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HIGHER HOMOTOPY COMMUTATIVITY OF H-SPACES AND HOMOTOPY LOCALIZATIONS

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Dedicated to Professor Takao Matumoto on his sixtieth birthday

In this paper, we prove that the homotopy localization of an AC_n-space is an AC_n-space so that the universal map is an AC_n-map. This result is used to study the higher homotopy commutativity of H-spaces with finitely generated cohomology over the Steenrod algebra $\mathscr{A}_p^*$. Our result shows that for any prime p, if X is a connected AC_p-space whose mod p cohomology $H^*(X; \mathbb{Z}/p)$ is finitely generated as an algebra over $\mathscr{A}_p^*$, then X has the mod p homotopy type of a Postnikov H-space.

1. Introduction

The theory of H-spaces has been studied in algebraic topology to understand homotopy properties of Lie groups. Given a prime p, a $\mathbb{Z}/p$-finite H-space is an H-space whose mod p cohomology is finite dimensional. In recent decades, many theorems have been proved about $\mathbb{Z}/p$-finite H-spaces [Kane 1988; Lin 1995], which suggest that they have many similar properties to those of Lie groups.

In this paper, we study an H-space which need not be $\mathbb{Z}/p$-finite but whose mod p cohomology is finitely generated as an algebra over the Steenrod algebra $\mathscr{A}_p^*$. For example, the n-connected covering $X\langle n\rangle$ of a $\mathbb{Z}/p$-finite H-space X is not $\mathbb{Z}/p$-finite for $n \geq 3$ but the mod p cohomology is finitely generated as an algebra over $\mathscr{A}_p^*$, by [Castellana et al. 2006, Corollary 4.3]. Eilenberg–Mac Lane spaces $K(\mathbb{Z}, n)$ and $K(\mathbb{Z}/p^i, n)$ are other examples for $n, i \geq 1$.

Using the homotopy localizations of Bousfield [1994] and Dror Farjoun [1996], Castellana, Crespo and Scherer have studied H-spaces with finitely generated cohomology over $\mathscr{A}_p^*$ [Castellana et al. 2007]. In their Theorem 7.3, these authors proved that if X is such an H-space, the $B\mathbb{Z}/p$-localization $L_{B\mathbb{Z}/p}(X)$ is a $\mathbb{Z}/p$-finite H-space and the homotopy fiber $F(\phi_X)$ of the universal map $\phi_X\colon X \to L_{B\mathbb{Z}/p}(X)$ is mod p homotopy equivalent to a Postnikov H-space (see Theorem

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5.1). Here an H-space is called Postnikov if the homotopy groups are finitely generated over the p-adic integers $\mathbb{Z}_p^\wedge$ which vanish above some dimension, and mod p homotopy equivalence means homotopy equivalence up to p-completion in the sense of [Bousfield and Kan 1972].

Moreover, by combining their main result with the mod 2 torus theorem by Hubbuck [1969] and Lin [1985], Castellana et al. generalized results of Slack [1991] and Lin and Williams [1991], as follows:

Theorem 1.1 [Castellana et al. 2007, Corollary 7.4]. *If X is a connected homotopy commutative H-space whose* mod 2 *cohomology $H^*(X;\mathbb{Z}/2)$ is finitely generated as an algebra over $\mathscr{A}_2^*$, then X is* mod 2 *homotopy equivalent to a Postnikov H-space.*

On the other hand, the odd prime version of Theorem 1.1 does not hold. In fact, Iriye and Kono [1985] showed that for an odd prime p, any connected H-space is mod p homotopy equivalent to a homotopy commutative H-space. Moreover, $Sp(2)_3^\wedge$ for $p=3$ and $(S^3)_p^\wedge$ for $p \geq 5$ are examples of homotopy commutative loop spaces which are not Postnikov H-spaces by McGibbon [1984], where $Y_p^\wedge$ denotes the p-completion of a space Y.

To describe an odd prime version of Theorem 1.1, we use the higher homotopy commutativity of the multiplication. Such notions are first considered by Sugawara [1960] and Williams [1969] in the case of topological monoids. (The higher homotopy commutativity of the third order in the sense of Williams is illustrated by the left hexagon in Figure 1.)

Williams' definition was generalized to the case of A_n-spaces in [Hemmi and Kawamoto 2004] (see also [Hemmi 1991]). An A_n-space with a multiplication admitting the higher homotopy commutativity of the n-th order is called an AC_n-space. By [Hemmi and Kawamoto 2004, Example 3.2(1)], an AC_2-space is the same as a homotopy commutative H-space. Let X be an A_3-space admitting an AC_2-structure. Then by using the associating homotopy $M_3\colon I \times X^3 \to X$ and the

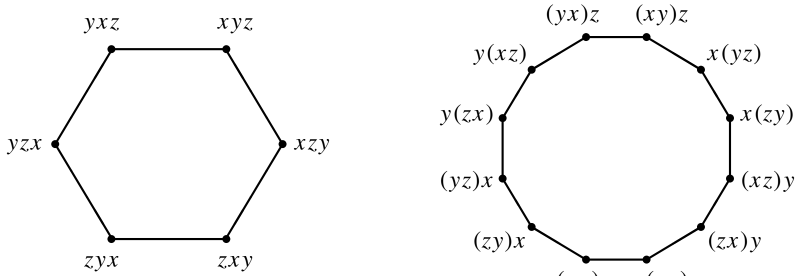

Figure 1. The higher homotopy commutativity of the third order.

commuting homotopy $Q_2 \colon I \times X^2 \to X$, we can define a map $\widetilde{Q}_3 \colon S^1 \times X^3 \to X$ illustrated by the right dodecagon in Figure 1. For example, the uppermost edge represents the commuting homotopy between xy and yx given by $Q_2(t, x, y)z$, and the next right edge is the associating homotopy between $(xy)z$ and $x(yz)$ given by $M_3(t, x, y, z)$. Then X is an AC_3-space if and only if $\widetilde{Q}_3$ is extended to a map $Q_3 \colon D^2 \times X^3 \to X$. In general, X is an AC_n-space if and only if there is a family of maps

$$\{Q_i \colon D^{i-1} \times X^i \to X\}_{1 \le i \le n}$$

with the relations in [Hemmi and Kawamoto 2004, Proposition 2.1].

To generalize Theorem 1.1 to the case of any prime p, we first show:

Theorem A. *Let A be a topological space and $n \ge 1$. If X is an AC_n-space, then the A-localization $L_A(X)$ is an AC_n-space so that the universal map $\phi_X \colon X \to L_A(X)$ is an AC_n-map.*

From Theorem A and [Castellana et al. 2007, Theorem 7.3], we can generalize the mod p torus theorem stated in [Hemmi and Kawamoto 2004, Corollary 1.1] to the case of AC_p-spaces with finitely generated cohomology over $\mathscr{A}_p^*$.

Theorem B. *Let p be a prime. If X is a connected AC_p-space whose* mod *p cohomology $H^*(X; \mathbb{Z}/p)$ is finitely generated as an algebra over $\mathscr{A}_p^*$, then X is* mod *p homotopy equivalent to a Postnikov H-space.*

Theorem B is a generalization of Theorem 1.1 to the case of any prime p since an AC_2-space is the same as a homotopy commutative H-space. In the above theorem, the assumption of AC_p-space cannot be relaxed to AC_{p-1}-space. In fact, by [Hemmi and Kawamoto 2004, Proposition 3.8], the $(2m-1)$-dimensional sphere $(S^{2m-1})^\wedge_p$ is an AC_{p-1}-space but not a Postnikov H-space for $m \ge 2$.

Moreover, since the loop space of an H-space admits an AC_∞-structure by [Hemmi and Kawamoto 2004, Example 3.2(3)], Theorem B implies:

Corollary 1.2 [Castellana et al. 2007, p. 17]. *Let p be a prime. Assume that X is a connected loop space whose* mod *p cohomology $H^*(X; \mathbb{Z}/p)$ is finitely generated as an algebra over $\mathscr{A}_p^*$. If the classifying space BX is an H-space, then X is* mod *p homotopy equivalent to a Postnikov H-space.*

There is an example of a Postnikov loop space Y admitting an AC_p-structure such that the classifying space BY is not an H-space by [McGibbon 1989, Example 5]. Corollary 1.2 is a generalization of results from [Aguadé and Smith 1986; Kawamoto 1999; Lin 1994].

Bousfield [2001] studied the $K(n)_*$-localizations of Postnikov H-spaces, where $K(n)_*$ denotes the Morava K-homology theory for $n \ge 1$. By Theorem B and [Bousfield 2001, Theorem 7.2], we have:

Corollary 1.3. *Let p be a prime and $n \geq 1$. If X is a connected AC_p-space whose* mod p *cohomology $H^*(X; \mathbb{Z}/p)$ is finitely generated as an algebra over $\mathscr{A}_p^*$, then the $K(n)_*$-localization $L_{K(n)_*}(X_p^\wedge)$ is* mod p *homotopy equivalent to the $\Sigma^n B\mathbb{Z}/p$-localization $L_{\Sigma^n B\mathbb{Z}/p}(X_p^\wedge)$. In particular, $X_p^\wedge$ is $K(n)_*$-local if and only if the n-fold loop space $\Omega^n X_p^\wedge$ is $B\mathbb{Z}/p$-local.*

We also generalize [Hemmi and Kawamoto 2007, Theorem B] to the case of A_p-spaces with finitely generated cohomology over $\mathscr{A}_p^*$.

Theorem C. *Let p be an odd prime. Assume that X is a connected A_p-space admitting an AC_n-structure with $n > (p-1)/2$ and the* mod p *cohomology $H^*(X; \mathbb{Z}/p)$ is finitely generated as an algebra over $\mathscr{A}_p^*$. If the Steenrod operations $\mathcal{P}^j$ act on the indecomposable module $QH^*(X; \mathbb{Z}/p)$ trivially for $j \geq 1$, then X is* mod p *homotopy equivalent to a finite product of $(S^1)_p^\wedge$s, $(\mathbb{C}P^\infty)_p^\wedge$s and $B\mathbb{Z}/p^i$s for $i \geq 1$.*

In Theorem C, the assumption $n > (p-1)/2$ is necessary. In fact, by [Hemmi 1991, Theorem 2.4], $(S^3)_p^\wedge$ is an A_p-space admitting an $AC_{(p-1)/2}$-structure for any odd prime p.

Outline of article. In Section 2, we recall the associahedra, the multiplihedra and the permuto-associahedra. Then we show that the permuto-associahedra are decomposed by using the multiplihedra in a combinatorial way (see Proposition 2.1). In Section 3, we give the definition of an AC_n-map between AC_n-spaces by using Proposition 2.1 (see Definition 3.1). Section 4 is devoted to the proof of Theorem A. We show that if $\phi\colon X \to Y$ is an A_n-map between A_n-spaces and Y is ϕ-local, then ϕ transmits an AC_n-structure from X to Y (see Proposition 4.1). By applying Proposition 4.1 to the universal map $\phi_X\colon X \to L_A(X)$ for the A-localization of X, we prove Theorem A. In Section 5, we first recall the result of [Castellana et al. 2007] on H-spaces with finitely generated cohomology over $\mathscr{A}_p^*$ (Theorem 5.1). From Theorem A, Theorem 5.1 and the results in [Hemmi 1991; Hemmi and Kawamoto 2004], we prove Theorem B. Next Corollary 1.3 is proved by Theorem B and the result from [Bousfield 2001] on the $K(n)_*$-localizations of Postnikov H-spaces. We finally give the proof of Theorem C by using Theorem A and [Hemmi and Kawamoto 2007, Theorem B].

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2. Decompositions of the permuto-associahedra

We first recall the associahedra $\{K_n\}_{n\geq 2}$ of Stasheff and the multiplihedra $\{J_n\}_{n\geq 1}$ of Iwase and Mimura.

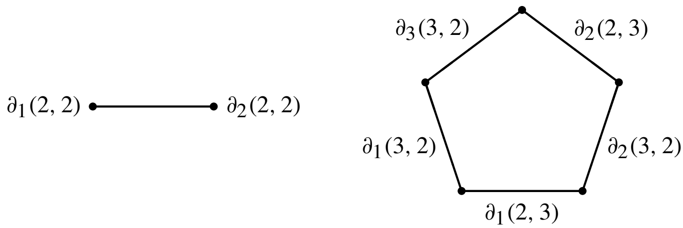

Figure 2. The associahedra K_3 and K_4.

Stasheff [1963, p. 283] constructed the associahedra $\{K_n\}_{n\geq 2}$ to introduce the concept of A_n-space (see Section 3). From the construction, the associahedron K_n is an $(n-2)$-dimensional polytope whose boundary ∂K_n is given by

$$\partial K_n = \bigcup_{r,s,k} K_k(r,s)$$

for $n \geq 2$, where $r, s \geq 2$ with $r+s = n+1$ and $1 \leq k \leq r$. Here the facet (codimension-one face) $K_k(r,s)$ is homeomorphic to the product $K_r \times K_s$ by a face operator $\partial_k(r,s)\colon K_r \times K_s \to K_k(r,s)$ with the relations in [Stasheff 1963, p. 278, 3(a),(b)]. There is a family of degeneracy operators $\{\theta_j : K_n \to K_{n-1}\}_{1\leq j\leq n}$ satisfying the relations in [Stasheff 1963, p. 278, Proposition 3].

The associahedra $\{K_n\}_{n\geq 2}$ are also used in [Stasheff 1970, Definition 11.9] to define an A_n-map from an A_n-space to a topological monoid.

Iwase and Mimura [1989, §2] introduced the multiplihedra $\{J_n\}_{n\geq 1}$ for the purpose of defining an A_n-map between A_n-spaces (see Section 3). From the properties in [Iwase and Mimura 1989, p. 200, (2-a) and (2-b)], the multiplihedron J_n is an $(n-1)$-dimensional polytope whose boundary ∂J_n is given by

$$\partial J_n = \bigcup_{k,r,s} J_k(r,s) \cup \bigcup_{q,r_1,\ldots,r_q} J(q, r_1, \ldots, r_q)$$

for $n \geq 1$, where $r \geq 1$, $s \geq 2$ with $r+s = n+1$ and $1 \leq k \leq r$, and $2 \leq q \leq n$, $r_1, \ldots, r_q \geq 1$ with $r_1+\cdots+r_q = n$. As in the case of the associahedra, we have face

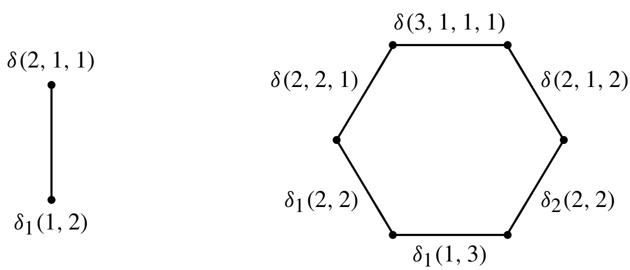

Figure 3. The multiplihedra J_2 and J_3.

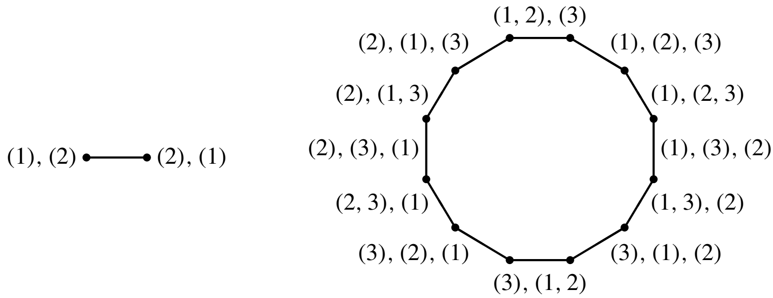

Figure 4. The permuto-associahedra Γ_2 and Γ_3.

operators $\delta_k(r,s)\colon J_r \times K_s \to J_k(r,s)$ and $\delta(q, r_1, \dots, r_q)\colon K_q \times J_{r_1} \times \cdots \times J_{r_q} \to J(q, r_1, \dots, r_q)$ with the relations in (2-c) of the same work. The degeneracy operators $\{\xi_j\colon J_n \to J_{n-1}\}_{1\le j\le n}$ satisfy the relations in (2-d).

We next recall the permuto-associahedra $\{\Gamma_n\}_{n\ge 1}$ constructed by Kapranov and by Reiner and Ziegler. By [Kapranov 1993, Theorem 2.5] and [Reiner and Ziegler 1994, Theorem 2], the permuto-associahedron Γ_n is an $(n-1)$-dimensional polytope whose faces are described in a combinatorial way for $n \ge 1$ (see also [Ziegler 1995, Definition 9.13, Example 9.14]). In particular, a facet of Γ_n is represented by a partition of the sequence $\mathbf{n} = (1, \dots, n)$ into at least two parts. Here a partition of $\mathbf{n}$ of type $(t_1, \dots, t_l)$ is an ordered sequence $(\alpha_1, \dots, \alpha_l)$ consisting of disjoint subsequences α_i of $\mathbf{n}$ of length t_i with $\alpha_1 \cup \cdots \cup \alpha_l = \mathbf{n}$. See [Hemmi and Kawamoto 2004; Ziegler 1995] for the full details of the partitions.

Let $\Gamma(\alpha_1, \dots, \alpha_l)$ denote the facet of Γ_n corresponding to a partition $(\alpha_1, \dots, \alpha_l)$. The boundary of Γ_n is given by

$$\partial \Gamma_n = \bigcup_{(\alpha_1,\dots,\alpha_l)} \Gamma(\alpha_1, \dots, \alpha_l), \tag{2-1}$$

where the union covers all partitions $(\alpha_1, \dots, \alpha_l)$ of $\mathbf{n}$ with $l \ge 2$. If $(\alpha_1, \dots, \alpha_l)$ is of type $(t_1, \dots, t_l)$, then the facet $\Gamma(\alpha_1, \dots, \alpha_l)$ is homeomorphic to the product $K_l \times \Gamma_{t_1} \times \cdots \times \Gamma_{t_l}$ by a face operator $\epsilon^{(\alpha_1,\dots,\alpha_l)}\colon K_l \times \Gamma_{t_1} \times \cdots \times \Gamma_{t_l} \to \Gamma(\alpha_1, \dots, \alpha_l)$ with the relations in Proposition 2.1 of [Hemmi and Kawamoto 2004]. Moreover, there are degeneracy operators $\{\omega_j\colon \Gamma_n \to \Gamma_{n-1}\}_{1\le j\le n}$ satisfying the relations in Proposition 2.3 of the same reference.

In Definition 3.1, we need the following result:

Proposition 2.1. *Let $n \ge 1$.*

(1) *The permuto-associahedron Γ_n is decomposed by*

$$\Gamma_n = \bigcup_{(\beta_1,\dots,\beta_m)} B(\beta_1, \dots, \beta_m),$$

where the union covers all partitions $(\beta_1, \dots, \beta_m)$ *of* $\mathbf{n}$ *with* $m \geq 1$.

(2) *If* $(\beta_1, \dots, \beta_m)$ *is a partition of* $\mathbf{n}$ *of type* $(u_1, \dots, u_m)$, *then* $B(\beta_1, \dots, \beta_m)$ *is homeomorphic to the product* $J_m \times \Gamma_{u_1} \times \cdots \times \Gamma_{u_m}$ *by an operator*

$$\iota^{(\beta_1,\dots,\beta_m)} \colon J_m \times \Gamma_{u_1} \times \cdots \times \Gamma_{u_m} \to B(\beta_1, \dots, \beta_m).$$

By an inductive argument, we can show:

Lemma 2.2 [Stasheff 1963, p. 288, Proposition 25]. *There is a family of homeomorphisms* $\{\zeta_m \colon I \times K_m \to J_m\}_{m\geq 2}$ *with the relations*

$$\zeta_m(0, \sigma) = \delta_1(1, m)(*, \sigma),$$
$$\zeta_m(t, \partial_k(r, s)(\rho, \sigma)) = \delta_k(r, s)(\zeta_r(t, \rho), \sigma)$$

for $r, s \geq 2$ *with* $r + s = m + 1$ *and* $1 \leq k \leq r$.

Proof of Proposition 2.1. We work by induction on n. Since $\Gamma_1 = J_1 = *$, the result is clear for $n = 1$. We put

$$\mathcal{U}_n = \Gamma_n \cup_{\{0\}\times\partial\Gamma_n} I \times \partial\Gamma_n, \tag{2-2}$$

where I is the unit interval and $\{0\} \times \partial\Gamma_n$ is identified with $\partial\Gamma_n \subset \Gamma_n$. It is clear that $\mathcal{U}_n$ is homeomorphic to the $(n-1)$-dimensional ball.

Let $B(\mathbf{n}) = \Gamma_n \subset \mathcal{U}_n$. Then an operator $\iota^{(\mathbf{n})} \colon J_1 \times \Gamma_n \to B(\mathbf{n})$ is defined by $\iota^{(\mathbf{n})}(*, \tau) = \tau$ for $\tau \in \Gamma_n$. If $m \geq 2$, then by Lemma 2.2, we can identify J_m with

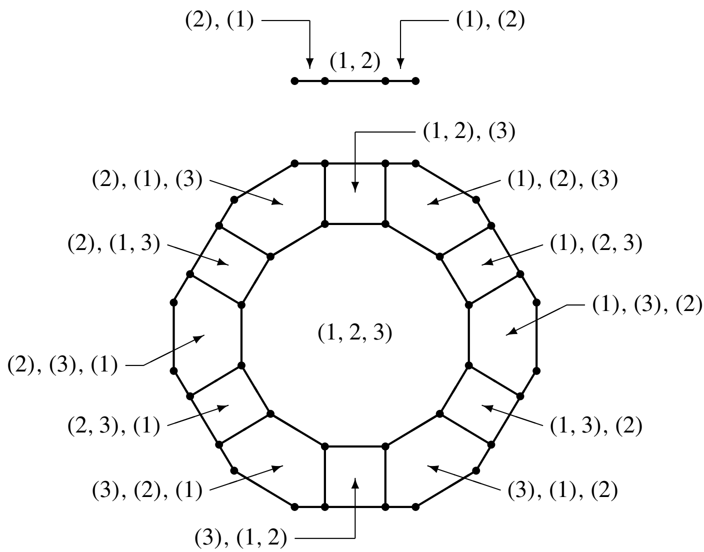

Figure 5. The decompositions of Γ_2 and Γ_3.

$I \times K_m$ by $\zeta_m \colon I \times K_m \to J_m$. Assume that $(\beta_1, \dots, \beta_m)$ is a partition of $\mathbf{n}$ of type $(u_1, \dots, u_m)$ with $m \geq 2$. Put

$$B(\beta_1, \dots, \beta_m) = I \times \Gamma(\beta_1, \dots, \beta_m) \subset \mathcal{U}_n,$$

and define an operator $\iota^{(\beta_1, \dots, \beta_m)} \colon J_m \times \Gamma_{u_1} \times \cdots \times \Gamma_{u_m} \to B(\beta_1, \dots, \beta_m)$ by

$$\iota^{(\beta_1, \dots, \beta_m)}(\zeta_m(t, \sigma), \tau_1, \dots, \tau_m) = (t, \epsilon^{(\beta_1, \dots, \beta_m)}(\sigma, \tau_1, \dots, \tau_m)).$$

Then by (2-1) and (2-2), we have that

$$\mathcal{U}_n = \bigcup_{(\beta_1, \dots, \beta_m)} B(\beta_1, \dots, \beta_m),$$

where the union covers all partitions $(\beta_1, \dots, \beta_m)$ of $\mathbf{n}$ with $m \geq 1$.

By Lemma 2.2, we see that

$$\zeta_m(\{1\} \times K_m) = \bigcup_{q, r_1, \dots, r_q} J(q, r_1, \dots, r_q)$$

for $2 \leq q \leq m$ and $r_1, \dots, r_q \geq 1$ with $r_1 + \cdots + r_q = m$. This implies that

$$\text{(2-3)} \quad \partial\mathcal{U}_n = \bigcup_{(\beta_1, \dots, \beta_m)} \iota^{(\beta_1, \dots, \beta_m)}\left(\left(\bigcup_{q, r_1, \dots, r_q} J(q, r_1, \dots, r_q)\right) \times \Gamma_{u_1} \times \cdots \times \Gamma_{u_m}\right)$$

for $2 \leq q \leq m$ and $r_1, \dots, r_q \geq 1$ with $r_1 + \cdots + r_q = m$, where $(\beta_1, \dots, \beta_m)$ are partitions of $\mathbf{n}$ of type $(u_1, \dots, u_m)$ with $m \geq 2$. If we define face operators on $\partial\mathcal{U}_n$ satisfying the relations in [Hemmi and Kawamoto 2004, Proposition 2.1], then $\partial\mathcal{U}_n$ is homeomorphic to $\partial\Gamma_n$, and it follows that $\mathcal{U}_n$ is homeomorphic to Γ_n, which implies the required conclusion.

Recall that $\partial\Gamma_n$ is given by

$$\partial\Gamma_n = \bigcup_{(\alpha_1, \dots, \alpha_l)} \Gamma(\alpha_1, \dots, \alpha_l),$$

where the union covers all partitions $(\alpha_1, \dots, \alpha_l)$ of $\mathbf{n}$ with $l \geq 2$.

Assume that $(\alpha_1, \dots, \alpha_l)$ is a partition of $\mathbf{n}$ of type $(t_1, \dots, t_l)$ with $l \geq 2$. Then by inductive hypothesis, we can assume that

$$\Gamma_{t_j} = \bigcup_{(\gamma_{j,1}, \dots, \gamma_{j,h_j})} B(\gamma_{j,1}, \dots, \gamma_{j,h_j})$$

for $1 \leq j \leq l$, where the union covers all partitions $(\gamma_{j,1}, \dots, \gamma_{j,h_j})$ of $(1, \dots, t_j)$ with $h_j \geq 1$. If $(\gamma_{j,1}, \dots, \gamma_{j,h_j})$ is a partition of $(1, \dots, t_j)$ of type $(v_{j,1}, \dots, v_{j,h_j})$, then by inductive hypothesis, we have the operator $\iota^{(\gamma_{j,1}, \dots, \gamma_{j,h_j})} \colon J_{h_j} \times \Gamma_{v_{j,1}} \times \cdots \times \Gamma_{v_{j,h_j}} \to B(\gamma_{j,1}, \dots, \gamma_{j,h_j})$ which is a homeomorphism. Put $m = h_1 + \cdots + h_l$.

We give a partition $(\beta_1, \dots, \beta_m)$ of $\mathbf{n}$ of type $(v_{1,1}, \dots, v_{1,h_1}, \dots, v_{l,1}, \dots, v_{l,h_l})$ by

$$\beta_i(t) = \alpha_j \gamma_{j,i-(h_1+\cdots+h_{j-1})}(t)$$

for $h_1 + \cdots + h_{j-1} + 1 \le i \le h_1 + \cdots + h_j$ and $1 \le t \le i - (h_1 + \cdots + h_{j-1})$. Define a face operator $\epsilon^{(\alpha_1,\dots,\alpha_l)} \colon K_l \times \Gamma_{t_1} \times \cdots \times \Gamma_{t_l} \to \partial \mathcal{U}_n$ by

$$\begin{aligned}\epsilon^{(\alpha_1,\dots,\alpha_l)}&(\sigma, \iota^{(\gamma_{1,1},\dots,\gamma_{1,h_1})}(\rho_1, \tau_{1,1}, \dots, \tau_{1,h_1}), \dots, \iota^{(\gamma_{l,1},\dots,\gamma_{l,h_l})}(\rho_l, \tau_{l,1}, \dots, \tau_{l,h_l})) \\ &= \iota^{(\beta_1,\dots,\beta_m)}(\delta(l, h_1, \dots, h_l)(\sigma, \rho_1, \dots, \rho_l), \tau_{1,1}, \dots, \tau_{1,h_1}, \dots, \tau_{l,1}, \dots, \tau_{l,h_l}).\end{aligned}$$

By (2-3) and the relation [Iwase and Mimura 1989, p. 201, (c-4)], the face operator satisfies the relations in [Hemmi and Kawamoto 2004, Proposition 2.1]. This implies that $\mathcal{U}_n$ is homeomorphic to Γ_n, and so we have the required conclusion. This completes the proof. □

Remark 2.3. The decomposition of Γ_n in Proposition 2.1 is compatible with the degeneracy operators $\{\omega_j \colon \Gamma_n \to \Gamma_{n-1}\}_{1\le j\le n}$. Assume that $(\beta_1, \dots, \beta_m)$ is a partition of $\mathbf{n}$ of type $(u_1, \dots, u_m)$ for $u_1, \dots, u_m \ge 1$ with $u_1 + \cdots + u_m = n$. Let $1 \le j \le n$. Then $\beta_k(t) = j$ for some $1 \le k \le m$ and $1 \le t \le u_k$.

(i) If $u_k \ge 2$, then

$$\omega_j \iota^{(\beta_1,\dots,\beta_m)}(\sigma, \tau_1, \dots, \tau_m) = \iota^{(\tilde\beta_1,\dots,\tilde\beta_m)}(\sigma, \tau_1, \dots, \tau_{k-1}, \omega_t(\tau_k), \tau_{k+1}, \dots, \tau_m),$$

where $(\tilde\beta_1, \dots, \tilde\beta_m)$ is the partition of $(1, \dots, n-1)$ of type $(u_1, \dots, u_{k-1}, u_k - 1, u_{k+1}, \dots, u_m)$ given by

$$\tilde\beta_k(s) = \begin{cases} \beta_k(s) & \text{if } \beta_k(s) < j, \\ \beta_k(s+1) - 1 & \text{if } \beta_k(s) \ge j \end{cases}$$

and for $1 \le i \le n$ with $i \ne k$,

$$\tilde\beta_i(s) = \begin{cases} \beta_i(s) & \text{if } \beta_i(s) < j, \\ \beta_i(s) - 1 & \text{if } \beta_i(s) > j. \end{cases} \tag{2-4}$$

(ii) If $u_k = 1$, then

$$\omega_j \iota^{(\beta_1,\dots,\beta_m)}(\sigma, \tau_1, \dots, \tau_m) = \iota^{(\tilde\beta_1,\dots,\tilde\beta_{k-1},\tilde\beta_{k+1},\dots,\tilde\beta_m)}(\xi_k(\sigma), \tau_1, \dots, \tau_{k-1}, \tau_{k+1}, \dots, \tau_m),$$

where $(\tilde\beta_1, \dots, \tilde\beta_{k-1}, \tilde\beta_{k+1}, \dots, \tilde\beta_m)$ is the partition of $(1, \dots, n-1)$ of type

$$(u_1, \dots, u_{k-1}, u_{k+1}, \dots, u_m)$$

given by (2-4) and ξ_k denotes the degeneracy operator of J_m.

3. Higher homotopy commutativity

We first recall the higher homotopy associativity of H-spaces and H-maps.

Sugawara [1957] gave a criterion for a topological space to have the homotopy type of a loop space. Later Stasheff [1963] expanded his definition, and introduced the concept of A_n-space by using the associahedra $\{K_i\}_{i\geq 2}$. Let X be an H-space whose multiplication is given by $\mu\colon X\times X\to X$ with $\mu(x,*)=\mu(*,x)=x$ for $x\in X$. Then an A_n-form on X is a family of maps $\{M_i\colon K_i\times X^i\to X\}_{2\leq i\leq n}$ with the relations

$$M_2(*,x,y)=\mu(x,y),$$

$$M_i(\partial_k(r,s)(\rho,\sigma),x_1,\dots,x_i)=M_r(\rho,x_1,\dots,x_{k-1},M_s(\sigma,x_k,\dots,x_{k+s-1}),x_{k+s},\dots,x_i)$$

for r, $s\geq 2$ with $r+s=i+1$ and $1\leq k\leq r$, and

$$M_i(\sigma,x_1,\dots,x_{j-1},*,x_{j+1},\dots,x_i)=M_{i-1}(\theta_j(\sigma),x_1,\dots,x_{j-1},x_{j+1},\dots,x_i)$$

for $1\leq j\leq i$.

An A_1-space is just a topological space, and an H-space which admits an A_n-form is called an A_n-space for $n\geq 2$. From the definition, an A_2-space and an A_3-space are an H-space and a homotopy associative H-space, respectively. Moreover, an A_∞-space X has the homotopy type of a loop space admitting the classifying space BX with $\Omega(BX)\simeq X$ (see [Kane 1988, §6-2]).

It is natural to consider the concept of A_n-map between A_n-spaces. Sugawara [1960] first considered such a concept for a map between topological monoids. Stasheff [1970] next studied an A_n-map from an A_n-space to a topological monoid by using the associahedra $\{K_i\}_{i\geq 2}$ used for the definition of an A_n-space.

The full generality was described by Iwase and Mimura [1989] by using the multiplihedra $\{J_i\}_{i\geq 1}$. Let X and Y be A_n-spaces with the A_n-forms $\{M_i\}_{2\leq i\leq n}$ and $\{N_i\}_{2\leq i\leq n}$, respectively. Assume that $\phi\colon X\to Y$ is a map. Then an A_n-form on ϕ is a family of maps $\{F_i\colon J_i\times X^i\to Y\}_{1\leq i\leq n}$ with the relations

$$F_1(*,x)=\phi(x),$$

$$F_i(\delta_k(r,s)(\rho,\sigma),x_1,\dots,x_i)=F_r(\rho,x_1,\dots,x_{k-1},M_s(\sigma,x_k,\dots,x_{k+s-1}),x_{k+s},\dots,x_i)$$

for $r\geq 1$, $s\geq 2$ with $r+s=i+1$ and $1\leq k\leq r$,

$$\begin{aligned}F_i(\delta(q,r_1,\dots,r_q)(\rho,\sigma_1,\dots,\sigma_q),x_1,\dots,x_i)\\ =N_q(\rho,F_{r_1}(\sigma_1,x_1,\dots,x_{r_1}),\dots,F_{r_q}(\sigma_q,x_{r_1+\cdots+r_{q-1}+1},\dots,x_i))\end{aligned}$$

for $q\geq 2$ and $r_1,\dots,r_q\geq 1$ with $r_1+\cdots+r_q=n$, and

$$F_i(\rho,x_1,\dots,x_{j-1},*,x_{j+1},\dots,x_i)=F_{i-1}(\xi_j(\rho),x_1,\dots,x_{j-1},x_{j+1},\dots,x_i)$$

for $1\leq j\leq i$.

A map which admits an A_n-form is called an A_n-map for $n \geq 1$. An A_1-map is just a map, and by [Iwase and Mimura 1989, p. 195, P8], an A_2-map and an A_3-map are an H-map and an H-map preserving the homotopy associativity, respectively. Moreover, an A_∞-map ϕ is homotopic to a loop map which induces a map between the classifying spaces $B\phi\colon BX \to BY$ with $\Omega(B\phi) \simeq \phi$ (see [Kane 1988, §6-4]).

We next recall the higher homotopy commutativity of H-spaces.

Sugawara [1960] gave a criterion for the classifying space of a topological monoid to have the homotopy type of an H-space. His criterion is a higher homotopy commutativity of the multiplication. Later Williams [1969] considered another type of higher homotopy commutativity which is weaker than the one of Sugawara, and defined C_n-spaces.

In [Hemmi and Kawamoto 2004], we generalized the definition of Williams to the case of A_n-spaces, and defined AC_n-spaces by using the permuto-associahedra $\{\Gamma_i\}_{i\geq 1}$. Let X be an A_n-space with the A_n-form $\{M_i\}_{2\leq i\leq n}$. Then an AC_n-form on X is a family of maps $\{Q_i\colon \Gamma_i \times X^i \to X\}_{1\leq i\leq n}$ with the relations

$$Q_1(*, x) = x, \tag{3-1}$$

$$\begin{aligned} Q_i(\epsilon^{(\alpha_1,\dots,\alpha_l)}&(\sigma, \tau_1, \dots, \tau_l), x_1, \dots, x_i) \\ &= M_l(\sigma, Q_{t_1}(\tau_1, x_{\alpha_1(1)}, \dots, x_{\alpha_1(t_1)}), \dots, Q_{t_l}(\tau_l, x_{\alpha_l(1)}, \dots, x_{\alpha_l(t_l)})) \end{aligned} \tag{3-2}$$

for a partition $(\alpha_1, \dots, \alpha_l)$ of $\mathbf{i}$ of type $(t_1, \dots, t_l)$ with $l \geq 2$, and

$$Q_i(\tau, x_1, \dots, x_{j-1}, *, x_{j+1}, \dots, x_i) = Q_{i-1}(\omega_j(\tau), x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_i) \tag{3-3}$$

for $1 \leq j \leq i$.

An A_n-space admitting an AC_n-form is called an AC_n-space for $n \geq 1$. By Example 3.2(1) in [Hemmi and Kawamoto 2004], X is an AC_2-space if and only if X is a homotopy commutative H-space. Moreover, if X is a topological monoid, then by Corollary 3.6 of the same work, X is an AC_n-space if and only if X is a C_n-space of Williams [1969].

Williams [1969, Definition 20], also considered the concept of C_n-map between C_n-spaces. We generalize his definition to the case of maps between AC_n-spaces.

Definition 3.1. Let X and Y be AC_n-spaces with the AC_n-forms $\{Q_i\}_{1\leq i\leq n}$ and $\{R_i\}_{1\leq i\leq n}$, respectively. Assume that $\phi\colon X \to Y$ is an A_n-map with the A_n-form $\{F_i\}_{1\leq i\leq n}$. Then an AC_n-form on ϕ is a family of maps

$$\{D_i\colon I \times \Gamma_i \times X^i \to Y\}_{1\leq i\leq n}$$

with the relations

$$D_1(t, *, x) = \phi(x), \tag{3-4}$$

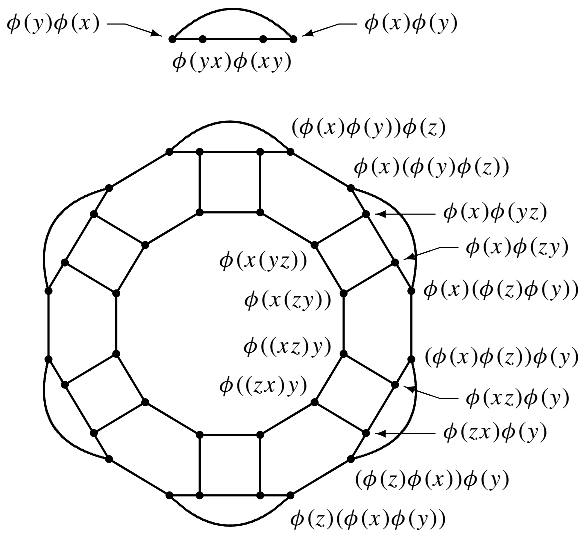

Figure 6. The AC_n-forms on ϕ for $n = 2, 3$.

$$D_i(t, \epsilon^{(\alpha_1,\dots,\alpha_l)}(\sigma, \tau_1, \dots, \tau_l), x_1, \dots, x_i) = N_l(\sigma, D_{t_1}(t, \tau_1, x_{\alpha_1(1)}, \dots, x_{\alpha_1(t_1)}), \dots, D_{t_l}(t, \tau_l, x_{\alpha_l(1)}, \dots, x_{\alpha_l(t_l)})) \tag{3-5}$$

for a partition $(\alpha_1, \dots, \alpha_l)$ of $\mathbf{i}$ of type $(t_1, \dots, t_l)$ with $l \geq 2$,

$$D_i(0, \iota^{(\beta_1,\dots,\beta_m)}(\sigma, \tau_1, \dots, \tau_m), x_1, \dots, x_i) = F_m(\sigma, Q_{u_1}(\tau_1, x_{\beta_1(1)}, \dots, x_{\beta_1(u_1)}), \dots, Q_{u_m}(\tau_m, x_{\beta_m(1)}, \dots, x_{\beta_m(u_m)})) \tag{3-6}$$

for a partition $(\beta_1, \dots, \beta_m)$ of $\mathbf{i}$ of type $(u_1, \dots, u_m)$ with $m \geq 1$,

$$D_i(1, \tau, x_1, \dots, x_i) = R_i(\tau, \phi(x_1), \dots, \phi(x_i)), \tag{3-7}$$

and

$$D_i(t, \tau, x_1, \dots, x_{j-1}, *, x_{j+1}, \dots, x_i) = D_{i-1}(t, \omega_j(\tau), x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_i) \tag{3-8}$$

for $1 \leq j \leq i$.

An A_n-map admitting an AC_n-form is called an AC_n-map for $n \geq 1$. If there is a family of maps $\{D_i\}_{i \geq 1}$ such that $\{D_i\}_{1 \leq i \leq n}$ is an AC_n-form on ϕ for any $n \geq 1$, then ϕ is called an AC_∞-map.

Example 3.2. (1) An AC_2-space is the same as a homotopy commutative H-space by [Hemmi and Kawamoto 2004, Example 3.2(1)]. Then an AC_2-map is a map between AC_2-spaces preserving the homotopy commutativity, and so it is the same as an HC-map of Zabrodsky [1976, p. 62].

(2) Let $\phi\colon X \to Y$ be a homomorphism for topological monoids X, Y. Then ϕ is an AC_n-map if and only if ϕ is a C_n-map of [Williams 1969, Definition 20].

(3) If $\phi\colon X \to Y$ is an H-map, then the loop map $\Omega\phi\colon \Omega X \to \Omega Y$ is an AC_∞-map.

4. Proof of Theorem A

Let $\mathcal{S}_*$ denote the category of pointed and connected topological spaces having the homotopy type of CW-complexes. Assume that $f\colon A \to B$ is a pointed map for $A, B \in \mathcal{S}_*$. According to Dror Farjoun [1996, p. 2, A.1], $Z \in \mathcal{S}_*$ is called f-local if the induced map

$$f^{\#}\colon \mathrm{Map}_*(B, Z) \longrightarrow \mathrm{Map}_*(A, Z)$$

is a homotopy equivalence. In the case that $B = *$ and $f\colon A \to *$ is the constant map, Z is called A-local, that is, the pointed mapping space $\mathrm{Map}_*(A, Z)$ is contractible.

Bousfield [1994, §2] and Dror Farjoun [1996, §1] constructed the A-localization $L_A(X)$ with the universal map $\phi_X\colon X \to L_A(X)$ for $X \in \mathcal{S}_*$ (see also [Chachólski 1996, §14]). By [Farjoun 1996, p. 4, A.4], $L_A(X)$ is A-local, and by [Bousfield 1994, Theorem 2.10(ii)], ϕ_X induces a homotopy equivalence

$$(\phi_X)^{\#}\colon \mathrm{Map}_*(L_A(X), Z) \longrightarrow \mathrm{Map}_*(X, Z) \tag{4-1}$$

for any A-local space Z (see also [Chachólski 1996, Theorem 14.1]).

To prove Theorem A, we first show:

Proposition 4.1. *Let $\phi\colon X \to Y$ be an A_n-map for A_n-spaces X, Y. If X is an AC_n-space and Y is ϕ-local, then Y is an AC_n-space so that ϕ is an AC_n-map.*

Lemma 4.2. *Let $\phi\colon X \to Y$ be a map. If Y is ϕ-local, then we have the homotopy equivalences*

$$(\phi^n)^{\#}\colon \mathrm{Map}_*(Y^n, Y) \longrightarrow \mathrm{Map}_*(X^n, Y) \tag{4-2}$$

$$(\phi^{(n)})^{\#}\colon \mathrm{Map}_*(Y^{(n)}, Y) \longrightarrow \mathrm{Map}_*(X^{(n)}, Y), \tag{4-3}$$

where $Z^{(n)}$ denotes the n-fold smash product of $Z \in \mathcal{S}_*$ for $n \geq 1$.

Proof. We first show (4-2). From the homotopy commutative diagram of fibrations

$$\begin{tikzcd}
\mathrm{Map}_*(Y^n, Y) \arrow[r, "(\phi^n)^\#"] \arrow[d] & \mathrm{Map}_*(X^n, Y) \arrow[d] \\
\mathrm{Map}(Y^n, Y) \arrow[r, "(\phi^n)^\#"] \arrow[d, "e"'] & \mathrm{Map}(X^n, Y) \arrow[d, "e'"] \\
Y \arrow[r, equal] & Y,
\end{tikzcd}$$

it is sufficient to show that the middle horizontal map is a homotopy equivalence for $n \geq 1$, where e and e' are the evaluation maps at the base points.

We work by induction on n. Since Y is ϕ-local, the result is clear for $n = 1$. Assume that $(\phi^{n-1})^\#\colon \mathrm{Map}(Y^{n-1}, Y) \to \mathrm{Map}(X^{n-1}, Y)$ is a homotopy equivalence. By [Farjoun 1996, p. 5, A.8, e.2], $\mathrm{Map}(Y^{n-1}, Y)$ is ϕ-local. From the homotopy commutative diagram

$$\begin{tikzcd}
\mathrm{Map}(Y^n, Y) \arrow[rr, "(\phi^n)^\#"] \arrow[d, "\simeq"'] & & \mathrm{Map}(X^n, Y) \arrow[d, "\simeq"] \\
\mathrm{Map}(Y, \mathrm{Map}(Y^{n-1}, Y)) \arrow[d, "\simeq"', "\phi^\#"] & & \mathrm{Map}(X, \mathrm{Map}(X^{n-1}, Y)) \arrow[d, equal] \\
\mathrm{Map}(X, \mathrm{Map}(Y^{n-1}, Y)) \arrow[rr, "((\phi^{n-1})^\#)_\#", "\simeq"'] & & \mathrm{Map}(X, \mathrm{Map}(X^{n-1}, Y)),
\end{tikzcd}$$

we have that $(\phi^n)^\#\colon \mathrm{Map}(Y^n, Y) \to \mathrm{Map}(X^n, Y)$ is a homotopy equivalence.

In the case of (4-3), by similar arguments to the case of (4-2) and a homotopy equivalence

$$\mathrm{Map}_*(Z \wedge W, U) \simeq \mathrm{Map}_*(Z, \mathrm{Map}_*(W, U))$$

for $Z, W, U \in \mathcal{S}_*$, we have the required conclusion. This completes the proof. $\square$

Lemma 4.3. *Let $\phi\colon Z \to W$ be a homotopy equivalence for $Z, W \in \mathcal{S}_*$, and let (K, L) be a relative CW-complex.*

(1) *If there are maps $f\colon K \to W$ and $g\colon L \to Z$ with $\phi g = f|_L$, then we have a map $h\colon K \to Z$ with $h|_L = g$ and $\phi h \simeq f$ rel L.*

(2) *If $h, k\colon K \to Z$ are maps with $h|_L = k|_L$ and $\phi h \simeq \phi k$ rel L, then $h \simeq k$ rel L.*

Proof of Proposition 4.1. Let $\{M_i\}_{2 \leq i \leq n}$ and $\{N_i\}_{2 \leq i \leq n}$ be the A_n-form on X and Y, respectively. Since $\phi\colon X \to Y$ is an A_n-map, there is an A_n-form $\{F_i\}_{1 \leq i \leq n}$ on ϕ. Moreover, we denote the AC_n-form on X by $\{Q_i\}_{1 \leq i \leq n}$.

We work by induction on n. By (3-1) and (3-4), the result is clear for $n = 1$. Assume that there are AC_{n-1}-forms $\{R_i\}_{1\le i\le n-1}$ and $\{D_i\}_{1\le i\le n-1}$ on Y and ϕ, respectively.

Put $\mathscr{V}_n(Z) = (I \times \partial\Gamma_n \cup \{0\} \times \Gamma_n) \times Z^n$ and $\mathscr{W}_n(Z) = I \times \Gamma_n \times Z^{[n]}$, where $Z^{[n]}$ denotes the n-fold fat wedge of $Z \in \mathscr{S}_*$ given by

$$Z^{[n]} = \{(z_1, \dots, z_n) \in Z^n \mid z_j = * \text{ for some } 1 \le j \le n\}.$$

Let $E_n\colon \mathscr{V}_n(X) \cup \mathscr{W}_n(X) \to Y$ be the map defined by

$$\begin{aligned} &E_n(t, \epsilon^{(\alpha_1,\dots,\alpha_l)}(\sigma, \tau_1, \dots, \tau_l), x_1, \dots, x_n) \\ &\qquad = N_l(\sigma, D_{t_1}(t, \tau_1, x_{\alpha_1(1)}, \dots, x_{\alpha_1(t_1)}), \dots, D_{t_l}(t, \tau_l, x_{\alpha_l(1)}, \dots, x_{\alpha_l(t_l)})) \end{aligned}$$

for a partition $(\alpha_1, \dots, \alpha_l)$ of $\mathbf{n}$ of type $(t_1, \dots, t_l)$ with $l \ge 2$,

$$\begin{aligned} &E_n(0, \iota^{(\beta_1,\dots,\beta_m)}(\sigma, \tau_1, \dots, \tau_m), x_1, \dots, x_n) \\ &\qquad = F_m(\sigma, Q_{u_1}(\tau_1, x_{\beta_1(1)}, \dots, x_{\beta_1(u_1)}), \dots, Q_{u_m}(\tau_m, x_{\beta_m(1)}, \dots, x_{\beta_m(u_m)})) \end{aligned}$$

for a partition $(\beta_1, \dots, \beta_m)$ of $\mathbf{n}$ of type $(u_1, \dots, u_m)$ with $m \ge 1$ and

$$E_n(t, \tau, x_1, \dots, x_{j-1}, *, x_{j+1}, \dots, x_n) = D_{n-1}(t, \omega_j(\tau), x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$$

for $1 \le j \le n$.

Since there is a map $\widetilde{E}_n\colon I \times \Gamma_n \times X^n \to Y$ with $\widetilde{E}_n|_{\mathscr{V}_n(X)\cup\mathscr{W}_n(X)} = E_n$ by the homotopy extension property, we define a map $S_n\colon \Gamma_n \times X^n \to Y$ by

$$S_n(\tau, x_1, \dots, x_n) = \widetilde{E}_n(1, \tau, x_1, \dots, x_n).$$

Let $\gamma_n\colon \Gamma_n \to \mathrm{Map}_*(X^n, Y)_{(\phi^n)^\#(\mu_n)}$ be the adjoint of S_n, where $\mu_n\colon Y^n \to Y$ is the map given by $\mu_n(y_1, \dots, y_n) = (\cdots(y_1 y_2)\cdots)y_n$. If a map $\kappa_n\colon \partial\Gamma_n \to \mathrm{Map}_*(Y^n, Y)_{\mu_n}$ is defined by

$$\begin{aligned} &\kappa_n(\epsilon^{(\alpha_1,\dots,\alpha_l)}(\sigma, \tau_1, \dots, \tau_l), y_1, \dots, y_n) \\ &\qquad = N_l(\sigma, R_{t_1}(\tau_1, y_{\alpha_1(1)}, \dots, y_{\alpha_1(t_1)}), \dots, R_{t_l}(\tau_l, y_{\alpha_l(1)}, \dots, y_{\alpha_l(t_l)})), \end{aligned}$$

then $(\phi^n)^\#(\kappa_n) = \gamma_n|_{\partial\Gamma_n}$, and so by (4-2) and Lemma 4.3 (1), we have a map $\lambda_n\colon \Gamma_n \to \mathrm{Map}_*(Y^n, Y)_{\mu_n}$ with $\lambda_n|_{\partial\Gamma_n} = \kappa_n$ and $(\phi^n)^\#(\lambda_n) \simeq \gamma_n$ rel $\partial\Gamma_n$.

To construct a map $R_n\colon \Gamma_n \times Y^n \to Y$ with the relations (3-1)–(3-3), we need to show that the induced map

$$(\phi^{[n]})^\#\colon \mathrm{Map}_*(Y^{[n]}, Y)_{\nu_n} \longrightarrow \mathrm{Map}_*(X^{[n]}, Y)_{(\phi^{[n]})^\#(\nu_n)} \tag{4-4}$$

is a homotopy equivalence, where $\nu_n\colon Y^{[n]} \to Y$ denotes the composite of μ_n with the inclusion $\iota_Y\colon Y^{[n]} \to Y^n$. Since Y is an H-space, it is sufficient to show the same homotopy equivalence on the components of the constant maps. Consider the following homotopy commutative diagram of fibrations:

$$\begin{tikzcd}
\mathrm{Map}_*(Y^{(n)}, Y)_{C_1} \arrow[r, "(\phi^{(n)})^\#", "\simeq"'] \arrow[d, "(\pi_Y)^\#"'] & \mathrm{Map}_*(X^{(n)}, Y)_{C_2} \arrow[d, "(\pi_X)^\#"] \\
\mathrm{Map}_*(Y^n, Y)_c \arrow[r, "(\phi^n)^\#", "\simeq"'] \arrow[d, "(\iota_Y)^\#"'] & \mathrm{Map}_*(X^n, Y)_c \arrow[d, "(\iota_X)^\#"] \\
\mathrm{Map}_*(Y^{[n]}, Y)_c \arrow[r, "(\phi^{[n]})^\#"] & \mathrm{Map}_*(X^{[n]}, Y)_c,
\end{tikzcd}$$

where $C_1 = \{h\colon Y^{(n)} \to Y \mid (\pi_Y)^\#(h) \simeq c\}$ and $C_2 = \{k\colon X^{(n)} \to Y \mid (\pi_X)^\#(k) \simeq c\}$. Since the vertical arrows are fibrations, the bottom horizontal arrow is a homotopy equivalence, which implies (4-4).

Define a map $\rho_n\colon \Gamma_n \to \mathrm{Map}_*(Y^{[n]}, Y)_{\nu_n}$ by

$$\rho_n(\tau)(y_1, \dots, y_{j-1}, *, y_{j+1}, \dots, y_n) = R_{n-1}(\omega_j(\tau), y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_n)$$

for $1 \le j \le n$. Then $(\phi^{[n]})^\#(\iota_Y)^\#(\lambda_n) \simeq (\phi^{[n]})^\#(\rho_n)$ rel $\partial\Gamma_n$, and so by (4-4) and Lemma 4.3 (2), we have $(\iota_Y)^\#(\lambda_n) \simeq \rho_n$ rel $\partial\Gamma_n$, which implies that there is a map $\psi_n\colon I \times \Gamma_n \to \mathrm{Map}_*(Y^{[n]}, Y)_{\nu_n}$ with

$$\psi_n(t, \tau) = \begin{cases} (\iota_Y)^\#(\lambda_n)(\tau) & \text{if } (t, \tau) \in \{0\} \times \Gamma_n \cup I \times \partial\Gamma_n, \\ \rho_n(\tau) & \text{if } (t, \tau) \in \{1\} \times \Gamma_n. \end{cases}$$

If a map $G_n\colon \mathscr{V}_n(Y) \cup \mathscr{W}_n(Y) \to Y$ is given by

$$G_n(t, \tau, y_1, \dots, y_n) = \begin{cases} \lambda_n(\tau)(y_1, \dots, y_n) & \text{if } (t, \tau, y_1, \dots, y_n) \in \mathscr{V}_n(Y), \\ \psi_n(t, \tau)(y_1, \dots, y_n) & \text{if } (t, \tau, y_1, \dots, y_n) \in \mathscr{W}_n(Y), \end{cases}$$

there is an extension $\widetilde{G}_n\colon I \times \Gamma_n \times Y^n \to Y$ with $\widetilde{G}_n|_{\mathscr{V}_n(Y)\cup\mathscr{W}_n(Y)} = G_n$. Let $R_n\colon \Gamma_n \times Y^n \to Y$ be the map defined by $R_n(\tau, y_1, \dots, y_n) = \widetilde{G}_n(1, \tau, y_1, \dots, y_n)$. Then R_n satisfies the relations (3-1)–(3-3).

Since $R_n(1_{\Gamma_n} \times \phi^n) \simeq S_n$ rel $\partial\Gamma_n \times X^n$, we have a map $H_n\colon I \times \Gamma_n \times X^n \to Y$ with $H_n|_{\mathscr{V}_n(X)} = E_n|_{\mathscr{V}_n(X)}$ and $H_n(1, \tau, x_1, \dots, x_n) = R_n(\tau, \phi(x_1), \dots, \phi(x_n))$. Moreover, $H_n|_{\partial(I\times\Gamma_n)\times X^{[n]}} = E_n|_{\partial(I\times\Gamma_n)\times X^{[n]}}$, and so by [Williams 1969, Remark 10], we can choose a map $D_n\colon I \times \Gamma_n \times X^n \to Y$ with $D_n|_{\partial(I\times\Gamma_n)\times X^n} = H_n|_{\partial(I\times\Gamma_n)\times X^n}$ and $D_n|_{\mathscr{W}_n(X)} = E_n|_{\mathscr{W}_n(X)}$. Then D_n satisfies the relations (3-4)–(3-8), and we have the required conclusion. This completes the proof. □

Let $\phi\colon X \to Y$ be a homotopy equivalence. Then Y is ϕ-local, and so by Proposition 4.1, we have:

Proposition 4.4. *Let X, Y be A_n-spaces. Assume that $\phi\colon X \to Y$ is an A_n-map which is a homotopy equivalence. If X is an AC_n-space, then Y is an AC_n-space so that ϕ is an AC_n-map.*

Remark 4.5. By Proposition 4.4, the property of being an AC_n-space is an invariant of A_n-homotopy type. This is a generalization of [Williams 1969, Proposition 8, Theorem 9] for C_n-spaces in the category of topological monoids.

Proof of Theorem A. If X is an AC_n-space, then the A-localization $L_A(X)$ is an A_n-space so that the universal map $\phi_X\colon X \to L_A(X)$ is an A_n-map, by [Kawamoto 2002, Theorem 2.1(1)]. Since $L_A(X)$ is ϕ_X-local by (4-1), we have the required conclusion by Proposition 4.1. This completes the proof of Theorem A. □

By [Farjoun 1996, p. 26, E.1], the S^{m+1}-localization $L_{S^{m+1}}(X)$ of X is the same as the m-th stage $P^m(X)$ of the Postnikov system of X for $m \geq 1$, where S^t denotes the t-dimensional sphere for $t \geq 1$. Then by Theorem A, we have:

Corollary 4.6. *Let $n \geq 1$. If X is an AC_n-space, then the m-th stage $P^m(X)$ of the Postnikov system of X is an AC_n-space so that the projection $\rho_m\colon X \to P^m(X)$ is an AC_n-map for $m \geq 1$.*

Let $A \in \mathcal{S}_*$. Dror Farjoun [1996, §2] constructed the A-colocalization $CW_A(X)$ with the universal map $\psi_X\colon CW_A(X) \to X$ for $X \in \mathcal{S}_*$ (see also [Chachólski 1996, §7]).

Theorem 4.7. *Let $A \in \mathcal{S}_*$. If X is an AC_n-space, then the A-colocalization $CW_A(X)$ is an AC_n-space so that the universal map $\psi_X\colon CW_A(X) \to X$ is an AC_n-map.*

Let $f\colon Z \to W$ be a pointed map for $Z, W \in \mathcal{S}_*$. According to [Farjoun 1996, p. 39, A.1], f is called an A-equivalence if the induced map

$$f_\#\colon \mathrm{Map}_*(A, Z) \longrightarrow \mathrm{Map}_*(A, W)$$

is a homotopy equivalence. From the proof of [Farjoun 1996, p. 53, E.1], the universal map $\psi_X\colon CW_A(X) \to X$ is a $CW_A(X)$-equivalence (see also [Chachólski 1996, p. 614]), and so we can prove Theorem 4.7 from the following result:

Proposition 4.8. *Let $\phi\colon X \to Y$ be an A_n-map for A_n-spaces X, Y. If Y is an AC_n-space and ϕ is an X-equivalence, then X is an AC_n-space so that ϕ is an AC_n-map.*

Proposition 4.8 is proved by similar arguments to the proof of Proposition 4.1, and so we omit the proof. In the proof of Proposition 4.8, we need the next lemma instead of Lemma 4.2:

Lemma 4.9. *Let $\phi\colon X \to Y$ be a map. If ϕ is an X-equivalence, then we have the homotopy equivalences*

$$\phi_\#\colon \mathrm{Map}_*(X^n, X) \longrightarrow \mathrm{Map}_*(X^n, Y),$$
$$\phi_\#\colon \mathrm{Map}_*(X^{(n)}, X) \longrightarrow \mathrm{Map}_*(X^{(n)}, Y)$$

for $n \geq 1$.

Proof. By [Farjoun 1996, p. 46, D.2.2],

$$\mathscr{E}(\phi) = \{A \in \mathscr{S}_* \mid \phi \text{ is an } A\text{-equivalence}\}$$

is a closed class.

Since ϕ is an X-equivalence, $X \in \mathscr{E}(\phi)$. If $A, B \in \mathscr{E}(\phi)$, then by [Farjoun 1996, p. 52, D.16], the product $A \times B \in \mathscr{E}(\phi)$. Since the wedge sum $A \vee B$ is represented by a homotopy colimit, we have $A \vee B \in \mathscr{E}(\phi)$ by [Farjoun 1996, p. 45, D.1] (see also [Chachólski 1996, Proposition 4.2]), and so $A \wedge B \in \mathscr{E}(\phi)$ by [Farjoun 1996, p. 45, D.1, 3,4]. From these properties, we have $X^n, X^{(n)} \in \mathscr{E}(\phi)$ for $n \geq 1$, which implies the required conclusion. This completes the proof. □

Dror Farjoun [1996, p. 39, A.3] proved that the S^{m+1}-colocalization $CW_{S^{m+1}}(X)$ of $X \in \mathscr{S}_*$ is the same as the m-connected covering $X\langle m\rangle$ of X for $m \geq 1$, and so by Theorem 4.7, we have:

Corollary 4.10. *Let $n \geq 1$. If X is an AC_n-space, then the m-connected covering $X\langle m\rangle$ of X is an AC_n-space so that the inclusion $\iota_m\colon X\langle m\rangle \to X$ is an AC_n-map for $m \geq 1$.*

Remark 4.11. In [Hemmi and Kawamoto 2004], we have shown that the universal covering inherits the property of being an AC_n-space. Corollary 4.10 is a generalization of Lemma 3.9 of that work to the case of any $m \geq 1$.

5. Proofs of Theorems B and C

Theorem 5.1 [Castellana et al. 2007, Theorem 7.3]. *Let p be a prime. If X is a connected H-space whose* mod p *cohomology $H^*(X; \mathbb{Z}/p)$ is finitely generated as an algebra over $\mathscr{A}_p^*$, then there is an H-fibration*

$$F(\phi_X) \longrightarrow X \xrightarrow{\phi_X} L_{B\mathbb{Z}/p}(X), \tag{5-1}$$

where $L_{B\mathbb{Z}/p}(X)$ is a connected $\mathbb{Z}/p$-finite H-space and $F(\phi_X)$ is mod p *homotopy equivalent to a Postnikov H-space.*

Remark 5.2. In Theorem 5.1, if $H^*(X; \mathbb{Z}/p)$ is finitely generated as an algebra over $\mathbb{Z}/p$, then $F(\phi_X)$ is mod p homotopy equivalent to a finite product of $(\mathbb{C}P^\infty)^\wedge_p$s and $B\mathbb{Z}/p^i$s for $i \geq 1$ by [Broto et al. 2001, Theorem 1.2, Theorem 1.3].

Proof of Theorem B. By Theorem A and Theorem 5.1, $L_{B\mathbb{Z}/p}(X)$ is a connected $\mathbb{Z}/p$-finite AC_p-space so that the universal map $\phi_X : X \to L_{B\mathbb{Z}/p}(X)$ is an AC_p-map. Then $H^*(L_{B\mathbb{Z}/p}(X); \mathbb{Z}/p)$ is an exterior algebra generated by odd dimensional generators by [Kane 1988, §12-3, Corollaries A and B]. Let Z be the universal covering of $L_{B\mathbb{Z}/p}(X)$. Then there is an H-fibration

$$Z \longrightarrow L_{B\mathbb{Z}/p}(X) \longrightarrow K(\pi_1(L_{B\mathbb{Z}/p}(X)), 1),$$

where $K(\pi_1(L_{B\mathbb{Z}/p}(X)), 1)$ has the mod p homotopy type of a torus. Since Z is a simply connected $\mathbb{Z}/p$-finite AC_p-space by [Hemmi and Kawamoto 2004, Lemma 3.9] and [Kane 1988, §3-1, Theorem B], we have $\widetilde{H}^*(Z; \mathbb{Z}/p) = 0$ by [Hemmi and Kawamoto 2004, Theorem A (1)] and [Hemmi 1991, Theorem 1.1]. Then $L_{B\mathbb{Z}/p}(X)$ has the mod p homotopy type of a torus, and so by Theorem 5.1, X is mod p homotopy equivalent to a Postnikov H-space. This completes the proof of Theorem B. □

Remark 5.3. From Theorem B, we have the mod p torus theorem stated in [Hemmi and Kawamoto 2004, Corollary 1.1] since a result of McGibbon and Neisendorfer [1984] on a conjecture of Serre implies that a connected Postnikov H-space which is also $\mathbb{Z}/p$-finite has the mod p homotopy type of a torus.

The proof of Corollary 1.3 is given as follows:

Proof of Corollary 1.3. By Theorem B, X is mod p homotopy equivalent to a Postnikov H-space. Put $Y = X^\wedge_p$. Then, by [Bousfield 2001, Theorem 7.2], $L_{K(n)_*}(Y)$ is homotopy equivalent to the $(n+1)$-st stage $\widetilde{P}^{n+1}(Y)$ of the modified Postnikov system of Y given by

$$\pi_j(\widetilde{P}^{n+1}(Y)) \cong \begin{cases} \pi_j(Y) & \text{for } 1 \le j \le n, \\ \pi_{n+1}(Y)/T_{n+1}(p) & \text{for } j = n+1, \\ 0 & \text{for } j > n+1, \end{cases} \tag{5-2}$$

where $T_{n+1}(p)$ denotes the p-torsion subgroup of $\pi_{n+1}(Y)$. Since $\Omega^n L_{K(n)_*}(Y)$ is $B\mathbb{Z}/p$-local by (5-2), there is a map $f : L_{\Sigma^n B\mathbb{Z}/p}(Y) \to L_{K(n)_*}(Y)$ with $f\phi_Y \simeq \kappa_Y$, where $\kappa_Y : Y \to L_{K(n)_*}(Y)$ denotes the universal map for the $K(n)_*$-localization of Y.

Since $\Omega^n L_{\Sigma^n B\mathbb{Z}/p}(Y)$ is $B\mathbb{Z}/p$-local, there is an H-fibration

$$F(\phi_{L_{\Sigma^n B\mathbb{Z}/p}(Y)}) \longrightarrow L_{\Sigma^n B\mathbb{Z}/p}(Y) \xrightarrow{\phi_{L_{\Sigma^n B\mathbb{Z}/p}(Y)}} L_{B\mathbb{Z}/p}(L_{\Sigma^n B\mathbb{Z}/p}(Y)),$$

by [Castellana et al. 2007, Theorem 3.2], where $F(\phi_{L_{\Sigma^n B\mathbb{Z}/p}(Y)})$ is mod p homotopy equivalent to a Postnikov H-space which satisfies that $\pi_{n+1}(F(\phi_{L_{\Sigma^n B\mathbb{Z}/p}(Y)})^\wedge_p)$ has no p-torsion and $\pi_j(F(\phi_{L_{\Sigma^n B\mathbb{Z}/p}(Y)})^\wedge_p) = 0$ for $j > n+1$.

By [Farjoun 1996, p. 139, B.6], $L_{B\mathbb{Z}/p}(L_{\Sigma^n B\mathbb{Z}/p}(Y)) \simeq L_{B\mathbb{Z}/p}(Y)$, and by the proof of Theorem B, we have that $L_{B\mathbb{Z}/p}(Y)$ has the mod p homotopy type of a torus. Then $L_{\Sigma^n B\mathbb{Z}/p}(Y)^\wedge_p$ is $K(n)_*$-local by (5-2), and so there is a map

$$g\colon L_{K(n)_*}(Y) \to L_{\Sigma^n B\mathbb{Z}/p}(Y)^\wedge_p$$

with $g\kappa_Y \simeq (\phi_Y)^\wedge_p$. From the universality of the localizations we see that $L_{K(n)_*}(Y)$ is mod p homotopy equivalent to $L_{\Sigma^n B\mathbb{Z}/p}(Y)$. This completes the proof. □

To prove Theorem C, we need a lemma:

Lemma 5.4. *Let p be an odd prime. Assume that X is a connected H-space whose* mod p *cohomology $H^*(X;\mathbb{Z}/p)$ is finitely generated as an algebra over $\mathbb{Z}/p$. If $x \in QH^{2p^t}(X;\mathbb{Z}/p)$ is a generator of infinite height with $t \geq 2$, then $\mathcal{P}^1\beta(x) \neq 0$ in $QH^{2p^t+2p-1}(X;\mathbb{Z}/p)$ or there is a generator $y \in QH^{2p^{t-1}+1}(X;\mathbb{Z}/p)$ with $\mathcal{P}^{p^{t-1}}(y) = \beta(x) \neq 0$ in $QH^{2p^t+1}(X;\mathbb{Z}/p)$.*

Proof. Let $\widetilde{X}$ be the universal covering of X. Then there is an H-fibration

$$\widetilde{X} \xrightarrow{\iota} X \xrightarrow{\rho} K(\pi_1(X),1),$$

where $K(\pi_1(X),1)$ has the mod p homotopy type of a finite product of $(S^1)^\wedge_p$s and $B\mathbb{Z}/p^i$s for $i \geq 1$. According to [Browder 1959], the mod p cohomology $H^*(\widetilde{X};\mathbb{Z}/p)$ is finitely generated as an algebra over $\mathbb{Z}/p$ and

$$\iota^*\colon QH^s(X;\mathbb{Z}/p) \to QH^s(\widetilde{X};\mathbb{Z}/p) \tag{5-3}$$

is an isomorphism if $s \neq 2, 2p^j - 1$ for $j \geq 1$.

Recall that the mod p cohomology of $B^2\mathbb{Z}/p$ is given by

$$H^*(B^2\mathbb{Z}/p;\mathbb{Z}/p) \cong \mathbb{Z}/p[u, \beta\mathcal{P}^1\beta(u), \ldots, \beta\mathcal{P}^{\Delta_t}\beta(u), \ldots] \otimes \Lambda(\beta(u), \mathcal{P}^1\beta(u), \ldots, \mathcal{P}^{\Delta_t}\beta(u), \ldots),$$

where $u \in QH^2(B^2\mathbb{Z}/p;\mathbb{Z}/p)$ denotes the generator and $\mathcal{P}^{\Delta_t} = \mathcal{P}^{p^t}\cdots\mathcal{P}^1$ for $t \geq 0$. Let $\tilde{x} = \iota^*(x) \in QH^{2p^t}(\widetilde{X};\mathbb{Z}/p)$. By [Crespo 2001, Theorem 2.10, Proposition 5.7], there is an H-space Y and an H-fibration

$$\widetilde{X} \longrightarrow Y \longrightarrow B^2\mathbb{Z}/p \tag{5-4}$$

such that $\tau(\tilde{x}) = \mathcal{P}^{\Delta_{t-1}}\beta(u) \in H^{2p^t+1}(B^2\mathbb{Z}/p;\mathbb{Z}/p)$ and $\tau(\beta(\tilde{x})) = \beta\mathcal{P}^{\Delta_{t-1}}\beta(u) \in H^{2p^t+2}(B^2\mathbb{Z}/p;\mathbb{Z}/p)$ in the spectral sequence associated to the H-fibration (5-4), where $\tau\colon QH^s(\widetilde{X};\mathbb{Z}/p) \to H^{s+1}(B^2\mathbb{Z}/p;\mathbb{Z}/p)$ denotes the transgression of the spectral sequence for $s \geq 2$. Then by [Crespo 2001, Theorem 1.5], there is a generator $\tilde{y} \in QH^{2p^{t-1}+1}(\widetilde{X};\mathbb{Z}/p)$ with

$$\tau(\tilde{y}) = \beta\mathcal{P}^{\Delta_{t-2}}\beta(u) \in H^{2p^{t-1}+2}(B^2\mathbb{Z}/p;\mathbb{Z}/p) \tag{5-5}$$

or there is a generator $\tilde{z} \in QH^{2p^t+2p-1}(\widetilde{X}; \mathbb{Z}/p)$ with

$$\tau(\tilde{z}) = (\beta \mathcal{P}^{\Delta_{t-2}} \beta(u))^p \in H^{2p^t+2p}(B^2\mathbb{Z}/p; \mathbb{Z}/p). \tag{5-6}$$

If we have (5-5), then it follows that $\mathcal{P}^{p^{t-1}}(\tilde{y}) = \beta(\tilde{x})$ in $QH^{2p^t+1}(\widetilde{X}; \mathbb{Z}/p)$ by the choice of the generators in [Crespo 2001, p. 126] since $\mathcal{P}^{p^{t-1}}(\beta \mathcal{P}^{\Delta_{t-2}} \beta(u)) = \beta \mathcal{P}^{\Delta_{t-1}} \beta(u)$ by Lemma 3.2 of the same reference. Choose $y \in QH^{2p^{t-1}+1}(X; \mathbb{Z}/p)$ with $\iota^*(y) = \tilde{y}$. Then $\mathcal{P}^{p^{t-1}}(y) = \beta(x)$ in $QH^{2p^t+1}(X; \mathbb{Z}/p)$ by (5-3).

In the case of (5-6), since $\mathcal{P}^1(\beta \mathcal{P}^{\Delta_{t-1}} \beta(u)) = (\beta \mathcal{P}^{\Delta_{t-2}} \beta(u))^p$ by [Crespo 2001, Lemma 3.3], we have $\mathcal{P}^1 \beta(\tilde{x}) = \tilde{z}$ in $QH^{2p^t+2p-1}(\widetilde{X}; \mathbb{Z}/p)$. Then by (5-3), we have $\mathcal{P}^1 \beta(x) \neq 0$ in $QH^{2p^t+2p-1}(X; \mathbb{Z}/p)$. This completes the proof. □

Proof of Theorem C. By Theorem A, [Kawamoto 2002, Theorem 2.1(1)] and Theorem 5.1, we have that $L_{B\mathbb{Z}/p}(X)$ is a connected $\mathbb{Z}/p$-finite A_p-space admitting an AC_n-form with $n > (p-1)/2$.

If $H^*(X; \mathbb{Z}/p)$ is finitely generated as an algebra over $\mathcal{A}_p^*$ and the operations $\mathcal{P}^j$ act on $QH^*(X; \mathbb{Z}/p)$ trivially for $j \geq 1$, then we see that $H^*(X; \mathbb{Z}/p)$ is finitely generated as an algebra over $\mathbb{Z}/p$, and so by Remark 5.2, $F(\phi_X)$ is mod p homotopy equivalent to a finite product of $(\mathbb{C}P^\infty)^\wedge_p$s and $B\mathbb{Z}/p^i$s for $i \geq 1$.

Consider the spectral sequence associated to the H-fibration (5-1) whose E_2-term is given by

$$E_2^{*,*} \cong H^*(L_{B\mathbb{Z}/p}(X); \mathbb{Z}/p) \otimes H^*(F(\phi_X); \mathbb{Z}/p).$$

Let us show that the spectral sequence collapses. If $w \in QH^1(F(\phi_X); \mathbb{Z}/p)$ is a generator, then $d_2(w) \in PH^2(L_{B\mathbb{Z}/p}(X); \mathbb{Z}/p)$ by [Kane 1988, §1-6], where PA denotes the primitive module of A. By [Kane 1988, §12-3, Corollary B], $H^*(L_{B\mathbb{Z}/p}(X); \mathbb{Z}/p)$ is an exterior algebra generated by odd dimensional generators. Since $PH^2(L_{B\mathbb{Z}/p}(X); \mathbb{Z}/p) = 0$, we have $d_2(w) = 0$. Assume that there is a generator $u \in QH^2(F(\phi_X); \mathbb{Z}/p)$ with $d_3(u) = v \neq 0$ in $QH^3(L_{B\mathbb{Z}/p}(X); \mathbb{Z}/p)$. Then $d_3(u^p) = \mathcal{P}^1(v) \neq 0$ in $QH^{2p+1}(L_{B\mathbb{Z}/p}(X); \mathbb{Z}/p)$ by [Hemmi and Kawamoto 2007, Theorem A (2)], and so by computing the spectral sequence, we have a generator $x \in QH^{2p^t}(X; \mathbb{Z}/p)$ with $t \geq 2$. Since the operations $\mathcal{P}^j$ act on $QH^*(X; \mathbb{Z}/p)$ trivially for $j \geq 1$, we have a contradiction by Lemma 5.4. Then the spectral sequence collapses, and so we have

$$H^*(X; \mathbb{Z}/p) \cong H^*(L_{B\mathbb{Z}/p}(X); \mathbb{Z}/p) \otimes H^*(F(\phi_X); \mathbb{Z}/p).$$

Since the operations $\mathcal{P}^j$ act on $QH^*(X; \mathbb{Z}/p)$ trivially for $j \geq 1$, they also act on $QH^*(L_{B\mathbb{Z}/p}(X); \mathbb{Z}/p)$ trivially, which implies that $L_{B\mathbb{Z}/p}(X)$ has the mod p homotopy type of a torus by [Hemmi and Kawamoto 2007, Theorem B] and Remark 5.5. Then there is a map $\zeta : L_{B\mathbb{Z}/p}(X) \times F(\phi_X) \to X$ which induces an isomorphism on the mod p cohomology, and so ζ is a mod p homotopy equivalence; compare [Mimura and Toda 1991, p. 157, Corollary 1.6]. This completes the proof of Theorem C. □

Remark 5.5. In [Hemmi and Kawamoto 2007], all spaces are assumed to be localized at p in the sense of [Bousfield and Kan 1972]. However, the proof of Theorem B in our paper with Hemmi is also available for $\mathbb{Z}/p$-finite A_p-spaces, even if they are not localized at p.

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