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IRREDUCIBLE REPRESENTATIONS FOR THE ABELIAN EXTENSION OF THE LIE ALGEBRA OF DIFFEOMORPHISMS OF TORI IN DIMENSIONS GREATER THAN 1

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We classify the irreducible weight modules of the abelian extension of the Lie algebra of diffeomorphisms of tori of dimension greater than 1, with finite-dimensional weight spaces.

1. Introduction

Let $W_{\nu+1}$ be the Lie algebra of diffeomorphisms of the $(\nu+1)$-dimensional torus. If $\nu = 0$, the universal central extension of the complex Lie algebra W_1 is the Virasoro algebra, which, together with its representations, plays a very important role in many areas of mathematics and physics [Belavin et al. 1984; Dotsenko and Fateev 1984; Di Francesco et al. 1997]. The representation theory of the Virasoro algebra has been studied extensively; see, for example, [Kac 1982; Kaplansky and Santharoubane 1985; Chari and Pressley 1988; Mathieu 1992].

If $\nu \geq 1$, however, the Lie algebra $W_{\nu+1}$ has no nontrivial central extension [Ramos et al. 1990]. But $W_{\nu+1}$ has abelian extensions whose abelian ideals are the central parts of the corresponding toroidal Lie algebras; see [Berman and Billig 1999], for example. There is a close connection between irreducible integrable modules of the toroidal Lie algebra and irreducible modules of the abelian extension $\mathscr{L}$; see [Berman and Billig 1999; Eswara Rao and Moody 1994; Jiang and Meng 2003], for instance. In fact, the classification of integrable modules of toroidal Lie algebras and their subalgebras depends heavily on the classification of irreducible representations of $\mathscr{L}$ and its subalgebras. See [Billig 2003] for the constructions of the abelian extensions for the group of diffeomorphisms of a torus.

In this paper we study the irreducible weight modules of $\mathscr{L}$, for $\nu \geq 1$. If V is an irreducible weight module of $\mathscr{L}$ some of whose central charges $c_0, \dots, c_\nu$ are nonzero, one can assume that $c_0, \dots, c_N$ are $\mathbb{Z}$-linearly independent and $c_{N+1} = \cdots = c_\nu = 0$, where $N \geq 0$. We prove that if $N \geq 1$, then V must have weight

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spaces which are infinite-dimensional. So if all the weight spaces of V are finite-dimensional, N vanishes. We classify the irreducible modules of $\mathcal{L}$ with finite-dimensional weight spaces and some nonzero central charges. We prove that such a module V is isomorphic to a highest weight module. The highest weight space T is isomorphic to an irreducible $(\mathcal{A}_\nu+W_\nu)$-module all of whose weight spaces have the same dimension, where $\mathcal{A}_\nu$ is the ring of Laurent polynomials in ν commuting variables, regarded as a commutative Lie algebra. An important step is to characterize the $\mathcal{A}_\nu$-module structure of T. It turns out that the action of $\mathcal{A}_\nu$ on T is essentially multiplication by polynomials in $\mathcal{A}_\nu$. Therefore T can be identified with Larsson's construction [1992] by a result in [Eswara Rao 2004]. That is, T is a tensor product of gl_ν-module with $\mathcal{A}_\nu$.

When all the central charges of V are zero, we prove that the abelian part acts on V as zero if V is a uniformly bounded $\mathcal{L}$-module. So the result in this case is not complete.

Throughout the paper, $\mathbb{C}$, $\mathbb{Z}_+$ and $\mathbb{Z}_-$ denote the sets of complex numbers, positive integers and negative integers.

2. Basic concepts and results

Let $\mathcal{A}_{\nu+1} = \mathbb{C}[t_0^{\pm 1}, t_1^{\pm 1}, \dots, t_\nu^{\pm 1}]$ $(\nu \geq 1)$ be the ring of Laurent polynomials in commuting variables $t_0, t_1, \dots, t_\nu$. For $\underline{n} = (n_1, n_2, \dots, n_\nu) \in \mathbb{Z}^\nu$, $n_0 \in \mathbb{Z}$, we denote $t_0^{n_0} t_1^{n_1} \cdots t_\nu^{n_\nu}$ by $t_0^{n_0} t^{\underline{n}}$. Let $\tilde{\mathcal{K}}$ be the free $\mathcal{A}_{\nu+1}$-module with basis $\{k_0, k_1, \dots, k_\nu\}$ and let $d\tilde{\mathcal{K}}$ be the subspace spanned by all elements of the form

$$\sum_{i=0}^{\nu} r_i t_0^{r_0} t^{\underline{r}} k_i, \qquad \text{for } (r_0, \underline{r}) = (r_0, r_1, \dots, r_\nu) \in \mathbb{Z}^{\nu+1}.$$

Set $\mathcal{K} = \tilde{\mathcal{K}}/d\tilde{\mathcal{K}}$ and denote the image of $t_0^{r_0} t^{\underline{r}} k_i$ still by itself. Then $\mathcal{K}$ is spanned by the elements $\{t_0^{r_0} t^{\underline{r}} k_p \mid p = 0, 1, \dots, \nu, r_0 \in \mathbb{Z}, \underline{r} \in \mathbb{Z}^\nu\}$ with relations

$$\sum_{p=0}^{\nu} r_p t_0^{r_0} t^{\underline{r}} k_p = 0. \tag{2-1}$$

Let $\mathcal{D}$ be the Lie algebra of derivations on $\mathcal{A}_{\nu+1}$. Then

$$\mathcal{D} = \left\{ \sum_{p=0}^{\nu} f_p(t_0, t_1, \dots, t_\nu) d_p \mid f_p(t_0, t_1, \dots, t_\nu) \in \mathcal{A}_{\nu+1} \right\},$$

where $d_p = t_p \partial/\partial t_p$, $p = 0, 1, \dots, \nu$. From [Berman and Billig 1999] we know that the algebra $\mathcal{D}$ admits two nontrivial 2-cocycles with values in $\mathcal{K}$:

$$\tau_1(t_0^{m_0} t^{\underline{m}} d_a, t_0^{n_0} t^{\underline{n}} d_b) = -n_a m_b \sum_{p=0}^{\nu} m_p t_0^{m_0+n_0} t^{\underline{m}+\underline{n}} k_p,$$

$$\tau_2(t_0^{m_0}t^{\underline{m}}d_a, t_0^{n_0}t^{\underline{n}}d_b) = m_a n_b \sum_{p=0}^{\nu} m_p t_0^{m_0+n_0} t^{\underline{m}+\underline{n}} k_p.$$

Let $\tau = \mu_1\tau_1 + \mu_2\tau_2$ be an arbitrary linear combination of τ_1 and τ_2. Then the corresponding abelian extension of $\mathcal{D}$ is

$$\mathcal{L} = \mathcal{D} \oplus \mathcal{K},$$

with the Lie bracket

$$\text{(2-2)}\quad \begin{aligned} [t_0^{m_0}t^{\underline{m}}d_a, t_0^{n_0}t^{\underline{n}}k_b] &= n_a t_0^{m_0+n_0}t^{\underline{m}+\underline{n}}k_b + \delta_{ab}\sum_{p=0}^{\nu} m_p t_0^{m_0+n_0}t^{\underline{m}+\underline{n}}k_p, \\ [t_0^{m_0}t^{\underline{m}}d_a, t_0^{n_0}t^{\underline{n}}d_b] &= n_a t_0^{m_0+n_0}t^{\underline{m}+\underline{n}}d_b - m_b t_0^{m_0+n_0}t^{\underline{m}+\underline{n}}d_a \\ &\qquad + \tau(t_0^{m_0}t^{\underline{m}}d_a, t_0^{n_0}t^{\underline{n}}d_b). \end{aligned}$$

The sum

$$\mathfrak{h} = \Big(\bigoplus_{i=0}^{\nu} \mathbb{C}k_i\Big) \oplus \Big(\bigoplus_{i=0}^{\nu} \mathbb{C}d_i\Big)$$

is an abelian Lie subalgebra of $\mathcal{L}$. An $\mathcal{L}$-module V is called a weight module if

$$V = \bigoplus_{\lambda\in\mathfrak{h}^*} V_\lambda,$$

where $V_\lambda = \{v \in V \mid h \cdot v = \lambda(h)v \text{ for all } h \in \mathfrak{h}\}$. Denote by $P(V)$ the set of all weights. Throughout the paper, we assume that V is an irreducible weight module of $\mathcal{L}$ with finite-dimensional weight spaces. Since V is irreducible, we have

$$k_i|_V = c_i,$$

where the constants c_i, for $i = 0, 1, \ldots, \nu$, are called the central charges of V.

Lemma 2.1. *Let $A = (a_{ij})$ $(0 \le i, j \le \nu)$ be a $(\nu+1) \times (\nu+1)$ matrix such that $\det A = 1$ and $a_{ij} \in \mathbb{Z}$. There exists an automorphism σ of $\mathcal{L}$ such that*

$$\sigma(t^{\bar{m}}k_j) = \sum_{p=0}^{\nu} a_{pj} t^{\bar{m}A^T} k_p, \quad \sigma(t^{\bar{m}}d_j) = \sum_{p=0}^{\nu} b_{jp} t^{\bar{m}A^T} d_p, \quad 0 \le j \le \nu,$$

where $t^{\bar{m}} = t_0^{m_0}t^{\underline{m}}$, $B = (b_{ij}) = A^{-1}$.

3. The structure of V with nonzero central charges

In this section, we discuss the weight module V which has nonzero central charges. It follows from Lemma 2.1 that we can assume that $c_0, c_1, \ldots, c_N$ are $\mathbb{Z}$-linearly independent, i.e., if $\sum_{i=0}^{N} a_i c_i = 0$, $a_i \in \mathbb{Z}$, then all $a_i (i = 0, \ldots, N)$ must be zero, and $c_{N+1} = c_{N+2} = \cdots = c_\nu = 0$, where $N \geq 0$. For $\bar{m} = (m_0, \underline{m})$, denote $t_0^{m_0} t^{\underline{m}}$ by $t^{\bar{m}}$ as in Lemma 2.1. It is easy to see that V has the decomposition

$$V = \bigoplus_{\bar{m} \in \mathbb{Z}^{\nu+1}} V_{\bar{m}},$$

where $V_{\bar{m}} = \{v \in V \mid d_i(v) = (\gamma_0(d_i) + m_i)v, i = 0, 1, \ldots, \nu\}$, with $\gamma_0 \in P(V)$ a fixed weight, and $\bar{m} = (m_0, m_1, \ldots, m_\nu) \in \mathbb{Z}^{\nu+1}$. If V has finite-dimensional weight spaces, the $V_{\bar{m}}$ are finite-dimensional, for $\bar{m} \in \mathbb{Z}^{\nu+1}$.

In Lemmas 3.1–3.6 we assume that V has finite-dimensional weight spaces.

Lemma 3.1. *For $p \in \{0, 1, \ldots, \nu\}$ and $0 \neq t^{\bar{m}} k_p \in \mathscr{L}$, if there is a nonzero element v in V such that $t^{\bar{m}} k_p v = 0$, then $t^{\bar{m}} k_p$ is locally nilpotent on V.*

Lemma 3.2. *Let $t_0^{m_0} t^{\underline{m}} k_p \in \mathscr{L}$ be such that $\bar{m} = (m_0, \underline{m}) \neq \bar{0}$, and there exists $0 \leq a \leq N$ such that $m_a \neq 0$ if $N < p \leq \nu$. If $t_0^{m_0} t^{\underline{m}} k_p$ is locally nilpotent on V, then* $\dim V_{\bar{n}} > \dim V_{\bar{n}+\bar{m}}$ *for all $\bar{n} \in \mathbb{Z}^{\nu+1}$.*

Proof. *Case 1:* $p \in \{0, 1, \ldots, N\}$. We first prove that $\dim V_{\bar{n}} \geq \dim V_{\bar{n}+\bar{m}}$ for all $\bar{n} \in \mathbb{Z}^{\nu+1}$. Suppose $\dim V_{\bar{n}} = m$, $\dim V_{\bar{n}+\bar{m}} = n$. Let $\{w_1, w_2, \ldots, w_n\}$ be a basis of $V_{\bar{n}+\bar{m}}$ and $\{w'_1, w'_2, \ldots, w'_m\}$ a basis of $V_{\bar{n}}$. We can assume that $m_a \neq 0$ for some $0 \leq a \leq \nu$ distinct from p, where $\bar{m} = (m_0, \underline{m}) = (m_0, m_1, \ldots, m_\nu)$. Since $t^{\bar{m}} k_p$ is locally nilpotent on V and $V_{\bar{n}+\bar{m}}$ is finite-dimensional, there exists $k > 0$ such that $(t^{\bar{m}} k_p)^k V_{\bar{n}+\bar{m}} = 0$. Therefore

$$(t^{-\bar{m}} d_a)^k (t^{\bar{m}} k_p)^k (w_1, w_2, \ldots, w_n) = 0.$$

On the other hand, by induction on k, we can deduce that

$$(t^{-\bar{m}} d_a)^k (t^{\bar{m}} k_p)^k = \sum_{i=0}^{k} \frac{k!\, k!}{i!\,(k-i)!\,(k-i)!} m_a^i c_p^i (t^{\bar{m}} k_p)^{k-i} (t^{-\bar{m}} d_a)^{k-i}.$$

Therefore

$$t^{\bar{m}} k_p \Big(\sum_{i=0}^{k-1} \frac{k!\, k!}{i!\,(k-i)!\,(k-i)!} m_a^i c_p^i (t^{\bar{m}} k_p)^{k-1-i} (t^{-\bar{m}} d_a)^{k-1-i} \Big) t^{-\bar{m}} d_a (w_1, w_2, \ldots, w_n) = -k!\, m_a^k c_p^k (w_1, w_2, \ldots, w_n).$$

Assume that

$$\Big(\sum_{i=0}^{k-1} \frac{k!\, k!}{i!\,(k-i)!\,(k-i)!} m_a^i c_p^i (t^{\bar{m}} k_p)^{k-1-i} (t^{-\bar{m}} d_a)^{k-1-i} \Big) t^{-\bar{m}} d_a (w_1, w_2, \ldots, w_n) = (w'_1, w'_2, \ldots, w'_m) C,$$

with $C \in \mathbb{C}^{m \times n}$, and that

$$t^{\bar{m}} k_p (w'_1, w'_2, \ldots, w'_m) = (w_1, w_2, \ldots, w_n) B, \tag{3-1}$$

with $B \in \mathbb{C}^{n\times m}$. Then

$$BC = -k!\, m_a^k c_p^k I.$$

This implies that $m \geq n$. So $\dim V_{\bar n} \geq \dim V_{\bar n+\bar m}$ for all $\bar n \in \mathbb{Z}^{\nu+1}$. Also, by (3-1) and the fact that $r(B) = n$, we know that $m > n$ if and only if there exists $v \in V_{\bar n}$ such that $t^{\bar m}k_p \cdot v = 0$. Since $t^{\bar m}k_p$ is locally nilpotent on V, there exist an integer $s \geq 0$ and $w \in V_{\bar n+s\bar m}$ such that

$$(t^{\bar m}k_p) \cdot w = 0.$$

Therefore $(t^{-\bar m}k_p)t^{\bar m}k_p \cdot w = t^{\bar m}k_p(t^{-\bar m}k_p \cdot w) = 0$. If $t^{-\bar m}k_p \cdot w = 0$, by the proof above, $\dim V_{\bar n+s\bar m-\bar m} < \dim V_{\bar n+s\bar m}$, contradicting the fact that $\dim V_{\bar n+s\bar m-\bar m} \geq \dim V_{\bar n+s\bar m}$. Therefore $(t^{-\bar m}k_p)^r \cdot w \neq 0$ for all $r \in \mathbb{N}$. Since

$$(t^{-\bar m}k_p)^s t^{\bar m}k_p \cdot w = t^{\bar m}k_p(t^{-\bar m}k_p)^s \cdot w = 0$$

and $(t^{-\bar m}k_p)^s \cdot w \in V_{\bar n}$, it follows that there is a nonzero element v in $V_{\bar n}$ such that $t^{\bar m}k_p \cdot v = 0$. Thus $n < m$.

Case 2: $N < p \leq \nu$. The proof is similar to that of case 1, but we have to consider $t^{-\bar m}d_p$ and $t^{\bar m}k_p$ instead and use the $\mathbb{Z}$-linear independence of $c_1, \dots, c_N$. □

Lemma 3.3. *Let $0 \neq t^{\bar m}k_p \in \mathcal{L}$ and $0 \neq t^{\bar n}k_p \in \mathcal{L}$ be such that $(m_0, \dots, m_N) \neq 0$, $(n_0, \dots, n_N) \neq 0$ if $N < p \leq \nu$, where $\bar m = (m_0, m_1, \dots, m_\nu)$.*

(1) *If $t^{\bar m}k_p$ is locally nilpotent on V, $t^{\bar m}k_q$ is locally nilpotent for $q = 0, 1, \dots, \nu$.*
(2) *If both $0 \neq t^{\bar m}k_p$ and $0 \neq t^{\bar n}k_p$ are locally nilpotent on V, then $t^{\bar m+\bar n}k_p$ is locally nilpotent.*
(3) *If $0 \neq t^{\bar m+\bar n}k_p$ is locally nilpotent on V and $(m_0+n_0, \dots, m_N+n_N) \neq 0$ if $N < p \leq \nu$, then $t^{\bar m}k_p$ or $t^{\bar n}k_p$ is locally nilpotent.*

Lemma 3.4. *For $0 \leq p \leq \nu$, let $0 \neq t^{\bar m}k_p \in \mathcal{L}$ be such that $(m_0, \dots, m_N) \neq 0$, where $\bar m = (m_0, m_1, \dots, m_\nu)$. Then $t^{\bar m}k_p$ or $t^{-\bar m}k_p$ is locally nilpotent on V.*

Proof. The proof occupies the next few pages. We first deal with the case $0 \leq p \leq N$. Without losing generality, we can take $p = 0$.

Suppose the lemma is false. By Lemma 3.2, for any $\bar r \in \mathbb{Z}^{\nu+1}$ we have

$$\dim V_{\bar r+\bar m} = \dim V_{\bar r} = \dim V_{\bar r-\bar m}, \quad t^{\bar m}k_0 V_{\bar r} = V_{\bar r+\bar m}, \quad t^{-\bar m}k_0 V_{\bar r} = V_{\bar r-\bar m}.$$

Fix $\bar r = (r_0, \underline{r}) \in \mathbb{Z}^{\nu+1}$ such that $V_{\bar r} \neq 0$. Let $\{v_1, \dots, v_n\}$ be a basis of $V_{\bar r}$ and set

$$v_i(k\bar m) = \frac{1}{c_0} t^{k\bar m}k_0 \cdot v_i, \quad i = 1, 2, \dots, n,$$

where $k \in \mathbb{Z} \setminus \{0\}$. Then $\{v_1(k\bar{m}), v_2(k\bar{m}), \ldots, v_n(k\bar{m})\}$ is a basis of $V_{\bar{r}+k\bar{m}}$. Let $B^{(0)}_{-\bar{m},\bar{m}}, B^{(0)}_{\bar{m},-\bar{m}} \in \mathbb{C}^{n\times n}$ be such that

$$\frac{1}{c_0} t^{\bar{m}} k_0 (v_1(-\bar{m}), v_2(-\bar{m}), \ldots, v_n(-\bar{m})) = (v_1, v_2, \ldots, v_n) B^{(0)}_{\bar{m},-\bar{m}},$$
$$\frac{1}{c_0} t^{-\bar{m}} k_0 (v_1(\bar{m}), v_2(\bar{m}), \ldots, v_n(\bar{m})) = (v_1, v_2, \ldots, v_n) B^{(0)}_{-\bar{m},\bar{m}}.$$

Since $t^{\bar{m}}k_0$ and $t^{-\bar{m}}k_0$ are commutative, it is easy to deduce that

$$B^{(0)}_{\bar{m},-\bar{m}} = B^{(0)}_{-\bar{m},\bar{m}}.$$

By Lemma 3.1, $B^{(0)}_{\bar{m},-\bar{m}}$ is an $n \times n$ invertible matrix.

Claim. *$B^{(0)}_{\bar{m},-\bar{m}}$ does not have distinct eigenvalues.*

Proof. Set $c = 1/c_0$. To prove the claim, we need to consider $ct^{\bar{m}}k_0ct^{-\bar{m}}k_0 - \lambda\,\mathrm{id}$, where $\lambda \in \mathbb{C}^*$. As in the proof of Lemma 3.1, we can deduce that if there is a nonzero element v in V such that $(ct^{\bar{m}}k_0ct^{-\bar{m}}k_0 - \lambda\,\mathrm{id})v = 0$, then $ct^{\bar{m}}k_0ct^{-\bar{m}}k_0 - \lambda\,\mathrm{id}$ is locally nilpotent on V. On the other hand, we have

$$(ct^{\bar{m}}k_0ct^{-\bar{m}}k_0 - \lambda\,\mathrm{id})^l (v_1, v_2, \ldots, v_n) = (v_1, v_2, \ldots, v_n)(B^{(0)}_{\bar{m},-\bar{m}} - \lambda\,\mathrm{id})^l.$$

Therefore the claim holds. □

For $p \in \{1, 2, \ldots, \nu\}$, let $C^{p}_{\bar{m},\bar{0}}, C^{p}_{\bar{m},-\bar{m}} \in \mathbb{C}^{n\times n}$ be such that

$$t^{\bar{m}}k_p(v_1, v_2, \ldots, v_n) = (v_1(\bar{m}), \ldots, v_n(\bar{m}))C^{(p)}_{\bar{m},\bar{0}},$$
$$t^{\bar{m}}k_p(v_1(-\bar{m}), \ldots, v_n(-\bar{m})) = (v_1, v_2, \ldots, v_n)C^{(p)}_{\bar{m},-\bar{m}}.$$

Since

$$\frac{1}{c_0} t^{-\bar{m}} k_0 t^{\bar{m}} k_p (v_1, v_2, \ldots, v_n) = t^{\bar{m}} k_p \frac{1}{c_0} t^{-\bar{m}} k_0 (v_1, v_2, \ldots, v_n),$$

we have

$$C^{(p)}_{\bar{m},-\bar{m}} = B^{(0)}_{-\bar{m},\bar{m}} C^{(p)}_{\bar{m},\bar{0}}. \tag{3-2}$$

Furthermore, by the fact that

$$\frac{1}{c_0} t^{\bar{m}} k_0 \frac{1}{c_0} t^{-\bar{m}} k_0 t^{\bar{m}} k_p (v_1, v_2, \ldots, v_n) = t^{\bar{m}} k_p \frac{1}{c_0} t^{\bar{m}} k_0 \frac{1}{c_0} t^{-\bar{m}} k_0 (v_1, v_2, \ldots, v_n)$$

and

$$t^{\bar{m}} k_q \frac{1}{c_0} t^{-\bar{m}} k_0 t^{\bar{m}} k_p = t^{\bar{m}} k_p \frac{1}{c_0} t^{-\bar{m}} k_0 t^{\bar{m}} k_q,$$

we deduce that

$$B^{(0)}_{-\bar m,\bar m}C^{(p)}_{\bar m,\bar 0}=C^{(p)}_{\bar m,\bar 0}B^{(0)}_{-\bar m,\bar m},\quad C^{(p)}_{\bar m,\bar 0}C^{(q)}_{\bar m,\bar 0}=C^{(q)}_{\bar m,\bar 0}C^{(p)}_{\bar m,\bar 0},\quad 1\le p,q\le \nu. \tag{3-3}$$

Hence there exists $D\in\mathbb{C}^{n\times n}$ such that $\{D^{-1}B^{(0)}_{-\bar m,\bar m}D,\ D^{-1}C^{(p)}_{\bar m,\bar 0}D \mid 1\le p\le\nu\}$ are all upper triangular matrices. If we set

$$(w_1,w_2,\dots,w_n)=(v_1,v_2,\dots,v_n)D$$

and

$$w_i(k\bar m)=\frac{1}{c_0}t^{k\bar m}k_0w_i,\ 1\le i\le n,\ k\in\mathbb{Z}\setminus\{0\},$$

then

$$\frac{1}{c_0}t^{k\bar m}k_0(w_1(-\bar m),w_2(-\bar m),\dots,w_n(-\bar m))=(w_1,\dots,w_n)D^{-1}B^{(0)}_{-\bar m,\bar m}D,$$
$$t^{\bar m}k_p(w_1,w_2,\dots,w_n)=(w_1(\bar m),\dots,w_n(\bar m))D^{-1}C^{(p)}_{\bar m,\bar 0}D.$$

So we can assume that $B^{(0)}_{-\bar m,\bar m}$, $C^{(p)}_{\bar m,\bar 0}$, and $C^{(p)}_{\bar m,-\bar m}$, for $1\le p\le\nu$ are all invertible upper triangular matrices. Furthermore, because

$$\Big(t^{\bar m}k_p\frac{1}{c_0}t^{-\bar m}k_0-\lambda\,\mathrm{id}\Big)^l(v_1,v_2,\dots,v_n)=(v_1,v_2,\dots,v_n)(C^{(p)}_{\bar m,-\bar m}-\lambda\,\mathrm{id})^l,$$

the argument used in the proof of the claim shows that $C^{(p)}_{\bar m,-\bar m}$ also does not have distinct eigenvalues. For $1\le p\le N$, set

$$B^{(p)}_{\bar m,-\bar m}=\frac{1}{c_p}C^{(p)}_{\bar m,-\bar m}$$

and for $0\le p\le N$ denote by λ_p the eigenvalue of $B^{(p)}_{\bar m,-\bar m}$.

Let $A^{(a)}_{k\bar m,\bar 0}$ and $A^{(a)}_{k_1\bar m,k_2\bar m}$, for $0\le a\le\nu$ and $k,k_1,k_2\in\mathbb{Z}\setminus\{0\}$, be such that

$$\begin{aligned}t^{k\bar m}d_a(v_1,v_2,\dots,v_n)&=(v_1(k\bar m),v_2(k\bar m),\dots,v_n(k\bar m))A^{(a)}_{k\bar m,\bar 0},\\ t^{k_1\bar m}d_a(v_1(k_2\bar m),&\,v_2(k_2\bar m),\dots,v_n(k_2\bar m))\\ &=(v_1(k_1\bar m+k_2\bar m),\dots,v_n(k_1\bar m+k_2\bar m))A^{(a)}_{k_1\bar m,k_2\bar m}.\end{aligned}$$

Case 1: $\nu>1$. Since $t^{\bar m}k_0=t_0^{m_0}t^{\bar m}k_0\ne 0$, it follows that there exists $1\le a\le\nu$ such that $m_a\ne 0$, where $\underline{m}=(m_1,m_2,\dots,m_\nu)$. Let $b\in\{1,\dots,\nu\}$ be such that $a\ne b$. Consider

$$[t^{-\bar m}d_a,\frac{1}{c_0}t^{\bar m}k_0]=m_a\frac{1}{c_0}k_0,\qquad [t^{-\bar m}d_a,t^{\bar m}k_b]=m_ak_b. \tag{3-4}$$

Case 1.1: There exists $b\in\{0,1,\dots,\nu\}$ such that $b\ne 0$, a and $c_b=0$. Then

$$A^{(a)}_{-\bar m,\bar m}=B^{(0)}_{\bar m,-\bar m}A^{(a)}_{-\bar m,\bar 0}+m_aI,\qquad A^{(a)}_{-\bar m,\bar m}C^{(b)}_{\bar m,\bar 0}=C^{(b)}_{\bar m,-\bar m}A^{(a)}_{-\bar m,\bar 0}.$$

By (3-2) and (3-3),

$$A^{(a)}_{-\bar{m},\bar{0}}+m_aB^{(0)}_{\bar{m},-\bar{m}}{}^{-1}=C^{(b)}_{\bar{m},\bar{0}}A^{(a)}_{-\bar{m},\bar{0}}C^{(b)}_{\bar{m},\bar{0}}{}^{-1}.$$

But the sum on the left-hand side cannot be similar to $A^{(a)}_{-\bar{m},\bar{0}}$, since $m_a\neq 0$ and $B^{(0)}_{\bar{m},-\bar{m}}{}^{-1}$ is an invertible upper triangular matrix and does not have different eigenvalues. Thus this case is excluded.

<u>*Case 1.2*</u>*:* $c_b\neq 0$ *for all* $b\in\{0,1,\dots,\nu\}$, $b\neq 0,a$. By (3-4) and (3-2), we have

$$B^{(0)}_{\bar{m},-\bar{m}}A^{(a)}_{-\bar{m},\bar{0}}B^{(0)}_{\bar{m},-\bar{m}}{}^{-1}+m_aB^{(0)}_{\bar{m},-\bar{m}}{}^{-1}-m_aB^{(b)}_{\bar{m},-\bar{m}}{}^{-1}\\ =B^{(0)}_{\bar{m},-\bar{m}}C^{(b)}_{\bar{m},\bar{0}}A^{(a)}_{-\bar{m},\bar{0}}C^{(b)}_{\bar{m},\bar{0}}{}^{-1}B^{(0)}_{\bar{m},-\bar{m}}{}^{-1}.$$

(I) There exists $b\neq 0$ and a such that $\lambda_0\neq\lambda_b$. Then $m_aB^{(0)}_{\bar{m},-\bar{m}}{}^{-1}-m_aB^{(b)}_{\bar{m},-\bar{m}}{}^{-1}$ is an invertible upper triangular matrix and does not have different eigenvalues. As in case 1.1, we deduce a contradiction.

(II) $\lambda_0=\lambda_b$ for all $b\in\{1,\dots,\nu\}$ distinct from a.

(II.1) Suppose first that $c_a=0$ (in this case $N=\nu-1$, $a=\nu$) or $c_a\neq 0$ and $\lambda_a=\lambda_0$ (in this case $N=\nu$). Since $\sum_{p=0}^{\nu}m_pt^{\bar{m}}k_p=0$, we have

$$\sum_{p=0}^{\nu}m_pt^{\bar{m}}k_p\frac{1}{c_0}t^{-\bar{m}}k_0=0.$$

So $\sum\limits_{p=0}^{\nu}m_pC^{(p)}_{\bar{m},-\bar{m}}=0$, and therefore

$$\sum_{p=0}^{\nu}m_pc_p=0,$$

which contradicts the assumption that $c_0,\dots,c_N$ are $\mathbb{Z}$-linearly independent.

(II.2) Now suppose $c_a\neq 0$, $\lambda_a\neq\lambda_0$ and there exists $b\neq 0$ and a such that $m_b\neq 0$. We deduce a contradiction as in case 1.2(I) by interchanging a by b.

(II.3) Suppose $c_a\neq 0$, $\lambda_a\neq\lambda_0$ and $m_b=0$ for all $b\in\{1,\dots,\nu\}$ distinct from a. Then $m_0c_0\lambda_0+m_ac_a\lambda_a=0$. The proof of this case is the same as in case 2.2 below.

<u>***Case 2.***</u>*:* $\nu=1$. In this case $a=1$.

<u>*Case 2.1*</u>*:* $c_a=0$. Since $[t^{-\bar{m}}d_0,t^{\bar{m}}k_0]=[t^{-\bar{m}}k_0,t^{\bar{m}}d_0]=0$, we have

$$A^{(0)}_{-\bar{m},\bar{m}}=B^{(0)}_{\bar{m},-\bar{m}}A^{(0)}_{-\bar{m},\bar{0}},\qquad A^{(0)}_{\bar{m},-\bar{m}}=B^{(0)}_{-\bar{m},\bar{m}}A^{(0)}_{\bar{m},\bar{0}}.$$

Therefore

$$[t^{-\bar{m}}d_0,t^{\bar{m}}d_0](v_1,v_2,\dots,v_n)=(v_1,v_2,\dots,v_n)B^{(0)}_{-\bar{m},\bar{m}}\big[A^{(0)}_{-\bar{m},\bar{0}},A^{(0)}_{\bar{m},\bar{0}}\big].$$

At the same time, we have

$$[t^{-\bar{m}}d_0, t^{\bar{m}}d_0] = 2m_0d_0 + m_0^2(-\mu_1+\mu_2)(m_0k_0+m_1k_1),$$

where $\tau = \mu_1\tau_1 + \mu_2\tau_2$ as above. So

$$B^{(0)}_{-\bar{m},\bar{m}}[A^{(0)}_{-\bar{m},\bar{0}}, A^{(0)}_{\bar{m},\bar{0}}] = \big(2m_0(\gamma_0(d_0)+r_0)+m_0^2(-\mu_1+\mu_2)(m_0c_0+m_1c_1)\big)I, \tag{3-5}$$

where γ_0 is the weight fixed above. Since γ_0 is arbitrary, we can choose it such that

$$2m_0(\gamma_0(d_0)+r_0) + m_0^2(-\mu_1+\mu_2)(m_0c_0+m_1c_1) \neq 0.$$

But $B^{(0)}_{-\bar{m},\bar{m}}$ is an invertible triangular matrix and does not have different eigenvalues, in contradiction with (3-5).

<u>*Case 2.2*</u>*:* $c_a \neq 0$. Since

$$[t^{-\bar{m}}d_0, t^{\bar{m}}k_0] = -m_1k_1, [t^{-\bar{m}}d_1, t^{\bar{m}}k_0] = m_1k_0 \text{ and}$$
$$[t^{\bar{m}}d_0, t^{-\bar{m}}k_0] = m_1k_1, [t^{\bar{m}}d_1, t^{-\bar{m}}k_0] = -m_1k_0,$$

we have

$$[k_0t^{-\bar{m}}d_0 + k_1t^{-\bar{m}}d_1, t^{\bar{m}}k_0] = [k_0t^{\bar{m}}d_0 + k_1t^{\bar{m}}d_1, t^{-\bar{m}}k_0] = 0.$$

Therefore

$$k_0A^{(0)}_{-\bar{m},\bar{m}} + k_1A^{(1)}_{-\bar{m},\bar{m}} = B^{(0)}_{\bar{m},-\bar{m}}\big(k_0A^{(0)}_{-\bar{m},\bar{0}} + k_1A^{(1)}_{-\bar{m},\bar{0}}\big),$$
$$k_0A^{(0)}_{\bar{m},-\bar{m}} + k_1A^{(1)}_{\bar{m},-\bar{m}} = B^{(0)}_{-\bar{m},\bar{m}}\big(k_0A^{(0)}_{\bar{m},\bar{0}} + k_1A^{(1)}_{\bar{m},\bar{0}}\big),$$

and

$$[k_0t^{-\bar{m}}d_0 + k_1t^{-\bar{m}}d_1, k_0t^{\bar{m}}d_0 + k_1t^{\bar{m}}d_1](v_1,\dots,v_n)$$
$$= (v_1,\dots,v_n)B^{(0)}_{\bar{m},-\bar{m}}\big[k_0A^{(0)}_{-\bar{m},\bar{0}} + k_1A^{(1)}_{-\bar{m},\bar{0}}, k_0A^{(0)}_{\bar{m},\bar{0}} + k_1A^{(1)}_{\bar{m},\bar{0}}\big].$$

At the same time, we have

$$[k_0t^{-\bar{m}}d_0 + k_1t^{-\bar{m}}d_1, k_0t^{\bar{m}}d_0 + k_1t^{\bar{m}}d_1]$$
$$= 2(m_0c_0+m_1c_1)(c_0d_0+c_1d_1) - (m_0c_0+m_1c_1)^3(\mu_1-\mu_2)\,\mathrm{id}\,.$$

Since c_0 and c_1 are $\mathbb{Z}$-linearly independent, we know that $m_0c_0+m_1c_1 \neq 0$. As in case 2.1, we deduce a contradiction.

This concludes the first part of the proof. We next turn to the second major case, $N < p \leq \nu$.

If $N \geq 1$ or $N = 0$, we have $(m_1,\dots,m_\nu) \neq 0$, and the lemma follows from the first part and Lemma 3.3. Otherwise, let $t^{\bar{m}}k_p = t_0^{m_0}k_p$. Set $\mathscr{L}_{\underline{0}} = \bigoplus_{m_0\in\mathbb{Z}} \mathbb{C}t_0^{m_0}d_0 \oplus \mathbb{C}k_0$ and $W = U(\mathscr{L}_{\underline{0}})v$, where $v \in V_{\bar{s}}$ is a homogeneous element. Since $c_0 \neq 0$, the sets $\{\dim W_{(n_0,0)+\bar{s}} \mid n_0 \in \mathbb{Z}\}$ are not uniformly bounded. But if neither $t_0^{m_0}k_p$

nor $t_0^{-m_0}k_p$ is locally nilpotent, then t_0k_p and $t_0^{-1}k_p$ are not locally nilpotent. So by Lemmas 3.2 and 3.1, $\dim V_{(n_0,0)+\bar{s}} = \dim V_{\bar{s}}$ for all $n_0 \in \mathbb{Z}$, which is impossible since $\dim V_{(n_0,0)+\bar{s}} \geq \dim W_{(n_0,0)+\bar{s}}$. This proves Lemma 3.4 □

For $0 \leq p \leq N$, consider the direct sum

$$\bigoplus_{m_p \in \mathbb{Z}} \mathbb{C} t_p^{m_p} d_p \oplus \mathbb{C} k_p,$$

which is a Virasoro Lie subalgebra of $\mathcal{L}$. Since $c_p \neq 0$, it follows from [Mathieu 1992] that there is a nonzero $v_p \in V_{\bar{r}}$ for some $\bar{r} \in \mathbb{Z}^{\nu+1}$ such that

$$t_p^{m_p} d_p v_p = 0 \quad \text{for all } m_p \in \mathbb{Z}_+ \tag{3-6}$$

or

$$t_p^{m_p} d_p v_p = 0 \quad \text{for all } m_p \in \mathbb{Z}_-. \tag{3-7}$$

Lemma 3.5. *If $v_p \in V_{\bar{r}}$ satisfies (3-6), the sets*

$$\{t_p^{m_p} k_q \mid m_p \in \mathbb{Z}_+, q = 0, 1, 2, \dots, \nu, q \neq p\}$$

are all locally nilpotent on V. Likewise for (3-7), with $\mathbb{Z}_+$ replaced by $\mathbb{Z}_-$.

Proof. We only prove the first statement. Suppose it is false; then by Lemma 3.3 t_pk_q is not locally nilpotent on V for some $q \in \{0, 1, \dots, \nu\}$, $q \neq p$. By Lemma 3.4, $t_p^{-1}k_q$ is locally nilpotent. Therefore there exists $k \in \mathbb{Z}_+$ such that

$$(t_p^{-1}k_q)^{k-1} v_p \neq 0, \quad (t_p^{-1}k_q)^k v_p = 0.$$

So

$$\begin{aligned} t_p^2 d_p (t_p^{-1}k_q)^k v_p &= -k t_p k_q (t_p^{-1}k_q)^{k-1} v_p + (t_p^{-1}k_q)^k t_p^2 d_p v_p \\ &= -k t_p k_q (t_p^{-1}k_q)^{k-1} v_p = 0. \end{aligned}$$

This implies that t_pk_q is locally nilpotent, a contradiction. □

Lemma 3.6. *If $v_p \in V_{\bar{r}}$ satisfies (3-6), the sets*

$$\{t^{\bar{m}} k_p \mid \bar{m} = (m_0, \dots, m_\nu) \in \mathbb{Z}^{\nu+1}, m_p \in \mathbb{Z}_+\}$$

are all locally nilpotent on V. Likewise for (3-7), with $\mathbb{Z}_+$ replaced by $\mathbb{Z}_-$.

Proof. Again we only prove the first statement. Without loss of generality, we assume that $p = 0$. Let $\mathcal{K}'$ be the subspace of $\mathcal{K}$ spanned by elements of $\mathcal{K}$ which are locally nilpotent on V. If $t^{\underline{m}}k_0$, for any $\underline{m} \in \mathbb{Z}^\nu \setminus \{0\}$, is not locally nilpotent on V, the lemma holds thanks to Lemmas 3.3 and 3.5. Suppose $\mathcal{K}' \cap \{t^{\underline{m}}k_0 \mid \underline{m} \in \mathbb{Z}^\nu\} \neq \{0\}$. By Lemmas 3.2, 3.3 and 3.5, if $t^{\underline{m}}k_0 \in \mathcal{K}'$, then $t^{-\underline{m}}k_0 \notin \mathcal{K}'$, and $t_0^{m_0} t^{\underline{m}} k_0 \in \mathcal{K}'$ for all $m_0 > 0$.

Case 1: Suppose $t_0^{m_0} t^{-\underline{m}} k_0 \in \mathcal{K}'$ for any $t^{\underline{m}}k_0 \in \mathcal{K}'$. Then the lemma is proved.

Case 2: Suppose there exists $0 \neq t^{\underline{m}}k_0 \in \mathcal{K}'$ such that $t_0 t^{-\underline{m}}k_0 \notin \mathcal{K}'$. Since $\underline{m} = (m_1, \dots, m_\nu) \neq 0$, we can assume that $m_a \neq 0$ for some $a \in \{1, 2, \dots, \nu\}$. Let $V_{\bar{r}_0}$ be such that

$$\dim V_{\bar{r}_0} = \min\{\dim V_{\bar{s}} \mid V_{\bar{s}} \neq 0, \bar{s} \in \mathbb{Z}^{\nu+1}\}.$$

Case 2.1: Assume $t_0^i t^{-\underline{m}}k_0 \notin \mathcal{K}'$ for any $i > 0$. Let $l \in \mathbb{Z}_+$ and consider

$$\sum_{i=0}^{l} a_i t_0^{-i} t^{-\underline{m}} k_0 t_0^i t^{-\underline{m}} k_0 v = 0, \tag{3-8}$$

where $v \in V_{\bar{r}_0} \setminus \{0\}$. By Lemma 3.4, $\{t_0^i t^{\underline{m}} k_0, t_0^{-i} t^{\underline{m}} k_0 \mid i \in \mathbb{Z}_+\} \subseteq \mathcal{K}'$. So by Lemma 3.2, we have

$$t_0^i t^{\underline{m}} k_0 V_{\bar{r}_0} = t_0^{-i} t^{\underline{m}} k_0 V_{\bar{r}_0} = t_0^i t^{\underline{m}} d_p V_{\bar{r}_0} = t_0^{-i} t^{\underline{m}} d_p V_{\bar{r}_0} = 0, i \in \mathbb{Z}_+, 0 \le p \le \nu.$$

Let $j \in \{0, 1, \dots, l\}$. From (3-8) we have

$$t_0^{-j} t^{\underline{m}} d_a t_0^j t^{\underline{m}} d_a \Big(\sum_{i=0}^{l} a_i t_0^{-i} t^{-\underline{m}} k_0 t_0^i t^{-\underline{m}} k_0\Big) v = 0.$$

Therefore

$$\sum_{i=0}^{l} a_i (-m_a) t_0^{j-i} k_0 (-m_a) t_0^{i-j} k_0 v = a_j m_a^2 c_0^2 v = 0.$$

So $a_j = 0$, $j = 0, 1, \dots, l$. This means $\{(t_0^{-i} t^{-\underline{m}} k_0 t_0^i t^{-\underline{m}} k_0) v \mid 0 \le i \le l\}$ are linearly independent. Since l can be any positive integer, it follows that $V_{\bar{r}_0 - (0, 2\underline{m})}$ is infinite-dimensional, a contradiction.

Case 2.2: Assume there exists $l \in \mathbb{Z}_+$ such that

$$t_0^{l-1} t^{-\underline{m}} k_0 \notin \mathcal{K}', \qquad t_0^l t^{-\underline{m}} k_0 \in \mathcal{K}'.$$

(I) Assume that $t_0^l t^{-i\underline{m}} k_0 \in \mathcal{K}'$ for any $i \in \mathbb{Z}_+$. Let $s > 0$ and consider

$$\sum_{i=1}^{s} a_i t_0^{-l} t^{i\underline{m}} k_0 t^{-i\underline{m}} k_0 v = 0.$$

Similar to the proof above, we can deduce that $V_{\bar{r}_0 - (l, \underline{0})}$ is infinite-dimensional, in contradiction with the assumption that V has finite-dimensional weight spaces.

(II) Assume there exists $s_1 \in \mathbb{Z}_+$ such that

$$t_0^l t^{-\underline{m}} k_0 \in \mathcal{K}', \quad t_0^l t^{-2\underline{m}} k_0 \in \mathcal{K}', \quad \dots, \quad t_0^l t^{-s_1\underline{m}} k_0 \in \mathcal{K}', \quad t_0^l t^{-(s_1+1)\underline{m}} k_0 \notin \mathcal{K}'.$$

Then there exist $s_2, s_3, \dots, s_k, \dots$ such that $s_i \ge s_1$ for $i = 2, 3, \dots, k, \dots$ and

$$t_0^{il} t^{(-s_1 - s_2 - \dots - s_{i-1} - 1)\underline{m}} k_0 \in \mathcal{K}', \ t_0^{il} t^{(-s_1 - s_2 - \dots - s_{i-1} - 2)\underline{m}} k_0 \in \mathcal{K}', \dots,$$

$$t_0^{il} t^{(-s_1-s_2-\cdots-s_{i-1}-s_i)\underline{m}} k_0 \in \mathcal{K}', \ \ t_0^{il} t^{(-s_1-s_2-\cdots-s_{i-1}-s_i-1)\underline{m}} k_0 \notin \mathcal{K}'.$$

Assume that

$$\Big(\sum_{i=1}^{s_1} a_i t_0^{-l} t^{i\underline{m}} k_0 t^{-i\underline{m}} k_0 + \sum_{i=1}^{s_2} a_{s_1+i} t_0^{-2l} t^{(s_1+i)\underline{m}} k_0 t_0^{l} t^{-(s_1+i)\underline{m}} k_0 + \sum_{i=1}^{s_3} a_{s_1+s_2+i} t_0^{-3l} t^{(s_1+s_2+i)\underline{m}} k_0 t_0^{2l} t^{-(s_1+s_2+i)\underline{m}} k_0 + \cdots + \sum_{i=1}^{s_k} a_{s_1+\cdots+s_{k-1}+i} t_0^{-kl} t^{(s_1+\cdots+s_{k-1}+i)\underline{m}} k_0 t_0^{(k-1)l} t^{-(s_1+\cdots+s_{k-1}+i)\underline{m}} k_0 \Big) v = 0.$$

Let

$$\begin{aligned} & t^{j\underline{m}} d_a t_0^{l} t^{-j\underline{m}} d_a, && 1 \le j \le s_1, \\ & t_0^{-l} t^{(s_1+j)\underline{m}} d_a t_0^{2l} t^{-(s_1+j)\underline{m}} d_a, && 1 \le j \le s_2, \\ & \ldots, \\ & t_0^{-(k-1)l} t^{(s_1+s_2+\cdots+s_{k-1}+j)\underline{m}} d_a t_0^{kl} t^{-(s_1+s_2+\cdots+s_{k-1}+j)\underline{m}} d_a, && 1 \le j \le s_k \end{aligned}$$

act on the two sides of the above equation respectively. By Lemma 3.4, we deduce that $a_i = 0$, for $i = 1, 2, \ldots, s_1$, and that

$$a_{s_1+\cdots+s_{j-1}+i} = 0 \quad \text{for } i = 1, 2, \ldots, s_j, \ 2 \le j \le k.$$

Since k can be any positive integer, it follows that $V_{\bar{r}_0-(l,\underline{0})}$ is infinite-dimensional, which contradicts our assumption. The lemma is proved. □

Lemmas 3.1 through 3.6 immediately yield the following result.

Theorem 3.7. *Let V be an irreducible weight module of $\mathcal{L}$ such that $c_0, \ldots, c_N$ are $\mathbb{Z}$-linearly independent and $N \ge 1$. Then V has weight spaces that are infinite-dimensional.*

Let

$$\begin{aligned} \mathcal{L}_+ &= \sum_{p=0}^{\nu} t_0 \mathbb{C}[t_0, t_1^{\pm 1}, \ldots, t_\nu^{\pm 1}] k_p \oplus \sum_{p=0}^{\nu} t_0 \mathbb{C}[t_0, t_1^{\pm 1}, \ldots, t_\nu^{\pm 1}] d_p, \\ \mathcal{L}_- &= \sum_{p=0}^{\nu} t_0^{-1} \mathbb{C}[t_0^{-1}, t_1^{\pm 1}, \ldots, t_\nu^{\pm 1}] k_p \oplus \sum_{p=0}^{\nu} t_0^{-1} \mathbb{C}[t_0^{-1}, t_1^{\pm 1}, \ldots, t_\nu^{\pm 1}] d_p, \\ \mathcal{L}_0 &= \sum_{p=0}^{\nu} \mathbb{C}[t_1^{\pm 1}, \ldots, t_\nu^{\pm 1}] k_p \oplus \sum_{p=0}^{\nu} \mathbb{C}[t_1^{\pm 1}, \ldots, t_\nu^{\pm 1}] d_p. \end{aligned}$$

Then

$$\mathcal{L} = \mathcal{L}_+ \oplus \mathcal{L}_0 \oplus \mathcal{L}_-.$$

Definition 3.8. Let W be a weight module of $\mathscr{L}$. If there is a nonzero vector $v_0 \in W$ such that

$$\mathscr{L}_+ v_0 = 0,\ W = U(\mathscr{L})v_0,$$

then W is called a highest weight module of $\mathscr{L}$. If there is a nonzero vector $v_0 \in W$ such that

$$\mathscr{L}_- v_0 = 0,\ W = U(\mathscr{L})v_0,$$

then W is called a lowest weight module of $\mathscr{L}$.

From Lemmas 3.2 and 3.6, we obtain:

Theorem 3.9. *Let V be an irreducible weight module of $\mathscr{L}$ with finite-dimensional weight spaces and with central charges $c_0 \neq 0$, $c_1 = c_2 = \cdots = c_\nu = 0$. Then V is a highest or lowest weight module of $\mathscr{L}$.*

In the remainder of this section we assume that V is an irreducible weight module of $\mathscr{L}$ with finite-dimensional weight spaces and with central charges $c_0 \neq 0$, $c_1 = \cdots = c_\nu = 0$.

Set

$$T = \begin{cases} \{v \in V \mid \mathscr{L}_+ v = 0\} & \text{if} V \text{ is a highest weight module of } \mathscr{L}, \\ \{v \in V \mid \mathscr{L}_- v = 0\} & \text{if} V \text{ is a lowest weight module of } \mathscr{L}. \end{cases}$$

Then T is a $\mathscr{L}_0$-module and

$$V = U(\mathscr{L}_-)T \quad \text{or} \quad V = U(\mathscr{L}_+)T.$$

Since V is an irreducible $\mathscr{L}$-module, T is an irreducible $\mathscr{L}_0$-module. T has the decomposition

$$T = \bigoplus_{\underline{m} \in \mathbb{Z}^\nu} T_{\underline{m}},$$

where $\underline{m} = (m_1, m_2, \ldots, m_\nu)$, $T_{\underline{m}} = \{v \in T \mid d_i v = (m_i + \mu(d_i))v,\ 1 \leq i \leq \nu\}$ and μ is a fixed weight of T. As in the proof in [Jiang and Meng 2003; Eswara Rao and Jiang 2005], we can deduce:

Theorem 3.10. (1) *For all $\underline{m}, \underline{n} \in \mathbb{Z}^\nu$, $p = 1, 2, \ldots, \nu$, we have*

$$\dim T_{\underline{m}} = \dim T_{\underline{n}},\ t^{\underline{m}} k_p \cdot T = 0,$$

$$t^{\underline{m}} k_0(v_1(\underline{n}), \ldots, v_m(\underline{n})) = c_0(v_1(\underline{m}+\underline{n}), v_2(\underline{m}+\underline{n}), \ldots, v_n(\underline{m}+\underline{n})),$$

$$t^{\underline{m}} d_0(v_1(\underline{n}), v_2(\underline{n}), \ldots, v_n(\underline{n})) = \mu(d_0)(v_1(\underline{m}+\underline{n}), v_2(\underline{m}+\underline{n}), \ldots, v_n(\underline{m}+\underline{n})),$$

where $\{v_1(\underline{0}), \ldots, v_m(\underline{0})\}$ is a basis of $T_{\underline{0}}$ and $v_i(\underline{m}) = \dfrac{1}{c_0} t^{\underline{m}} k_0 v_i(\underline{0})$, for $i = 1, 2, \ldots, m$.

(2) *As an* $(\mathcal{A}_\nu \oplus \mathcal{D}_\nu)$*-module*, T *is isomorphic to*

$$F^\alpha(\psi, b) = V(\psi, b) \otimes \mathbb{C}[t_1^{\pm 1}, \dots, t_\nu^{\pm 1}]$$

for some $\alpha = (\alpha_1, \dots, \alpha_\nu)$, ψ, *and* b, *where* $\mathcal{A}_\nu = \mathbb{C}[t_1^{\pm 1}, \dots, t_\nu^{\pm 1}]$, $\mathcal{D}_\nu$ *is the derivation algebra of* $\mathcal{A}_\nu$, *and* $V(\psi, b)$ *is an* m*-dimensional, irreducible* $gl_\nu(\mathbb{C})$*-module satisfying* $\psi(I) = b\,\mathrm{id}_{V(\psi,b)}$ *and*

$$t^{\underline{r}} d_p(w \otimes t^{\underline{m}}) = (m_p + \alpha_p) w \otimes t^{\underline{r}+\underline{m}} + \sum_{i=1}^{\nu} r_i \psi(E_{ip}) w \otimes t^{\underline{r}+\underline{m}}$$

for $w \in V(\psi, b)$.

Let

$$M = \mathrm{Ind}_{\mathcal{L}_+ + \mathcal{L}_0}^{\mathcal{L}} T \quad \text{or} \quad M = \mathrm{Ind}_{\mathcal{L}_- + \mathcal{L}_0}^{\mathcal{L}} T.$$

Theorem 3.11. *Among the submodules of* M *intersecting* T *trivially, there is a maximal one, which we denote by* M^{rad}. *Moreover* $V \cong M/M^{\mathrm{rad}}$.

4. The structure of V with $c_0 = \cdots = c_\nu = 0$

Assume that V is an irreducible weight module of $\mathcal{L}$ with finite-dimensional weight spaces and $c_0 = \cdots = c_\nu = 0$.

Lemma 4.1. *For any* $t^{\bar{r}} k_p \in \mathcal{K}$, $t^{\bar{r}} k_p$ *or* $t^{-\bar{r}} k_p$ *is locally nilpotent on* V.

Lemma 4.2. *If* V *is uniformly bounded*, $t^{\bar{r}} k_p$ *is locally nilpotent on* V *for any* $t^{\bar{r}} k_p \in \mathcal{K}$.

Proof. For $t^{\bar{r}} k_p \in \mathcal{K}$, by Lemma 4.1, $t^{\bar{r}} k_p$ or $t^{-\bar{r}} k_p$ is nilpotent on $V_{\bar{m}}$ for all $\bar{m} \in \mathbb{Z}^{\nu+1}$. Since V is uniformly bounded, i.e., $\max\{\dim V_{\bar{m}} \mid \bar{m} \in \mathbb{Z}^{\nu+1}\} < \infty$, there exists $N \in \mathbb{Z}_+$ such that

$$(t^{\bar{r}} k_p t^{-\bar{r}} k_p)^N V = 0, (t^{\bar{r}} k_p t^{-\bar{r}} k_p)^{N-1} V \neq 0$$

If the lemma is false, we can assume that $t^{-\bar{r}} k_p$ is not locally nilpotent on V. Therefore for any $0 \neq v \in V$, we have $t^{-\bar{r}} k_p v \neq 0$. So

$$(t^{\bar{r}} k_p)^N V = 0.$$

Let $t^{-2\bar{r}} d_q \in \mathcal{K}$ be such that $p \neq q$ and $r_q \neq 0$. By the fact that $[t^{-2\bar{r}} d_q, t^{\bar{r}} k_p] = r_q t^{-\bar{r}} k_p$, we deduce that $t^{-\bar{r}} k_p (t^{\bar{r}} k_p)^{N-1} V = 0$, a contradiction. □

Lemma 4.3. *If there exists* $0 \neq v \in V$ *such that* $t^{\bar{m}} k_p v = 0$ *for all* $\bar{m} \in \mathbb{Z}^{\nu+1}$ *and* $0 \leq p \leq \nu$. *Then* $\mathcal{K}(V) = 0$.

Proof. This follows from (2-2), since $\mathcal{K}$ is commutative and V is an irreducible $\mathcal{L}$-module. □

Theorem 4.4. *If* V *is uniformly bounded*, $t^{\bar{r}} k_p V$ *vanishes for any* $t^{\bar{r}} k_p \in \mathcal{K}$.

Proof. Let $0 \neq t_i k_p \in \mathcal{K}$. If $t_i k_p V = 0$, it is easy to prove that $\mathcal{K}(V) = 0$. If $t_i k_p V \neq 0$. Since V is uniformly bounded, by Lemma 4.2, there exists $l \in \mathbb{Z}_+$ such that

$$(t_i k_p t_i^{-1} k_p)^l V = 0, \quad (t_1 k_p t_1^{-1} k_p)^{l-1} V \neq 0. \tag{4-1}$$

If there exists $s \in \mathbb{Z}_+$ such that $(t_i^{-1} k_p)^s V = 0$, $(t_i^{-1} k_p)^{s-1} V \neq 0$. By the fact that $[t^{\bar{m}} d_i, t_i^{-1} k_p] = -t_i^{-1} t^{\bar{m}} k_p$ and $[t^{\bar{m}} d_p, t_i^{-1} k_p] = t_i^{-1} t^{\bar{m}} k_i$, we have

$$t^{\bar{r}} k_p (t_i^{-1} k_p)^{s-1} V = t^{\bar{r}} k_i (t^{-\bar{r}} k_p)^{s-1} V = 0 \quad \text{for all } \bar{r} \in \mathbb{Z}^{\nu+1}.$$

If $(t_i^{-1} k_p)^s V \neq 0$ for all $s \in \mathbb{Z}_+$. Then by (4-1) there is $r \geq 0$ such that $(t_i k_p)^{l-i} (t_i^{-1} k_p)^{l+i} V = 0$ for all $0 \leq i \leq r$, and $(t_i k_p)^{l-r-1} (t_i^{-1} k_p)^{l+r+1} V \neq 0$. So for any $\bar{m} \in \mathbb{Z}^{\nu+1}$, we have

$$t^{-\bar{m}} d_i (t_i k_p)^{l-r} (t_i^{-1} k_p)^{l+r+1} V = 0, \quad t^{-\bar{m}} d_p (t_i k_p)^{l-r} (t_i^{-1} k_p)^{l+r+1} V = 0.$$

Therefore

$$\begin{aligned} t^{\bar{r}} k_p (t_i k_p)^{l-r-1} (t_i^{-1} k_p)^{l+r+1} V &= 0, \\ t^{\bar{r}} k_i (t_i k_p)^{l-r-1} (t_i^{-1} k_p)^{l+r+1} V &= 0, \end{aligned}$$

for all $\bar{r} \in \mathbb{Z}^{\nu+1}$.

Case 1: $\nu \in 2\mathbb{Z}_+ + 1$. By the preceding discussion, there exist nonnegative integers l_i and r_i, for $i = 0, 2, 4, \ldots, \nu - 1$, such that

$$(t_\nu k_{\nu-1})^{l_{\nu-1}} (t_\nu^{-1} k_{\nu-1})^{r_{\nu-1}} (t_{\nu-2} k_{\nu-3})^{l_{\nu-3}} (t_{\nu-2}^{-1} k_{\nu-3})^{r_{\nu-3}} \cdots (t_1 k_0)^{l_0} (t_1^{-1} k_0)^{r_0} V \neq 0$$

and

$$t^{\bar{m}} k_p (t_\nu k_{\nu-1})^{l_{\nu-1}} (t_\nu^{-1} k_{\nu-1})^{r_{\nu-1}} (t_{\nu-2} k_{\nu-3})^{l_{\nu-3}} (t_{\nu-2}^{-1} k_{\nu-3})^{r_{\nu-3}} \cdots (t_1 k_0)^{l_0} (t_1^{-1} k_0)^{r_0} V$$

vanishes for all $0 \leq p \leq \nu$ and $\bar{m} \in \mathbb{Z}^{\nu+1}$. By Lemma 4.3, the conclusion of the theorem holds.

Case 2: $\nu \in 2\mathbb{Z}$. Then there exist nonnegative integers l_i and r_i, for $i = 0, 2, 4, \ldots,$ $\nu - 2$, such that

$$W = (t_{\nu-1} k_{\nu-2})^{l_{\nu-2}} (t_{\nu-1}^{-1} k_{\nu-2})^{r_{\nu-2}} (t_{\nu-3} k_{\nu-4})^{l_{\nu-4}} (t_{\nu-3}^{-1} k_{\nu-4})^{r_{\nu-4}} \cdots (t_1 k_0)^{l_0} (t_1^{-1} k_0)^{r_0} V$$

is nonzero and

$$t^{\bar{m}} k_p W = 0 \tag{4-2}$$

for all $0 \leq p \leq \nu - 1$ and $\bar{m} \in \mathbb{Z}^{\nu+1}$. By (2-1), we know that

$$t^{\bar{m}} k_\nu W = 0, \tag{4-3}$$

for $\bar{m} \in \mathbb{Z}^{\nu+1}$ such that $m_\nu \neq 0$. If there exists $t^{\bar{r}_0}k_\nu$ satisfying $t^{\bar{r}_0}k_\nu W \neq 0$, let

$$\begin{aligned}
&\mathscr{L}_\nu = \operatorname{span}\{t^{\underline{m}}d_i, t^{\bar{m}}d_\nu, t^{\underline{m}}k_\nu \mid t^{\underline{m}} = t_0^{m_0}t_1^{m_1}\cdots t_{\nu-1}^{m_{\nu-1}}, 0 \leq i \leq \nu-1,\\
&\underline{m} = (m_0, \ldots, m_{\nu-1}) \in \mathbb{Z}^\nu, \bar{m} \in \mathbb{Z}^{\nu+1}\},\\
&W' = U(\mathscr{L}_\nu)W.
\end{aligned}$$

Then $W' \neq 0$ and

$$t^{\bar{m}}k_p W' = 0, \qquad t^{\bar{n}}k_\nu W' = 0,$$

for all $0 \leq p \leq \nu-1$, $\bar{m} \in \mathbb{Z}^{\nu+1}$, and $\bar{n} \in \mathbb{Z}^{\nu+1}$ such that $n_\nu \neq 0$. If there exists $0 \neq t^{\underline{m}}k_\nu$ such that $t^{\underline{m}}k_\nu W' \neq 0$, we have

$$(t^{-\underline{m}}k_\nu)^l(t^{\underline{m}}k_\nu)^l W' = 0 \quad \text{and} \quad (t^{-\underline{m}}k_\nu)^{l-1}(t^{\underline{m}}k_\nu)^{l-1} W' \neq 0$$

for some $l \in \mathbb{Z}_+$. As in the preceding proof, we can deduce that there exists a nonzero $v \in W'$ such that

$$t^{\underline{n}}k_\nu v = 0$$

for all $\underline{n} \in \mathbb{Z}^\nu$. Therefore

$$t^{\bar{m}}k_p v = 0$$

for all $\bar{m} \in \mathbb{Z}^{\nu+1}$ and $0 \leq p \leq \nu$. We have proved that $\mathscr{K}(V) = 0$. □

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