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We prove that the Euclidean rank of any 3-dimensional Hilbert geometry (D, h_D) is 1; that is, (D, h_D) does not admit an isometric embedding of the Euclidean plane. We show that for higher dimensions this remains true if the boundary ∂D of D is C^1.

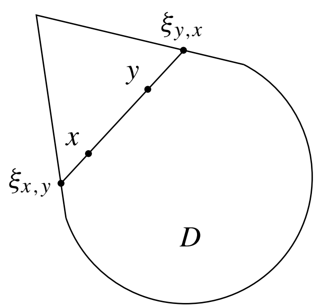
1. Introduction

In the late nineteenth century D. Hilbert informed F. Klein in a letter about his discovery of a method to construct metric spaces, generalizing Klein's model of the real hyperbolic space [Hilbert 1895]:

Let $\mathbb{E}^n$ denote the n-dimensional Euclidean space. The Euclidean distance of $x, y \in \mathbb{E}^n$ is written $|xy|$, the line segment between x and y is $[x, y]$, whereas $L(x, y) = \langle x, y \rangle$ denotes the whole affine line through x and y in $\mathbb{E}^n$.

Given an open bounded convex domain $D \subset \mathbb{E}^n$ with boundary $\partial D \subset \mathbb{E}^n$, the Hilbert metric $h_D : D \times D \to \mathbb{R}_0^+$ is defined via the cross-ratio: Given distinct points $x, y \in D$, take the points $\xi_{x,y}, \xi_{y,x}$ where $L(x, y)$ intersects ∂D and set

$$h_D(x, y) = \log \frac{|y\xi_{x,y}|\,|x\xi_{y,x}|}{|x\xi_{x,y}|\,|y\xi_{y,x}|}.$$



Formally, $\xi_{x,y} \in \langle x, y \rangle \cap \partial D$ is uniquely determined by the condition $|\xi_{x,y}x| < |\xi_{x,y}y|$. The cross ratio, of course, is invariant under projective transformations. For the basic properties of h_D see [Busemann 1955] and [de la Harpe 1993]; for example, the topology induced by h_D on D coincides with the subspace topology inherited from $\mathbb{E}^n$. We refer to the metric space (D, h_D) as a *Hilbert geometry*.

Hilbert proved that the straight line segments in (D, h_D) indeed are geodesics. In general, however, other geodesic segments also exist (see Lemma 2.1).

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It is natural to ask what metric spaces can be realized as Hilbert geometries. Hilbert himself pointed out that his construction, applied to an open n-dimensional Euclidean ball (or ellipsoid), yields Klein's model of n-dimensional real hyperbolic space.

Kelly and Strauss [1958] have proved that these are the only globally nonpositively Busemann curved Hilbert geometries, and therefore also the only Hilbert geometries which are CAT(0). (For these notions see [Bridson and Haefliger 1999; Jost 1997], for instance.)

More recently, Karlsson and Noskov [2002] showed that, nevertheless, there exist many Gromov-hyperbolic Hilbert geometries. (A geodesic metric space is called δ-hyperbolic, for a given $\delta \geq 0$, if each side of any geodesic triangle is contained in the union of the δ-neighborhoods of the other two sides; a δ-hyperbolic space is also called Gromov-hyperbolic.) Benoist [2003] then presented a necessary and sufficient condition for a Hilbert geometry to be Gromov-hyperbolic in terms of the boundary ∂D of D. Another characterization in terms of the spectrum of the Laplacian of (D, h_D) was recently obtained in [Colbois and Vernicos 2006].

A necessary — but by far not sufficient — condition for the Hilbert geometry on D to be Gromov-hyperbolic is that the boundary ∂D be C^1 and that $\bar{D}$ be strictly convex (see [Karlsson and Noskov 2002] or [Benoist 2003]). Thus there exist plenty of examples of Hilbert geometries that are *not* Gromov-hyperbolic.

Nonetheless, these examples still show certain "hyperbolic" features. This raises the question: *How close to being hyperbolic are Hilbert geometries in general?*

One particularly interesting class of non-Gromov-hyperbolic Hilbert geometries are those defined on the interior of simplices (the convex hull of $n+1$ points in $\mathbb{E}^n$ in general position). Although such geometries have been studied in detail for decades — see [Phadke 1974/75; Busemann and Phadke 1987], for example — it was not until much later that de la Harpe [1993] proved that, surprisingly, Hilbert geometries defined on the interior of simplices are isometric to normed vector spaces (see also [Nussbaum 1988]). In [Foertsch and Karlsson 2005] it was proved that these normed vector spaces are the only ones that can be realized as Hilbert geometries.

Whereas the papers mentioned above were concerned with the question of which metric spaces admit a realization as a Hilbert geometry, here we will be interested, more generally, in isometric embeddings into Hilbert geometries. This includes Hilbert geometries that are not uniquely geodesic, in particular those arising from nonstrictly convex domains — these cases are, in our view, even more interesting.

(Recall that a map $f : (X, d_X) \to (Y, d_Y)$ between metric spaces (X, d_X) and (Y, d_Y) is a *quasiisometric embedding* if there exist $\lambda \geq 1$ and $k \geq 0$ such that

$$\text{(1-1)}\quad \frac{1}{\lambda} d_X(x, x') - k \leq d_Y\big(f(x), f(x')\big) \leq \lambda\, d_X(x, x') + k \quad \text{for any } x, x' \in X.$$

If, moreover, $f(X)$ is a k-net in (Y, d_Y) (that is, Y is the k-neighborhood of $f(X)$), then f is called a quasiisometry. Recall further that if the inequalities in (1-1) hold with $\lambda = 1$, then f is called a rough-isometric embedding.)

This article is somehow motivated by the following result, which we quote from [Bridson and Haefliger 1999, Theorem III.H 1.9]: *There is no* (*quasi*)*isometric embedding of the Euclidean plane in a geodesic Gromov-hyperbolic metric space*.

Obviously, since every n-dimensional normed vector space is bilipschitz to $\mathbb{E}^n$, the Hilbert geometry defined on the interior of any simplex in $\mathbb{E}^n$ is quasiisometric to $\mathbb{E}^n$. However, as we will show, at least every 3-dimensional Hilbert geometry does not admit an isometric embedding of the Euclidean plane. To state the corresponding theorem, recall that the Euclidean rank, $\mathrm{rank}_E(X, d)$, of a metric space (X, d) is defined to be the supremum over all dimensions of Euclidean spaces that admit isometric embeddings into (X, d). Since affine line segments in a Hilbert geometry are geodesics, any Hilbert geometry has Euclidean rank at least 1.

Towards answering the question of how close to hyperbolic spaces a Hilbert geometry is, we will prove that Hilbert geometries have Euclidean rank 1 — at least in the cases mentioned in the abstract (though we believe this holds true for all cases):

Theorem 1.1. *Let* (D, h_D) *be a* 3*-dimensional Hilbert geometry. Then*

$$\mathrm{rank}_E(D, h_D) = 1.$$

We believe that Theorem 1.1 generalizes to arbitrary dimensions. For higher dimensions we can prove:

Theorem 1.2. *Let* (D, h_D) *be an* n*-dimensional Hilbert geometry. Then*

$$\mathrm{rank}_E(D, h_D) = 1$$

if ∂D *does not contain a line segment*, *or if* ∂D *is* C^1.

The main idea of the proof is to compare the Gromov products in $\mathbb{E}^2$ with those in the Hilbert geometry: in $\mathbb{E}^2$ the Gromov product of pairs of points on two geodesic rays — emanating from a fixed base point on different affine lines — is unbounded if the points tend (on the rays) to infinity. Now consider an isometric embedding of $\mathbb{E}^2$ into a Hilbert geometry and the geodesic rays which are the images of the two rays in $\mathbb{E}^2$. By a result of Karlsson and Noskov (Theorem 2.9), the endpoints of these geodesic rays span an affine line segment in the border of D (that is, this line segment does not intersect with D itself). Since this holds for all rays in $\mathbb{E}^2$ which are not collinear, this gives a lot of information about ∂D. The main difficulty is that the endpoints of geodesic rays do not depend continuously on the rays themselves (but compare Lemma 2.6). Therefore, there are many configurations to consider for how the endpoints may be spread over ∂D. In dimension 3, we are able to exploit the facts mentioned to prove Theorem 1.1. In higher dimensions the geometry is much richer, and the variety of imaginable geometric configurations is much broader; this has kept us from solving the problem in higher dimensions. But we get Theorem 1.2 as a byproduct of our considerations.

We do not know whether there exist Hilbert geometries that admit isometric embeddings of open subsets of $\mathbb{E}^2$. To address this problem, the asymptotic methods used in this paper will not be of any help. However, an advantage of the asymptotic methods presented here is that they might even yield a stronger nonembedding result: To us it seems likely that Theorem 1.1 can be sharpened to prove that there does not exist even a rough-isometric embedding of the Euclidean plane into a Hilbert geometry. But it seems that new ideas are necessary to obtain such a result.

2. Preliminaries

A *geodesic segment* in a metric space X is an isometric embedding $\gamma : I \to X$ of an interval $I \subset \mathbb{E}^1$ into X. If $I = [0, \infty)$, we also call γ a *geodesic ray* (originating at $\gamma(0)$). The images of such maps γ can also be called geodesic segments and rays. A metric space X is *geodesic* if for every $x, y \in X$ there exists a geodesic segment connecting x to y, i.e., a geodesic $\gamma : [a, b] \to X$ such that $\gamma(a) = x$ and $\gamma(b) = y$. If for every $x, y \in X$ in a geodesic metric space X the image of a geodesic segment connecting x to y is unique, X is called *uniquely geodesic*.

Let D be an open bounded convex domain in $\mathbb{E}^n$. The closure $\overline{D}$ is also convex, and every affine line containing a point of D intersects ∂D in exactly two points. For distinct $a, b \in \partial D$, the open Euclidean segment from a to b, denoted $[a, b]^\circ := [a, b] - \{a, b\}$, lies either entirely in D or entirely in ∂D.

Convex hulls in Hilbert geometries. As mentioned in the introduction, Hilbert geometries are geodesic spaces, but are not, in general, uniquely geodesic. The following well-known lemma states this more precisely. In it, we use the notation $[x, z]_d$ for the *metric convex hull* of two points x and z in a metric space (X, d):

$$[x, z]_d = \big\{ y \in X \;\big|\; d(x, z) = d(x, y) + d(y, z) \big\}.$$

Lemma 2.1 [de la Harpe 1993, Proposition 2]. *Let (D, h_D) be a Hilbert geometry and let $x, z \in D$ be distinct. Then*

$$[x, z]_{h_D} = \big\{ y \in D \;\big|\; [\xi_{z,y}, \xi_{y,x}] \cup [\xi_{x,y}, \xi_{y,z}] \subset \partial D \big\};$$

in other words, there exists $y \in D \setminus \langle x, z\rangle$ such that $h_D(x, z) = h_D(x, y) + h_D(y, z)$ if and only if there exist open line segments $I, J \subset \partial D$ spanning an affine 2-plane in $\mathbb{E}^n$ and such that $\xi_{x,z} \in I$, $\xi_{z,x} \in J$.

In particular, if D is strictly convex (that is, if every affine line intersecting ∂D in at least two points also intersects D), the Hilbert geometry (D, h_D) is uniquely geodesic.

If Σ is an affine subspace of $\mathbb{E}^n$ intersecting ∂D but not D, then $F = \Sigma \cap \overline{D} = \Sigma \cap \partial D$ will be called a *boundary flat* of the Hilbert geometry (D, h_D); the border of F with respect to the subspace topology in Σ will be denoted by ∂F and its (relative) interior by $F^\circ = F \setminus \partial F$. A boundary flat is a convex set in $\mathbb{E}^n$.

We make some simple remarks on convexity, to be used several times later.

Remark 2.2. **(a)** ***Continuity of the boundary projection:*** Consider the map

$$(x, y) \mapsto (\xi_{x,y}, \xi_{y,x})$$

defined on $\{(x, y) \in D \times D \mid x \neq y\}$. Extend it to $\{(x, y) \in \overline{D} \times \overline{D} \mid x \neq y\}$ by setting $\xi_{x,y}$ and $\xi_{y,x}$ such that $[\xi_{x,y}, \xi_{y,x}] = \langle x, y\rangle \cap \overline{D}$ with $|x\xi_{x,y}| < |y\xi_{x,y}|$ and $|y\xi_{y,x}| < |x\xi_{y,x}|$. Then $(x, y) \mapsto (\xi_{x,y}, \xi_{y,x})$ is continuous on the set

$$\big\{(x, y) \in \overline{D} \times \overline{D} \;\big|\; x \neq y \text{ and } [x, y] \not\subset \partial D\big\};$$

to be more precise, the maximal set on which the map is continuous consists exactly of the pairs of distinct points of $\overline{D}$ for which there is no line segment in the boundary of D containing both of them and even one of them in its relative interior.

(b) ***Edges converge to edges:*** If $\{p_n\}_{n\in\mathbb{N}}$, $\{q_n\}_{n\in\mathbb{N}}$ are sequences in ∂D converging to $p, q \in \partial D$ and such that $[p_n, q_n] \subset \partial D$ for all $n \in \mathbb{N}$, then $[p, q] \subset \partial D$.

(c) ***Triangles on the boundary:*** Assume that $a, b, c \in \partial D$ are not collinear, and denote by $\Delta(a, b, c)$ the closed affine triangle with vertices a, b and c, and by $\Delta^\circ(a, b, c) = \Delta(a, b, c) \setminus \big([a, b] \cup [a, c] \cup [b, c]\big)$ the relative interior of $\Delta(a, b, c)$. Then either $\Delta(a, b, c) \subset \partial D$ or $\Delta(a, b, c)^\circ \subset D$.

Geodesic rays in Hilbert geometries. Now we turn to geodesic rays in a Hilbert geometry (D, h_D). As one would expect, such rays do converge at infinity:

Theorem 2.3 [Foertsch and Karlsson 2005, Theorem 3]. *Let (D, h_D) be a Hilbert geometry.*

(i) *Every geodesic ray r in (D, h_D) converges to a point in ∂D, written $r(\infty)$.*

(ii) *Every complete geodesic γ in (D, h_D) has precisely two accumulation points $\gamma(\infty)$ and $\gamma(-\infty)$ in ∂D.*

Remark 2.4. Let r and $\tilde{r}$ be geodesic rays in a Hilbert geometry (D, h_D) such that $h_D(r(t), \tilde{r}(t))$ is a bounded function in t. If $r(\infty)$ lies in the (relative) interior of a boundary flat, the same is true of $\tilde{r}(\infty)$.

Lemma 2.5. *Let* (D, h_D) *be an* n*-dimensional Hilbert geometry and let*

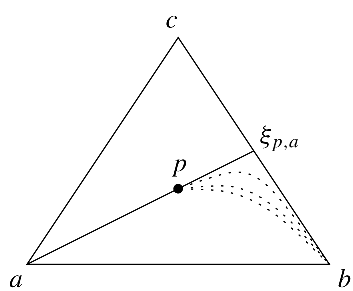
$$\gamma : (-\infty, \infty) \to D$$

be a geodesic in (D, h_D)*. Suppose that for* $t_1 < t_2$ *the straight line* $L(\gamma(t_1), \gamma(t_2))$ *intersects* ∂D *in the interior of two* $(n-1)$*-dimensional boundary flats. Then for all* $t_0 < t_1$ *there exists a neighborhood* U *of* $\gamma(t_2)$ *such that each point* $p \in U$ *lies on a geodesic through* $\gamma(t_0)$ *and* $\gamma(t_1)$*.*

The lemma immediately implies that two points satisfying the condition on $\gamma(t_1)$ and $\gamma(t_2)$ cannot occur as image points of an isometric embedding $\varphi : \mathbb{E}^2 \to (D, h_D)$. For suppose $\varphi : \mathbb{E}^2 \to (D, h_D)$ were an isometric embedding and let γ denote the image of a straight line in $\mathbb{E}^2$ under φ, with $\varphi(x_1) = \gamma(t_1)$ and $\varphi(x_2) = \gamma(t_2)$. Then for any x_0 on the straight line through x_1 and x_2 in $\mathbb{E}^2$ satisfying $|x_0x_1| < |x_0x_2|$ we have $\varphi(x_0) = \gamma(t_0)$ for some $t_0 < t_1$. Now for any given neighborhood U of $\gamma(t_2)$ in (D, h_D), sufficiently small neighborhoods of x_2 in $\mathbb{E}^2$ are mapped via φ into U. This, however, is not possible for the neighborhood U as in Lemma 2.5, since $h_D(\gamma(t_0), \gamma(t_1)) + h_D(\gamma(t_1), u) = h_D(\gamma(t_0), u)$ for all $u \in U$, whereas in any arbitrarily small neighborhood V of x_2 in $\mathbb{E}^2$ there exists $v \in V$ with $|x_0x_1| + |x_1v| > |x_0v|$. This yields the desired contradiction.

Proof of Lemma 2.5. Fix $t_0 < t_1$. Then, by Lemma 2.1, $[\xi_{\gamma(t_1),\gamma(t_0)}, \xi_{\gamma(t_2),\gamma(t_1)}] \subset \partial D$ and $[\xi_{\gamma(t_0),\gamma(t_1)}, \xi_{\gamma(t_1),\gamma(t_2)}] \subset \partial D$. Since $\xi_{\gamma(t_1),\gamma(t_2)}$ and $\xi_{\gamma(t_2),\gamma(t_1)}$ lie in the interior of $(n-1)$-dimensional boundary flats F_1 and F_2, there exists a neighborhood U of $\gamma(t_2)$ in D such that for all $p \in U$ the projection $\xi_{\gamma(t_1),p}$ lies in the interior of F_1 and $\xi_{p,\gamma(t_1)}$ lies in the interior of F_2. The claim follows from Lemma 2.1. □

Consider sequences of geodesic rays r_i, $i \in \mathbb{N}$, in the Hilbert geometry (D, h_D) converging pointwise to some geodesic ray r in (D, h_D). In general their limit points $r_i(\infty)$ do not converge to $r(\infty)$! This can be observed in the Hilbert geometry $(\Delta^\circ, h_{\Delta^\circ})$ defined inside an open triangle $\Delta^\circ = \Delta^\circ(a, b, c)$ in $\mathbb{E}^2$ with vertices a, b and c. Fix $p \in \Delta^\circ$. Then $[p, \xi_{p,a}]$ is a geodesic ray converging to $\xi_{p,a} \in [b, c]^\circ$, yet $[p, \xi_{p,a}]$ is the pointwise limit of geodesic rays r_i in $(\Delta^\circ, h_{\Delta^\circ})$, which all converge to b.



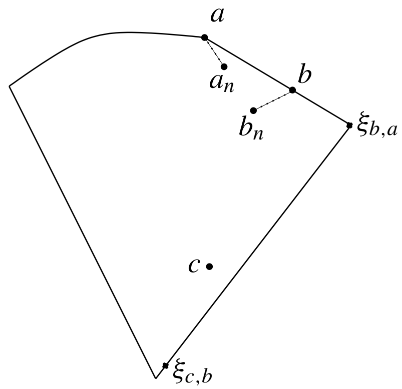
The Hilbert geometry of a 2-simplex is not uniquely geodesic.

The next lemma and its corollary investigate such behavior to some extent.

Lemma 2.6 (Jump Lemma). *Let* (D, h_D) *be a Hilbert geometry. Suppose* $\{a_n\}_n$*,* $\{b_n\}_n$ *and* $\{c_n\}_n$ *are sequences in* D *converging to distinct points* $a, b \in \partial D$ *and* $c \in \overline{D}$*, respectively, such that*

$$h_D(a_n, c_n) = h_D(a_n, b_n) + h_D(b_n, c_n) \quad \text{for all } n \in \mathbb{N}.$$

(I) *If* $c \in D$, *then* $[a, b] \subsetneq [a, \xi_{b,a}] \subset \partial D$ *and* $[\xi_{b,a}, \xi_{c,b}] \subset \partial D$. *In particular,* $c \notin \langle a, b\rangle$ *and* $b \neq \xi_{b,a}$. (See figure.)



(II) *If* $c \in \partial D$, *one of the following* (*nondisjoint*) *cases occurs*:

(i) $[a, b] \cup [b, c] \subset \partial D$.

(ii) $[a, b] \subsetneq [a, \xi_{b,a}] \subset \partial D$ *and* $[c, \xi_{b,a}] \subset \partial D$, *for* $[b, c] \not\subset \partial D$.

(iii) $[c, b] \subsetneq [c, \xi_{b,c}] \subset \partial D$ *and* $[a, \xi_{b,c}] \subset \partial D$, *for* $[a, b] \not\subset \partial D$.

Proof. (I) Let $\widetilde{\xi}_{a,b}$ be an accumulation point of $\{\xi_{a_n,b_n}\}_{n\in\mathbb{N}}$; it might be different from $\xi_{a,b}$. Since $a_n \to a$ and $b_n \to b$ as $n \to \infty$, it follows that $\widetilde{\xi}_{a,b} \in \langle a, b\rangle \cap \partial D$ and $a \in [\widetilde{\xi}_{a,b}, b]$. Hence, with $[\xi_{a_n,b_n}, \xi_{b_n,c_n}] \subset \partial D$ and $b = \lim_{n\to\infty} \xi_{b_n,c_n}$, Remark 2.2 yields $[a, b] \subset \partial D$ and thus $c \notin \langle a, b\rangle$.

Now let $\widetilde{\xi}_{b,a}$ be an accumulation point of $\{\xi_{b_n,a_n}\}_{n\in\mathbb{N}}$. Then $\widetilde{\xi}_{b,a} \in \langle a, b\rangle \cap \partial D$ and $b \in [\widetilde{\xi}_{b,a}, a]$. Since $[\xi_{c,b}, \widetilde{\xi}_{b,a}] \subset \partial D$ (by Remark 2.2(b) and Lemma 2.1), it follows from Remark 2.2(c) (or directly from convexity) that $\widetilde{\xi}_{b,a} = \xi_{b,a}$, and therefore $[\xi_{b,a}, \xi_{c,b}] \subset \partial D$. Since $c \in [\xi_{c,b}, b] \cap D$, we deduce $b \neq \xi_{b,a}$ and thus $[a, b] \subsetneq [a, \xi_{b,a}] \subset \partial D$.

(II) Now suppose that $[b, c] \not\subset \partial D$. By Lemma 2.1 and Remark 2.2(a), $b = \xi_{b,c} = \lim_{n\to\infty} \xi_{b_n,c_n}$ and $c = \xi_{c,b} = \lim_{n\to\infty} \xi_{c_n,b_n}$. Again from the same lemma and remark, it follows that $[a, b] \subset \partial D$, since for every accumulation point $\widetilde{\xi}_{a,b}$ of $\{\xi_{a_n,b_n}\}_{n\in\mathbb{N}}$ we have $\widetilde{\xi}_{a,b} \in \langle a, b\rangle$.

Since $[b, c] \not\subset \partial D$, it follows just as in part (I) that $\widetilde{\xi}_{b,a} = \xi_{b,a}$, $[c, \xi_{b,a}] \subset \partial D$ and $b \neq \xi_{b,a}$, hence $[a, b] \subsetneq [a, \xi_{b,a}] \subset \partial D$. This proves (ii). Interchanging the roles of a and c, we get (iii). □

Corollary 2.7. *Let* r *and* r_i, $i \in \mathbb{N}$, *be geodesic rays in the Hilbert geometry* (D, h_D) *such that the* r_i *converge pointwise to* r, *and suppose that* $\lim_{i\to\infty} r_i(\infty)$ *exists. Then*

$$\left[\lim_{i\to\infty} r_i(\infty), r(\infty)\right] \subset \partial D.$$

If, moreover, $\lim_{i\to\infty} r_i(\infty) \neq r(\infty)$, *there exists* $\xi \in \partial D$ *such that*

$$r(\infty) \in \left[\xi, \lim_{i\to\infty} r_i(\infty)\right]^\circ.$$

Lemma 2.8. *Let* $\gamma : (-\infty, \infty) \to (D, h_D)$ *be a geodesic in the Hilbert geometry* (D, h_D) *such that*

$$[\gamma(-\infty), \gamma(\infty)] \subset \partial D.$$

Then

$$[\gamma(-\infty), \gamma(\infty)] = \langle \gamma(-\infty), \gamma(\infty)\rangle \cap \partial D.$$

Proof. From Lemma 2.1 we deduce

$$\left[\xi_{\gamma(-n),\gamma(0)}, \xi_{\gamma(0),\gamma(n)}\right] \cup \left[\xi_{\gamma(n),\gamma(0)}, \xi_{\gamma(0),\gamma(-n)}\right] \subset \partial D \quad \text{for all } n \in \mathbb{N}.$$

Hence, Remark 2.2(b) yields

$$\left[\xi_{\gamma(-\infty),\gamma(0)}, \xi_{\gamma(0),\gamma(\infty)}\right] \cup \left[\xi_{\gamma(\infty),\gamma(0)}, \xi_{\gamma(0),\gamma(-\infty)}\right] \subset \partial D,$$

from which the claim follows, since $\gamma(0) \in D$. □

Gromov products. Let (X, d) be a metric space. We use the standard notation

$$(x \cdot y)_o = \tfrac{1}{2}\left[d(x, o) + d(y, o) - d(x, y)\right],$$

where $x, y, o \in X$. The expression $(x \cdot y)_o$ is called the Gromov product of x and y with respect to the basepoint o. The following theorem will be used to study the boundary of images of isometric embeddings of the Euclidean plane $\mathbb{E}^2$ into Hilbert geometries:

Theorem 2.9 [Karlsson and Noskov 2002]. *Let D be a bounded convex domain. Let* $\{x_n\}_{n\in\mathbb{N}}$, $\{z_n\}_{n\in\mathbb{N}}$ *be two sequences of points in D. Assume that*

$$x_n \to \bar{x} \in \partial D, \quad z_n \to \bar{z} \in \partial D, \quad [\bar{x}, \bar{z}] \not\subset \partial D.$$

Then, for any fixed p_0, there is a constant $K = K(p_0, \bar{x}, \bar{z})$ *such that*

$$\limsup_{n\to\infty} (x_n \cdot z_n)_{p_0} \leq K.$$

For Gromov products in the Euclidean plane, we have:

Lemma 2.10 (Gromov products in $\mathbb{E}^2$). *Let r_1, r_2 be two geodesic rays in $\mathbb{E}^2$, parameterized by arc length, starting at $o \in \mathbb{E}^2$ and forming there an angle* $\alpha = \angle_o(r_1, r_2) \neq \pi$. *Then*

$$\lim_{n\to\infty} \big(r_1(n) \cdot r_2(n)\big)_o = \infty.$$

3. The proof of Theorem 1.2

Notation. Suppose that (D, h_D) is a Hilbert geometry admitting an isometric embedding $\varphi : \mathbb{E}^2 \to (D, h_D)$ of the Euclidean plane.

Fix an origin $o \in \mathbb{E}^2$. Consider the set $\mathcal{R}_o$ of geodesic rays in D that are images (under φ) of geodesic rays in $\mathbb{E}^2$ starting at o. We coordinatize $\mathcal{R}_o$ by the unit sphere $\mathbb{S}^1 \subset \mathbb{C} \cong \mathbb{E}^2$ around o: for $\alpha \in \mathbb{S}^1$ we denote by r_α the image in D of the geodesic ray starting at o and passing through α. We call the point $r_\alpha(\infty)$ in ∂D to which r_α converges the *endpoint* of r_α, and so get the *endpoint map*

$$\mathbb{S}^1 \to \partial D, \quad \alpha \mapsto r_\alpha(\infty),$$

whose image is the set $\mathcal{E}$ of endpoints. The endpoint map is not necessarily continuous — see remarks before the Jump Lemma 2.6 — but by that lemma or its corollary, the map *is* continuous at every $\alpha \in \mathbb{S}^1$ having the property that $r_\alpha(\infty)$ is not contained in the (relative) interior of a boundary flat. Indeed, if the endpoint map is not continuous at $\alpha \in \mathbb{S}^1$, then $r_\alpha(\infty)$ is contained in the (relative) interior of a line segment in ∂D. We may express this also as follows.

Proposition 3.1 (Endpoint Alternative). *If e is an endpoint*,

- *the point e is the limit of a sequence of distinct endpoints*, *or*
- *there is a line segment in ∂D containing e such that e or another endpoint is contained in the relative interior of the segment.*

(These alternatives are not mutually exclusive.)

Proof. Suppose a sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ in $\mathbb{S}^1$ converges to $\alpha \in \mathbb{S}^1$ and that $\{r_{\alpha_n}(\infty)\}$ converges to some $e' := \lim_{n\to\infty} r_{\alpha_n}(\infty) \neq r_\alpha(\infty) =: e$. We show that

$$e \text{ lies in the relative interior of } \langle e, e'\rangle \cap \partial D. \tag{3-1}$$

Indeed, set $c_n := \varphi(o)$ for all $n \in \mathbb{N}$. The geodesic rays r_{α_n} converge pointwise to r_α with $r_\alpha(\infty) = e$, but $\lim_{n\to\infty} = e' \neq e$. By passing to a suitable subsequence of $\{\alpha_n\}_{n\in\mathbb{N}}$ (which, for simplicity, we denote again by $\{\alpha_n\}_{n\in\mathbb{N}}$), we can pick $b_n, a_n \in r_{\alpha_n}$ such that

$$h_D(a_n, c_n) = h_D(a_n, b_n) + h_D(b_n, c_n) \quad \text{for all } n \in \mathbb{N}$$

as well as $\lim_{n\to\infty} b_n = e$ and $\lim_{n\to\infty} a_n = e'$. Now apply part (I) of the Jump Lemma 2.6, with $b = e$ and $a = e'$, to obtain (3-1).

We will apply this fact in different cases. Suppose that $e = r_\alpha(\infty)$ is not the limit of a sequence of distinct endpoints. Then either

(a) $\delta \mapsto r_\delta(\infty)$ is locally constant at α, i.e., there exists $\varepsilon > 0$ such that $r_\beta(\infty) = e$ for all $\beta \in (\alpha - \varepsilon, \alpha + \varepsilon)$, or

(b) there exists a sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} \alpha_n = \alpha$ such that $\{r_{\alpha_n}(\infty)\}_{n\in\mathbb{N}}$ converges to some $e' := \lim_{n\to\infty} r_{\alpha_n}(\infty) \neq e$.

In case (b), we see immediately that the conditions leading to (3-1) are satisfied, and we are done.

Now suppose that (a) holds and let

$$\varepsilon_0 := \sup\{\varepsilon > 0 \mid r_\beta(\infty) = e \text{ for all } \beta \in (\alpha - \varepsilon, \alpha + \varepsilon)\}.$$

From Theorem 2.3(ii) it follows that $r_\beta(\infty)$ is not globally constant, which implies $\varepsilon_0 \leq \pi$.

Let $\alpha' := \alpha \pm \varepsilon_0$ be such that $r_\beta(\infty)$ is not locally constant at α'. Then either

$$r_{\alpha'}(\infty) = e \quad \text{or} \quad r_{\alpha'}(\infty) \neq e.$$

In the first case, since ε_0 was chosen maximal, the choice of α' guarantees that there exists a sequence $\{\alpha'_n\}_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} \alpha'_n = \alpha$ such that $\{r_{\alpha'_n}(\infty)\}_{n\in\mathbb{N}}$ converges to some $e' := \lim_{n\to\infty} r_{\alpha'_n}(\infty) \neq e$. This follows since e is assumed not to be a limit of a sequence of distinct endpoints. Therefore, since $r_{\alpha'}(\infty) = e$, we are essentially back in alternative (b).

If instead $r_{\alpha'}(\infty) \neq e$, since $r_\beta(\infty) = e$ for all $\beta \in (\alpha - \varepsilon_0, \alpha + \varepsilon_0)$, there exists a sequence $\{\alpha'_n\}_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} \alpha'_n = \alpha'$ such that $r_{\alpha'_n}(\infty) = e \neq e' = r_{\alpha'}(\infty)$ for all $n \in \mathbb{N}$. Then again the conditions leading to (3-1) hold, with e and e' interchanged, showing that e' lies in the relative interior of $\langle e, e' \rangle \cap \partial D$. □

Corollary 3.2. *The set $\mathscr{E}$ is not contained in a single affine line. In particular, $\mathscr{E}$ contains more than two points.*

Proof. Suppose for a contradiction that there exists an affine line L such that $\mathscr{E} \subset L \cap \partial D$. Either $l := L \cap \partial D$ consists of exactly two points e and e', or l is a connected affine line segment.

In the first case, Theorem 2.3(ii) yields $\mathscr{E} = \{e, e'\}$. Thus the argument in the proof of the Endpoint Alternative implies that e or e' lie in the relative interior of $\langle e, e' \rangle \cap \partial D$; a contradiction.

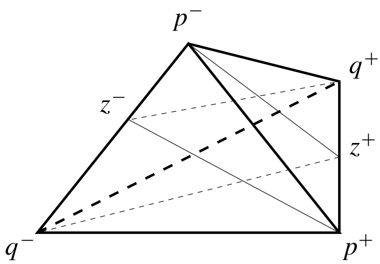
In the second case, when $l = L \cap \partial D$ is a connected affine line segment, let $e, e' \in \partial D$ be such that $l = [e, e']$. Lemma 2.8 implies that $r_\alpha(\infty) \in \{e, e'\}$ for all $\alpha \in \mathbb{S}^1$. Thus $\mathscr{E} \subset \{e, e'\}$, and Theorem 2.3(ii) yields $\mathscr{E} = \{e, e'\}$. Once again the argument in the proof of the Endpoint Alternative implies that e or e' lie in the relative interior of $\langle e, e' \rangle \cap \partial D$: a contradiction. □

Here is an immediate consequence of Theorem 2.9 and Lemma 2.10:

Corollary 3.3. *For $\alpha, \beta \in \mathbb{S}^1$ with $\alpha \neq -\beta$ we have $[r_\alpha(\infty), r_\beta(\infty)] \subset \partial D$.*

We will show that if there is an $\alpha \in \mathbb{S}^1$ with $[r_\alpha(\infty), r_{-\alpha}(\infty)] \not\subset \partial D$, then there is a tetrahedron (i.e. a 3-simplex) bordering D. Corollary 3.3 will be generalized in Lemma 4.2 for three-dimensional Hilbert geometries by showing that if D is the (interior of a) tetrahedron, then the case $[r_\alpha(\infty), r_{-\alpha}(\infty)] \not\subset \partial D$ does not occur.

Lemma 3.4. *Let (D, h_D) be an n-dimensional Hilbert geometry. Suppose there exists a geodesic $\gamma : (-\infty, \infty) \to D$ in $\mathbb{E}^2$ whose image satisfies $[\gamma(-\infty), \gamma(\infty)] \not\subset \partial D$. Then there exists a 3-dimensional affine space Σ in $\mathbb{E}^n$ such that $D \cap \Sigma^3$ is (bordered by) a 3-simplex.*



Proof. We can suppose that $\gamma|_{[0,\infty)} = r_1$, and we set

$$z^+ = r_1(\infty), \quad z^- = r_{-1}(\infty) = \gamma(-\infty).$$

(See the figure above, where the points p^+, p^-, q^+, and q^- are the vertices of a tetrahedron whose surface is contained in ∂D.)

Let $\{\alpha_n\}_n$ be a sequence in $\mathbb{S}^1 \setminus \{1, -1\}$ converging to 1 such that

$$\lim_{n\to\infty} r_{\alpha_n}(\infty) = q^+ \in \partial D \quad \text{and} \quad \lim_{n\to\infty} r_{-\alpha_n}(\infty) = q^- \in \partial D$$

exist. Thanks to Corollary 3.3, we have

$$[z^-, r_{\alpha_n}(\infty)],\ [z^+, r_{-\alpha_n}(\infty)],\ [z^-, r_{-\alpha_n}(\infty)],\ [z^+, r_{\alpha_n}(\infty)] \subset \partial D,$$

for all n, and since by Remark 2.2(b) edges converge to edges, we get

$$[z^-, q^+],\ [z^+, q^-],\ [z^-, q^-],\ [z^+, q^+] \subset \partial D.$$

Hence $q^+, q^- \in \partial D \setminus \{z^+, z^-\}$, since $[z^-, z^+] \not\subset \partial D$ by assumption. Furthermore, Remark 2.2(b) also implies $[q^+, q^-] \subset \partial D$.

Set $p^+ = \xi_{z^+,q^+}$ and $p^- = \xi_{z^-,q^-}$. From Corollary 2.7 it follows that $p^- \neq z^-$ and $p^+ \neq z^+$.

Now suppose $q^- = p^+$; since $[q^+, z^-] \subset \partial D$ as we saw above, Remark 2.2 yields $\Delta(q^+, q^-, p^-) \subset \partial D$, in particular $[z^+, z^-] \subset \partial D$; a contradiction. Thus we have $q^- \neq p^+$ and, similarly, $q^+ \neq p^-$.

From the Jump Lemma 2.6 we get $[z^-, p^+]$, $[z^+, p^-] \subset \partial D$, which implies that $q^+ \neq q^-$, for otherwise $[z^-, z^+] \subset \partial D$, which contradicts the assumption $[\gamma(-\infty), \gamma(\infty)] \not\subset \partial D$. Now Remark 2.2(c) finally implies

$$\Delta(q^+, q^-, p^-), \Delta(q^+, q^-, p^+), \Delta(p^+, p^-, q^+), \Delta(p^+, p^-, q^-) \subset \partial D;$$

that is, the surface of the tetrahedron spanned by q^+, q^-, p^+ and p^- is the part of ∂D contained in the affine space spanned by these four points. $\square$

Proof of Theorem 1.2. Suppose there is an isometric embedding $\varphi : \mathbb{E}^2 \to (D, h_D)$; we will use the notation of page 265.

Lemma 2.8 and the Endpoint Alternative (Proposition 3.1) show there exists $\alpha \in \mathbb{S}^1 \setminus \{1, -1\}$ such that $r_\alpha(\infty) \neq r_1(\infty)$. Therefore, the existence of a line segment in ∂D follows from Corollary 3.3.

To prove that ∂D is not C^1, it obviously suffices to show that there exists an affine plane Σ in $\mathbb{E}^n$ intersecting D and such that there exist distinct points $a, b, c \in \Sigma \cap \partial D$ with $[a, b],\ [b, c] \subset \partial D$ and $a \notin \langle b, c\rangle$.

Suppose first that there exists a geodesic γ in D such that $[\gamma(-\infty), \gamma(\infty)] \subset \partial D$ and which is the image of a geodesic in $\mathbb{E}^2$ under φ. Then

$$\left[\xi_{\gamma(0),\gamma(-\infty)}, \xi_{\gamma(\infty),\gamma(0)}\right] \cup \left[\xi_{\gamma(0),\gamma(\infty)}, \xi_{\gamma(-\infty),\gamma(0)}\right] \subset \partial D.$$

Therefore, $a = \gamma(-\infty)$, $b = \gamma(\infty)$, $c = \xi_{\gamma(0),\gamma(-\infty)}$, and the affine plane Σ spanned by a, b and c are as claimed; note that $\gamma(0) \in \Sigma \cap D$.

Now suppose there is a geodesic $\gamma : (-\infty, \infty) \to D$ with $[\gamma(-\infty), \gamma(\infty)]^\circ \subset D$ that is the image of a geodesic in $\mathbb{E}^2$ under φ. With the notation as in the proof of Lemma 3.4, we set $a = z^-$, $b = q^+$ and $c = z^+$ and consider the affine plane Σ spanned by a, b and c. These points have the desired properties since $[z^+, z^-]^\circ \subset \Sigma \cap D$. $\square$

4. The proof of Theorem 1.1

We continue to use the notation introduced in the last section, and we additionally assume $D \subset \mathbb{E}^3$.

To prove Lemma 4.2, a generalization of Corollary 3.3 in dimension 3 to the case $\alpha = -\beta$, we will use Lemma 4.1.

Lemma 4.1. *Let (D, h_D) be a 3-dimensional Hilbert geometry that admits an isometric embedding $\varphi : \mathbb{E}^2 \to (D, h_D)$. Suppose there exist 2-dimensional boundary faces $F_0, F_1, F_2, \subset \partial D$ such that $F_1 \cap F_0$ and $F_2 \cap F_0$ are one-dimensional boundary flats. Then the (relative) interior of $F_1 \cap F_0$ and $F_2 \cap F_0$ satisfy*

$$\mathscr{E} \cap (F_1 \cap F_0)^\circ = \varnothing \quad \text{or} \quad \mathscr{E} \cap (F_2 \cap F_0)^\circ = \varnothing.$$

Proof. Suppose $\mathscr{E} \cap (F_1 \cap F_0)^\circ \neq \varnothing \neq \mathscr{E} \cap (F_2 \cap F_0)^\circ$. We denote the affine plane containing F_0 by Σ_0. For $t \in (0, dist(\varphi(o), \Sigma_0))$ let Σ_t be the affine plane in Euclidean distance $dist(\Sigma_0, \Sigma_t) = t$ to Σ_0 which intersects D. For t sufficiently small, we can choose $e_t, f_t \in \Sigma_t \cap \varphi(\mathbb{E}^2)$ such that $\xi_{e_t, f_t} \in F_1$ and $\xi_{f_t, e_t} \in F_2$, which by Lemma 2.5 is not possible; this contradiction proves the claim. $\square$

The generalization of Corollary 3.3 in dimension 3 mentioned above reads as follows:

Lemma 4.2. *Let (D, h_D) be a 3-dimensional Hilbert geometry that admits an isometric embedding $\varphi : \mathbb{E}^2 \to (D, h_D)$. Then*

$$\big[r_\alpha(\infty), r_\beta(\infty)\big] \subset \partial D \quad \textit{for all } \alpha, \beta \in \mathbb{S}^1.$$

Slightly more generally: Any two points $e, f \in \overline{\mathscr{E}}$ span a line segment $[e, f] \subset \partial D$ in the boundary of D.

Proof. Due to Corollary 3.3, all we have to prove is that

$$\big[r_\alpha(\infty), r_{-\alpha}(\infty)\big] \subset \partial D \quad \text{for all } \alpha \in \mathbb{S}^1.$$

Suppose for a contradiction that there exists a geodesic through $o \in \mathbb{E}^2$, such that for its image γ under φ we get $[\gamma(-\infty), \gamma(\infty)] \not\subset \partial D$. Then, since D is three-dimensional, it follows from Lemma 3.4, that ∂D is a tetrahedron such that $z^- := \gamma(-\infty)$ and $z^+ := \gamma(\infty)$ are points on the relative interiors of two opposite edges.

Let $\alpha \in \mathbb{S}^1$ be such that $r_\alpha(\infty) = z^+$ and $r_{-\alpha}(\infty) = z^-$. We deduce from Corollary 3.3 that $[r_\beta(\infty), r_\alpha(\infty)] \cup [r_\beta(\infty), r_{-\alpha}(\infty)] \subset \partial D$ for any $\beta \in \mathbb{S}^1 \setminus \{-\alpha, \alpha\}$; this implies

$$\mathscr{E} \subset [q^+, q^-] \cup [q^+, p^-] \cup [p^+, q^-] \cup [p^+, p^-] \cup \{z^-, z^+\},$$

where we use the same notation as in the proof of Lemma 3.4. Thus the Jump Lemma 2.6 implies

$$\mathscr{E} \cap \big([q^+, q^-]^\circ \cup [q^+, p^-]^\circ \cup [p^+, q^-]^\circ \cup [p^+, p^-]^\circ\big) \neq \varnothing.$$

This, however, contradicts Lemma 4.1. To see this, suppose, for instance, that there exists $e \in [q^+, q^-]^\circ \cap \mathscr{E}$. The contradiction to Lemma 4.1 follows in this case by setting for instance $F_0 := \Delta(q^-, q^+, p^+)$, $F_1 := \Delta(p^+, p^-, q^+)$ and $F_2 := \Delta(q^+, q^-, p^-)$. The other cases follow in the same way. □

Being interested in the endpoint set $\mathscr{E}$, we have so far restricted our attention to those geodesic rays r_α in $\varphi(\mathbb{E}^2)$ the endpoints of which define $\mathscr{E}$, that is, the images of rays emanating from o. In the following we will also have to consider other geodesic rays in $\varphi(\mathbb{E}^2)$. The next lemma examines the relation of endpoints $r_\alpha(\infty)$ and those of geodesic rays r in $\varphi(\mathbb{E}^2)$ which are parallel to r_α.

Lemma 4.3. *Let r_α and r be geodesic rays in $\varphi(\mathbb{E}^2)$ which are in finite Hausdorff distance to each other. The following alternatives are possible.*

(1) *If $r_\alpha(\infty)$ is not contained in the interior of an affine line segment $[e, e'] \subset \partial D$, then $r_\alpha(\infty) = r(\infty)$.*

(2) *If $r_\alpha(\infty)$ is contained in the interior of an affine line segment $[e, e'] \subset \partial D$, then $r(\infty) \in [L(e, e') \cap \partial D]^\circ$.*

Proof. Suppose $r_\alpha(\infty) \neq r(\infty)$ and $\xi_{r_\alpha(\infty),r(\infty)} = r_\alpha(\infty)$. For all $\varepsilon > 0$ there exist Euclidean neighborhoods $U^\varepsilon_{r_\alpha}$ of $r_\alpha(\infty)$ and U^ε_r of $r(\infty)$ such that $|x\xi_{xy}| < \varepsilon$ for all $x \in U^\varepsilon_{r_\alpha} \cap D$ and $y \in U^\varepsilon_r \cap D$. The conclusion of the lemma follows from the definition of the Hilbert distance, since $|r(\infty)r_\alpha(\infty)| > 0$ and since the rays are supposed to be in finite Hausdorff distance. □

Lemma 4.4. *Let (D, h_D) be a 3-dimensional Hilbert geometry admitting an isometric embedding $\varphi : \mathbb{E}^2 \to (D, h_D)$ of the Euclidean plane. Suppose there are $\alpha, \beta \in \mathbb{S}^1$ such that the affine space $\Sigma = \langle r_\alpha(\infty), r_\beta(\infty), r_{-\beta}(\infty)\rangle$ induced by $r_\alpha(\infty), r_\beta(\infty)$ and $r_{-\beta}(\infty)$ in $\mathbb{E}^3$ intersects ∂D in a two-dimensional boundary flat $F = \Sigma \cap \partial D$, and assume that $r_\alpha(\infty)$ is not contained in an open line segment of ∂D. Then there exist $u, v \in \partial F$ with*

$$[u, r_\alpha(\infty)] \cup [v, r_\alpha(\infty)] \subset \partial F,$$

such that u, v and $r_\alpha(\infty)$ are affinely independent. Here ∂F denotes the boundary of F in Σ.

Proof. From Lemma 2.8 and our assumptions it follows that $r_\alpha(\infty)$, $r_{-\beta}(\infty)$, and $r_\beta(\infty)$ lie in ∂F.

Suppose that $r_{-\beta}(\infty)$ or $r_\beta(\infty)$ are contained in the interior of affine line segments which are contained in ∂D. Then those line segments are subsets of F, since otherwise, Remark 2.2(c) implies that ∂D is 3-dimensional. Since $r_{-\beta}(\infty)$ and $r_\beta(\infty)$ lie in ∂F, the same remark also yields that such a line segment in F containing $r_{-\beta}(\infty)$ or $r_\beta(\infty)$ in its relative interior has to be contained in ∂F.

(1) First assume that $r_{-\beta}(\infty)$ and $r_\beta(\infty)$ are not contained in the relative interior of line segments in ∂D.

Let γ denote the geodesic in $\varphi(\mathbb{E}^2)$ through $\varphi(o)$ determined by $\gamma(1) = r_\beta(1)$, i.e. $\gamma|_{[0,\infty]} = r_1$ and $\gamma(-t) = r_{-\beta}(t)$ for all $t > 0$. Let further $\gamma_n : (-\infty, \infty) \to D$ for $n \in \mathbb{N}$ be the geodesic in $\varphi(\mathbb{E}^2)$ determined by $|\gamma_n(t)\gamma(t)| = n$ for all $t \in (-\infty, \infty)$ and by $\mathrm{im}(r_\alpha) \cap \mathrm{im}(\gamma_n) \neq \varnothing$.

Then Lemma 4.3 implies $\gamma_n(-\infty) = r_{-\beta}(\infty)$ and $\gamma_n(\infty) = r_\beta(\infty)$ for all $n \in \mathbb{N}$. Set $b_n = \mathrm{im}(r_\alpha) \cap \mathrm{im}(\gamma_n)$; then $r_\alpha(\infty) = \lim_{n\to\infty} b_n$. Finally let Σ_n be the affine plane in $\mathbb{E}^3$ spanned by $r_{-\beta}(\infty)$, b_n and $r_\beta(\infty)$. Then $[\xi_{b_n r_\beta(\infty)}, r_{-\beta}(\infty)] \cup [\xi_{b_n r_{-\beta}(\infty)}, r_\beta(\infty)] \subset \Sigma_n \cap \partial D$. Since the b_n converge to $r_\alpha(\infty)$, it follows that

$$[r_\alpha(\infty), r_{-\beta}(\infty)] \cup [r_\alpha(\infty), r_\beta(\infty)] \subset \partial F,$$

and the claim follows with $u = r_{-\beta}(\infty)$ and $v = r_\beta(\infty)$.

(2) Now assume $r_{-\beta}(\infty)$ or $r_\beta(\infty)$ lie in the interior of an open line segment which is contained in ∂D. Then, as explained above, such a line segment has to be contained in ∂F.

To treat the remaining cases at once, we use the following notation. Let $e, e', f, f' \in \partial F$ be such that $r_{-\beta}(\infty) \in [e, e']^\circ \subset \partial F$ and $r_\beta(\infty) \in [f, f']^\circ \subset \partial F$. In case such points, say e and e', do not exist, we use the conventions $[e, e']^\circ = \{r_{-\beta}(\infty)\}$ and $[L(e, e') \cap \partial D]^\circ = \{r_{-\beta}(\infty)\} = L(e, e') \cap \partial D$.

From Lemma 4.3 it follows that for γ_n as defined in (1) we obtain $\gamma_n(-\infty) \in [L(e, e') \cap \partial D]^\circ$ as well as $\gamma_n(\infty) \in [L(f, f') \cap \partial D]^\circ$ for all $n \in \mathbb{N}$. By compactness of $L(e, e') \cap \partial D$ and $L(f, f') \cap \partial D$ there exists a subsequence of $\{\gamma_n\}_{n\in\mathbb{N}}$, which we again denote by $\{\gamma_n\}_{n\in\mathbb{N}}$, such that the $\gamma_n(-\infty)$ and the $\gamma_n(\infty)$ converge to some $u' \in L(e, e') \cap \partial D$ and some $v' \in L(f, f') \cap \partial D$.

Now let b_n be as in part (1) of the proof and let Σ_n denote the affine plane spanned by $\gamma_n(-\infty)$, b_n and $\gamma_n(\infty)$. Then

$$[\xi_{b_n,\gamma_n(-\infty)}\xi_{\gamma_n(\infty),b_n}] \quad \text{and} \quad [\xi_{b_n,\gamma_n(-\infty)}\xi_{\gamma_n(\infty),b_n}]$$

lie in $\Sigma_n \cap \partial D$.

If $\lim_{n\to\infty} \gamma_n(-\infty) = r_\alpha(\infty)$ or $\lim_{n\to\infty} \gamma_n(\infty) = r_\alpha(\infty)$, we deduce respectively that $[r_\alpha(\infty), r_{-\beta}(\infty)] \subset \partial F$ or $[r_\alpha(\infty), r_\beta(\infty)] \subset \partial F$, and we set $u = r_{-\beta}(\infty)$ or $v = r_\beta(\infty)$. Otherwise $u \neq r_\alpha(\infty) \neq v'$ and with $u := u'$ and $v := v'$ we conclude that

$$[r_\alpha(\infty), u] \cup [r_\alpha(\infty), v] \subset \partial F. \qquad \square$$

Theorem 1.1 follows as a corollary of the following two propositions, which restrict the possibilities for the configuration of the points in $\mathscr{E}$.

Proposition 4.5. *Let (D, h_D) be a 3-dimensional Hilbert geometry admitting an isometric embedding $\varphi : \mathbb{E}^2 \to (D, h_D)$ of the Euclidean plane. Then $\mathscr{E}$ is not contained in a single boundary flat.*

Proof. By Lemma 2.8 and the Endpoint Alternative (Proposition 3.1), the set $\mathscr{E}$ cannot be contained in a 1-dimensional boundary flat.

Now suppose that $\mathscr{E}$ is contained in a 2-dimensional boundary flat F. Let Σ be the affine plane in $\mathbb{E}^3$ containing F, and let ∂F denote the boundary of F in Σ. Then Lemma 2.8 already implies that $\mathscr{E} \subset \partial F$.

Claim. There exist two noncollinear line segments in ∂D each of which contains a point of $\mathscr{E}$ in its relative interior.

Proof. This follows from the Jump Lemma 2.6 and Lemma 4.4, but the argument requires distinguishing several cases. Fix $\beta \in \mathbb{S}^1$.

(1) If $r_{-\beta}(\infty)$ and $r_\beta(\infty)$ are both contained in (necessarily distinct) open line segments of ∂F, there is nothing to prove.

(2) Suppose next that $r_{-\beta}(\infty)$ is contained in an open line segment $[e, e']^\circ = [L(e, e') \cap \partial D]^\circ = [L(e, e') \cap \partial F]^\circ$, but $r_\beta(\infty)$ is not. Then, since e, e' and $r_\beta(\infty)$ are affinely independent, it follows from our analysis of the continuity properties

of $\delta \mapsto r_\delta(\infty)$ that there exists $\alpha \in \mathbb{S}^1$ with $r_\alpha(\infty) \neq r_\beta(\infty)$ and $r_\alpha(\infty) \notin [e, e']$. Without loss of generality we may assume that $r_\alpha(\infty)$ is not contained in an open line segment which is contained in ∂F, since otherwise $r_{-\beta}(\infty)$ and $r_\alpha(\infty)$ are points as desired. Lemma 4.4 then yields the existence of $u, v \in \partial F$ with $[u, r_\alpha(\infty)] \cup [v, r_\alpha(\infty)] \subset \partial F$. Again due to our analysis of the continuity properties of $\delta \mapsto r_\delta(\infty)$, there exists some $p \in ([u, r_\alpha(\infty)]^\circ \cup [v, r_\alpha(\infty)]^\circ) \cap \mathscr{E}$. Since $[u, r_\alpha(\infty)]$ and $[v, r_\alpha(\infty)]$ are not collinear with $[e, e']$ (by our choice of α), we see that p and $r_\alpha(\infty)$ form a tuple of points in $\mathscr{E}$ as desired.

(3) Now assume that neither $r_{-\beta}(\infty)$ nor $r_\beta(\infty)$ are contained in open line segments which are contained in ∂F. By Corollary 3.2 there exists $\alpha \in \mathbb{S}^1$ such that $r_\alpha(\infty)$, $r_{-\beta}(\infty)$ and $r_\beta(\infty)$ span the two-dimensional affine space Σ.

(3a) If $r_\alpha(\infty)$ and $r_{-\alpha}(\infty)$ are both contained in (necessarily distinct) open line segments which are contained in ∂F, again there is nothing to prove.

(3b) Suppose $r_\alpha(\infty)$ is contained in an open affine line segment which is contained in ∂F, but $r_{-\alpha}(\infty)$ is not. Then, by exactly the same arguments as above in (2) with $r_{-\beta}(\infty)$ and $r_\beta(\infty)$ replaced by $r_{-\alpha}(\infty)$ and $r_\alpha(\infty)$, the claim follows.

(3c) Finally suppose that $r_\alpha(\infty)$ is not contained in an open affine line segment contained in ∂F. Then, by Lemma 4.4, $[r_{-\beta}(\infty), r_\alpha(\infty)] \cup [r_\beta(\infty), r_\alpha(\infty)] \subset \partial F$. From our analysis of the continuity properties of $\delta \mapsto r_\delta(\infty)$, it follows that there exists

$$p \in ([r_{-\beta}(\infty), r_\alpha(\infty)]^\circ \cup [r_\beta(\infty), r_\alpha(\infty)]^\circ) \cap \mathscr{E}.$$

Assume $p \in [r_{-\beta}(\infty), r_\alpha(\infty)]^\circ$. If there also exists $q \in [r_\beta(\infty), r_\alpha(\infty)]^\circ \cap \mathscr{E}$, then p and q are points as desired. If not, then, due to the continuity properties of $\delta \mapsto r_\delta(\infty)$, there exists $q \in (\Sigma^- \cap \partial F \cap \mathscr{E}) \setminus \{r_{-\beta}(\infty), r_\beta(\infty)\}$, where Σ^- denotes the closure of the connected component of $\Sigma \setminus L(r_{-\beta}(\infty), r_\beta(\infty))$, which does not contain $r_\alpha(\infty)$.

If such a q is contained in an open line segment which is contained in ∂F, then p and q are points as desired. Otherwise Lemma 4.4 yields $[r_{-\beta}(\infty), q] \cup [r_\beta(\infty), q] \subset \partial F$ and there exists

$$q' \in ([r_{-\beta}(\infty), q]^\circ \cup [r_\beta(\infty), q]^\circ) \cap \mathscr{E}.$$

In this case p and q' are the points as desired, which completes the proof of the claim. $\square$

To complete the proof of Proposition 4.5, we will apply Lemma 4.1 for a contradiction. Pick $\alpha, \beta \in \mathbb{S}^1$ so that $r_\alpha(\infty)$ and $r_\beta(\infty)$ are contained in the relative interior of noncollinear line segments l_α, l_β of ∂F. Then $[\xi_{r_\alpha(n), r_\alpha(0)}, \xi_{r_\alpha(0), r_{-\alpha}(n)}] \subset \partial D$ for all $n \in \mathbb{N}$, and therefore also $[\xi_{r_\alpha(\infty), r_\alpha(0)}, \xi_{r_\alpha(0), r_{-\alpha}(\infty)}] \subset \partial D$. Hence, by convexity, l_α and $[\xi_{r_\alpha(\infty), r_\alpha(0)}, \xi_{r_\alpha(0), r_{-\alpha}(\infty)}] = [r_\alpha(\infty), \xi_{r_\alpha(0), r_{-\alpha}(\infty)}]$ span a boundary flat F_1 of ∂D which intersects F in l_α; see Remark 2.2(c). Similarly, l_β is part of the boundary of another flat F_2 contained in ∂D.

Setting $F_0 := F$, this contradicts Lemma 4.1. Hence, $\mathscr{E}$ is not contained in a 2-dimensional boundary flat. □

Proposition 4.6. *Let (D, h_D) be a 3-dimensional Hilbert geometry admitting an isometric embedding $\varphi : \mathbb{E}^2 \to (D, h_D)$ of the Euclidean plane. Then $\mathscr{E}$ is contained in a single 2-dimensional affine space $\Sigma \subset \mathbb{E}^3$.*

Proof. In order to reach a contradiction, suppose there exist endpoints $a, b, c, d \in \mathscr{E}$ not contained in a common affine plane. Then any three of them are not contained in a common affine line. The four points a, b, c, d span four triangles the edges of which lie in ∂D. In the following we distinguish several cases by the number of those triangles, which are contained in boundary flats, i.e. the interiors of which are not contained in D. Note that the border of such a triangle belongs to ∂D and that such a triangle is either contained in ∂D, or its intersection with ∂D is the boundary of the triangle; see Remark 2.2(c).

Case 0: *All four triangles are boundary flats.* In this case, D is the interior of the tetrahedron with vertices a, b, c, and d. The Endpoint Alternative yields that at least two of the edges carry a point of $\mathscr{E}$ in their relative interiors, while Lemma 4.2 implies that those edges belong to the boundary of the same boundary flat. This, however, is not possible, due to Lemma 4.1.

Case 1: *Exactly three triangles are boundary flats.* Assume that the triangles $\Delta(d, a, b)$, $\Delta(d, a, c)$ and $\Delta(d, b, c)$ are boundary flats, whereas $\Delta^\circ(a, b, c)$ is contained in D. Let $\mathscr{C}_{\mathrm{hull}}(a, b, c, d)$ be the (Euclidean) convex hull of a, b, c and d. Then its interior $\mathscr{C}^\circ_{\mathrm{hull}}(a, b, c, d)$ satisfies $\mathscr{C}^\circ_{\mathrm{hull}}(a, b, c, d) \subset D$.

Now consider the points $\xi_{a,d}$, $\xi_{b,d}$ and $\xi_{c,d}$. At least two of them must correspond to the points a, b and c, since otherwise we get a contradiction due to Lemma 2.5. So assume that $\xi_{b,d} = b$ and $\xi_{c,d} = c$. If also $\xi_{a,d} = a$, there exists

$$e \in \mathscr{E} \cap \{[a, d]^\circ \cup [b, d]^\circ \cup [c, d]^\circ\},$$

as can be seen by combining the inclusion $\mathscr{C}^\circ_{\mathrm{hull}}(a, b, c, d) \subset D$ with Lemma 4.2, since $\delta \mapsto r_\delta(\infty)$ has to leave d somehow. Suppose that $e \in [a, d]$, say; then we can just replace a by e and obtain $e \neq \xi_{e,d}$. Thus we can assume without loss of generality that $a \neq \xi_{a,d}$.

Again from Lemmas 4.2 and 2.5 we deduce that $\mathscr{E} \cap \langle b, c, d\rangle \cap \partial D = \{b, c, d\}$. Also, since $\mathscr{C}^\circ_{\mathrm{hull}}(\xi_{a,d}, b, c, d) \subset D$, it follows that $\mathscr{E} \subset (\langle d, a, b\rangle \cup \langle d, a, c\rangle) \cap \partial D$.

Suppose next that there exists $e \in ([\xi_{a,d}, b]^\circ \cup [\xi_{a,d}, c]^\circ) \cap \mathscr{E}$. Then $\Delta(\xi_{a,d}, b, c) \subset \partial D$ and we reach a contradiction with Lemma 2.5, since $a \in [\xi_{a,d}, d]^\circ$.

Since $\delta \mapsto r_\delta(\infty)$ has to leave both b and c somehow, the Endpoint Alternative (Proposition 3.1) yields $\partial F^-(\xi_{a,d}, b, d) \cap \mathscr{E} \neq \varnothing$ as well as $\partial F^-(\xi_{a,d}, c, d) \cap \mathscr{E} \neq \varnothing$, where, for instance, $F^-(\xi_{a,d}, b, d)$ denotes the intersection of the boundary flat $\langle \xi_{a,d}, b, d\rangle \cap \partial D$ with the open half-plane in $\langle \xi_{a,d}, b, d\rangle \setminus \langle \xi_{a,d}, b\rangle$ not containing d, and $\partial F^-(\xi_{a,d}, b, d)$ denotes its boundary in $\langle \xi_{a,d}, b, d\rangle \setminus \langle \xi_{a,d}, b\rangle$.

Set $\partial[F_b]^- := \partial F^-(\xi_{a,d}, b, d)$ and $\partial[F_c]^- := \partial F^-(\xi_{a,d}, c, d)$. We claim that

$$\bigcup_{q\in[\partial F_c]^+} [b, q] \subset \partial D \quad \text{and} \quad \bigcup_{q\in[\partial F_c]^-} [b, q] \subset \partial D. \tag{4-1}$$

Indeed, $\delta \mapsto r_\delta(\infty)$ has to move from c to b somehow. If $\delta \mapsto r_\delta(\infty)$ is continuous, the claim is clear from Lemma 4.2. The general case follows, since whenever $\delta \mapsto r_\delta(\infty)$ is not continuous, there exists $e \in [\partial F_b]^- \cap \mathscr{E}$ (or $e \in [\partial F_c]^- \cap \mathscr{E}$) in the relative interior of a line segment in $[\partial F_b]^-$ (or $[\partial F_b]^-$, respectively), and then Lemma 4.2 and Remark 2.2(c) lead to $[b, q] \subset \partial D$ (or $[b, q] \subset \partial D$) for all q contained in such a line segment.

There exist $e \in \mathscr{E} \cap [\partial F_b]^-$ and $e' \in \mathscr{E} \cap [\partial F_c]^-$. From Lemma 4.2 we find that $[e, e'] \subset \partial D$; hence, since $[\partial F_b]^- \cap [\xi_{ad}, b] = \varnothing$ and $[\partial F_b]^- \cap [\xi_{ad}, b] = \varnothing$, the segment $[e, e']$ is contained in the open half-space of $\mathbb{E}^3 \setminus \langle \xi_{a,d}, b, c\rangle$ that does not contain d.

Now take a sequence $\{q_n\}_{n\in\mathbb{N}}$ in $[\partial F_b]^-$ converging to ξ_{ad}. Then $[b, q_n]$ lies in ∂D for all $n \in \mathbb{N}$, and for n sufficiently large and $p \in [q_n, b]^\circ$ the straight line $\langle e, p\rangle$ intersects $\Delta^\circ(\xi_{a,d}, b, c)$. We reach a contradiction, since $e \in \partial D$ and $e \neq p \in \partial D$ imply that for the 3-dimensional $\mathscr{C}^\circ_{\text{hull}}(\xi_{a,d}, b, c, e)$ we obtain $\mathscr{C}^\circ_{\text{hull}}(\xi_{a,d}, b, c, e) \subset \partial D$ due to Remark 2.2(c).

Note that this argument also works if $d \in \partial D \setminus \mathscr{E}$ and there exists some $e \in \mathscr{E}$ in the relative interior of one of the edges of the closed triangles meeting in d.

<u>Case 2:</u> *Exactly two triangles are boundary flats.* We assume without loss of generality that $\Delta(a, b, d), \Delta(a, c, d) \subset \partial D$. We first fix some notation: $\Sigma(a, b, c)$ will denote the affine plane in $\mathbb{E}^3$ spanned by a, b and c, and $H^-(a, b, c; d)$, $H^+(a, b, c; d)$ will denote the open half-spaces of $\mathbb{E}^3 \setminus \Sigma(a, b, c)$ characterized by $d \notin H^-(a, b, c; d)$ and $d \in H^+(a, b, c; d)$. Also define the boundary flats

$$F_b = \Sigma(a, b, d) \cap \partial D, \qquad F_c = \Sigma(a, c, d) \cap \partial D,$$

and let ∂F_b and ∂F_c be their boundaries in $\Sigma(a, b, d)$ and $\Sigma(a, c, d)$, respectively.

We now seek a contradiction with the existence of an isometric embedding of $\mathbb{E}^2$ into D.

(1) We first verify that $\mathscr{E} \subset \partial F_b \cup \partial F_c$.

Suppose there exists $p \in \mathscr{E} \cap (L(a, d) \cap \partial D)^\circ$. Since ∂D is 2-dimensional, Lemma 4.2 and Remark 2.2(c) yield $\mathscr{E} \cap \partial D \subset F_b \cup F_c$. Thus $\mathscr{E} \subset \partial F_b \cup \partial F_c$, since for any $p \in \mathscr{E} \cap F_b^\circ$ (or $p \in \mathscr{E} \cap F_c^\circ$), the condition $[c, p] \subset \partial D$ (or $[b, p] \subset \partial D$) contradicts the 2-dimensionality of ∂D, due to Remark 2.2(c).

Thus we may assume that $L(a,d)\cap\partial D=[a,d]$ and $[a,d]^\circ\cap\mathscr{E}=\varnothing$. From Remark 2.2(c) it follows that $C^\circ_{\text{hull}}(a,b,c,d)\cap\partial D$ is nonempty. Therefore, since $\Delta(a,b,d)\cup\Delta(a,c,d)\subset\partial D$, we conclude that

$$\partial D\cap\big(H^-(a,c,d;b)\cup H^-(a,b,d;c)\big)=\varnothing. \tag{4-2}$$

Moreover, we claim that

$$\mathscr{E}\cap H^+(a,c,d;b)\cap H^+(a,b,d;c)=\varnothing. \tag{4-3}$$

To verify this, we set $Q:=H^+(a,c,d;b)\cap H^+(a,b,d;c)$, $Q_0:=\Sigma(b,c,d)\cap Q$, $Q_+:=Q\cap H^+(b,c,d;a)$ and $Q_-:=Q\cap H^-(b,c,d;a)$. Then $Q=Q_0\cup Q_+\cup Q_-$. Now $Q_0\cap\mathscr{E}=\varnothing$, since otherwise $\Delta(b,c,d)\subset\partial D$. We also deduce $Q_+\cap\mathscr{E}=\varnothing$; indeed, if this intersection contained a point p, we would have $[p,d]\subset\partial D$ from Lemma 4.2, but since $p\notin\mathscr{C}_{\text{hull}}(a,b,c,d)$, this would imply that $[p,d]$ and $\Delta^\circ(a,b,c)$ are disjoint, contradicting $\Delta^\circ(a,b,c)\subset D$. Finally, $Q_-\cap\mathscr{E}=\varnothing$, since for $p\in Q_-\cap\mathscr{E}$, either $[p,a]\cap\Delta^\circ(b,c,d)\neq\varnothing$ and therefore $\Delta^\circ(b,c,d)\subset\partial D$, or $[p,d]\cap\Sigma(a,b,c)\cap Q_-\neq\varnothing$ and therefore $\Delta^\circ(a,b,c)\subset\partial D$. This proves our claim (4-3).

From (4-2) and (4-3) it follows $\mathscr{E}\subset F_b\cup F_c$, and just as above we deduce that $\mathscr{E}\subset\partial F_b\cup\partial F_c$.

(2) We will reach the desired contradiction by proving that $\Delta(\xi_{a,d},b,c)\subset\partial D$ or $\Delta(\xi_{d,a},b,c)\subset\partial D$; indeed, by our remark at the end of Case 1, either of these two inclusions precludes the existence of an isometric embedding $\varphi:\mathbb{E}^2\to D$.

Fix $p\in[\xi_{a,d},\xi_{d,a}]^\circ$. Define Σ^+, Σ^- as the components of $\Sigma(a,c,d)\setminus L(p,c)$ containing respectively $\xi_{a,d}$, $\xi_{d,a}$, and set $[\partial F_c]^+:=\big((\partial F_c\cap\Sigma^+)\setminus[\xi_{a,d},p]^\circ\big)\cup\{c\}$ and likewise for $[\partial F_c]^-$. Thus $[\partial F_c]^+$ is the piece of ∂F_c connecting $\xi_{a,d}$ to c in $\Sigma^+\cup\{c\}$. Define $[\partial F_b]^+$ and $[\partial F_b]^-$ similarly, replacing c by b. Then

$$\mathscr{E}\subset\partial F_b\cup\partial F_c=[\partial F_c]^+\cup[\partial F_c]^-\cup[\partial F_b]^+\cup[\partial F_b]^-\cup[\xi_{a,d},\xi_{d,a}].$$

An argument analogous to the one following (4-1) shows that

$$\bigcup_{q\in[\partial F_c]^+}[b,q]\subset\partial D\quad\text{or}\quad\bigcup_{q\in[\partial F_c]^-}[b,q]\subset\partial D.$$

Similarly,

$$\text{(a)}\ \bigcup_{q\in[\partial F_b]^+}[c,q]\subset\partial D\quad\text{or}\quad\text{(b)}\ \bigcup_{q\in[\partial F_b]^-}[c,q]\subset\partial D, \tag{4-4}$$

depending on whether $\delta\mapsto r_\delta(\infty)$ moves along $[\partial F_b]^+$ or $[\partial F_b]^-$. Without loss of generality we may assume that $\delta\mapsto r_\delta(\infty)$ moves along $[\partial F_c]^+$, that is to say, $\bigcup_{q\in[\partial F_c]^+}[b,q]\subset\partial D$. We have to consider cases (a) and (b) of (4-4).

(a) If there exists $q \in [\partial F_c]^+$ with $[\xi_{a,d}, q] \subset [\partial F_c]^+$, then, since $\delta \mapsto r_\delta(\infty)$ moves along $[\partial F_c]^+$ there exists $c' \in \mathscr{E} \cap [\xi_{a,d}, \xi_{q,\xi_{a,d}}]^\circ$. Replacing c by c', we obtain $\Delta(b, c', \xi_{a,d}) \subset \partial D$ by Remark 2.2(c), and we are done by our remark at the end of Case 1. Thus there exists a sequence $\{e_n\}_{n\in\mathbb{N}}$ of endpoints $e_n \in (\mathscr{E} \cap [\partial F_c]^+) \setminus \{\xi_{a,d}\}$ converging to $\xi_{a,d}$. Moreover, $[\partial F_c]^+ \cap [\xi_{a,d}, c]^\circ = \varnothing$. Since the $[c, e_n]^\circ$ converge to $[c, \xi_{a,d}]^\circ$ in $H^-(b, c, \xi_{a,d}; \xi_{d,a})$, it follows that, for n sufficiently large, there exist straight lines in $\mathbb{E}^3$ through distinct points $p \in [\partial F_c]^+ \subset \partial D$ and $q \in [c, e_n]^\circ \subset \partial D$, which intersect $\Delta^\circ(b, c, \xi_{a,d})$. Therefore, these intersection points do not belong to D and $\Delta^\circ(b, c, \xi_{a,d}) \subset \partial D$, due to Remark 2.2(c). Thus we are also done in this case.

(b) In this case there exists an endpoint $e \in \mathscr{E} \cap [\partial F_b]^- \setminus \{b, \xi_{a,d}\}$. For this endpoint we have $e \in \mathscr{E} \cap \Sigma(b, \xi_{a,d}, \xi_{d,a}) \cap H^-(b, c, \xi_{d,a}; \xi_{a,d})$, for otherwise $e \in [b, \xi_{a,d}]^\circ$ and therefore $\Delta^\circ(b, c, \xi_{a,d}) \subset \partial D$. Now just as for b, we also get for e the inclusion $\bigcup_{q\in[\partial F_c]^+}[e, q] \subset \partial D$. From the choice of e we further obtain $[e, \xi_{a,d}] \cap \Sigma(b, c, \xi_{d,a}) \in [b, \xi_{d,a}]^\circ$ as well as $[e, c] \cap \Sigma(b, c, \xi_{d,a}) = \{c\}$. Moving from $\xi_{a,d}$ to c along $[\partial F_c]^+$, the intersection of $[e, f]$ with $\Sigma(b, c, \xi_{d,a})$ moves continuously with f in $\Sigma(b, c, \xi_{d,a})$. But since $f \in [\partial F_c]^+$ implies $[e, f] \cap \Sigma(b, c, \xi_{d,a}) \notin \{b, \xi_{d,a}\}$, this point of intersection has to leave $[b, \xi_{a,d}]$ continuously in its relative interior. Since all these intersection points belong to ∂D, we deduce $\Delta^\circ(b, c, \xi_{d,a}) \subset \partial D$, which completes Case 2.

Case 3: *At most one of the triangles is a boundary flat.* We assume without loss of generality that none of the triangles with vertex d is a boundary flat. ($\Delta(a, b, c)$ might be a boundary flat or not.) Let $H = H^+(a, b, c; d)$ be the connected component of $\mathbb{E}^3 \setminus \Sigma(a, b, c)$ containing d.

(1) We first prove that $H \cap \mathscr{E} = \{d\}$.

In order to reach a contradiction, suppose that there exists $d' \in H \cap \mathscr{E}$ with $d' \neq d$. Since none of the three triangles with vertex d is a boundary flat, we deduce that $d' \notin \mathscr{C}_{\text{hull}}(a, b, c, d)$, the Euclidean convex hull of the points a, b, c, and d. In particular, d' is separated from one of the points a, b and c by the plane spanned by the two others and d.

Assume without loss of generality that d' is separated from a by $\Sigma(b, c, d)$. Then $[a, d']$ intersects $\Sigma(b, c, d)$ in one of the sides $[c, d]$ or $[d, b]$, since $\Delta(b, c, d)$ is not a boundary flat, and $[a, d'] \subset \partial D$, due to Lemma 4.2.

Without loss of generality, assume $[b, d] \cap [a, d'] \neq \varnothing$. Then $\Delta(a, b, d)$ is a boundary flat, by Remark 2.2(c); a contradiction.

(2) From $H \cap \mathscr{E} = \{d\}$ it follows by the Endpoint Alternative that the endpoint map jumps from d to some $f \in \mathbb{E}^3 \setminus H$. Let f' be the single point in $[d, f] \cap \langle a, b, c \rangle$. Due to Lemma 4.2 we have $[d, f] \subset \partial D$, and since none of the triangles with

vertex d is contained in a boundary flat, f' is separated in $\Sigma(a, b, c)$ from one of the points a, b and c by the line through the other two points.

Assume without loss of generality that f' and a are on different sides of $L(b, c)$ in $\Sigma(a, b, c)$. Since the endpoint map jumps from d to f, the claim in our proof of the Endpoint Alternative yields $f \in [d, \xi_{f,d}]^\circ$. Hence it follows from Remark 2.2 that $\Delta(a, \xi_{f,d}, d) \subset \partial D$.

Now $l = \Delta(a, d, \xi_{f,d}) \cap \Sigma(b, c, d) \subset \partial D$ is a line segment with vertex d. But, since $\Delta(b, c, d)$ is not a boundary flat, we have

$$\Sigma(b, c, d) \cap \partial D = [b, c] \cup [c, d] \cup [b, d],$$

and it follows that $[d, b] \subset l$ or $[c, d] \subset l$, contradicting the fact that neither $\Delta(a, b, d)$ nor $\Delta(a, c, d)$ are boundary flats. □

Proof of Theorem 1.1. From Proposition 4.5 and Proposition 4.6 it follows that all we have to show is that $\mathscr{E}$ is not contained in a two-dimensional plane which intersects D, i.e. that $\mathscr{E}$ does not consist of exactly three points a, b, and c with $\Delta^\circ(a, b, c) \subset D$, where $\Delta^\circ(a, b, c) = \Delta(a, b, c) \setminus ([a, b] \cup [a, c] \cup [b, c])$ is the relative interior of $\Delta(a, b, c)$. This, however, follows from our claim in the proof of the Endpoint Alternative, since none of the three points lies in the interior of an open line segment in ∂D contained in the line spanned by this and any of the other endpoints. □

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