Big)\mathscr{F}_i^W(a,z_1)\mathscr{G}_i^W(b,z_2)v - z_0^{-1}\delta\Big(\frac{z_2-z_1}{-z_0}\Big)\mathscr{F}_i^W(b,z_2)\mathscr{G}_i^W(a,z_1)v = z_2^{-1}\delta\Big(\frac{z_1-z_0}{z_2}\Big)\mathscr{G}_i^W(\mathscr{J}(a,z_0)b,z_2)v. \tag{5-28}$$

Comparing Equation (5-28) for $i = 1$ and $i = 2$, we have

$$\gamma(\mu-1)z_2^{-1}\delta\Big(\frac{z_1-z_0}{z_2}\Big)\mathscr{G}_1^W(\mathscr{J}(a,z_0)b,z_2)v = 0,$$

since $\mathscr{F}_2^M(\,\cdot\,,z) = \lambda\mathscr{F}_1^M(\,\cdot\,,z)$, $\mathscr{F}_2^W(\,\cdot\,,z) = \mu\mathscr{F}_1^W(\,\cdot\,,z)$, and $\mathscr{G}_2^W(\,\cdot\,,z) = \gamma\mathscr{G}_1^W(\,\cdot\,,z)$. Now, $z_2^{-1}\delta\big(\frac{z_1-z_0}{z_2}\big) = z_1^{-1}\delta\big(\frac{z_2+z_0}{z_1}\big)$ by [Frenkel et al. 1988, Proposition 8.8.5] and so the above equation is equivalent to the following assertion.

$$\gamma(\mu-1)(z_2+z_0)^k\mathscr{G}_1^W(\mathscr{J}(a,z_0)b,z_2)v = 0 \quad \text{for all} \quad k \in \mathbb{Z}.$$

This implies that

$$\gamma(\mu-1)\mathscr{G}_1^W(\mathscr{J}(a,z_0)b,z_2)v = 0,$$

since $\mathscr{G}_1^W(\mathscr{J}(a,z_0)b,z_2)v \in W((z_0))[[z_2,z_2^{-1}]]$. Then since $\mathscr{J}(\,\cdot\,,z)$ and $\mathscr{G}_1^W(\,\cdot\,,z)$ are nonzero, we conclude that $\mu = 1$.

Next, we use Equation (5-27). Since $\mathscr{I}(a,z_0)b \in M^0((z_0))$, we have

$$Y_1(\mathscr{I}(a,z_0)b,z_2)v = Y_2(\mathscr{I}(a,z_0)b,z_2)v$$

by our assumption. Then it follows from (5-27) for $i = 1, 2$ that

$$(\lambda\gamma-1)z_2^{-1}\delta\Big(\frac{z_1-z_0}{z_2}\Big)Y_1(\mathscr{I}(a,z_0)b,z_2)v = 0.$$

Since $\mathscr{I}(\,\cdot\,,z) \neq 0$ and M is a simple (M^0, Y_1)-module, a similar argument as above gives that $\lambda\gamma = 1$.

For $a \in M^0$, $b \in W^0$, $v \in M$, and $w \in W$,

$$Y_i(a+b, z)(v+w) = Y_i(a, z)v + Y_i(a, z)w + \mathscr{G}_i^W(b, z)v + \bigl(\mathscr{F}_i^M(b, z) + \mathscr{F}_i^W(b, z)\bigr)w.$$

Note that $Y_i(a, z)v$, $\mathscr{F}_i^M(b, z)w \in M((z))$ and $Y_i(a, z)w$, $\mathscr{G}_i^W(b, z)v$, $\mathscr{F}_i^W(b, z)w \in W((z))$. Define $\varphi : U \to U$ by $\varphi(u) = \lambda u$ if $u \in M$ and $\varphi(u) = u$ if $u \in W$. Since $\mu = 1$ and $\lambda\gamma = 1$, we can verify that

$$Y_2(a + b, z)\varphi(v + w) = \varphi\bigl(Y_1(a + b, z)(v + w)\bigr).$$

Thus φ is an isomorphism of V_L^τ-modules from (U, Y_1) onto (U, Y_2). This completes the proof. □

Remark 5.8. The proof of the above lemma is essentially the same as that of [Lam et al. 2005, Lemma C.3]. Consider the Jacobi identity for $a, b \in W^0$ and $w \in W$ and take the images of both sides of the identity under the projections p_M and p_W, respectively. Then

$$\begin{aligned} z_0^{-1}\delta\Bigl(\frac{z_1 - z_2}{z_0}\Bigr)\mathscr{F}_i^M(a, z_1)\mathscr{F}_i^W(b, z_2)w - z_0^{-1}\delta\Bigl(\frac{z_2 - z_1}{-z_0}\Bigr)\mathscr{F}_i^M(b, z_2)\mathscr{F}_i^W(a, z_1)w \\ = z_2^{-1}\delta\Bigl(\frac{z_1 - z_0}{z_2}\Bigr)\mathscr{F}_i^M(\mathscr{I}(a, z_0)b, z_2)w, \end{aligned}$$

$$\begin{aligned} z_0^{-1}\delta\Bigl(\frac{z_1 - z_2}{z_0}\Bigr)\bigl(\mathscr{G}_i^W(a, z_1)\mathscr{F}_i^M(b, z_2) + \mathscr{F}_i^W(a, z_1)\mathscr{F}_i^W(b, z_2)\bigr)w \\ - z_0^{-1}\delta\Bigl(\frac{z_2 - z_1}{-z_0}\Bigr)\bigl(\mathscr{G}_i^W(b, z_2)\mathscr{F}_i^M(a, z_1) + \mathscr{F}_i^W(b, z_2)\mathscr{F}_i^W(a, z_1)\bigr)w \\ = z_2^{-1}\delta\Bigl(\frac{z_1 - z_0}{z_2}\Bigr)\bigl(Y_i(\mathscr{I}(a, z_0)b, z_2) + \mathscr{F}_i^W(\mathscr{I}(a, z_0)b, z_2)\bigr)w. \end{aligned}$$

Each of these two equations gives the identical equations in case of $i = 1$ and $i = 2$ provided that $\mu = 1$ and $\lambda\gamma = 1$.

Theorem 5.9. *There are exactly* 30 *inequivalent simple V_L^τ-modules. They are represented by the* 30 *simple V_L^τ-modules listed in Lemma 3.2.*

Proof. Let U be a simple V_L^τ-module. Then by Lemma 5.5, U contains a simple M^0-submodule M isomorphic to a member of $\mathscr{M}$. Since U is a simple V_L^τ-module, Lemma 5.6 implies that $U = M \oplus W$ for some simple M^0-submodule W isomorphic to a member of $\mathscr{W}$. In fact, the isomorphism class of W is uniquely determined by M. By Lemma 5.7, U admits a unique V_L^τ-module structure. Since $\mathscr{M}$ consists of 30 members, it follows that there are at most 30 inequivalent simple V_L^τ-modules. Hence the assertion holds. □

Theorem 5.10. *V_L^τ is a rational vertex operator algebra.*

Proof. It is sufficient to show that every $\mathbb{N}$-graded weak V_L^τ-module U is a sum of simple V_L^τ-modules. Since M^0 is rational, U is a direct sum of simple M^0-modules. Thus by Lemma 5.2, we may assume that $U = \big(\bigoplus_{\gamma\in\Gamma} S^\gamma\big) \oplus \big(\bigoplus_{\lambda\in\Lambda} S^\lambda\big)$, where S^γ is isomorphic to a member of $\mathcal{M}$ and S^λ is isomorphic to a member of $\mathcal{W}$. We know that $V_L^\tau \cdot S^\gamma$ is a simple V_L^τ-module by Lemma 5.6. Set $N = \sum_{\gamma\in\Gamma} V_L^\tau \cdot S^\gamma$. Since U/N has no simple M^0-submodule isomorphic to a member of $\mathcal{M}$, it follows from Lemma 5.5 that $U = N$ and the proof is complete. □

Corollary 5.11. *The Zhu algebra $A(V_L^\tau)$ of V_L^τ is a 51 dimensional semisimple associative algebra isomorphic to a direct sum of 23 copies of the one dimensional algebra $\mathbb{C}$ and 7 copies of the algebra $\mathrm{Mat}_2(\mathbb{C})$ of 2×2 matrices. Moreover, $A(V_L^\tau)$ is generated by $[\tilde\omega^1]$, $[\tilde\omega^2]$, $[J]$, $[K]$, and $[P]$.*

Proof. Since V_L^τ is rational, $A(V_L^\tau)$ is a finite dimensional semisimple associative algebra [Dong et al. 1998a, Theorem 8.1; Zhu 1996, Theorem 2.2.3]. We know all the simple V_L^τ-modules and the action of $[\tilde\omega^1]$, $[\tilde\omega^2]$, $[J]$, $[K]$, and $[P]$ on their top levels in Section 4. Hence we can determine the structure of $A(V_L^\tau)$ as in the assertion. □

Appendix: Some fusion rules for $M(0)$

We give a proof of the fusion rules

$$W(0) \times M_T(\tau^i)(\varepsilon) = W_T(\tau^i)(\varepsilon),$$
$$W(0) \times W_T(\tau^i)(\varepsilon) = M_T(\tau^i)(\varepsilon) + W_T(\tau^i)(\varepsilon),$$

$i = 1, 2$, $\varepsilon = 0, 1, 2$ of simple $M(0)$-modules in (3-25).

Recall that $V_L^\tau \cong M^0 \oplus W^0$, where $M^0 = M(0)\otimes M_t^0$ and $W^0 = W(0)\otimes W_t^0$. Set $\hat M_T(\tau^i)(\varepsilon) = M_T(\tau^i)(\varepsilon)\otimes M_t^0$ and $\hat W_T(\tau^i)(\varepsilon) = W_T(\tau^i)(\varepsilon)\otimes W_t^0$, which are simple M^0-modules. Then

$$V_L^0(\tau)(\varepsilon) \cong \hat M_T(\tau)(\varepsilon) \oplus \hat W_T(\tau)(\varepsilon),$$
$$V_L^0(\tau^2)(\varepsilon) \cong \hat M_T(\tau^2)(\varepsilon) \oplus \hat W_T(\tau^2)(\varepsilon)$$

as M^0-modules by (3-23) and (3-24). Denote by $Y_1(\,\cdot\,, z)$ (resp. $Y_2(\,\cdot\,, z)$) the vertex operator of the simple V_L^τ-module $V_L^0(\tau)(\varepsilon)$ (resp. $V_L^0(\tau^2)(\varepsilon)$). Let $p_M : V_L^0(\tau)(\varepsilon) \to \hat M_T(\tau)(\varepsilon)$ and $p_W : V_L^0(\tau)(\varepsilon) \to \hat W_T(\tau)(\varepsilon)$ be projections. We also use the same symbol p_M or p_W to denote a projection from $V_L^0(\tau^2)(\varepsilon)$ onto $\hat M_T(\tau^2)(\varepsilon)$ or onto $\hat W_T(\tau^2)(\varepsilon)$. We fix $i = 1, 2$ and $\varepsilon = 0, 1, 2$. For simplicity of notation, set $\hat M = \hat M_T(\tau^i)(\varepsilon)$ and $\hat W = \hat W_T(\tau^i)(\varepsilon)$.

Let $\mathcal{F}_i^M(a, z)w = p_M Y_i(a, z)w$ and $\mathcal{F}_i^W(a, z)w = p_W Y_i(a, z)w$ for $a \in W^0$ and $w \in \hat W$. Then $\mathcal{F}_i^M(\,\cdot\,, z)$ and $\mathcal{F}_i^W(\,\cdot\,, z)$ are intertwining operators of type $\binom{\hat M}{W^0\ \hat W}$ and

$\binom{\hat{W}}{W^0\ \hat{W}}$, respectively. Likewise, let $\mathcal{G}_i^W(a,z)v = Y_i(a,z)v$ for $a \in W^0$ and $v \in \hat{M}$. Then $\mathcal{G}_i^W(\,\cdot\,,z)$ is an intertwining operator of type $\binom{\hat{W}}{W^0\ \hat{M}}$, since the fusion rule $W_t^0 \times M_t^0 = W_t^0$ of M_t^0-modules implies that $W^0 \cdot \hat{M} = \operatorname{span}\{a_n\hat{M} \mid a \in W^0, n \in \mathbb{Z}\}$ is contained in $\hat{W}$. If $\mathcal{G}_i^W(\,\cdot\,,z) = 0$, then $V_L^\tau \cdot \hat{M} = (M^0 + W^0)\cdot \hat{M} \subset \hat{M}$. This is a contradiction, since $V_L^0(\tau)(\varepsilon)$ and $V_L^0(\tau^2)(\varepsilon)$ are simple V_L^τ-modules. Thus $\mathcal{G}_i^W(\,\cdot\,,z) \neq 0$. Similarly, $\mathcal{F}_i^M(\,\cdot\,,z) \neq 0$. Indeed, if $\mathcal{F}_i^M(\,\cdot\,,z) = 0$, then $V_L^\tau \cdot \hat{W} \subset \hat{W}$, which is a contradiction. Assume that $\mathcal{F}_i^W(\,\cdot\,,z) = 0$. Then $W^0 \cdot \hat{W} \subset \hat{M}$ and so $W^0 \cdot (W^0 \cdot \hat{W}) \subset \hat{W}$. However, $W^0 \cdot (W^0 \cdot \hat{W}) \supset (W^0 \cdot W^0)\cdot \hat{W} = V_L^\tau \cdot \hat{W}$ by Lemma 2.6 and (5-18). This contradiction implies that $\mathcal{F}_i^W(\,\cdot\,,z) \neq 0$.

Restricting the three nonzero intertwining operators $\mathcal{F}_i^M(\,\cdot\,,z)$, $\mathcal{F}_i^W(\,\cdot\,,z)$, and $\mathcal{G}_i^W(\,\cdot\,,z)$ to the first component of each of the tensor products $W^0 = W(0) \otimes W_t^0$, $\hat{M} = M_T(\tau^i)(\varepsilon) \otimes M_t^0$, and $\hat{W} = W_T(\tau^i)(\varepsilon) \otimes W_t^0$, we obtain nonzero intertwining operators of type

$$\binom{M_T(\tau^i)(\varepsilon)}{W(0)\ \ W_T(\tau^i)(\varepsilon)}, \quad \binom{W_T(\tau^i)(\varepsilon)}{W(0)\ \ W_T(\tau^i)(\varepsilon)} \quad \binom{W_T(\tau^i)(\varepsilon)}{W(0)\ \ M_T(\tau^i)(\varepsilon)}$$

for $M(0)$-modules, respectively.

Let N^2 be one of $M_T(\tau^i)(\varepsilon)$, $W_T(\tau^i)(\varepsilon)$, $i = 1, 2$, $\varepsilon = 0, 1, 2$ and let N^3 be any of the 20 simple $M(0)$-modules. Then the top level $N^j_{(0)}$ of N^j is of dimension one. By [Dong et al. 2004], the Zhu algebra $A(M(0))$ of $M(0)$ is generated by $[\tilde{\omega}^1]$ and $[J]$. Moreover, we know the action of $o(\tilde{\omega}^1)$ and $o(J)$ on $N^j_{(0)}$. Thus, by an argument as in [Tanabe 2005, pp. 192–193], we can calculate that the dimension of

$$\operatorname{Hom}_{A(M(0))}(A(W(0)) \otimes_{A(M(0))} N^2_{(0)}, N^3_{(0)})$$

is at most one and it is equal to one if and only if the pair (N^2, N^3) is one of

$$(M_T(\tau^i)(\varepsilon), W_T(\tau^i)(\varepsilon)), \quad (W_T(\tau^i)(\varepsilon), M_T(\tau^i)(\varepsilon)), \quad (W_T(\tau^i)(\varepsilon), W_T(\tau^i)(\varepsilon))$$

for $i = 1, 2$, $\varepsilon = 0, 1, 2$. Note that $W(0)$ was denoted by $W_k^{0(0)}$ in [Tanabe 2005]. Now, the desired fusion rules are obtained by [Li 1999a, Proposition 2.10 and Corollary 2.13].

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KENICHIRO TANABE
DEPARTMENT OF MATHEMATICS
HOKKAIDO UNIVERSITY
KITA 10, NISHI 8, KITA-KU
SAPPORO, HOKKAIDO 060-0810
JAPAN

ktanabe@math.sci.hokudai.ac.jp

HIROMICHI YAMADA
DEPARTMENT OF MATHEMATICS
HITOTSUBASHI UNIVERSITY
NAKA 2-1, KUNITACHI
TOKYO 186-8601
JAPAN

yamada@math.hit-u.ac.jp