Pacific Journal of Mathematics

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Volume 231 No. 1 May 2007

PACIFIC JOURNAL OF MATHEMATICS
Vol. 231, No. 1, 2007

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We introduce a notion of "hopfish algebra" structure on an associative algebra, allowing the structure morphisms (coproduct, counit, antipode) to be bimodules rather than algebra homomorphisms. We prove that quasi-Hopf algebras are hopfish algebras. We find that a hopfish structure on the algebra of functions on a finite set G is closely related to a "hypergroupoid" structure on G. The Morita theory of hopfish algebras is also discussed.

1. Introduction

When the multiplication on a (discrete, topological, smooth, algebraic) group G is encoded in an appropriate algebra $A = A(G)$ of functions on G with values in a commutative ring k, it becomes a coproduct, that is, an algebra homomorphism $\Delta : A \to A \otimes_k A$. The inclusion of the unit and the inversion map are also encoded as homomorphisms: the counit $\epsilon : A \to k$ and the antipode $S : A \to A$. The group properties (associativity, unit, inverse) become statements about these homomorphisms which constitute the axioms for a (commutative) *Hopf algebra*; any noncommutativity of the underlying group appears as non<u>co</u>commutativity of the coproduct.

In noncommutative geometry, a noncommutative algebra A is thought of as the functions on a "noncommutative space" or "quantum space" X. If X is to be a "quantum group", the algebra A should have the additional structure of a Hopf algebra. We note that, for noncommutative Hopf algebras, the antipode has to be an antihomomorphism rather than a homomorphism of algebras. For this reason, a Hopf algebra is not quite a group in the category of algebras; this anomaly will come back to haunt us later.

One type of quantum space is a quantum torus, whose function algebra is the crossed product algebra A_α associated to an action of $\mathbb{Z}$ on the circle $S^1 = \mathbb{R}/\mathbb{Z}$ generated by an irrational rotation r_α. This irrational rotation algebra is generally taken as a surrogate for the algebra of continuous functions on the "bad quotient

MSC2000: primary 16W30; secondary 81R50.
Keywords: Hopf algebra, hopfish algebra, groupoid, bimodule, Morita equivalence, hypergroupoid.
Weinstein's research is partially supported by NSF Grant DMS-0204100. Tang and Zhu worked respectively at UC Davis and ETH Zürich during the period of this research.

space" $S^1/\alpha\mathbb{Z}$ because, for nice quotients, the crossed product algebra is Morita equivalent to the algebra of functions on the quotient. Since $S^1/\alpha\mathbb{Z}$ is a group, one might expect A_α to have a Hopf algebra structure, but this is not so. In particular, there can be no counit, since there are no algebra homomorphisms $A_\alpha \to \mathbb{C}$. In geometric language, "the quantum torus has no points".

Additionally, in noncommutative geometry, Morita equivalent algebras are often thought of as representing the "same space", but the notion of Hopf algebra, and even that of biunital bialgebra, is far from Morita invariant.

In this paper, we propose a new algebraic approach to "group structure" based on the idea that the appropriate morphisms between algebras are bimodules (perhaps with extra structure, or satisfying extra conditions) rather than algebra homomorphisms. Our immediate inspiration to use bimodules was [Tseng and Zhu 2006], in which leaf spaces of foliations are treated as differential stacks for the purpose of putting group(oid)-like structures on them. This means that the structure morphisms of the groupoids are themselves bibundles [Mrčun 1996] (with respect to foliation groupoids, which play in this geometric story the role of the crossed product algebras above) rather than ordinary mappings of leaf spaces. We were also motivated by previous uses of bimodules as generalized morphisms of algebras, C^*-algebras, groupoids, and Poisson manifolds, a point of view which has been extensively developed by Landsman and others (see, for instance, [Bursztyn and Weinstein 2005; Landsman 2001a; 2001b]).

We call our new objects *hopfish algebras*, the suffix "oid" and prefixes like "quasi" and "pseudo" having already been appropriated for other uses. Also, our term retains a hint of the Poisson geometry which inspired some of our work.

Outline of the paper. We begin with a discussion of the category in which objects are algebras and morphisms are bimodules, emphasizing the functor, which we call modulation, from the usual category to this one. We then look at the analogues of semigroups and groups in this category, which we call sesquialgebras and hopfish algebras. What turns out to be especially delicate is the definition of the antipode. We next show that Hopf algebras, and the more general quasi-Hopf algebras, become hopfish algebras upon modulation. In the following section, we study the Morita invariance of the hopfish property, showing that a sufficient condition for this to hold is that a Morita equivalence bimodule be compatible with the antipode of a hopfish algebra. Finally, we study hopfish structures on finite dimensional commutative algebras. We show that these correspond to "multiple-valued groupoid structures" and give examples of hopfish algebras which do not correspond under Morita equivalence to Hopf algebras.

Outlook. In this present paper, we restrict ourselves to the purely algebraic situation; in particular, our tensor products do not involve any completion. We do not require finite dimensionality of our algebras, although some of our examples do have this property. We hope to develop a theory of hopfish C^*-algebras in the future, with a treatment of irrational rotation algebras as a first goal. Even without this theory, it has been possible in [Blohmann et al. 2006] to construct a sesquiunital sesquialgebra structure on the "polynomial part" of the irrational rotation algebras. These algebras are not quite hopfish, since the candidate antiautomorphism satisfies only a weakened version of our antipode axiom. (We hope that this will be remedied when we go on to the C^*-algebras.) Nevertheless, our structure is sufficient to induce an interesting monoid structure on isomorphism classes of modules.

Finally, we remark that all of our examples of hopfish examples are either weak Hopf algebras or Morita equivalent to quasi-Hopf algebras. It would be interesting to find completely new examples. The irrational rotation algebras are probably not of either of these special types, but, as we have already noted, they are not quite hopfish.

2. The modulation functor

Fixing a commutative ring k as our ring of scalars, we will work mostly in a category Alg whose objects are unital k-algebras. The morphism space $\mathrm{Hom}(A, B)$ is taken to be the set of isomorphism classes of biunital (A, B)-bimodules. We will almost always consider these morphisms as going from right to left, i.e.- from B to A (or, better, "to A from B"). The composition $XY \in \mathrm{Hom}(A, C)$ of $X \in \mathrm{Hom}(A, B)$ and $Y \in \mathrm{Hom}(B, C)$ is defined (on representative bimodules) as $X \otimes_B Y$, with the residual actions of A and C providing the bimodule structure.

We will frequently fail to distinguish between morphisms in Alg and their representative bimodules, as long as we can do so without causing confusion. It is also possible to work in the more refined 2-category whose morphisms are bimodules and whose 2-morphisms are bimodule isomorphisms, but we leave this for the future.

We will denote by Alg_0 the "usual" category whose objects are again unital k algebras but whose morphisms are unital homomorphisms. Thus, $\mathrm{Hom}_0(A, B)$ will denote the homomorphisms to A from B. There is an important functor from Alg_0 to Alg which we will call *modulation*.[1] The modulation of $f \in \mathrm{Hom}_0(A, B)$ is the isomorphism class of A_f, which is the k-module A with the (A, B)-bimodule structure

$$a \cdot x \cdot b = axf(b). \tag{1}$$

[1] We are indebted to Yvette Kosmann-Schwarzbach for suggesting this apt name for a functor which is ubiquitous in the literature on Morita equivalence, but which does not seem to have acquired a standard designation.

We will often denote the modulation of a morphism by the same symbol, but in bold face, e.g. $\boldsymbol{f} \in \mathrm{Hom}(A, B)$. The modulation functor is not necessarily faithful, as the next lemma shows.

Lemma 2.1. *For $f, g \in \mathrm{Hom}_0(A, B)$, their modulations $\boldsymbol{f}$ and $\boldsymbol{g}$ are equal (i.e. the bimodules A_f and A_g are isomorphic) if and only if $f = \phi g \phi^{-1}$ for some invertible $\phi \in A$.*

Proof. If $f = \phi g \phi^{-1}$, $\boldsymbol{f}$ and $\boldsymbol{g}$ are both represented by A, with the same left A-module structures. To correct for the difference between the right actions of B, we introduce the bijective map $\boldsymbol{\Phi} : A_f \to A_g$ defined by $x \mapsto x\phi$, which is a bimodule isomorphism because

$$\boldsymbol{\Phi}(axf(b)) = axf(b)\phi = ax\phi\phi^{-1}f(b)\phi = ax\phi g(b) = a\boldsymbol{\Phi}(x)g(b).$$

For the converse, given a bimodule isomorphism $\boldsymbol{\Phi} : A_f \to A_g$, we define ϕ to be $\boldsymbol{\Phi}(1_A)$. By setting $x = 1_A$ in the bimodule morphism identities $\boldsymbol{\Phi}(ax) = a\boldsymbol{\Phi}(x)$ and $\boldsymbol{\Phi}(xf(b)) = \boldsymbol{\Phi}(x)g(b)$, we find first that $\boldsymbol{\Phi}(a) = a\phi$, so that ϕ is invertible because $\boldsymbol{\Phi}$ is, and then that $f(b)\phi = \phi g(b)$, or $f = \phi g \phi^{-1}$. □

Lemma 2.2. *A morphism $X \in \mathrm{Hom}(A, B)$ is the modulation of $f \in \mathrm{Hom}_0(A, B)$ if and only if it is isomorphic to A as a left A module.*

Proof. If X represents f, then clearly X is isomorphic to A as a left A module. For the converse, if $X = A$ as a left A module then X is isomorphic to A_f where $f(b) = 1_A \cdot b$. □

An invertible morphism in $\mathrm{Hom}(A, B)$ is called a *Morita equivalence* between A and B, and the group of Morita self-equivalences of A is called its *Picard group*. The modulation functor clearly takes algebra isomorphisms to Morita equivalences. In fact, we have:

Lemma 2.3. *The modulation of $f \in \mathrm{Hom}_0(A, B)$ is invertible if and only if f is invertible.*

Proof. A standard fact about Morita equivalence is that, if $X \in \mathrm{Hom}(A, B)$ is invertible, the natural homomorphisms from A and B to the B- and A-endomorphisms of X are isomorphisms. When $X = A_f$, the map which takes $b \in B$ to the operator of right multiplication by $f(b)$ is injective if and only if f is injective. On the other hand, all of the left A-module endomorphisms of A are the right multiplications, so they are all realized by the action of B if and only if f is surjective. □

Remark 2.4. It is also possible to modulate a nonunital f. In this case, the underlying k-module should be taken to be the left ideal I in A generated by $f(1_B)$, so that the bimodule structure (1) is still biunital. The three lemmas above change to the following statements, whose proofs are similar, so we only sketch them.

Lemma 2.1′. *If f and g are algebra homomorphisms $A \leftarrow B$ not necessarily unital, then their modulations $\boldsymbol{f}$ and $\boldsymbol{g}$ are equal if and only if there are elements $\phi \in A \cdot f(1_B)$ and $\psi \in A \cdot g(1_B)$ such that $\phi\psi = g(1_B)$, $\psi\phi = f(1_B)$, $g = \phi f \psi$, satisfying the two additional conditions that $x\phi\psi = 0$ implies $x\phi = 0$ and $x\psi\phi = 0$ implies $x\psi = 0$.*

Sketch of proof. Given an isomorphism Φ to $\boldsymbol{f}$ from $\boldsymbol{g}$, let $\phi = \Phi(g(1_B))$ and $\psi = \Phi^{-1}(f(1_B))$. Then $\Phi(xg(1_B)) = x\phi$ and $\Phi^{-1}(xf(1_B)) = x\psi$. All this gives us the desired equations and properties. For the converse, the morphism $\Phi(xg(1_B)) := x\phi$ is an isomorphism from $A \cdot g(1_B)$ to $A \cdot f(1_B)$ with inverse $\Phi^{-1}(xf(1_B)) := x\psi$. The two additional conditions make Φ and Φ^{-1} well defined. □

Lemma 2.2′. *A morphism $X \in \mathrm{Hom}(A, B)$ is the modulation of a (not necessarily unital) map $f : A \leftarrow B$ if and only if it is represented by a principal left ideal in A.*

Sketch of proof. If X is the modulation of f, then $X = A \cdot f(1_B)$. For the converse, if X is isomorphic to a left A ideal $A \cdot c$, then X is the modulation of $f : b \mapsto c \cdot b$, where $b \in B$ and $\cdot$ is the right action of B on $X = A \cdot c$. □

Lemma 2.3′. *When $f(1_B)$ is in the center of A, the modulation of a morphism $f : A \leftarrow B$ (not necessarily unital) is invertible if and only if f is an isomorphism from B to $A \cdot f(1_B)$ and $f(1_B)$ is not a zero divisor.*

Sketch of proof. One applies the same argument. If $\boldsymbol{f}$ is invertible, notice that $A \to \mathrm{End}_B(X)$ by $a \mapsto a\cdot$ is an isomorphism, therefore $af(1_B) \neq a' f(1_B)$ if $a \neq a'$. This implies that $f(1_B)$ is not a zero divisor. As before f has to be injective. For any $a \in A$, right multiplication by a is in $\mathrm{End}_A(X)$, therefore there is $b \in B$ such that $f(1_B)a = f(b)$. It is not hard to prove the converse. □

Finally, we recall that every (A, B) bimodule gives rise (via tensor product over B) to a k-linear functor from the category of left B-modules to that of left A-modules, that isomorphisms between bimodules produce naturally equivalent functors, and that invertible elements of $\mathrm{Hom}(A, B)$ correspond to homotopy classes of equivalences of categories. (The Eilenberg–Watts theorem characterizes the functors arising from bimodules as those which commute with finite limits and colimits.)

Sesquialgebras. To make the notion of biunital bialgebra Morita invariant, we introduce the following definition. For simplicity of notation, we omit the subscript $_k$ on tensor products over k, and the unadorned asterisk * will denote the k-dual.

Definition 2.4. A *sesquiunital sesquialgebra* over a commutative ring k is a unital k-algebra A equipped with an $(A \otimes A, A)$-bimodule $\boldsymbol{\Delta}$ (the *coproduct*) and a (k, A)-module (that is, a right A module) $\boldsymbol{\epsilon}$ (the *counit*), satisfying the following properties.

(1) (coassociativity) The $(A \otimes A \otimes A,\, A)$-bimodules

$$(A \otimes \mathbf{\Delta}) \otimes_{A\otimes A} \mathbf{\Delta} \quad \text{and} \quad (\mathbf{\Delta} \otimes A) \otimes_{A\otimes A} \mathbf{\Delta}$$

are isomorphic.

(2) (counit) The $(k \otimes A,\, A) = (A \otimes k,\, A) = (A,\, A)$-bimodules

$$(\boldsymbol{\epsilon} \otimes A) \otimes_{A\otimes A} \mathbf{\Delta} \quad \text{and} \quad (A \otimes \boldsymbol{\epsilon}) \otimes_{A\otimes A} \mathbf{\Delta}$$

are isomorphic to A.

For example, if (A, Δ, ϵ) is a biunital bialgebra, then its modulation $(A, \mathbf{\Delta}, \boldsymbol{\epsilon})$ is a sesquiunital sesquialgebra. If we have a Morita equivalence X between A and another algebra B, we can use composition with X and $X \otimes X$ to put a biunital sesquialgebra structure on B. See Section 5 below for more details.

3. The antipode and hopfish algebras

Our definition of sesquiunital sesquialgebra expresses (with arrows reversed) the usual axioms of a monoid (semigroup with identity) in the category Alg. A monoid is a group when all its elements have inverses, so it is natural to look for a sesquialgebraic analogue of the inverse. In a Hopf algebra, the antipode, which encodes inversion, is an algebra *anti*homomorphism $S : A \to A$. The properties of inversion ($gg^{-1} = e = g^{-1}g$ for every group element) are then expressed as commutativity of two diagrams, or equality of compositions

$$1 \circ \epsilon = \mu \circ \beta \circ \Delta, \tag{2}$$

where $1 : k \to A$ is inclusion of the scalars, $\mu : A \otimes A \to A$ is algebra multiplication, and $\beta : A \otimes A \to A \otimes A$ is either $I \otimes S$ or $S \otimes I$ (I being the identity morphism on A).

When A is noncommutative, the maps μ and β are k-linear but not algebra homomorphisms. One can consider S as a homomorphism from A to the opposite algebra A^{op}, or vice versa, but there is no way to correct μ in such a manner. As a result, we see no way to rewrite (2) in the category Alg. Instead, we take an alternate approach, which may also be useful elsewhere in the theory of Hopf algebras.

We keep in mind the example where A is the algebra of k-valued functions on a group G.

One way to characterize groups among monoids without explicitly postulating the existence of inverses is to consider the subset

$$J = \{(g, h) \mid gh = e\} \subset G \times G$$

and require that it project bijectively to one factor in the product. To represent J algebraically, even when A is noncommutative, we borrow an idea from Poisson geometry [Lu 1993], where coisotropic submanifolds become one-sided ideals when a Poisson manifold is quantized to become a noncommutative algebra.

We begin, then, with the space $Z' = \mathrm{Hom}_A(\boldsymbol{\epsilon}, \boldsymbol{\Delta})$ of right module homomorphisms. (In the group case, Z' plays the role of measures on $G \times G$ which are supported on J.) Using the left $A \otimes A$ module structure on $\boldsymbol{\Delta}$, we define a right $A \otimes A$ module structure on Z' by $(gb)(u) = g(bu)$ for g in Z', b in $A \otimes A$ and u in $\boldsymbol{\Delta}$. Note that Z' is completely determined by $\boldsymbol{\epsilon}$ and $\boldsymbol{\Delta}$ and is not an extra piece of data.

For the algebraic model of functions on J, we must take a *predual* of Z', that is, a left $A \otimes A$-module Z whose k-dual Z^* is equipped with a right $A \otimes A$-module isomorphism with Z'.

Definition 3.1. A *preantipode* for a sesquiunital sesquialgebra A over k is a left $A \otimes A$ module $\boldsymbol{S}$ together with an isomorphism of its k-dual with the right $A \otimes A$ module $\mathrm{Hom}_A(\boldsymbol{\epsilon}, \boldsymbol{\Delta})$.

Since a left A module is also a right A^{op} module, we may consider $\boldsymbol{S}$ as an (A, A^{op}) bimodule, where $(A, \cdot)$ is from the left A in $A \otimes A$ and $(\cdot, A^{\mathrm{op}})$ is from the right one, i.e. as an Alg morphism in $\mathrm{Hom}(A, A^{\mathrm{op}})$.

The following is our way of expressing algebraically that the first projection from J to G is bijective.

Definition 3.2. Let A be a sesquiunital sesquialgebra. If a preantipode $\boldsymbol{S}$, considered as an (A, A^{op}) bimodule, is a free left A module of rank 1, we call $\boldsymbol{S}$ an *antipode* and say that A along with $\boldsymbol{S}$ is a *hopfish algebra*.

By Lemma 2.2, $\boldsymbol{S}$ is the modulation of an algebra homomorphism $A \leftarrow A^{\mathrm{op}}$. Thus, the definition is effectively that there is a homomorphism S to A from A^{op} such that the full k-dual of the modulation of S is isomorphic to $\mathrm{Hom}_A(\boldsymbol{\epsilon}, \boldsymbol{\Delta})$.

4. Hopf and quasi-Hopf algebras as hopfish algebras

As we observed earlier, the modulation of a biunital bialgebra is a sesquiunital sesquialgebra. In this section, we will give an explicit description of a preantipode in this case, and we will show that the modulation of a Hopf algebra is hopfish. Although this is a special case of the quasi-Hopf algebras treated later in this section, we deal separately with the Hopf case because the proof is much simpler.

Let (A, Δ, ϵ) be a biunital bialgebra. Considering the modulations $\boldsymbol{\epsilon} = k$ and $\boldsymbol{\Delta} = A \otimes A$ as right A modules respectively, one may identify Z' with the subspace of $(A \otimes A)^* = \mathrm{Hom}_k(k, A \otimes A)$ consisting of those linear functionals which annihilate the left ideal W generated by

$$\{\epsilon(a)(1 \otimes 1) - \Delta(a) | a \in A\},$$

i.e. with the k-module dual to $(A \otimes A)/W$. We may therefore take the (cyclic) left $A \otimes A$ module $\boldsymbol{S}_1 = (A \otimes A)/W$ as a preantipode.

We will use the following lemma later. Its straightforward proof is left to the reader.

Lemma 4.1. *W is equal to the left ideal generated by* $\Delta(\ker \epsilon)$.

Now suppose that A is equipped with an antipode S making it into a Hopf algebra. We will consider S as a homomorphism $A \leftarrow A^{\mathrm{op}}$, with modulation $\boldsymbol{S}$. As a k-module, $\boldsymbol{S}$ is A; its (A, A^{op}) bimodule structure is $a \cdot x \cdot b = axS(b)$.

If we can show that the preantipode $\boldsymbol{S}_1$ is isomorphic to $\boldsymbol{S}$ as a bimodule, then since $\boldsymbol{S}$ is isomorphic to A as a left A-module, $\boldsymbol{S} = \boldsymbol{S}_1$ is an antipode, making the modulation of A into a hopfish algebra.

We define a map $\phi : A \otimes A \to A$ by

$$a \otimes b \mapsto aS(b),$$

This map is obviously a morphism of (A, A^{op}) bimodules because

$$\phi(c \cdot (a \otimes b)) = \phi(ca \otimes b) = caS(b) = c \cdot (aS(b)),$$
$$\phi((a \otimes b) \cdot c) = \phi(a \otimes cb)) = aS(b)S(c) = (aS(b)) \cdot c.$$

Hence this map descends to $\boldsymbol{S}_1 = (A \otimes A)/W$ because

$$\phi(\epsilon(a)(1 \otimes 1) - \Delta(a)) = 1 \circ \epsilon(a) - (\mathrm{id} \otimes S) \circ \Delta(a) = 0.$$

The induced map from $\boldsymbol{S}_1$ to A, which we also denote by ϕ, is also a morphism of (A, A^{op}) bimodules.

Moreover ϕ is surjective, since it has a left inverse $a \mapsto [a \otimes 1]$, where [ ] denotes the equivalence class modulo W. This map is also a right inverse, and ϕ is injective, if and only if the equation

$$1 \otimes a - S(a) \otimes 1 \in W \tag{3}$$

is satisfied for all $a \in A$. Notice that $aS(b) \otimes 1 - a \otimes b = (a \otimes 1)(S(b) \otimes 1 - 1 \otimes b)$ and W is a left ideal. Since $\mathrm{id} = m \circ (\mathrm{id} \otimes \epsilon) \circ \Delta$, composing with S we have $\sum S(a_1)\epsilon(a_2) = \sum S(a_1\epsilon(a_2)) = S(a)$. (Here we use Sweedler's notation $\Delta(a) = \sum a_1 \otimes a_2$ and $\sum \Delta(a_1) \otimes a_2 = \sum a_{1,1} \otimes a_{1,2} \otimes a_2$, etc.) On the other hand, we have

$$\begin{aligned}\sum (S(a_1) \otimes 1) \cdot \Delta(a_2) &= \sum (S(a_1)a_{2,1}) \otimes a_{2,2} \\ &= \sum (S(a_{1,1})a_{1,2}) \otimes a_2 = \sum 1 \otimes \epsilon(a_1)a_2 = 1 \otimes a.\end{aligned}$$

We explain the equalities above as follows. The first equality just comes from the notation and the multiplication in the tensor product algebra. For the second, we consider the map $s : A \otimes A \otimes A \to A \otimes A$ defined by $s(a \otimes b \otimes c) = S(a)b \otimes c$. Coassociativity and evaluation of s give

$$\begin{aligned}\sum s(a_1 \otimes a_{2,1} \otimes a_{2,2}) &= \sum s(a_{1,1} \otimes a_{1,2} \otimes a_2) \\ &= \sum (S(a_1)a_{2,1}) \otimes a_{2,2} = \sum (S(a_{1,1})a_{1,2}) \otimes a_2.\end{aligned}$$

For the third equality, we have used the property of S that $\mu \circ (S \otimes \mathrm{id}) \circ \Delta = 1 \circ \epsilon$. Therefore,

$$1 \otimes a - S(a) \otimes 1 = \sum (S(a_1) \otimes 1)(-\epsilon(a_2) + \Delta(a_2)) \in W.$$

So (3) is proved, hence $\boldsymbol{S}$ and $\boldsymbol{S}_1$ are isomorphic as (A, A^{op}) bimodules.

We have thus proved the following theorem.

Theorem 4.2. *Let (A, Δ, ϵ) be a biunital bialgebra. Then $(A \otimes A)/W$, where W is the left ideal generated by*

$$\{\epsilon(a)(1 \otimes 1) - \Delta(a) \mid a \in A\},$$

is a preantipode for the modulation of A. If A is a Hopf algebra, with antipode S, then $(A \otimes A)/W$ is isomorphic to the modulation $\boldsymbol{S}$, and $(A, \boldsymbol{\Delta}, \boldsymbol{\epsilon}, \boldsymbol{S})$ is a hopfish algebra.

Remark 4.3. The hopfish antipode $\boldsymbol{S}$ is also isomorphic to A^{op} as a right A^{op}-module if and only if the Hopf antipode S is invertible. This is why we use a "one sided" criterion for a preantipode to be an antipode.

We turn now to quasi-Hopf algebras. Recall that a quasibialgebra (A, ϵ, Δ, S) is nearly a bialgebra, except that the coproduct does not satisfy associativity exactly; instead, there is an invertible element $\Phi \in A \otimes A \otimes A$ (the coassociator), satisfying

$$(\mathrm{id} \otimes \Delta)(\Delta(a)) = \Phi^{-1}(\Delta \otimes \mathrm{id})(\Delta(a))\Phi \quad \text{for all } a \in A, \tag{4}$$

and further coherence conditions,

$$\begin{aligned}(\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\Phi) \cdot (\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\Phi) &= (\Phi \otimes 1) \cdot (\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\Phi) \cdot (1 \otimes \Phi), \\ (\epsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} &= (\mathrm{id} \otimes \epsilon) \circ \Delta, \\ (\mathrm{id} \otimes \epsilon \otimes \mathrm{id})(\Phi) &= 1.\end{aligned} \tag{5}$$

Since the modulation functor "kills" inner automorphisms (Lemma 2.1), the modulation of a quasibialgebra is a sesquialgebra.

Now A is a quasi-Hopf algebra if there is an anti-homomorphism $S : A \to A$ and elements α, β in A, such that

$$\sum S(a_1)\alpha a_2 = \epsilon(a)\alpha, \quad \sum a_1\beta S(a_2) = \epsilon(a)\beta \quad \text{for all } a \in A, \tag{6}$$

where we use Sweedler's notation: $\Delta(a) = \sum a_1 \otimes a_2$. There are also higher coherence conditions for α and β; see [Drinfel'd 1989] for details.

The following proposition is a slight modification of [Drinfel'd 1989, Proposition 1.5]. Unlike Drinfel'd, we do not assume that S is invertible, so we can not obtain the "right" part of his proposition, but his "left" part can be proved under weaker hypotheses.

Proposition 4.4. *Let $(A, \Delta, \epsilon, \Phi, S, \alpha, \beta)$ be a quasi-Hopf algebra, with $\Phi = \sum_i X_i \otimes Y_i \otimes Z_i$ and $\Phi^{-1} = \sum_j P_j \otimes Q_j \otimes R_j$. Define*

$$\omega = \sum_j S(P_j)\alpha Q_j \otimes R_j \in A \otimes A.$$

Denote by W the left ideal of $A \otimes A$ generated by $\Delta(\ker \epsilon)$. Then

(1) *the k-linear mappings $\phi, \psi : A \otimes A \to A \otimes A$, given by*

$$\phi(a \otimes b) = (a \otimes 1)\omega\Delta(b), \quad \psi(a \otimes b) = \sum_i aX_i\beta S(Y_i)S(b_1) \otimes b_2 Z_i,$$

are bijective, where we have used Sweedler's notation $\Delta b = b_1 \otimes b_2$;

(2) *the mapping $a \otimes b \mapsto (\mathrm{id} \otimes \epsilon)(\phi^{-1}(a \otimes b))$ induces a bijection $(A \otimes A)/W \to A$, and $(\mathrm{id} \otimes \epsilon)(\phi^{-1}(a \otimes b)) = a\beta S(b)$;*

Proof. First, $\phi\psi = \mathrm{id} = \psi\phi$. We will prove only that $\phi\psi = \mathrm{id}$; the other equation can be derived by the same method, as in [Drinfel'd 1989]. We have

$$\begin{aligned}
\phi\psi(a \otimes b) &= \sum_i \phi(aX_i\beta S(Y_i)S(b_1) \otimes b_2 Z_i) \\
&= \sum_i (aX_i\beta S(Y_i)S(b_1) \otimes 1)\omega\Delta(b_2)\Delta(Z_i) \\
&= \sum_i (a \otimes 1)(X_i\beta S(Y_i) \otimes 1)\big((S(b_1) \otimes 1)\omega\Delta b_2\big)\Delta Z_i \\
&= \sum_i (a \otimes 1)(X_i\beta S(Y_i) \otimes 1)(B)\Delta(Z_i),
\end{aligned}$$

where $B = (S(b_1) \otimes 1)\omega\Delta b_2$.

We insert the definition of ω in B, and have

$$
\begin{aligned}
&(S(b_1)\otimes 1)\,\omega\Delta b_2\\
&\qquad = \sum_j \big(S(b_1)S(P_j)\alpha Q_j \otimes R_j\big)\Delta b_2\\
&\qquad = \sum_j (m\otimes \mathrm{id})\big((S\otimes\alpha\cdot\otimes \mathrm{id})\big((P_j b_1\otimes Q_j\otimes R_j)(1\otimes\Delta b_2)\big)\big)\\
&\qquad = \sum_j (m\otimes \mathrm{id})\big((S\otimes\alpha\cdot\otimes \mathrm{id})\big((P_j\otimes Q_j\otimes R_j)(b_1\otimes 1\otimes 1)(1\otimes\Delta b_2)\big)\big)\\
&\qquad = (m\otimes \mathrm{id})\big((S\otimes\alpha\cdot\otimes \mathrm{id})\big(\Phi^{-1}(1\otimes\Delta)\Delta(b)\big)\big),
\end{aligned}
$$

where $m : A\otimes A \to A$ is the multiplication on A and $\alpha\cdot\ : A\to A$ is the left multiplication by α.

Using the twisted coassociativity $(\mathrm{id}\otimes\Delta)\Delta = \Phi(\Delta\otimes \mathrm{id})(\Delta)\Phi^{-1}$ we continue the calculation above to find that B is equal to

$$
\begin{aligned}
&(m\otimes \mathrm{id})\big((S\otimes\alpha\cdot\otimes \mathrm{id})\big((\Delta\otimes \mathrm{id})\Delta(b)\Phi^{-1}\big)\big)\\
&\qquad = \sum_j (m\otimes \mathrm{id})\big((S\otimes\alpha\cdot\otimes \mathrm{id})\big(b_{11}P_j\otimes b_{12}Q_j\otimes b_2R_j\big)\big)\\
&\qquad = \sum_j (m\otimes \mathrm{id})\big(S(P_j)S(b_{11})\otimes \alpha b_{12}Q_j\otimes b_2R_j\big)\\
&\qquad = \sum_j S(P_j)S(b_{11})\alpha b_{12}Q_j\otimes b_2R_j\\
&\qquad = \sum_j S(P_j)\alpha\epsilon(b_1)Q_j\otimes b_2R_j = \sum_j S(P_j)\alpha Q_j\otimes \epsilon(b_1)b_2R_j\\
&\qquad = \sum_j S(P_j)\alpha Q_j\otimes bR_j = (1\otimes b)\sum_j (S(P_j)\alpha Q_j\otimes R_j),
\end{aligned}
$$

where in the fourth equality we have used a property of the antipode S, and at the fifth we have used a property of ϵ.

Substituting the expression above for B in the calculation of $\phi\psi$, we have

$$
\begin{aligned}
\phi\psi(a\otimes b) &= \sum_{i,j}(a\otimes 1)(X_i\beta S(Y_i)\otimes 1)(1\otimes b)(S(P_j)\alpha Q_j\otimes R_j)\Delta(Z_i)\\
&= (a\otimes b)\sum_{i,j}(X_i\beta S(Y_i)\otimes 1)(S(P_j)\alpha Q_j\otimes R_j)\Delta(Z_i).
\end{aligned}
$$

Next, we show that $U = \sum_{i,j}(X_i\beta S(Y_i)\otimes 1)(S(P_j)\alpha Q_j\otimes R_j)\Delta(Z_i)$ is equal to 1. We define the k-linear map $\Psi : A\otimes A\otimes A\otimes A \to A\otimes A$ by

$$
\Psi(a\otimes b\otimes c\otimes d) = a\beta S(b)\alpha c\otimes f,
$$

so U can be written as

$$\begin{aligned}\sum_{i,j}&(X_i\beta S(Y_i)\otimes 1)(S(P_j)\alpha Q_j\otimes R_j)\Delta(Z_i)\\&=\sum_{i,j}X_i\beta S(Y_i)S(P_j)\alpha Q_jZ_{i1}\otimes R_jZ_{i2}\\&=\sum_{i,j}\Psi((1\otimes P_j\otimes Q_j\otimes R_j)(X_i\otimes Y_i\otimes Z_{i1}\otimes Z_{i2}))\\&=\Psi\big((1\otimes\Phi^{-1})(\mathrm{id}\otimes\mathrm{id}\otimes\Delta)(\Phi)\big).\end{aligned}$$

Using the coherence condition

$$(\mathrm{id}\otimes\mathrm{id}\otimes\Delta)(\Phi)(\Delta\otimes\mathrm{id}\otimes\mathrm{id})(\Phi)=(1\otimes\Phi)(\mathrm{id}\otimes\Delta\otimes\mathrm{id})(\Phi)(\Phi\otimes 1),\tag{7}$$

we get

$$\begin{aligned}(1\otimes\Phi^{-1})(\mathrm{id}\otimes\mathrm{id}\otimes\Delta)(\Phi)&=(\mathrm{id}\otimes\Delta\otimes\mathrm{id})(\Phi)(\Phi\otimes 1)(\Delta\otimes\mathrm{id}\otimes\mathrm{id})(\Phi^{-1})\\&=\sum_{i,j,k}X_iX_jP_{k1}\otimes Y_{i1}Y_jP_{k2}\otimes Y_{i2}Z_jQ_k\otimes Z_iR_k.\end{aligned}$$

Hence $\Psi\big((1\otimes\Phi^{-1})(\mathrm{id}\otimes\mathrm{id}\otimes\Delta)(\Phi)\big)$ is equal to

$$\begin{aligned}\sum_{i,j,k}&\Psi(X_iX_jP_{k1}\otimes Y_{i1}Y_jP_{k2}\otimes Y_{i2}Z_jQ_k\otimes Z_iR_k)\\&=\sum_{i,j,k}X_iX_jP_{k_1}\beta S(P_{k2})S(Y_j)S(Y_{i1})\alpha Y_{i2}Z_jQ_k\otimes Z_iR_k\\&=\sum_{i,j,k}X_iX_j\beta\epsilon(P_k)S(Y_j)\epsilon(Y_i)\alpha Z_jQ_k\otimes Z_iR_k\\&=\sum_{i,j,k}X_i\epsilon(P_k)(X_j\beta S(Y_j)\alpha Z_j)\epsilon(Y_i)Q_k\otimes Z_iR_k\\&=\sum_{i,k}X_i\epsilon(P_k)\epsilon(Y_i)Q_k\otimes Z_iR_k.\end{aligned}$$

In the second equality, we used properties of the antipode: $P_{k1}\beta S(P_{k2})=\beta\epsilon(P_k)$ and $S(Y_{i1})\alpha Y_{i2}=\alpha\epsilon(Y_i)$. In the last equality, we used $\sum_j X_j\beta S(Y_j)\alpha Z_j=1$.

We evaluate $\mathrm{id}\otimes\epsilon\otimes\mathrm{id}\otimes\mathrm{id}$ on both sides of (7), and since ϵ is an homomorphism from A to k, we obtain

$$\begin{aligned}&(\mathrm{id}\otimes\epsilon\otimes\Delta)(\Phi)((\mathrm{id}\otimes\epsilon)\Delta\otimes\mathrm{id}\otimes\mathrm{id})(\Phi)\\&\quad=(\mathrm{id}\otimes((\epsilon\otimes\mathrm{id}\otimes\mathrm{id})(\Phi)))(\mathrm{id}\otimes(\epsilon\otimes\mathrm{id})\Delta\otimes\mathrm{id})(\Phi)((\mathrm{id}\otimes\epsilon\otimes\mathrm{id})(\Phi)\otimes\mathrm{id}).\end{aligned}\tag{8}$$

In the definition of a quasi-Hopf algebra, we assumed that $\mathrm{id}\otimes\epsilon\otimes\mathrm{id}(\Phi)=1$. Therefore, $(\mathrm{id}\otimes\epsilon\otimes\Delta)(\Phi)=(\mathrm{id}\otimes\mathrm{id}\otimes\Delta)(\mathrm{id}\otimes\epsilon\otimes\mathrm{id})(\Phi)=1$. Hence, by

$(\mathrm{id}\otimes\epsilon))\Delta=\mathrm{id}\otimes 1$, the left-hand side of (8) is equal to

$$((\mathrm{id}\otimes\epsilon)\Delta\otimes\mathrm{id}\otimes\mathrm{id})(\Phi)=\sum_i X_i\otimes 1\otimes Y_i\otimes Z_i.$$

The right-hand side of (8) is equal to

$$(\epsilon\otimes\mathrm{id}\otimes\mathrm{id})(\Phi)\Big(\sum_i X_i\otimes 1\otimes Y_i\otimes Z_j\Big).$$

Therefore, we have

$$\sum_i X_i\otimes 1\otimes Y_i\otimes Z_i=(\epsilon\otimes\mathrm{id}\otimes\mathrm{id})(\Phi)\Big(\sum_i X_i\otimes 1\otimes Y_i\otimes Z_j\Big). \tag{9}$$

We multiply both sides of (9) by $\sum_i P_j\otimes 1\otimes Q_j\otimes R_j$ and obtain

$$\epsilon\otimes\mathrm{id}\otimes\mathrm{id}(\Phi)=1.$$

So we have $\epsilon\otimes\mathrm{id}\otimes\mathrm{id}(\Phi^{-1})=\epsilon\otimes\mathrm{id}\otimes\mathrm{id}(\Phi^{-1}\Phi)=1$.

Finally,

$$\begin{aligned}\sum_{i,j} X_i\epsilon(P_k)\epsilon(Y_i)Q_k\otimes Z_iR_k&=\sum_{i,k}(m\otimes\mathrm{id})(X_i\otimes\epsilon(Y_i)\otimes Z_i)(\epsilon(P_k)\otimes Q_k\otimes R_k)\\&=(m\otimes\mathrm{id})\big((\mathrm{id}\otimes\epsilon\otimes\mathrm{id})(\Phi)(\epsilon\otimes\mathrm{id}\otimes\mathrm{id})(\Phi^{-1})\big)\\&=1.\end{aligned}$$

In conclusion, we have shown that $\phi\psi(a\otimes b)=a\otimes b$ and similarly $\psi\phi(a\otimes b)=a\otimes b$. Therefore, ϕ and ψ are invertible. This completes the proof of the first statement of Proposition 4.4.

Now we calculate $(\mathrm{id}\otimes\epsilon)\phi^{-1}(a\otimes b)$. By the proof above, ψ is the inverse of ϕ, and

$$\begin{aligned}(\mathrm{id}\otimes\epsilon)\phi^{-1}(a\otimes b)&=(\mathrm{id}\otimes\epsilon)\Big(\sum_i aX_i\beta S(Y_i)S(b_1)\otimes b_2Z_i\Big)\\&=\sum_i aX_i\beta S(Y_i)S(b_1)\epsilon(b_2)\epsilon(Z_i)\\&=\sum_i aX_i\beta S(Y_i)S(b_1\epsilon(b_2))\epsilon(Z_i)\\&=\sum_i aX_i\beta S(Y_i)S(b)\epsilon(Z_i).\end{aligned}$$

To show that the last term is equal to $a\beta S(b)$, we consider the k-linear map $\Upsilon: A\otimes A\otimes A\to A$ defined by $\Upsilon(a_1\otimes a_2\otimes a_3)=a_1\beta S(a_2)a_3$. Accordingly, we have $\sum_i X_i\beta S(Y_i)\epsilon(Z_i)=\Upsilon((\mathrm{id}\otimes\mathrm{id}\otimes\epsilon)(\Phi))$. By applying $\mathrm{id}\otimes\mathrm{id}\otimes\epsilon\otimes\mathrm{id}$ to (8), we have similarly $(\mathrm{id}\otimes\mathrm{id}\otimes\epsilon)(\Phi)=1\otimes 1\otimes 1$. So $\sum_i X_i\beta S(Y_i)\epsilon(Z_i)=\Upsilon(1)=\beta$, and $\sum_i aX_i\beta S(Y_i)S(b)\epsilon(Z_i)$ is equal to $a\beta S(b)$.

Therefore if there is an element in W, which can be written as $\Delta(\mu)$, where μ is in the kernel of ϵ, then $(\mathrm{id}\otimes\epsilon)\phi^{-1}(\Delta(\mu))=\mu_1\beta S(\mu_2)=\epsilon(\mu)\beta=0$. This shows that W is contained in the kernel of the map $(\mathrm{id}\otimes\epsilon)\phi^{-1}: A\otimes A\to A$. Finally, we show that $(\mathrm{id}\otimes\epsilon)\phi^{-1}$ is a bijection from $A\otimes A/W$ to A. If $\sum_i x_i\otimes y_i$ is in the kernel of $(\mathrm{id}\otimes\epsilon)\phi^{-1}$. We define $\sum_j a_j\otimes b_j$ to be equal to $\phi\big(\sum_i x_i\otimes y_i\big)$, and $(\mathrm{id}\otimes\epsilon)\big(\sum_j a_j\otimes b_j\big)=\sum_j a_j\epsilon(b_j)=0$. Then $\sum_i x_i\otimes y_i$ is equal to

$$\begin{aligned}\sum_i x_i\otimes y_i=\sum_j \phi(a_j\otimes b_j)&=\sum_j(a_j\otimes 1)\omega\Delta(b_j)\\&=\sum_j(a_j\otimes 1)\omega(\Delta(b_j)-\epsilon(b_j))\in W,\end{aligned}$$

where in the third equality, we have used that

$$\sum_j(a_j\otimes 1)\omega\epsilon(b_j)=\sum_j(a_j\epsilon(b_j)\otimes 1)\omega=0. \qquad\square$$

By the same arguments used in Theorem 4.2, we obtain:

Theorem 4.5. *Let (A,Δ,ϵ,Φ) be a biunital quasibialgebra, and let W be the left ideal generated by $\{\epsilon(a)(1\otimes 1)-\Delta(a)\mid a\in A\}$. Then $(A\otimes A)/W$ is a preantipode for the modulation of A.*

If A is a quasi-Hopf algebra, with antipode (S,α,β), then $(A\otimes A)/W$ is isomorphic to the modulation $\mathbf{S}$, and $(A,\boldsymbol{\Delta},\boldsymbol{\epsilon},\mathbf{S})$ is a hopfish algebra.

5. Morita invariance

The following theorem shows that, with our definition of hopfish algebra, we are on the right track toward defining a Morita invariant notion.

Theorem 5.1. *Let A be a quasi-Hopf algebra and B an algebra Morita equivalent to A. Then B is a sesquiunital sesquialgebra with a preantipode.*

Proof. Let P be an (A,B)-bimodule, and Q a (B,A)-bimodule, inverse to one another in the category Alg. We recall the hopfish structure on A defined in Theorem 4.5, with

$$\boldsymbol{\epsilon}^A=k,\quad \boldsymbol{\Delta}^A=A\otimes A,\quad \mathbf{S}^A=A\otimes A/W.$$

We use the bimodules P and Q to define

$$\boldsymbol{\epsilon}^B:=\boldsymbol{\epsilon}^A\otimes_A P,\qquad \boldsymbol{\Delta}^B:=\big(Q\otimes Q\big)\otimes_{A\otimes A}\boldsymbol{\Delta}^A\otimes_A P,$$

These data make B into a sesquiunital sesquilinear algebra.

Now we define

$$\boldsymbol{S}^B := \big(Q \otimes Q\big) \otimes_{A\otimes A} \boldsymbol{S}^A.$$

Remark 5.2. Our definition of the antipode $\boldsymbol{S}^B$ only uses the bimodule Q, not P. This is because Q is a (B, A) bimodule, and therefore is also an $(A^{\mathrm{op}}, B^{\mathrm{op}})$ bimodule naturally. Since $\boldsymbol{S}^A$ is an (A, A^{op}) bimodule, $Q \otimes_A \boldsymbol{S}^B \otimes_{A^{\mathrm{op}}} Q$ defines a (B, B^{op}) bimodule, which is isomorphic to $(Q \otimes Q) \otimes_{A\otimes A} \boldsymbol{S}^A$.

In the following, we will show that $\boldsymbol{S}^B$ is a preantipode:

$$\mathrm{Hom}_k(k, \boldsymbol{S}^B) \cong \mathrm{Hom}_B(\boldsymbol{\epsilon}^B, \boldsymbol{\Delta}^B).$$

According to our definitions, we have

$$\mathrm{Hom}_B(\boldsymbol{\epsilon}^B, \boldsymbol{\Delta}^B) = \mathrm{Hom}_B\big(\boldsymbol{\epsilon}^A \otimes_A P, \big(Q \otimes Q\big) \otimes_{A\otimes A} \boldsymbol{\Delta}^A \otimes_A P\big).$$

Since the Morita equivalence between A and B defines an equivalence of right-module categories, we have a natural isomorphism

$$\mathrm{Hom}_B\big(\boldsymbol{\epsilon}^A \otimes_A P, \big(Q \otimes Q\big) \otimes_{A\otimes A} \boldsymbol{\Delta}^A \otimes_A P\big) \cong \mathrm{Hom}_A\big(\boldsymbol{\epsilon}^A, \big(Q \otimes Q\big) \otimes_{A\otimes A} \boldsymbol{\Delta}^A\big).$$

The space $\mathrm{Hom}_A\big(\boldsymbol{\epsilon}^A, \big(Q \otimes Q\big) \otimes_{A\otimes A} \boldsymbol{\Delta}^A\big)$ consists of k-linear morphisms from $\big(Q \otimes Q\big) \otimes_{A\otimes A} \boldsymbol{\Delta}^A$ to k, vanishing on the A-submodule $\widetilde{W}$ spanned by

$$(q_1 \otimes q_2) \otimes_{A\otimes A} (a_1 \otimes a_2)(\epsilon(a)1 \otimes 1 - \Delta(a)), \; q_i \in Q, \; a, \; a_i \in A, \; i = 1, 2.$$

The A-submodule $\widetilde{W}$ is isomorphic to $(Q \otimes Q) \otimes_{A\otimes A} W$, where W is defined as in Theorem 4.5. Therefore, $\mathrm{Hom}_A\big(\boldsymbol{\epsilon}^A, \big(Q \otimes Q\big) \otimes_{A\otimes A} \boldsymbol{\Delta}^A\big)$ is isomorphic to the k-dual of the quotient

$$(Q \otimes Q) \otimes_{A\otimes A} \boldsymbol{\Delta}^A/\widetilde{W} \cong (Q \otimes Q) \otimes_{A\otimes A} (A \otimes A/W). \tag{10}$$

Replacing $A \otimes A/W$ by $\boldsymbol{S}^A$ in (10), we have

$$\mathrm{Hom}_B(\boldsymbol{\epsilon}^B, \boldsymbol{\Delta}^B) \cong \big((Q \otimes Q) \otimes_{A\otimes A} \boldsymbol{S}^A\big)^* \cong (\boldsymbol{S}^B)^*.$$

Therefore, $\boldsymbol{S}^B$ defines a preantipode on $(B, \boldsymbol{\Delta}^B, \boldsymbol{\epsilon}^B)$. □

Now we study when the sesquiunital sesquialgebra just defined is a hopfish algebra, i.e. when $\boldsymbol{S}^B$ is isomorphic to B as a left B-module.

We introduce the following special type of module over a hopfish algebra.

Definition 5.3. Let be A be a hopfish algebra with antipode bimodule $\boldsymbol{S}$, and let X be a right A-module and therefore a left A^{op}-module. Then X is *self-conjugate* if $\mathrm{Hom}_A(A, X)$ is isomorphic to $\boldsymbol{S} \otimes_{A^{\mathrm{op}}} X$ as a left A-module.

Remark 5.4. The category of finite dimensional left modules over a quasi-Hopf algebra is a rigid monoidal category. A self-dual module X of a quasi-Hopf algebra A is a self-dual object in the category of finite dimensional modules, i.e. $\mathrm{Hom}_k(k, X)$ is isomorphic to $\boldsymbol{S} \otimes_{A^{\mathrm{op}}} X$.

We can understand the definition of a self-conjugate module geometrically as follows. A hopfish algebra A can be thought as functions on a "noncommutative space with group structure" G. If we view a finite projective right A-module X as the space of sections of a "vector bundle" E over G, $\mathrm{Hom}_A(A, X)$ corresponds to the space of sections of the dual bundle E^*, and $\boldsymbol{S} \otimes_{A^{\mathrm{op}}} X$ is the pullback of the bundle E by the "inversion" map ι of G. The self-conjugacy condition on E says that E^* is isomorphic to $\iota^* E$.

Proposition 5.5. *With the same assumptions and notation as in Theorem 5.1, if the (B, A)-Morita equivalence bimodule Q is self-conjugate as a right A-module, then B is a hopfish algebra with antipode $\boldsymbol{S}^B$ defined in Theorem 5.1.*

Proof. Recall that the preantipode on B defined in Theorem 5.1 is equal to $(Q \otimes Q) \otimes_{A\otimes A} \boldsymbol{S}^A$. Since Q is a right A-module, it is also a left A^{op}-module, and the preantipode $\boldsymbol{S}^B$ can be rewritten as $Q \otimes_A \boldsymbol{S}^A \otimes_{A^{\mathrm{op}}} Q$.

Since Q is self-conjugate, we have $\boldsymbol{S}^A \otimes_{A^{\mathrm{op}}} Q \cong \mathrm{Hom}_A(A, Q)$, and so

$$Q \otimes_A \boldsymbol{S}^A \otimes_{A^{\mathrm{op}}} Q \cong Q \otimes_A \mathrm{Hom}_A(A, Q).$$

When Q is a Morita equivalence bimodule between A and B, Q is a finitely generated projective A-module and $B \cong \mathrm{Hom}_A(Q, Q) = Q \otimes_A \mathrm{Hom}_A(A, Q)$. This shows that $Q \otimes_A \boldsymbol{S}^A \otimes_{A^{\mathrm{op}}} Q$ is isomorphic to B as a left B-module. □

The following example is a special case of Proposition 5.5. We remark that given a (quasi)-Hopf algebra A, the matrix algebra $M_n(A)$ of $n \times n$ matrices with coefficients in A is not a (quasi-)Hopf algebra when $n \geq 2$.

Example 5.6. Let A be a quasi-Hopf algebra with $\boldsymbol{\epsilon}^A = k$, $\boldsymbol{\Delta}^A = A \otimes A$, and $\boldsymbol{S}^A = A$. Then the $n \times n$ matrix algebra $M_n(A) = B$ with coefficients in A is a hopfish algebra. We consider $Q = A^n$ as a space of column vectors, so that it has the structure of an $(M_n(A), A)$-bimodule, The counit $\boldsymbol{\epsilon}^B$ is A^n viewed as row vectors, i.e. as a $(k, M_n(A))$-bimodule. The coproduct $\boldsymbol{\Delta}^B$ is isomorphic to

$$\left(A^n \otimes A^n\right) \otimes_{A\otimes A} (A \otimes A) \otimes_A (A^n)^T = \left(A^n \otimes A^n\right) \otimes_{A\otimes A} (A^n)^T.$$

$\boldsymbol{S}^B$ is equal to $A^n \otimes_A A \otimes_{A^{\mathrm{op}}} A^n$. $A^n \otimes_A A \otimes_{A^{\mathrm{op}}} A^n$ is isomorphic to $M_n(A)$ as an $(M_n(A), M_n(A)^{\mathrm{op}})$-bimodule, where the left $M_n(A)$ module structure is from the standard left multiplication, while the right $M_n(A)^{\mathrm{op}}$ module structure is the composition of the left multiplication, transposition of matrices, and the antipode on A. Therefore, $B = M_n(A)$ is a hopfish algebra.

The following example shows that the self-conjugacy condition in Proposition 5.5 can not be eliminated.

Example 5.7. Consider the cyclic group $\mathbb{Z}/3\mathbb{Z}$ with elements 0, 1, 2. The algebra A of functions on $\mathbb{Z}/3\mathbb{Z}$ is a commutative Hopf algebra spanned by the characteristic functions e_0, e_1, and e_2. We notice that the e_i's are projections in A, and denote the submodule $e_i A$ by A_i. Now consider the following projective module over A

$$Q = A_0^r \oplus A_1^s \oplus A_2^t,$$

where r, s, t are nonnegative integers. Then

$$B = \mathrm{Hom}_A(Q, Q) = A_0^{r^2} \oplus A_1^{s^2} \oplus A_2^{t^2}.$$

It is not difficult to see that Q is self-conjugate if and only if $s = t$.

We calculate the expression for $\boldsymbol{S}^B$ in Theorem 5.1:

$$\begin{aligned}(Q \otimes Q) \otimes_{A\otimes A} \boldsymbol{S}^A &= \big((A_0^r \oplus A_1^s \oplus A_2^t) \otimes (A_0^r \oplus A_1^s \oplus A_2^t)\big) \otimes_{A\otimes A} \boldsymbol{S}^A \\ &= \big(A_0^r \otimes (A_0^r \oplus A_1^s \oplus A_2^t)\big) \otimes_{A\otimes A} \boldsymbol{S}^A \\ &\qquad \oplus \big(A_1^s \otimes (A_0^r \oplus A_1^s \oplus A_2^t)\big) \otimes_{A\otimes A} \boldsymbol{S}^A \\ &\qquad\qquad \oplus \big(A_2^t \otimes (A_0^r \oplus A_1^s \oplus A_2^t)\big) \otimes_{A\otimes A} \boldsymbol{S}^A.\end{aligned}$$

We look at the tensor product $(A_i \otimes A_j) \otimes_{A\otimes A} \boldsymbol{S}^A$. By Theorem 4.2, the antipode bimodule $\boldsymbol{S}^A$ is isomorphic to A. Therefore $(A_i \otimes A_j) \otimes_{A\otimes A} \boldsymbol{S}^A$ is equal to

$$(A_i \otimes A_j) \otimes_{A\otimes A} A = A_i \otimes_A A_j,$$

where the left A-module structure on A_j is the composition of the right multiplication with the antipode map $S : A \to A$.

We notice that $S(e_i)e_j = 0$ if $S(e_i) \neq e_j$. Therefore,

$$A_i \otimes_A A_j = \begin{cases} 0 & \text{if } S(e_i) \neq e_j, \\ A_i & \text{if } S(e_i) = e_j. \end{cases}$$

We conclude that $\boldsymbol{S}^B = A_0^{r^2} \oplus A_1^{st} \oplus A_2^{st}$. We observe that $\boldsymbol{S}^B$ is isomorphic to B as a left B module if and only if $s = t$.

Therefore, $\boldsymbol{S}^B$ is isomorphic to B if and only if Q is a self-conjugate A-module.

We define a notion of Morita equivalence between hopfish algebras.

Definition 5.8. Let $(A, \boldsymbol{\epsilon}^A, \boldsymbol{\Delta}^A, \boldsymbol{S}^A)$ and $(B, \boldsymbol{\epsilon}^B, \boldsymbol{\Delta}^B, \boldsymbol{S}^B)$ be two hopfish algebras. Then $(A, \boldsymbol{\epsilon}^A, \boldsymbol{\Delta}^A, \boldsymbol{S}^A)$ is Morita equivalent to $(B, \boldsymbol{\epsilon}^B, \boldsymbol{\Delta}^B, \boldsymbol{S}^B)$ if there is an (A, B)-bimodule ${}_A P_B$ and a (B, A)-bimodule ${}_B Q_A$ satisfying

(1) $P \otimes_B Q = A$, and $Q \otimes_A P = B$.

(2) $\boldsymbol{\epsilon}^B = \boldsymbol{\epsilon}^A \otimes_A P$,

(3) $\mathbf{\Delta}^B = (Q \otimes Q) \otimes_{A\otimes A} \mathbf{\Delta}^A \otimes_A P$,

(4) $\boldsymbol{S}^B = (Q \otimes Q) \otimes_{A\otimes A} \boldsymbol{S}^A$.

Proposition 5.9. *Definition 5.8 defines an equivalence relation among hopfish algebras.*

The proof is a straightforward check.

6. Hopfish structures on k^n

In this section, we give examples of hopfish algebras which are not Morita equivalent to modulations of Hopf algebras. In particular, we will describe hopfish structures on the commutative algebra k^G of k-valued functions on a finite set G which do not correspond to group structures on G.

We may identify the r-th tensor power of k^G with k^{G^r}. Since this algebra is commutative, we can also identify $(k^g)^{\mathrm{op}}$ with k^g.

If G is a semigroup, k^G is a bialgebra with coproduct $\Delta(a)(g,h) = a(gh)$, with a counit $\epsilon(a) = a(e)$ when G has an identity element e. When G is a group, we get a Hopf algebra structure by setting $S(a)(g) = a(g^{-1})$.

Now let G be a groupoid. We may make the same definitions as above, adding that $\Delta(a)(g,h)$ should be 0 when gh is not defined, and $\epsilon(a)$ is the sum of the values of a on all the identity elements. When G is not a group, k^G is no longer a Hopf algebra, but rather a *weak Hopf algebra* [Nikshych 2002, Example 2.3], since Δ is not unital and ϵ is not even an algebra homomorphism. When G is a groupoid, we have two algebra morphisms $\alpha, \beta: k^{G^0} \to k^G$ as the lifts of the source and target maps. The coproduct Δ is defined on $k^G \otimes_{k^{G^0}} k^G$ by $\Delta(a)(g,h) = a(gh)$, and counit $\epsilon : k^G \to k^{G^0}$ by $\epsilon(a)(e) = a(e)$, and the antipode S is defined by $S(a)(g) = a(g^{-1})$. $(k^G, \alpha, \beta, \Delta, \epsilon, S)$ is a *quantum groupoid* [Lu 1996]. It turns out that we can still form the modulation of these operators, and we still get a hopfish algebra because of the commutativity of the algebras k^G and k^{G^0}. To prove this, we will look at a more general situation.

Any sesquialgebra coproduct on $A = k^G$ is an $(A \otimes A, A)$-bimodule, i.e. a module $\mathbf{\Delta}$ over $k^{G\times G\times G}$. Such a module decomposes into submodules supported at the points of G^3. For our purposes, we will assume that these are free modules of finite rank. Then $\mathbf{\Delta}$ is determined up to isomorphism by the dimensions d^g_{hk} of the components $\mathbf{\Delta}^g_{hk}$, for $(g,h,k) \in G^3$. It is straightforward to check that the condition for coassociativity is precisely that the d^g_{hk}'s be the structure constants of an associative algebra $A' = \mathbb{Z}^G$ over $\mathbb{Z}$, i.e. a ring. Namely, identifying each element of G with its characteristic function, we have $gh = \sum_k d^k_{gh} k$. Similarly, a (k, A)-bimodule $\boldsymbol{\epsilon}$ with free submodules $\boldsymbol{\epsilon}^g$ as components is determined by the dimensions e^g of $\boldsymbol{\epsilon}^g$, and this bimodule is a counit precisely when $e := \sum_g e^g g$ is a unit element for A'. We say that such sesquiunital sesquialgebras are of *finite free type*. Thus we have shown:

Proposition 6.1. *There is a one to one correspondence between sesquiunital sesquialgebra structures of finite free type on k^G and unital ring structures on $\mathbb{Z}^G$ for which the structure constants and the components of the unit are nonnegative.*

The best known examples of such rings are the monoid algebras. If G is a monoid, then we may define δ^g_{hk} to be the characteristic function of the graph $g = hk$ of multiplication and e^g to be the characteristic function of the identity element. The corresponding sesquialgebra is just the modulation of the dual to the monoid bialgebra A'.

With this example in mind, we may think of a general structure of convolution type on $\mathbb{Z}^G$ as corresponding to a "product" operation on G in which the product of any two elements is a (possibly empty) subset of G whose elements are provided with positive integer "multiplicities". We will call such a subset a "multiple element"; the identity is also such a multiple element. (Of course, any ring structure may be viewed in this way, if we allow the multiplicities to be arbitrary integers).

To begin our analysis of these structures, we show that there are restricted possibilities for the unit.

Proposition 6.2. *Each e^g is either* 0 *or* 1.

Proof. Given g, by the counit property $\sum_k e^k d^g_{gk} = \delta_{gg} = 1$, we see that there is at least one $k \in G$ such that $d^g_{gk} \neq 0$. By the counit property again, we have

$$e^g \leq e^g d^g_{gk} \leq \sum_h e^h d^g_{hk} = \delta_{gk} \leq 1. \qquad \square$$

We will denote by G^0 the support of the unit, that is, the set of $g \in G$ for which $e^g = 1$. This set will play the role of identity elements in G.

As long as G is nonempty, so is G^0. In fact, we have the following:

Proposition 6.3. *Given any g in G, there are unique elements $l(g)$ and $r(g)$ in G^0 such that, for all $h \in G^0$, $d^k_{hg} = \delta_{hl(g)}\delta_{gk}$ and $d^k_{gh} = \delta_{r(g)h}\delta_{gk}$.*

Proof. This is again a straightforward corollary of the counit property. We obtain from $\sum_k e^k d^h_{gk} = \delta_{gh}$, $\sum_g e^g d^h_{gk} = \delta_{kh}$ that $\sum_{g \in G^0} d^k_{gh} = \delta_{kh}$. So $d^k_{gh} = 0$ when $k \neq h$ and there exists a unique element $g_0 \in G^0$ such that d^h_{gh} equals 1 for $g = g_0$ and 0 for all other g. We let $l(h)$ be this g_0. This proves the first equation; the second is proved by a similar argument. $\square$

Since the sum of the elements of G^0 is the unit of k^G, it is idempotent, from which it follows that k^{G_0} is a subalgebra. In fact, one may show:

Proposition 6.4. *The elements of G^0 form a set of orthogonal idempotents in $\mathbb{Z}^G$. In other words, the algebra structure on the subalgebra $\mathbb{Z}^{G^0}$ of A' is just pointwise multiplication.*

Proof. This follows from uniqueness in Proposition 6.3. □

Proposition 6.5. *For all g and h in G, if $d^k_{gh} \neq 0$, $l(k) = l(g)$ and $r(k) = r(h)$. If $r(g)$ is not equal to $l(h)$, then $gh = 0$ in G. In particular, $l(h) = h = r(h)$ for all $h \in G^0$.*

Proof. Coassociativity gives us

$$\sum_s d^k_{l(g)s} d^s_{gh} = \sum_s d^s_{l(g)g} d^k_{sh}.$$

By Proposition 6.3, $d^s_{l(g)g} = \delta_{gs}$. Therefore, the right-hand side of the equation is equal to $d^k_{gh} \neq 0$.

On the left-hand side, according to Proposition 6.3, $d^k_{l(g)s} \neq 0$ only when $l(s) = l(g)$ and $k = s$. Therefore, if $d^k_{gh} \neq 0$, then $d^k_{l(g)k} = 1$, so $l(k) = l(g)$. Similar arguments show that $r(k) = r(h)$.

If $r(g) \neq l(h)$, again by coassociativity, we have

$$d^k_{gh} = \sum_s d^s_{gr(g)} d^k_{sh} = \sum_s d^k_{gs} d^s_{r(g)h}.$$

According to Proposition 6.3, if $r(g) \neq l(h)$, $d^s_{r(g)h} = 0$ for all s; therefore, $d^k_{gh} = 0$. □

We now have retractions l and r from G onto G^0 which are like the target and source maps from a category to its set of identity elements. In fact, in terms of the multiplicative structure on G corresponding to the algebra structure on A', we have $l(g)g = gr(g) = g$; in particular, these products are single valued and without multiplicities. We might call G a "hypercategory". The composition of morphisms is a "multiple morphism" between two definite objects.

We will show next that, when k^G has an antipode and is hence a hopfish algebra, the underlying multiplicative structure on G has inverses and the property that gh is nonzero whenever $r(g) = l(h)$. We will call such a structure a "hypergroupoid" (see Definition 6.9).[2]

According to Definition 3.2, an antipode is a (k^G, k^G)-bimodule $\boldsymbol{S}$ whose dual is isomorphic to $\mathrm{Hom}_{k^G}(\boldsymbol{\epsilon}, \boldsymbol{\Delta})$.

We recall the definition of $\boldsymbol{\epsilon}$ and $\boldsymbol{\Delta}$:

$$\boldsymbol{\epsilon} = \oplus_g \boldsymbol{\epsilon}^g, \quad \boldsymbol{\Delta} = \oplus_{g,h,t} \boldsymbol{\Delta}^t_{gh}.$$

[2]The notion of group with multiple-valued multiplication has a long history. The reader may start with [Kuntzmann 1939], which cites even earlier work.

Therefore we can write

$$\mathrm{Hom}_{k^G}(\boldsymbol{\epsilon}, \boldsymbol{\Delta}) = \big(\oplus_s \boldsymbol{\epsilon}^s\big) \otimes_{k^G} \big(\oplus_{g,h,t} \boldsymbol{\Delta}^t_{gh}\big)^*.$$

We notice that k^G acts via the upper indices of $\boldsymbol{\epsilon}$ and $\boldsymbol{\Delta}$ by componentwise multiplication. Therefore, this expression for $\mathrm{Hom}_{k^G}(\boldsymbol{\epsilon}, \boldsymbol{\Delta})$ can be simplified to

$$\oplus_{g,h}\big(\oplus_t \boldsymbol{\epsilon}^{t*} \otimes \boldsymbol{\Delta}^t_{gh}\big)^*,$$

which is isomorphic to

$$\oplus_{g,h}\big(\oplus_t \mathrm{Hom}_k(k, \boldsymbol{\epsilon}^{t*} \otimes \boldsymbol{\Delta}^t_{gh})\big) \cong \mathrm{Hom}_k\big(k, \oplus_{g,h}(\oplus_t \boldsymbol{\epsilon}^{t*} \otimes \boldsymbol{\Delta}^t_{gh})\big).$$

Therefore, $\boldsymbol{S}$ is isomorphic to $\oplus_{g,h}(\oplus_t \boldsymbol{\epsilon}^{t*} \otimes \boldsymbol{\Delta}^t_{gh})$ as a (k^G, k^G) bimodule.

When $\boldsymbol{S}$ is an antipode, $\boldsymbol{S}$ is by definition isomorphic to k^G as a left k^G-module. Therefore, if we write $\boldsymbol{S}$ as $\oplus_{g,h}\boldsymbol{S}_{gh}$, for any fixed g there exists a unique element $h \in G$ such that $\dim(\boldsymbol{S}_{gh}) = 1$, and $\dim(\boldsymbol{S}_{gh'}) = 0$ for $h' \neq h$. Hence, we may define a map $\sigma : G \to G$ by $g \mapsto h$.

Since $\mathrm{Hom}_k(k, \boldsymbol{S}) \cong \mathrm{Hom}_{k^G}(\boldsymbol{\epsilon}, \boldsymbol{\Delta})$, we know that $\dim\big(\bigoplus_t \boldsymbol{\epsilon}^{t*} \otimes \boldsymbol{\Delta}^t_{gh}\big) = \delta_{\sigma(g)h}$, that is,

$$\sum_t e^t d^t_{gh} = \delta_{\sigma(g)h}. \tag{11}$$

Proposition 6.6. *For any $g \in G$, define g^{-1} to be $\sigma(g)$. Then there is a unique $s \in G^0$ such that*

$$d^s_{gg^{-1}} = 1.$$

In fact, $s = l(g) = r(g^{-1})$.

Proof. By (11), we have

$$\sum_t e^t d^t_{gg^{-1}} = \delta_{\sigma(g)g^{-1}} = 1.$$

Thus, there is a unique $s \in G^0$ such that $d^s_{gg^{-1}} = 1$ and $d^t_{gg^{-1}} = 0$ for all other $t \in G^0$. By Proposition 6.5, $s = l(g) = r(g^{-1})$. □

We also have another characteristic property of categories (though here we can only prove it in the presence of an antipode).

Proposition 6.7. *If $r(g) = l(h)$, there exists $s \in G$, such that $d^s_{gh} \neq 0$.*

Proof. Using coassociativity, we have

$$\sum_s d^g_{gs} d^s_{hh^{-1}} = \sum_s d^s_{gh} d^g_{sh^{-1}}.$$

Since $d^{l(h)}_{hh^{-1}} = 1$ and $d^g_{gl(h)} = 1$(since $r(g) = l(h)$), the left-hand side of the preceding equation is not equal to 0. This implies that, on the right-hand side, there is at least one term which is not equal to 0. Therefore, there exists $s \in G$, such that $d^s_{gh} \neq 0$. □

Question 6.8. Is the inversion map $\sigma : G \to G$ involutive?

To summarize the arguments above, we define the combinatorial objects associated to hopfish structures on k^G:

Definition 6.9. A *hypergroupoid* is a set G with data $(\,\cdot\,, l, r, ^{-1}\,)$ as follows.

(1) There is a multivalued associative binary operation $\cdot$ on G. More precisely, for all $g, h \in G$, $g \cdot h$ is an element of $\mathbb{Z}_+^G$, where $\mathbb{Z}_+$ is the semiring of nonnegative integers. When this product is linearly extended to a product on $\mathbb{Z}_+^G$, we have $g \cdot (h \cdot k) = (g \cdot h) \cdot k$.

(2) There is a subset $G^0 \subset G$ with maps $l, r : G \to G^0$ such that $l(g) \cdot g = g \cdot r(g) = g$, for all $g \in G$ The product of g and h is nonzero if and only if $r(g) = l(h)$.

(3) There is an inverse operation $g \mapsto g^{-1}$ on G such that $g \cdot g^{-1}|_{G^0} = l(g)$ for all $g \in G$, and further $g \cdot h|_{G^0} = 0$ if $h \neq g^{-1}$.

Note that the inverse operation is determined by the product operation and G^0.

We now look at the commutative algebra k^G of k-valued functions on a hypergroupoid G. The duals of the maps $l, r : G \to G^0$ define morphisms from k^G to k^{G^0}. The multiplication on G defines a (nonunital) coproduct homomorphism $k^{G\times G} \leftarrow k^G$ whose modulation is a coproduct bimodule, the embedding map from G^0 to G makes k^{G^0} into a counit bimodule, and the inversion map defines an antipode. These define a hopfish algebra structure on k^G.

Theorem 6.10. *The hopfish structures of free finite type on k^G are in one to one correspondence with the hypergroupoid structures on G.*

Example 6.11. Here is an example of a hypergroupoid which is not a groupoid, based on [Etingof et al. 2005, Example 8.19].[3] Let $G = \{e, g\}$, with multiplication and inversion given by

$$eg = ge = g, \quad ee = e, \quad gg = e + ng, \quad e^{-1} = e, \quad g^{-1} = g,$$

where n is a nonnegative integer. $G^0 = \{e\}$ and $l(g) = r(g) = e$. The algebra A' associated to this hypergroupoid is $\mathbb{Z}[x]/\{x^2 = 1 + nx\}$. The corresponding hopfish algebra k^G is not a quasi-Hopf algebra when $n = 1$ and k a field. We explain the reason below.

The hopfish algebra structure of k^G is in fact a weak Hopf algebra, with $\epsilon(e) = 1$, $\epsilon(g) = 0$, $\Delta(e) = g \otimes g + e \otimes e$ and $\Delta(g) = e \otimes g + g \otimes e + g \otimes g$. Since a k^G module can be decomposed into submodules supported at points of G, the representation ring of k^G is generated by two elements 1 and X corresponding to

[3]The hypergroupoid itself, when $n = 1$, already appears as the first example in [Kuntzmann 1939]!

the 1-dimensional k^G module supported at e and g respectively. Since k is a field, 1 and X are just 1-dimensional k-vector spaces. Using the formulas for the coproduct and counit, it is easy to check that this representation ring is the Grothendieck ring of what is called Yang–Lee fusion rules in [Ostrik 2003], namely it is generated by 1 and X with the relation $X \otimes X = 1 \oplus X$. The Frobenius–Perron dimension of the element 1 is 1, while the Frobenius–Perron dimension of the element X is the irrational number $(1+\sqrt{5})/2$. According to Theorem 8.33 of [Etingof et al. 2005], the Frobenius–Perron dimension of any finite dimensional module over a finite dimensional quasi-Hopf algebra must be a positive integer, which is equal to the dimension of the module. This shows that k^G is not Morita equivalent to a quasi-Hopf algebra.

Acknowledgements

This work began in July, 2004, when Tang and Weinstein were participants in the trimester on K-Theory and Noncommutative Geometry at the Centre Émile Borel, and was carried to completion while Tang was affiliated with the University of California, Davis and Zhu with the Eidgenössische Technische Hochschule, Zürich. We had further opportunities for collaboration, and received further stimulation, during a number of other conferences and short-term visits.

We would like to thank Max Karoubi, Yvette Kosmann-Schwarzbach, Ryszard Nest, Tudor Ratiu, Pierre Schapira, and Michel van den Bergh for their invitations and hospitality. For useful comments in conversation and correspondence, we thank Paul Baum, Christian Blohmann, Micho Durdevic, Piotr Hajac, Yvette Kosmann-Schwarzbach, Olivier Mathieu, Ryszard Nest, Radu Popescu, Jean Renault, Earl Taft, and Boris Tsygan. We would especially like to thank Pavel Etingof for helping us to overcome a stumbling block in the characterization of the antipode, as well as Noah Snyder for providing the example in the last section.

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Received November 2, 2005.

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