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We obtain the isoperimetric profile for the standard initial slices in the Reissner–Nordstrom and Schwarzschild anti-de Sitter spacetimes, following recent work of Bray and Morgan on isoperimetric comparison. We then discuss these results in the context of Bray’s isoperimetric approach to the Penrose inequality.

1. Introduction

One of the major recent developments in mathematical relativity is the resolution of the Riemannian case of the Penrose conjecture, by Huisken and Ilmanen [2001] and by Bray [2001]. Bray had obtained earlier partial results in his thesis [1997] by using isoperimetric surface techniques. As a key step, Bray established that the isoperimetric profile of the time-symmetric Schwarzschild initial data (of positive mass) is given by the radially symmetric spheres, the method of proof of which has been codified in [Bray and Morgan 2002]. The main idea is that one can deduce the isoperimetric profile of a given metric if one can construct an appropriate map to a model space (for instance, Euclidean space or hyperbolic space) in which the profile is known. We obtain below as a direct corollary the isoperimetric profile for the Reissner–Nordstrom initial data. We then carry out an extension of the method to derive the isoperimetric profile for the Schwarzschild anti-de Sitter (AdS) data; unlike the previous two families, which are asymptotically flat, Schwarzschild AdS is asymptotically hyperbolic. In all these cases, the spaces are rotationally symmetric, and the rotationally symmetric spheres give the isoperimetric profile. For contrast, in the negative mass Schwarzschild, the analogous family of spheres is unstable, as we discuss below.

We will review Bray’s isoperimetric surface approach to the Penrose inequality and discuss its extension to certain asymptotically flat solutions of the Einstein–Maxwell constraint equations. We also include computations relevant to a form

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of the Penrose inequality for a class of asymptotically hyperbolic spaces. For background and references on the Penrose inequality, see [Bray 2002; Bray and Chruściel 2004]; for recently announced work by Huisken which explores the relation between isoperimetric inequalities and the mass of asymptotically flat metrics, see [Huisken 2005; 2006].

2. Preliminaries

We recall the isoperimetric problem and introduce the families of metrics (Schwarzschild, Reissner–Nordstrom, and Schwarzschild AdS) whose isoperimetric profiles we will discuss.

Isoperimetric problems. The isoperimetric problem is the classical problem of how to enclose a given volume V with a surface of least area. In Euclidean and hyperbolic space, homogeneity allows one to conclude that if a volume V can be enclosed with a surface of area A , a volume $V_0 < V$ can be enclosed with an area $A_0 < A$. In general spaces, one can pose an isoperimetric problem to find the minimum area that encloses a volume of at least V . It is a classical result that in Euclidean and hyperbolic spaces, the most efficient way to enclose a volume V is by using a sphere [Chavel 1993; Howards et al. 1999]. We will in fact consider the problem of minimizing volume *against* a (two-sided) hypersurface Σ_0 ; that is, we consider the problem of finding least-area enclosures in the homology class of Σ_0 of net volume (at least) V with Σ_0 .

The metrics of interest. We will focus on three families of spherically symmetric metrics which appear as constant time slices in well-known solutions of the Einstein equations of general relativity. Let \mathbb{S}^2 be the two-dimensional sphere and let $d\Omega^2$ be the standard round metric on the unit two-sphere. Each of the following metrics is defined on the smooth manifold $(r_0, +\infty) \times \mathbb{S}^2$, where $m > 0$, and $r_0 > 0$ is specified below:

(1) Schwarzschild metric:

$$\left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \quad r_0 = 2m.$$

(2) Schwarzschild AdS metric:

$$\left(1 + r^2 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \quad \text{where } r_0 \text{ satisfies } 1 + r_0^2 - \frac{2m}{r_0} = 0.$$

(3) Reissner–Nordstrom metric:

$$\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2,$$

where $m^2 > Q^2$ and r_0 is the larger solution of

$$1 - \frac{2m}{r_0} + \frac{Q^2}{r_0^2} = 0.$$

The parameter m measures the deviation (in the third example, the top-order deviation) of the metrics from the model Euclidean or hyperbolic metrics. It is called the *mass*, and indeed it has an interpretation in terms of the energy of isolated gravitational systems [Bray 2002; Bray and Chruściel 2004]. See the [Appendix](#) for useful formulas (Christoffel symbols, curvatures) for these metrics. Each metric extends to r_0 , where there is a minimal sphere (*horizon*) S_{r_0} . This minimal sphere is in fact totally geodesic, and the metrics can be smoothly reflected across the horizon (using inversion in the horizon sphere with respect to the metric distance along radial geodesics) to produce complete metrics with two ends. These metrics are conformally flat; for example, the extended Schwarzschild metric in appropriate coordinates is precisely $(1 + m/(2r))^4 \delta$, where δ is the Euclidean metric. In these coordinates the horizon is located at $r = m/2$, and the inversion $r \mapsto m^2/(4r)$ is an isometry.

The Schwarzschild metric with $m > 0$ can be isometrically embedded into the Euclidean space \mathbb{R}^4 as the set $\{(x, y, z, w) : r = w^2/(8m) + 2m\}$, where $r^2 = x^2 + y^2 + z^2$. To see this, we look for an embedding which in terms of spherical coordinates on Schwarzschild is of the form $(r, \omega) \mapsto (r\omega, \xi(r)) \in \mathbb{R}^4$. Using the above form of the Schwarzschild metric, we see that the map is an isometry if and only if $(\xi'(r))^2 + 1 = (1 - 2m/r)^{-1}$, which can be rewritten (choosing $\xi'(r) > 0$) as

$$\xi'(r) = \sqrt{\frac{2m}{r-2m}}.$$

We note that $\xi(r) = \sqrt{8m(r-2m)}$ does indeed satisfy this equation. Interestingly enough, this derivation breaks down for $m < 0$; however (as was pointed out to us by Greg Galloway and Hubert Bray), the same idea can be pushed through in the negative mass case to obtain an isometric embedding of the negative mass Schwarzschild into *Minkowski* space.

The Einstein constraint equations. The three families of metrics above give particular solutions to the Einstein constraint equations, as we now recall. The Einstein equations for the corresponding four-dimensional Lorentzian spacetimes (\mathcal{S}, \bar{g}) in which these three-dimensional Riemannian spaces embed as totally geodesic spacelike slices are $\text{Ric}(\bar{g}) = 0$, $\text{Ric}(\bar{g}) = -3\bar{g}$, and $\text{Ric}(\bar{g}) - \frac{1}{2}R(\bar{g})\bar{g} = 8\pi T$, respectively, where T is the stress-energy tensor of a Maxwell field [Wald 1984]. Consider in general any spacetime (\mathcal{S}, \bar{g}) satisfying one of these Einstein equations; then the Gauss and Codazzi equations (together with the Einstein equation)

imply constraint equations on the geometry (intrinsic and extrinsic) of spacelike slices. If g is the induced metric and \mathbb{II} the second fundamental form (with trace H) of a spacelike slice, then using the Einstein equation along with the Gauss equation, we obtain the Hamiltonian constraint, which in the first two cases yields $R(g) - \|\mathbb{II}\|^2 + H^2 = 0$ and $R(g) - \|\mathbb{II}\|^2 + H^2 = -6$, respectively. In the totally geodesic case ($\mathbb{II} = 0$), these constraints reduce to the condition of constant scalar curvature $R(g) = 0$ or $R(g) = -6$, respectively; in the case of a maximal slice ($H = 0$), the constraints imply the inequalities $R(g) \geq 0$ and $R(g) \geq -6$. Similarly, the (totally geodesic) Einstein–Maxwell constraint equations for a metric g and an electromagnetic field E are given by the Hamiltonian constraint $R(g) = 2|E|^2$ coupled with the Maxwell field equation $\operatorname{div}_g E = 0$. If we let \mathbf{e}_r be the unit outward radial vector, and couple the field $E = (Q/r^2)\mathbf{e}_r$ to the Reissner–Nordstrom metric, we produce a solution to the Einstein–Maxwell constraints.

On the isoperimetric inequality and the mass. In Euclidean space, the isoperimetric inequality for a closed surface Σ of area A enclosing a volume V can be written $V \leq A^{3/2}/(6\sqrt{\pi})$, with equality precisely when Σ is a round sphere. We compare this to Schwarzschild, where (using [Corollary 3.5](#)) it is easy to compute the volume $V(\sigma)$ enclosed by the isoperimetric sphere of area σ . In fact, if we use the conformally flat coordinates for Schwarzschild, in which the metric is $(1 + m/(2r))^4\delta$, we have

$$A(S_r) = 4\pi r^2 \left(1 + \frac{m}{2r}\right)^4.$$

Thus

$$\frac{A(S_r)^{3/2}}{6\sqrt{\pi}} = \frac{4\pi}{3} r^3 \left(1 + \frac{3m}{r} + mO\left(\frac{1}{r^2}\right)\right).$$

The net volume enclosed by S_r has the expansion

$$4\pi \int_{m/2}^r \left(1 + \frac{m}{2t}\right)^6 t^2 dt = \frac{4\pi}{3} r^3 \left(1 + \frac{9m}{2r} + mO\left(\frac{1}{r^2}\right)\right).$$

From this it is easy to see that the volume enclosed by the isoperimetric sphere of area σ has the expansion

$$V(\sigma) = \frac{\sigma^{3/2}}{6\pi^{1/2}} \left(1 + \frac{(3\sqrt{\pi})m}{\sqrt{\sigma}} + mO\left(\frac{1}{\sigma}\right)\right).$$

This is yet another quantitative way in which the mass m measures the deviation of the geometry from that of Euclidean, which is explored in the recent work of Huisken [[2005](#); [2006](#)].

3. Isoperimetric profiles by comparison

We review the isoperimetric comparison theorem of Bray and Morgan and apply it to the Schwarzschild and Reissner–Nordstrom spaces. Let $I \subset \mathbb{R}$ be an interval. Suppose we have a rotationally symmetric model space $M_0 = I \times \mathbb{S}^2$ with the twisted product metric $dr^2 + \varphi_0^2(r) d\Omega^2$ for which we know the isoperimetric surfaces are the radially symmetric spheres $S_c = \{r = c\}$. We consider another rotationally symmetric space $M = I \times \mathbb{S}^2$ with the metric $dr^2 + \varphi^2(r) d\Omega^2$. Bray and Morgan showed that under certain geometric conditions, the isoperimetric surfaces in M are also the radially symmetric spheres. We now recall their argument, which as in [Bray and Morgan 2002] can be more generally applied to twisted products $I \times N$ with a closed manifold fiber N .

Let $F : M \rightarrow M_0$ map radially symmetric spheres in M to radially symmetric spheres in M_0 , so that $F(r, \omega) = (\psi(r), \omega)$. We assume that ψ is increasing, so that F is orientation-preserving. We define the area stretch AS_Σ for a surface $\Sigma \subset M$ by the equation $F^*(dA_{F(\Sigma)}) = AS_\Sigma dA_\Sigma$, where dA_Σ and $dA_{F(\Sigma)}$ are the area forms of $\Sigma \subset M$ and $F(\Sigma) \subset M_0$, respectively. The volume stretch VS is defined similarly by $F^*(dV_{M_0}) = VS dV_M$, where dV_M and dV_{M_0} are the volume forms of M and M_0 , respectively. By symmetry, VS depends only on r . Finally, let $A(\Sigma)$ be the area of the surface $\Sigma \subset M$, and let $A_0(\Sigma_0)$ be the area of the surface $\Sigma_0 \subset M_0$.

Let $a = A(S_{r_1})/A_0(F(S_{r_1}))$. Suppose the map F can be constructed so that the area stretch under F satisfies $AS_\Sigma \leq 1/a$, so the volume stretch satisfies $VS(r) \leq b$ for $r < r_1$, and $VS(r) \geq b$ for $r > r_1$. Now suppose there were a surface $\Sigma \subset M$ bounding nonnegative net volume against S_{r_1} (that is, Σ bounds no less volume against S_{r_0} than S_{r_1} does), so that Σ has the same or less surface area as S_{r_1} . We will show that in fact $A(\Sigma) = A(S_{r_1})$, which will then imply that S_{r_1} is an isoperimetric surface. Since the volume stretch for $r > r_1$ is no less than the volume stretch for $r < r_1$, the net volume bounded by $F(\Sigma)$ contained in $\{r > \psi(r_1)\}$ is no less than the net volume bounded by $F(\Sigma)$ contained in $\{r < \psi(r_1)\}$. Thus the net volume bounded by $F(\Sigma)$ is greater than or equal to the volume bounded by $F(S_{r_1})$. Since the area stretch $AS_\Sigma \leq 1/a = A_0(F(S_{r_1}))/A(S_{r_1})$, and $A(\Sigma) \leq A(S_{r_1})$, we obtain

$$\begin{aligned} A_0(F(\Sigma)) &= \int_{F(\Sigma)} dA_{F(\Sigma)} = \int_\Sigma F^* dA_{F(\Sigma)} = \int_\Sigma AS_\Sigma dA_\Sigma \\ &\leq \frac{1}{a} A(\Sigma) = A_0(F(S_{r_1})) \frac{A(\Sigma)}{A(S_{r_1})} \leq A_0(F(S_{r_1})). \end{aligned}$$

Since $F(S_{r_1}) = S_{\psi(r_1)}$ is an isoperimetric surface in M_0 , $F(\Sigma)$ and $F(S_{r_1})$ must thus bound the same amount of volume and have the same surface areas, $A_0(F(\Sigma)) = A_0(F(S_{r_1}))$. Thus the inequalities above must be equalities, and we see that indeed

$A(\Sigma) = A(S_{r_1})$. Therefore we have shown by comparison that S_{r_1} is an isoperimetric surface in M ; if we have uniqueness for the isoperimetric surfaces, we can go further to assert $\Sigma = S_{r_1}$.

To put this observation to work, one identifies concrete geometric conditions that allow such a map F to be constructed. Indeed the main theorem in [Bray and Morgan 2002] is stated in geometric terms from which the following is readily established as a corollary. We note that the comparison space M_0 for this corollary is Euclidean space, so the comparison metric is $dr^2 + r^2 d\Omega^2$.

Theorem 3.1 [Bray and Morgan 2002]. *Consider a rotationally symmetric three-manifold $M = I \times \mathbb{S}^2$ with the metric $dr^2 + \varphi^2(r) d\Omega^2$. Suppose (1) φ' is non-decreasing for all r , and (2) $0 \leq \varphi' \leq 1$ for all $r \geq r_0$. Then for all $r \geq r_0$, the radially symmetric spheres S_r minimize surface area among smooth surfaces enclosing the same volume with S_{r_0} , where volume inside $\{r < r_0\}$ is counted as negative. Furthermore, these spheres are unique minimizers if $\varphi'(r) < 1$.*

Condition (1) holds if and only if M has nonpositive radial Ricci curvature. For any r , condition (2) holds if and only if S_r has nonnegative (inward) mean curvature and M has nonnegative tangential sectional curvature, or equivalently, S_r has nonnegative Hawking mass.

We take the mean curvature to be the trace of the second fundamental form (the sum of the principal curvatures), not the average of the principal curvatures as in [Bray and Morgan 2002]. We recall the Hawking mass of a surface Σ is

$$m_H(\Sigma) = \sqrt{\frac{A(\Sigma)}{16\pi}} \left(1 - \frac{1}{16\pi} \int_{\Sigma} H^2 dA \right).$$

We will see the Hawking mass play a role in the Penrose inequality below; in fact, the underlying motivation for the Huisken–Ilmanen inverse mean curvature flow is the monotonicity of the Hawking mass under the flow [Geroch 1973].

We are interested in spaces (M, g) with $M = I \times \mathbb{S}^2$ and with $g = f(r) dr^2 + r^2 d\Omega^2$, where f is a positive function. The metrics which we study here, in both the forms given above and for the metrics suitably extended by reflection, all have this form. In order to apply Theorem 3.1 to such spaces, we note:

Lemma 3.2. *The metric $g = f(r) dr^2 + r^2 d\Omega^2$ can be written as a twisted product metric $g = dt^2 + \varphi^2(t) d\Omega^2$, where $\varphi(t) > 0$.*

Proof. The result is equivalent to $dt = \sqrt{f(r)} dr$, for $r = \varphi(t)$. We integrate to find $t = t(r)$; by the equation t is increasing, and we write the inverse as $r = \varphi(t)$. \square

Theorem 3.3. *Consider the space $M = I \times \mathbb{S}^2$ with metric $g = f(r) dr^2 + r^2 d\Omega^2$. Suppose $f'(r) \leq 0$ for all r and $f(r) \geq 1$ for $r \geq r_0$. Then every sphere of revolution S_r for $r \geq r_0$ minimizes perimeter among smooth surfaces enclosing fixed volume with S_{r_0} , uniquely if $f(r) > 1$ for $r \geq r_0$.*

Proof. It suffices to show that M satisfies the conditions for [Theorem 3.1](#), in particular that M has nonpositive radial Ricci curvature, S_r has nonnegative mean curvature (with respect to the inward unit normal), and M has nonnegative tangential sectional curvature. For indices, let $(1, 2, 3)$ represent (r, ϕ, θ) . We find $R_{12} = R_{13} = 0$ and $R_{11} = f'/(rf)$ (see the [Appendix](#)), so the radial Ricci curvature is nonpositive if and only if $f' \leq 0$. We know that $H_{S_r} = 2/(r\sqrt{f}) > 0$ as required. We compute the sectional curvature K of the plane containing ∂_ϕ and ∂_θ as

$$K = \frac{g_{33}R_{232}^3}{g_{22}g_{33} - (g_{23})^2} = \frac{\left(1 - \frac{1}{f(r)}\right)r^2 \sin^2 \phi}{r^2 r^2 \sin^2 \phi - 0^2} = r^{-2} \left(1 - \frac{1}{f(r)}\right)$$

Thus $K \geq 0$ if and only if $f \geq 1$. The spheres S_r are uniquely minimizing provided $f > 1$. \square

Remark 3.4. It is often convenient to consider the function $1/f$ instead of f . If $h = 1/f$, $f' = -h'/h^2$, so $f' \leq 0$ if and only if $h' \geq 0$. To check if $f \geq 1$, we check if $h \leq 1$ and similarly for strict inequality, in which case the tangential sectional curvature is strictly positive.

The Schwarzschild profile. We let g be the Schwarzschild metric with $m > 0$, which we recall has the form $(1 - 2m/r)^{-1} dr^2 + r^2 d\Omega^2$ on $(2m, +\infty) \times \mathbb{S}^2$. We recall the following result from [\[Bray 1997\]](#), proved as in [\[Bray and Morgan 2002\]](#).

Corollary 3.5 [\[Bray 1997\]](#). *In the Schwarzschild metric with positive mass $m > 0$, every sphere of revolution S_r for $r \geq 2m$ uniquely minimizes perimeter among smooth surfaces enclosing fixed volume with the horizon S_{2m} .*

Proof. Let $h(r) = 1 - 2m/r$. We note $h(r) = 1 - 2m/r < 1$ for positive mass. Also, $h'(r) = 2m/r^2 > 0$, so by [Theorem 3.3](#), every sphere of revolution S_r for $r \geq 2m$ uniquely minimizes perimeter among smooth surfaces enclosing fixed volume with the horizon S_{2m} . \square

Remark 3.6. Of course if we consider the full Schwarzschild space with reflection symmetry, then uniqueness is with respect to one chosen end. Similar considerations apply to Reissner–Nordstrom and Schwarzschild AdS below.

The Reissner–Nordstrom profile. Let g be the Reissner–Nordstrom metric, which on $(r_0, \infty) \times \mathbb{S}^2$ takes the form $g = h(r)^{-1} dr^2 + r^2 d\Omega^2$, with

$$h(r) = 1 - \frac{2m}{r} + \frac{Q^2}{r^2}.$$

We shall assume $m^2 > Q^2$, so that h has two positive roots, and we take r_0 to be the larger of the two. Then $r_0 > m > Q^2/m$.

Corollary 3.7. *In Reissner–Nordstrom with $m^2 > Q^2$, every sphere of revolution S_r for $r \geq r_0$ uniquely minimizes perimeter among smooth surfaces enclosing fixed volume with S_{r_0} .*

Proof. We have $h(r) < 1$ for $r > Q^2/(2m)$. We also have $h'(r) = 2m/r^2 - 2Q^2/r^3$, so that $h'(r) \geq 0$ for $r \geq Q^2/m$. Both conditions of [Theorem 3.3](#) hold for $r \geq r_0$, so every sphere of revolution S_r for $r \geq r_0$ uniquely minimizes perimeter among smooth surfaces enclosing fixed volume with S_{r_0} . \square

4. Isoperimetric profile for Schwarzschild AdS

We now let the comparison space M_0 be hyperbolic three-space with hyperbolic metric $(1+r^2)^{-1} dr^2 + r^2 d\Omega^2$. Consider $M = (r_0, \infty) \times \mathbb{S}^2$ with the Schwarzschild AdS metric $g = (1+r^2 - 2m/r)^{-1} dr^2 + r^2 d\Omega^2$. We will construct a comparison map $F : M \rightarrow M_0$ given by $F(r, \omega) = (\psi(r), \omega)$ to show that the radially symmetric spheres are the isoperimetric surfaces in Schwarzschild AdS space.

We will be concerned with two particular types of area stretches. The first one encodes the area stretch for a radially symmetric sphere, $F^*(dA_{F(S_r)}) = \text{AS}_1 dA_{S_r}$:

$$\text{AS}_1(r) = \frac{\int_{\mathbb{S}^2} \psi^2(r) dA_{\mathbb{S}^2}}{\int_{\mathbb{S}^2} r^2 dA_{\mathbb{S}^2}} = \frac{\psi^2(r)}{r^2}.$$

For example, in the previous section, we had $\text{AS}_1 = 1/a$.

The second stretch factor encodes the area stretch for an annular surface $\Sigma = J \times \mathbb{S}^1$ ($J \subset (r_0, +\infty)$), obtained by flowing some great circle \mathbb{S}^1 (with element of arclength ds) along the radial direction field ∂_r , $F^*(dA_{F(\Sigma)}) = \text{AS}_2 dA_{\Sigma}$:

$$\text{AS}_2(r) = \frac{\frac{d}{dr} \int_{\psi(r_0)}^{\psi(r)} \int_{\mathbb{S}^1} \rho(1 + \rho^2)^{-1/2} ds d\rho}{\frac{d}{dr} \int_{r_0}^r \int_{\mathbb{S}^1} \rho(1 + \rho^2 - 2m/\rho)^{-1/2} ds d\rho} = \frac{\psi(r)(1 + \psi^2(r))^{-1/2} \psi'(r)}{r(1 + r^2 - 2m/r)^{-1/2}}.$$

The volume stretch VS , where $F^*(dV_{M_0}) = \text{VS} dV_M$, is given by

$$\text{VS}(r) = \frac{\frac{d}{dr} \int_{\psi(r_0)}^{\psi(r)} \int_{\mathbb{S}^2} \rho(1 + \rho^2)^{-1/2} dA_{\mathbb{S}^2} d\rho}{\frac{d}{dr} \int_{r_0}^r \int_{\mathbb{S}^2} \rho^2(1 + \rho^2 - 2m/\rho)^{-1/2} dA_{\mathbb{S}^2} d\rho} = \frac{\psi^2(r)(1 + \psi^2(r))^{-1/2} \psi'(r)}{r^2(1 + r^2 - 2m/r)^{-1/2}}.$$

Note that $\text{VS} = \sqrt{\text{AS}_1 \text{AS}_2}$.

Lemma 4.1. *The area stretch AS_{Σ} for any surface does not exceed the maximum of AS_1 and AS_2 .*

Proof. By dimension considerations, if Σ is any smooth surface, $T_p\Sigma$ contains at least one tangent direction to the radial sphere through p . We let E_1 be such a unit vector; let E_2 be an orthogonal unit vector tangent to the radial sphere, and let E_3 be the unit outward radial vector. There exist α and β with $\alpha^2 + \beta^2 = 1$ so that $\alpha E_2 + \beta E_3 \in T_p\Sigma$. We have by orthogonality

$$\begin{aligned} \text{AS}_\Sigma &= dA_{F(\Sigma)}(F_*(E_1), \alpha F_*(E_2) + \beta F_*(E_3)) = |F_*(E_1)| |\alpha F_*(E_2) + \beta F_*(E_3)| \\ &= \frac{\psi(r)}{r} \sqrt{\alpha^2 \frac{\psi(r)^2}{r^2} + \beta^2 \frac{(\psi'(r))^2 ((1 + \psi(r))^{-1})}{(1 + r^2 - 2m/r)^{-1}}}. \end{aligned}$$

Thus $\text{AS}_\Sigma^2 = \alpha^2 \text{AS}_1^2 + \beta^2 \text{AS}_2^2$, from which the claim follows. \square

As above, we will produce a map $F : M \rightarrow M_0$ with the following properties: at $r = r_1$, the area stretch $\text{AS}_1(r_1) = 1/a$ and the volume stretch $\text{VS}(r_1) = b$, for some $a, b > 0$; for all Σ , $\text{AS}_\Sigma \leq 1/a$; $\text{VS}(r) \geq b$ for $r > r_1$, and $\text{VS}(r) \leq b$ for $r < r_1$. By the lemma, it suffices to show that $\text{AS}_1, \text{AS}_2 \leq 1/a$ everywhere. (The construction in [Bray and Morgan 2002] uses only the parameter a , in which case $\text{VS}(r_1) = 1/a$; this suffices for the asymptotically flat cases above, but we require slightly more flexibility in constructing the map F in the asymptotically hyperbolic case, and so we introduce the parameter b .) As above, it follows that for any competitor surface Σ bounding at least as much volume as S_{r_1} with equal or less surface area, the image $F(\Sigma)$ will bound no less volume with no more surface area than $F(S_{r_1})$. (In hyperbolic space M_0 , we can shrink $F(\Sigma)$ to produce a surface Σ' bounding the same volume as $F(S_{r_1})$ with less or equal surface area.) But the radially symmetric spheres are isoperimetric surfaces in hyperbolic space M_0 , so all previously mentioned area and volume inequalities must be equalities; hence radially symmetric spheres are isoperimetric surfaces in M . Furthermore, if the maximal area stretch is strictly tangential ($\text{AS}_2 < 1/a$), radially symmetric spheres are the unique isoperimetric surfaces in M .

Theorem 4.2. *In Schwarzschild AdS, every sphere of revolution S_r uniquely minimizes perimeter among smooth surfaces enclosing fixed volume with S_{r_0} .*

Proof. First we consider $r_1 > 2m$. Let $a = 1 - 2m/r_1 < 1$, and define F using $\psi(r) = a^{-1/2}r$ for all $r \geq r_0$. Then $\text{AS}_1 = 1/a$ everywhere. Also

$$\text{AS}_2(r) = \frac{\sqrt{1 + r^2 - 2m/r}}{a\sqrt{1 + a^{-1}r^2}}.$$

Hence $\text{AS}_2(r) < 1/a$ is equivalent to $1 - 2mr^{-3} < 1/a$. Since $1 - 2mr^{-3} < 1 < 1/a$ for all $r > 0$, the maximal area stretch equals $\text{AS}_1 = 1/a$, is strictly tangential, and occurs on S_{r_1} .

At $r = r_1$, we have $\text{AS}_2(r_1) = a^{-1/2}$, while $\text{AS}_2(r) \rightarrow a^{-1/2}$ as $r \rightarrow \infty$. We have

$$\begin{aligned} \frac{d}{dr} \left(\frac{1+r^2-2m/r}{1+a^{-1}r^2} \right) &= \frac{(2r+2m/r^2)(1+a^{-1}r^2) - (1+r^2-2m/r)(2a^{-1}r)}{(1+a^{-1}r^2)^2} \\ &= \frac{2(a-1)r^3 + 6mr^2 + 2am}{ar^2(1+a^{-1}r^2)^2}. \end{aligned}$$

The cubic numerator has a positive local minimum at $r = 0$ and one other critical point at some $r > 0$, so in particular it has only one root (which is positive). Thus AS_2 has a unique maximum on the set $r \geq r_0$. Since AS_2 decreases to $a^{-1/2}$ as $r \rightarrow \infty$, the maximum occurs on (r_1, ∞) , and on this interval

$$\text{AS}_2(r) > \text{AS}_2(r_1) = a^{-1/2}.$$

Hence $\text{VS}(r) = \sqrt{\text{AS}_1(r)} \text{AS}_2(r) \leq a^{-1/2} a^{-1/2} = 1/a$ for $r \leq r_1$, and $\text{VS}(r) = \sqrt{\text{AS}_1(r)} \text{AS}_2(r) \geq a^{-1/2} a^{-1/2} = 1/a$ for $r \geq r_1$. Since areas and volumes stretch in the required manner, S_r are the unique isoperimetric surfaces for $r > 2m$.

Now suppose $r_1 \leq 2m$. Choose $a \in (0, 1)$ and let $\psi(r) = a^{-1/2}r$ for all $r \geq r_0$. Then $\text{AS}_1 = 1/a$ and $\text{AS}_2 < 1/a$ everywhere as before. Note that at $r = r_1$, $\text{AS}_2(r_1) < a^{-1/2}$ since $1 - 2m/r_1 \leq 0$. As before, AS_2 has a unique maximum for $r \geq r_0$ and AS_2 decreases to $a^{-1/2}$ as $r \rightarrow \infty$. Hence the maximum occurs for some $r_{\max} > r_1$, and AS_2 is increasing on (r_0, r_{\max}) . Thus the volume stretch $\text{VS} = a^{-1/2}\text{AS}_2$ is also increasing on (r_0, r_{\max}) , and so $\text{VS}(r) \leq b := \text{VS}(r_1)$ for $r < r_1$, and $\text{VS}(r) \geq b$ for $r \in [r_1, r_{\max}]$. Furthermore, $b = \text{VS}(r_1) = \sqrt{\text{AS}_1(r_1)}\text{AS}_2(r_1) < 1/a$, so for $r > r_{\max}$, $\text{VS}(r) = \sqrt{\text{AS}_1(r)}\text{AS}_2(r) \geq 1/a > b$. Since areas and volumes stretch in the required manner, S_r are the unique isoperimetric surfaces for $r \leq 2m$. \square

5. Remarks on the negative mass Schwarzschild

If we let the mass m be negative in the formula $(1 - 2m/r)^{-1} dr^2 + r^2 d\Omega^2$ for the Schwarzschild metric, we obtain an inextendible metric with no minimal sphere. The coordinates are only singular at the origin; in fact the metric is incomplete, as radial geodesics have finite length as $r \rightarrow 0^+$, but the Ricci tensor blows up on approach to the origin. The Bray–Morgan construction for the positive-mass Schwarzschild does not extend to the negative mass case; in fact we will show below that radial spheres are unstable.

Instability of the radial spheres. We consider now the variations of area and volume enclosed by the coordinate spheres, and compute the second variation of area with respect to volume-preserving perturbations. The variation formulas are standard [Chavel 1993; Taylor 1996]. We note that H below is the trace of the second

fundamental form computed with respect to the *inward* unit normal $-v$, which accounts for a sign difference from some versions of the variation formulas.

We consider a smooth family of surfaces Σ_t obtained from $\Sigma = \Sigma_0$ using the variation field given by $V(x, t) = \eta(x, t)v(x, t)$. Then we have the first variation $A'(t) = \int_{\Sigma_t} H\eta dA$, and the second variation

$$A''(0) = \int_{\Sigma} \left(\eta(-\Delta_{\Sigma}\eta - \eta\|\text{II}\|^2 - \eta \text{Ric}(v, v)) + H\frac{\partial\eta}{\partial t} + H^2\eta^2 \right) dA.$$

The first variation of volume $V(t)$ inside Σ_t is given by $V'(t) = \int_{\Sigma_t} \eta dA$, so the second variation is $V''(t) = \int_{\Sigma_t} (H\eta^2 + \partial\eta/\partial t) dA$.

The radial spheres $\Sigma = S_r$ have constant mean curvature, and hence they are critical points for the area functional with respect to volume-preserving perturbations. Indeed, from the variation of volume formula, we have $0 = V'(0) = \int_{\Sigma} \eta dA$, which implies that $A'(0) = 0$ too. If we now consider the second variation at S_r , since the mean curvature H is *constant* we have $0 = HV''(0) = \int_{S_r} (H^2\eta^2 + H\partial\eta/\partial t) dA$. Thus the second variation formula simplifies; if we also apply the divergence theorem to the first term, we then have

$$(5-1) \quad A''(0) = \int_{S_r} (|\nabla^{\Sigma}\eta|^2 - \eta^2\|\text{II}\|^2 - \eta^2 \text{Ric}(v, v)) dA.$$

From the [Appendix](#) we have $v = \sqrt{1 - 2m/r} \partial_r$, $\text{Ric}(v, v) = -2m/r^3$ and $\|\text{II}\|^2 = (2/r^2)(1 - 2m/r)$. When we plug this into the preceding equation we get

$$A''(0) = \int_{S_r} \left(|\nabla^{\Sigma}\eta|^2 - \eta^2 \frac{2}{r^2} \left(1 - \frac{3m}{r} \right) \right) dA.$$

It is well known [[Axler et al. 1992](#)] that the lowest nonzero eigenvalue λ_1 for the Laplacian on a round two-sphere S_{κ}^2 of curvature κ is $\lambda_1 = 2\kappa$, with eigenspace spanned by the restriction of the coordinate functions x, y, z to the sphere (isometrically embedded in \mathbb{R}^3 centered at the origin): e.g., $\Delta_{S_{\kappa}^2}(x) = -2\kappa x$. We now invoke the Poincaré inequality we obtain from the decomposition of $L^2(\Sigma)$ by the eigenspaces of the Laplacian [[Chavel 1993](#)]: $\lambda_1 \int_{\Sigma} \eta^2 dA \leq \int_{\Sigma} |\nabla^{\Sigma}\eta|^2 dA$, for all η with $\int_{\Sigma} \eta dA = 0$; equality holds precisely for functions in the λ_1 -eigenspace. Applying this with $\Sigma = S_r$ we have $\lambda_1 = 2/r^2$, so that

$$(5-2) \quad A''(0) \geq \int_{S_r} \frac{6m}{r^3} \eta^2 dA,$$

with equality if and only if η is in the λ_1 -eigenspace. We see from this that in the *positive* mass Schwarzschild case, the second variation must be positive for (nontrivial) volume-preserving deformations (which we knew already from the isoperimetric profile). But in the negative mass case, we see that for η a coordinate

function, the right-hand side of (5-2) is negative. We note that $\eta(x, t) = x$ does not satisfy $V''(0) = 0$. To satisfy this condition, we can let $\eta_0(x, t) = x + \alpha t$, where α is a constant chosen precisely so that $V''(0) = 0$. Then η_0 generates a deformation that preserves volume to second order; from here it is not hard to modify the variation by a scaling to preserve volume, and so that the corresponding η has first-order Taylor expansion η_0 . Another way to see that the spheres do not minimize area for a given volume is by considering the variation $\eta_0 \nu$. Since this variation leaves the volume unchanged to second-order in t , the change in volume is $O(t^3)$. Now, the volume $V(S_r)$ enclosed by the radial spheres satisfies

$$\frac{dV(S_r)}{dr} = \frac{4\pi r^2}{\sqrt{1-2m/r}} > 0,$$

so the radius $r(t)$ of the radial sphere with volume $V(t)$ is such that $(r(t) - r) = O(t^3)$. So the area $A(S_{r(t)}) = A(S_r) + O(t^3)$, and thus $A''(0) < 0$ implies that for some C and small $t > 0$, the area $A(t)$ of Σ_t satisfies $A(t) < A(S_r) - Ct^2 < A(S_{r(t)})$. This should not be surprising by considering the growth of the volume for small r :

$$V(S_r) = 4\pi \int_0^r t^2 \sqrt{\frac{1}{1-2m/t}} dt < 4\pi \int_0^r t^2 \sqrt{\frac{t}{2|m|}} dt = O(r^{7/2}).$$

This volume growth is slower than for the Euclidean metric $dr^2 + r^2 d\Omega^2$, but the radial spheres have the same area as in the Euclidean metric, so that it is more efficient to slide them off-center. It might be interesting to consider the isoperimetric problem in this singular space, and whether optimizing shapes tend to singular varieties that go *through* the singular point.

6. The Penrose inequality from isoperimetric techniques

The Riemannian Penrose inequality is a lower bound on the ADM mass of an asymptotically flat metric of nonnegative scalar curvature in terms of the areas of certain horizons. There are a host of partial results, including the isoperimetric approach of [Bray 1997], and then there are the proofs of [Huisken and Ilmanen 2001] and [Bray 2001]. We state the version from the latter reference.

Theorem 6.1 (Penrose Inequality). *Let (M, g) be asymptotically flat with $R(g) \geq 0$. Let m be the ADM mass of an end, and let A be the total surface area of the outermost minimal spheres with respect to this end. Then $m \geq \sqrt{A/(16\pi)}$.*

Various analogues of this inequality have been sought [Bray and Chruściel 2004], including asymptotically hyperbolic versions and versions with charge. We discuss an example each for both types, to illustrate that the beautiful arguments of Bray [1997] which connect the isoperimetric profiles to the Penrose inequality extend to the context of the isoperimetric profiles obtained above.

Variation of area along an isoperimetric profile. We again consider the isoperimetric problem of minimizing area for volume V between a horizon and competitor surfaces in the homology class of the horizon. We assume we have an isoperimetric profile $\Sigma(V)$, each surface of which is connected. The objective in the next sections will be to establish that a mass function $m(V)$ associated with the Hawking mass function $m_H(\Sigma(V))$ determined by the isoperimetric profile is nondecreasing, for which we now derive a key inequality. We compute the variation of the area function $A(V)$ of the profile, where we employ the harmless abuse of notation, $A(V) := A(\Sigma(V))$, and we note that $A(0) = A(\Sigma_0)$. The area function of the isoperimetric profile may not be smooth in V , so that this fact is established in a weak but sufficient form. To be precise, for each $V_0 > 0$, we let $A_{V_0}(V)$ be the area of the surface obtained by flowing $\Sigma(V_0)$ in the outward normal direction at unit speed until the volume enclosed with the horizon is V . A_{V_0} will be smooth for V near V_0 . Moreover, $A_{V_0}(V_0) = A(V_0)$ and $A_{V_0}(V) \geq A(V)$. Thus if A were smooth, then $A'(V_0) = A'_{V_0}(V_0)$ and $A''(V_0) \leq A''_{V_0}(V_0)$; so an inequality for the derivatives of A_{V_0} at V_0 can be interpreted as a weak (distributional) inequality for the derivatives of A . We let Σ'_{V_0} be the surface obtained by flowing $\Sigma(V_0)$ for time t , and let $V(t)$ be the volume this surfaces encloses with the horizon. Then, by the equations of variation (as recalled in the preceding section), we have

$$\frac{d}{dt}(A_{V_0}(V(t))) = \int_{\Sigma'_{V_0}} H dA, \quad \frac{dV}{dt} = A_{V_0}(V(t)),$$

so that

$$\frac{d}{dV}(A_{V_0}(V)) = A'_{V_0}(V) = \frac{\int_{\Sigma'_{V_0}} H dA}{A_{V_0}(V(t))}.$$

By the second variation of area formula we obtain (since $\eta = 1$)

$$A_{V_0}(V_0)^2 A''_{V_0}(V_0) = \int_{\Sigma(V_0)} (-\|\mathbb{I}\|^2 - \text{Ric}(v, v) + H^2) dA.$$

Taking the trace of the Gauss equation gives $\text{Ric}(v, v) = \frac{1}{2}R - K + \frac{1}{2}(H^2 - \|\mathbb{I}\|^2)$, where $R = R(g)$ is the scalar curvature of the ambient three-space and K is the Gauss curvature of the surface. We obtain

$$A_{V_0}(V_0)^2 A''_{V_0}(V_0) = \int_{\Sigma(V_0)} \left(-\frac{1}{2}R + K - \frac{1}{2}H^2 - \frac{1}{2}\|\mathbb{I}\|^2\right) dA.$$

Since $\Sigma(V_0)$ has only one component by assumption,

$$\int_{\Sigma(V_0)} K dA = 2\pi \chi(\Sigma(V_0)) \leq 4\pi$$

by the Gauss–Bonnet theorem. Since $\|\Pi\|^2 \geq \frac{1}{2}H^2$, we arrive at the inequality

$$(6-1) \quad A_{V_0}(V_0)^2 A''_{V_0}(V_0) \leq 4\pi + \int_{\Sigma(V_0)} \left(-\frac{1}{2}R - \frac{3}{4}H^2\right) dA,$$

which we apply below.

Penrose inequality for some solutions of the Einstein–Maxwell constraints. We now discuss the Penrose Inequality in the context of a certain class of solutions to the Einstein–Maxwell constraints. As noted in [Weinstein and Yamada 2005], in the case of a connected horizon, the Huisken–Ilmanen proof can be carried through to prove the Penrose inequality that we discuss below, under less restrictive assumptions. We also remark that Weinstein and Yamada [2005] showed that for multiple-component horizons, a natural related Penrose inequality fails.

Proposition 6.2. *Assume (M, g, E) is an asymptotically flat solution of the Einstein–Maxwell constraints $R(g) = 2|E|^2$, $\operatorname{div}_g(E) = 0$, which outside a compact set agrees with Reissner–Nordstrom data on the exterior of a ball, with mass m and charge Q , and $m > |Q|$. Suppose furthermore that M has only one horizon Σ_0 and admits a connected isoperimetric profile (with respect to Σ_0) $\Sigma(V)$, so that for sufficiently large V , $\Sigma(V)$ is a spherically symmetric sphere in Reissner–Nordstrom. Then*

$$m \geq \sqrt{\frac{A(\Sigma_0)}{16\pi}} + \frac{Q^2}{2r_0} = \frac{1}{2} \left(r_0 + \frac{Q^2}{r_0} \right),$$

where r_0 is defined by $A(\Sigma_0) = 4\pi r_0^2$.

Proof. We have established the isoperimetric profile for Reissner–Nordstrom in Corollary 3.7. We discuss the calculations that relate the mass to the Hawking mass of the isoperimetric surfaces for the model. Since solutions (g, E) of the Einstein–Maxwell constraints have nonnegative scalar curvature $R(g) = 2|E|^2$, we have from (6-1)

$$\begin{aligned} A_{V_0}(V_0)^2 A''_{V_0}(V_0) &\leq 4\pi - \int_{\Sigma(V_0)} \left(|E|^2 + \frac{3}{4}H^2\right) dA \\ &= 4\pi - \frac{3}{4}H^2 A_{V_0}(V_0) - \int_{\Sigma(V_0)} |E|^2 dA. \end{aligned}$$

Since E is divergence-free, the flux integral $\int_{\Sigma} E^i v_i dA$ is a homological invariant, and thus is just $4\pi Q$. The preceding inequality thus yields (using Cauchy–Schwarz)

$$A_{V_0}(V_0)^2 A''_{V_0}(V_0) \leq 4\pi - \frac{3}{4}H^2 A_{V_0}(V_0) - \frac{(4\pi Q)^2}{A(V_0)}.$$

Since $A'_{V_0}(V_0) = H$, this can, as noted above, be interpreted as a weak formulation of

$$A''(V) \leq \frac{4\pi}{A(V)^2} - \frac{3A'(V)^2}{4A(V)} - \frac{(4\pi Q)^2}{A(V)^3}.$$

Equivalently, for $F = A^{3/2}$ we have

$$(6-2) \quad F''(V) \leq \frac{36\pi - F'(V)^2 - 144\pi^2 Q^2 F(V)^{-2/3}}{6F(V)}.$$

We will work with the mass function $m(V)$, defined by

$$m(V) = \frac{F(V)^{1/3}}{144\pi^{3/2}} (36\pi - F'(V)^2) + \sqrt{\pi} Q^2 F(V)^{-1/3}.$$

If $F(V)$ were smooth, we would have

$$m'(V) = \frac{\frac{1}{3}F'(V)(F(V))^{-2/3}}{144\pi^{3/2}} \times ((36\pi - (F'(V))^2) - 6F(V)F''(V) - \sqrt{\pi}Q^2(F(V))^{-2/3}).$$

In view of (6-2), and since $F(V)$ is nondecreasing (there being only one horizon), $m(V)$ is a nondecreasing function. Actually this statement requires some care to prove, since the function $F(V)$ may fail to be smooth, so one would need to check directly that $m'(V) \geq 0$ in the sense of distributions, that is, by pairing with appropriate test functions. We omit the details.

If $Q = 0$, the mass function is the Hawking mass of the isoperimetric surface bounding a volume V , since $F'(V) = \frac{3}{2}A(V)^{1/2}A'(V) = \frac{3}{2}A(V)^{1/2}H$ implies

$$\begin{aligned} m(V) &= \frac{A(V)^{1/2}}{144\pi^{3/2}} (36\pi - \frac{9}{4}A(V)H^2) + \sqrt{\pi} Q^2 F(V)^{-1/3} \\ &= \sqrt{\frac{A(V)}{16\pi}} \left(1 - \int_{\Sigma(V)} \frac{H^2}{16\pi} \right) + \sqrt{\pi} Q^2 F(V)^{-1/3}. \end{aligned}$$

Since $H = 0$ at the horizon, we have

$$m(0) = \sqrt{\frac{A(0)}{16\pi}} + \frac{Q^2}{2r_0}.$$

For V sufficiently large $\Sigma(V)$ is a radial sphere S_r in Reissner–Nordstrom, so $m(V)$ is the Hawking mass of S_r plus the charge term:

$$m(V) = \sqrt{\frac{4\pi r^2}{16\pi}} \left(1 - 4\pi r^2 \frac{1 - 2m/r + Q^2/r^2}{4\pi r^2} \right) + \frac{Q^2}{2r} = m.$$

Hence $m = \lim_{V \rightarrow +\infty} m(V) \geq m(0)$, giving us a Penrose Inequality with charge:

$$m \geq \sqrt{\frac{A(0)}{16\pi}} + \frac{Q^2}{2r_0} = \frac{1}{2} \left(r_0 + \frac{Q^2}{r_0} \right). \quad \square$$

We now briefly sketch how to use this result to conclude the Penrose inequality holds for more general asymptotically flat solutions (M, g, E) of the Einstein–Maxwell constraints. We cite a condition (C1) from [Bray 1997]: there is only one horizon, and for $V > 0$, if one or more isoperimetric surfaces exists for this volume V , then at least one of these surfaces has only one component. This condition is not required in [Bray 2001], [Huisken and Ilmanen 2001], for which if there is more than one horizon, one considers the *outermost* horizons in any end. We have an approximation result from [Corvino \geq 2007] which allows us to normalize the asymptotics: asymptotically flat solutions (M, g, E) of the Einstein–Maxwell constraints admit approximations by data which agree with suitable Reissner–Nordstrom data in each end, where the perturbation is localized near infinity. Assuming condition (C1) holds after this perturbation, one shows that the isoperimetric surfaces $\Sigma(V)$ exist and agree with those of Reissner–Nordstrom for sufficiently large V . The proof of these claims should actually follow from the proofs in [Bray 1997] for the Schwarzschild case; much of the construction relies on the geometry being asymptotically flat and spherically symmetric near infinity, and a main technical theorem which is used in the proof is an inequality in Euclidean space, which carries over to Schwarzschild (as used by Bray) and Reissner–Nordstrom for large radii by perturbation. Since the Penrose inequality

$$m \geq \sqrt{\frac{A(0)}{16\pi}} + \frac{Q^2}{2r_0}$$

in this case also follows from [Huisken and Ilmanen 2001], we omit the technical details.

On the Penrose inequality for asymptotically Schwarzschild AdS spaces. We now show that the analogous mass function $m(V)$ (if it exists) will be nondecreasing in an asymptotically Schwarzschild AdS space. In general, the mass of asymptotically hyperbolic spaces is more subtle than for asymptotically flat spaces; compare [Chruściel and Herzlich 2003; Wang 2001; Zhang 2004]. We are only discussing below a class of asymptotically hyperbolic spaces with a spherical infinity and with special asymptotics.

In the next proposition, we mean by *horizon* that Σ_0 has (inward) mean curvature $H = 2$ [Bray and Chruściel 2004].

Proposition 6.3. *Assume (M, g) is a three-manifold with $R(g) \geq -6$, which outside a compact set is isometric to an exterior of a ball in Schwarzschild AdS space*

of mass $m > 0$. Suppose furthermore that M has only one horizon Σ_0 and admits a connected isoperimetric profile (with respect to Σ_0) $\Sigma(V)$, so that for sufficiently large V , $\Sigma(V)$ is the spherically symmetric sphere in Schwarzschild AdS of volume V . Then

$$m \geq \sqrt{\frac{A(\Sigma_0)}{16\pi}}.$$

Proof. Schwarzschild AdS is asymptotic to hyperbolic three-space, so the definitions and computations change slightly from above. We begin by putting $R(g) \geq -6$ into inequality (6-1) to obtain

$$A_{V_0}(V_0)^2 A''_{V_0}(V_0) \leq 3A_{V_0}(V_0) + 4\pi - \int_{\Sigma(V_0)} \frac{3}{4} H^2 dA = 3A_{V_0}(V_0) + 4\pi - \frac{3}{4} H^2 A_{V_0}(V_0).$$

Hence

$$A''_{V_0}(V_0) \leq \frac{3}{A_{V_0}(V_0)} + \frac{4\pi}{A_{V_0}(V_0)^2} - \frac{3A'_{V_0}(V_0)^2}{4A_{V_0}(V_0)}.$$

Since by definition $A(V_0) = A_{V_0}(V_0)$ and $A(V) \leq A_{V_0}(V)$, we have the weak inequality

$$A''(V) \leq \frac{3}{A(V)} + \frac{4\pi}{A(V)^2} - \frac{3A'(V)^2}{4A(V)},$$

or equivalently, for $F = A^{3/2}$,

$$(6-3) \quad F''(V) \leq \frac{27F(V)^{2/3} + 36\pi - F'(V)^2}{6F(V)}.$$

We modify the Hawking mass in this setting with one extra term which accounts for the nonminimal horizon, so we get a corresponding $m(V)$ for the isoperimetric surfaces as follows:

$$\begin{aligned} m(V) &= \sqrt{\frac{A(V)}{16\pi}} \left(1 + \frac{A(V)}{4\pi} - \int_{\Sigma(V)} \frac{H^2}{16\pi} dA \right) \\ &= \frac{A(V)^{1/2}}{16\pi^{3/2}} \left(4\pi + A(V) - \frac{1}{4} A(V) (A'(V))^2 \right) \\ &= \frac{F(V)^{1/3}}{16\pi^{3/2}} \left(4\pi + F(V)^{2/3} - \frac{1}{4} F(V)^{2/3} \left(\frac{2}{3} F(V)^{-1/3} F'(V) \right)^2 \right) \\ &= \frac{F(V)^{1/3}}{144\pi^{3/2}} \left(36\pi + 9F(V)^{2/3} - F'(V)^2 \right). \end{aligned}$$

The reason for the modification is that since the (inward) mean curvature of Σ_0 is 2, we again have $m(0) = \sqrt{A(0)/(16\pi)}$.

Since (6-3) holds (and again, $F(V)$ is nondecreasing since there is only one horizon), we have the distributional inequality

$$m'(V) = \frac{2}{144\pi^{3/2}} F(V)^{1/3} F'(V) \left(-F''(V) + \frac{36\pi + 27F(V)^{2/3} - F'(V)^2}{6F(V)} \right) \geq 0.$$

For V sufficiently large, $m(V)$ is the Hawking mass of some radially symmetric sphere $S_r = \Sigma(V)$ and thus

$$m(V) = \sqrt{\frac{4\pi r^2}{16\pi}} \left(1 + \frac{4\pi r^2}{4\pi} - \frac{4\pi r^2}{16\pi} \frac{4(1+r^2-2m/r)}{r^2} \right) = m.$$

Hence

$$m = \lim_{V \rightarrow +\infty} m(V) \geq m(0) = \sqrt{\frac{A(0)}{16\pi}},$$

giving us the desired Penrose Inequality. \square

7. Conclusions

We conjecture that there exists a reasonable class of spaces with $R(g) \geq -6$ which are asymptotically Schwarzschild AdS for which the above analysis will yield a Penrose Inequality. We hope to report on this in a future work. Although the class would be limited in several respects, it is interesting problem, following the work of Bray and in light of the recent work of Huisken [2005; 2006], to understand better the relationship of the mass to the isoperimetric properties of the space.

We mention that foliations near infinity of constant mean curvature (CMC) have appeared in the context of relativity; see [Huisken and Yau 1996; Metzger 2004; Qing and Tian 2004; Ye 1996]. It is tempting to conjecture that these uniquely determined foliations near infinity by constant mean curvature spheres give the isoperimetric profiles.

Appendix: Metric formulas

Consider a metric of the form

$$g = f(r) dr^2 + r^2 d\Omega^2 = f(r) dr^2 + r^2 d\phi^2 + r^2 \sin^2(\phi) d\theta^2,$$

with $f(r) > 0$. We collect here the basic geometric formulas which we apply to our three families of metrics above. We use the Einstein summation convention below, and the indices (1, 2, 3) correspond to the variables (r, ϕ, θ) .

Christoffel symbols. We display the metric and its inverse in matrix form:

$$(g_{ij}) = \begin{pmatrix} f(r) & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \phi \end{pmatrix}, \quad (g^{ij}) = \begin{pmatrix} \frac{1}{f(r)} & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2 \sin^2 \phi} \end{pmatrix}.$$

We recall the formula $\Gamma_{ij}^k = \frac{1}{2}g^{mk}(g_{jm,i} + g_{mi,j} - g_{ij,m})$ for the Christoffel symbols. We can simplify our calculations by making two observations. Since g_{ij} and g^{ij} are diagonal, we have $\Gamma_{ij}^k = \frac{1}{2}g^{kk}(g_{jk,i} + g_{ki,j} - g_{ij,k})$, and $\Gamma_{ij}^k = 0$ when i, j , and k are all distinct. For reference here are the nonzero Christoffel symbols:

$$\begin{aligned} \Gamma_{11}^1 &= \frac{1}{2}g^{11}(g_{11,1} + g_{11,1} - g_{11,1}) = \frac{f'(r)}{2f(r)}, \\ \Gamma_{22}^1 &= \frac{1}{2}g^{11}(g_{21,2} + g_{12,2} - g_{22,1}) = -\frac{r}{f(r)}, \\ \Gamma_{33}^1 &= \frac{1}{2}g^{11}(g_{31,3} + g_{13,3} - g_{33,1}) = -\frac{r \sin^2 \phi}{f(r)}, \\ \Gamma_{33}^2 &= \frac{1}{2}g^{22}(g_{32,3} + g_{23,3} - g_{33,2}) = -\sin \phi \cos \phi, \\ \Gamma_{12}^2 &= \Gamma_{21}^2 = \frac{1}{2}g^{22}(g_{22,1} + g_{21,2} - g_{12,2}) = \frac{1}{r}, \\ \Gamma_{13}^3 &= \Gamma_{31}^3 = \frac{1}{2}g^{33}(g_{33,1} + g_{31,3} - g_{13,3}) = \frac{1}{r}, \\ \Gamma_{23}^3 &= \Gamma_{32}^3 = \frac{1}{2}g^{33}(g_{33,2} + g_{32,3} - g_{23,3}) = \cot \phi. \end{aligned}$$

Second fundamental form and mean curvature of radial spheres S_r . We compute the second fundamental form and mean curvature of the coordinate spheres of constant r . Let Z^N be the normal projection of a vector Z . We have

$$\begin{aligned} B(\partial_\phi, \partial_\phi) &= (\nabla_{\partial_\phi} \partial_\phi)^N = (\Gamma_{22}^1 \partial_r + \Gamma_{22}^2 \partial_\phi + \Gamma_{22}^3 \partial_\theta)^N = -\frac{r}{f(r)} \partial_r, \\ B(\partial_\phi, \partial_\theta) &= (\nabla_{\partial_\phi} \partial_\theta)^N = (\Gamma_{23}^1 \partial_r + \Gamma_{23}^2 \partial_\phi + \Gamma_{23}^3 \partial_\theta)^N = 0, \\ B(\partial_\theta, \partial_\theta) &= (\nabla_{\partial_\theta} \partial_\theta)^N = (\Gamma_{33}^1 \partial_r + \Gamma_{33}^2 \partial_\phi + \Gamma_{33}^3 \partial_\theta)^N = -\frac{r \sin^2 \phi}{f(r)} \partial_r. \end{aligned}$$

Let $N = -\nu$ denote the inward unit normal vector field to S_r . Then $g(\partial_r, N) = -\|\partial_r\| = -\sqrt{f(r)}$, so the second fundamental form Π , defined by $\Pi(V, W) = g(B(V, W), N)$, is given by

$$\Pi(\partial_\phi, \partial_\phi) = \frac{r}{\sqrt{f(r)}}, \quad \Pi(\partial_\phi, \partial_\theta) = 0, \quad \Pi(\partial_\theta, \partial_\theta) = \frac{r \sin^2(\phi)}{\sqrt{f(r)}}.$$

Thus the mean curvature for S_r , which is constant by symmetry, is

$$\begin{aligned} H_{S_r} &= g^{\phi\phi} \Pi(\partial_\phi, \partial_\phi) + g^{\theta\theta} \Pi(\partial_\theta, \partial_\theta) = \left(\frac{r/\sqrt{f(r)}}{r^2} + \frac{(r \sin^2 \phi)/\sqrt{f(r)}}{r^2 \sin^2 \phi} \right) \\ &= \frac{2}{r\sqrt{f(r)}}. \end{aligned}$$

Ricci and scalar curvature. We use the formulas

$$R_{ij} = R_{ilj}^l \quad \text{and} \quad R_{ikj}^l = \Gamma_{ij,k}^l - \Gamma_{ik,j}^l + \Gamma_{ij}^m \Gamma_{km}^l - \Gamma_{ik}^m \Gamma_{jm}^l.$$

A simple computation shows that the Ricci tensor is diagonal in this coordinate system, and the diagonal entries are given by

$$\begin{aligned} R_{11} &= R_{111}^1 + R_{121}^2 + R_{131}^3 = 0 + \frac{f'(r)}{2rf(r)} + \frac{f'(r)}{2rf(r)}, \\ R_{22} &= R_{212}^1 + R_{222}^2 + R_{232}^3 = \frac{rf'(r)}{2f(r)^2} + 0 + \left(1 - \frac{1}{f(r)}\right), \\ R_{33} &= R_{313}^1 + R_{323}^2 + R_{333}^3 = \sin^2(\phi) \left(\frac{rf'(r)}{2f(r)^2} + \left(1 - \frac{1}{f(r)}\right) + 0 \right). \end{aligned}$$

Thus we find that the scalar curvature is

$$R(g) = g^{ij} R_{ij} = g^{11} R_{11} + g^{22} R_{22} + g^{33} R_{33} = \frac{2f'(r)}{rf(r)^2} + \frac{2}{r^2} - \frac{2}{r^2 f(r)}.$$

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