Pacific Journal of Mathematics

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Volume 231 No. 2 June 2007

PACIFIC JOURNAL OF MATHEMATICS
Vol. 231, No. 2, 2007
dx.doi.org/10.2140/pjm.2007.231.445

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Let V^n be a compact, oriented Riemannian manifold and S^n the standard sphere. We study the problem of obtaining upper bounds for the dilatation invariants of maps $V^n \to S^n$ of nonzero degree. The dilatation upper bounds are then used to estimate Gromov's filling volume from below.

1. Introduction

Let V be a compact, connected, oriented n-manifold. The famous Hopf theorem says that there is an one-to-one correspondence between the degree and the homotopy class of continuous maps from V to S^n. If V is equipped with a Riemannian metric and S^n is equipped the canonical metric (of sectional curvature $+1$), we wish to measure the geometrical complexity of a map $V \to S^n$ of degree q. A natural measure of the geometrical complexity $f : X \to Y$ of a map between metric spaces is its dilatation, defined by

$$\operatorname{dil} f = \sup_{\substack{x,x' \in X \\ x \neq x'}} \frac{d_Y(f(x), f(x'))}{d_X(x, x')},$$

where d_X, d_Y are the respective metrics. If f is a Lipschitz map, $\operatorname{dil} f$ is the minimal Lipschitz constant C satisfying $d_Y(f(x), f(x')) \leq C d_X(x, x')$. If $\operatorname{Dil}_q V$ denotes the infimum of the dilatation of maps $V \to S^n$ with degree $\pm q$, an upper bound for $\operatorname{Dil}_q V$ can be regarded as a quantitative version of the Hopf theorem. For V a flat torus, M. Gromov proved such a bound in terms of the length $\operatorname{sys}_1 V$ of the systole (shortest noncontractible closed curve) in V.

Theorem 1.1 [Gromov 1999b, 2.12, p. 33]. *Let $\mathbb{T}^n$ be a flat torus. There exists a map $f : \mathbb{T}^n \to (S^n, \text{can})$ with nonzero degree and dilatation at most* 1 *if and only if* $\operatorname{sys}_1 \mathbb{T}^n \geq 2\pi$.

It has been Gromov's constant concern to study quantitative problems in algebraic topology and differential topology [1978; 1999a; 1999b, Chapters 2 and 7]).

MSC2000: primary 53C23, 53C20; secondary .
Keywords: dilatation, degree, filling volume.

In this direction, A. Nabutovsky and R. Rotman [2003] have studied a quantitative Hurewicz theorem. For estimates of dilatation invariants of maps between spheres, see [Peng and Tang 2002] and references therein.

Here is a slight generalization of a problem posed by Gromov after the theorem just quoted.

Problem [Gromov 1999b, 2.13, p. 35]. *Let V be a compact, connected, oriented n-manifold. What condition on the metric of V guarantees that there exist mappings $f : V \to S^n$ with degree q and dilatation 1? In other words, given a metric on V, for which values of q do there exist mappings $V \to S^n$ with degree q and dilatation less than some constant D?*

L. Guth [2005] has obtained important results concerning the dilatation of a nonzero degree map from a Riemannian surface to the standard 2-sphere. Following Guth, we denote by $\mathrm{HS}(V)$ the hypersphericity of a Riemannian n-manifold, defined as the maximal R such that there is a contracting map ($\mathrm{dil} \le 1$) of nonzero degree from V to the n-sphere of radius R. Replacing nonzero degree by degree q, one defines the degree-q hypersphericity $\mathrm{HS}_q(V)$ of a Riemannian manifold. It is clear that

$$\mathrm{HS}(V) \ge \mathrm{HS}_q(V) = (\mathrm{Dil}_q\ V)^{-1} \quad \text{for } q \neq 0.$$

In this paper we prove several results concerning dilatation. Our method is to construct directly some maps $V \to S^n$ of nonzero degree, and then estimate their dilatation using (geo)metric invariants of V.

Recall that the q-th packing radius, $\mathrm{pack}_q(X)$, of a compact metric space X in the sense of K. Grove and S. Markvorsen [1995] is the largest $r \ge 0$ for which X contains q disjoint open r-balls. It is clear that

$$\tfrac{1}{2} \operatorname{diam} X = \mathrm{pack}_2 X \ge \cdots \ge \mathrm{pack}_q X \ge \cdots \to 0.$$

Theorem A. (Proved in Section 2.) *Let V be a compact, oriented Riemannian n-manifold and let K be the supremum of the sectional curvature of V.*

(1) *In the case $K > 0$, if q is large enough to satisfy*

$$\mathrm{pack}_q V < \min\{\pi/\sqrt{K},\ \mathrm{Inj}\, V\},$$

there exists a map $f : V \to S^n$ of degree q such that

$$\mathrm{dil}\, f \le \frac{\pi\sqrt{K}}{\sin\bigl(\sqrt{K}\ \mathrm{pack}_q V\bigr)}.$$

Hence

$$\mathrm{HS}_q(V) \ge \sin\bigl(\sqrt{K}\ \mathrm{pack}_q V\bigr)/(\pi\sqrt{K}).$$

(2) *In the case* $K \le 0$, *if* q *is large enough to satisfy*

$$\operatorname{pack}_q V \le \operatorname{Inj} V,$$

there exists a map $f : V \to S^n$ *of degree* q *and*

$$\operatorname{dil} f \le \frac{\pi}{\operatorname{pack}_q V}.$$

Hence $\operatorname{HS}_q(V) \ge (\operatorname{pack}_q V)/\pi$.

Here $\operatorname{Inj} V$, the injectivity radius of V, is the infimum of the injectivity radius at any point in V. For a flat torus $\mathbb{T}^n$ we have $\operatorname{Inj} \mathbb{T}^n = \frac{1}{2} \operatorname{sys}_1 \mathbb{T}^n$.

We denote by $\operatorname{Inj}_{\max} V = \max\{\operatorname{Inj}_x V : x \in V\}$ the largest injectivity radius of a compact Riemannian manifold V. The following corollary may be regarded as a slight generalization of the "if" part of Theorem 1.1.

Corollary 1.2. *Let* V *be a compact Riemannian* n*-manifold.*

(1) *If* $\sup \sec(V) \le 1/\pi^2$ *and* $\operatorname{Inj}_{\max} V \ge \frac{1}{2}\pi^2$, *there is a map* $f : V \to S^n$ *with degree* 1 *and dilatation* 1.

(2) *If* $\sup \sec(V) \le 0$ *and* $\operatorname{Inj}_{\max} V \ge \pi$, *there is a map* $f : V \to S^n$ *with degree* 1 *and dilatation* 1.

By deepening the results of D. Burago and S. Ivanov [1995], Gromov [1999b, p. 259] proved a sharp inequality relation between $\operatorname{Vol}(V)$ and $\operatorname{stsys}_1 V$ (the stable 1-systole of V — see page 452) under certain topological restrictions on V. In [Ivanov and Katz 2004; Katz and Lescop 2005], this inequality is stated as follows:

Theorem 1.3 (Burago–Ivanov–Gromov Inequality). *Assume that a compact, oriented Riemannian manifold* V *satisfies* $\dim(V) =$ *first Betti number* $b_1(V) =$ *real cuplength of* $V = n$. *Then*

$$\operatorname{Vol}(V) \ge (\gamma_n)^{-n/2} (\operatorname{stsys}_1 V)^n$$

where γ_n *denotes the classical Hermite constant.*

Theorem 1.3 and [Ivanov and Katz 2004, Lemma 7.3] lead us to:

Theorem B. *Assume that a compact, oriented Riemannian manifold* V *satisfies* $dim(V) = b_1(V) =$ *real cuplength of* $V = n$. *Then there exists a map* $f : V \to S^n$ *of nonzero degree such that*

$$\operatorname{dil} f \le \frac{2\pi\sqrt{n}}{\operatorname{stsys}_1 V}.$$

Hence $\operatorname{HS}(V) \ge (\operatorname{stsys}_1 V)/(2\pi\sqrt{n})$.

The proof of Theorem B, which we give in Section 3, makes heavy use of the construction in [Ivanov and Katz 2004, Section 7], itself based on the techniques of [Burago and Ivanov 1995].

An immediate application of the quantitative Hopf theorem is that we can estimate volume from below by the formula

$$\mathrm{Vol}(V) \geq \frac{|\deg f|}{(\mathrm{dil}\, f)^n} \mathrm{Vol}(S^n, \mathrm{can}).$$

However, in general, using estimates for the Jacobian of a Lipschitz map, one can obtain better lower volume bounds [Ivanov and Katz 2004; Gromov 1999b, pp. 255, 257].

A second goal of this paper is to use quantitative Hopf theorem to estimate the filling volume of a Riemannian manifold. We believe the quantitative versions of the Hopf theorem to be of independent interest. For example, we have used them to estimate lower bounds for the filling radius of a Riemannian manifold in [Liu 2005].

Let V be a compact n-manifold equipped with a distance d (not necessarily Riemannian), and let $L^\infty(V)$ be the Banach space of bounded Borel functions f on V with the norm $\|f\| = \sup_x |f(x)|$. The map $i : V \to L^\infty(V)$ defined by $x \mapsto f_x(\cdot) = \mathrm{dist}(x, \cdot)$ is an isometric (distance-preserving) embedding. The filling volume $\mathrm{FillVol}(V)$ of (V, d) in the Gromov's sense is, roughly speaking, the infimum of the volumes of $(n+1)$-dimensional submanifolds in $L^\infty(V)$ whose boundary is $i(V)$. For details, see [Gromov 1983] or Section 4 below.

Our main tool to estimate filling volume from below is the following mapping property:

Theorem C. (Proved in Section 5.) *Let V be a compact, oriented n-manifold with a metric d and let $f : V \to S^n$ a map of nonzero degree. Then*

$$\mathrm{FillVol}(V) > \frac{|\deg f|}{(\mathrm{dil}\, f)^{n+1}} \left(\arccos \frac{n-1}{n+1}\right)^{n+1}.$$

In the definition of hypersphericity, it is not necessary that V be Riemannian, merely that V have a metric d. Hence an equivalent statement of Theorem C is that for any compact, oriented n-manifold V with a metric d,

$$\mathrm{FillVol}(V, d) > \left(\arccos \frac{n-1}{n+1}\right)^{n+1} |q|\ \mathrm{HS}(V, d)^{n+1}.$$

Define

$$c_n = \left(\frac{1}{2\pi} \arccos \frac{n-1}{n+1}\right)^{n+1}.$$

From Theorem C we derive corollaries of Theorems 1.1, A and B, respectively:

Corollary 1.4. *Let $\mathbb{T}^n$ be a flat torus. Then*

$$\mathrm{FillVol}(\mathbb{T}^n) > c_n\,(\mathrm{sys}_1\,\mathbb{T}^n)^{n+1}.$$

Corollary 1.5. *Let V be a compact, oriented Riemannian n-manifold.*

(1) *If* $K = \sup \sec(V) > 0$ *and* $\mathrm{pack}_q V < \min\{\pi/\sqrt{K}, \mathrm{Inj}\, V\}$, *then*

$$\mathrm{FillVol}(V) > q\left(\frac{\sin\bigl(\sqrt{K}\,\mathrm{pack}_q V\bigr)}{\pi\sqrt{K}}\ \arccos\frac{n-1}{n+1}\right)^{n+1}.$$

(2) *If* $K = \sup \sec(V) \le 0$ *and* $\mathrm{pack}_q V \le \mathrm{Inj}\, V$, *then*

$$\mathrm{FillVol}(V) > 2^{n+1} c_n\, q\,(\mathrm{pack}_q V)^{n+1}.$$

Corollary 1.6. *Let V be a compact Riemannian manifold of equal dimension, first Betti number and real cuplength. Then*

$$\mathrm{FillVol}(V) > n^{-(n+1)/2} c_n\,(\mathrm{stsys}_1 V)^{n+1}.$$

The following statements are alternative forms of Corollaries 1.4 and 1.6, relating the volumes of Riemannian manifolds with boundary to the boundary metric invariants. Their proofs appear in Section 4. (We omit a similar Corollary 1.5’.)

Corollary 1.4′. *Let W be a $(n+1)$-dimensional solid torus with boundary $\partial W = \mathbb{T}^n$. Given a flat metric g_0 on $\mathbb{T}^n$, g is any Riemannian metric on W which satisfies $d_g|_{\partial W} \ge d_{g_0}$, then*

$$\mathrm{Vol}(W, g) > c_n \cdot \mathrm{sys}_1(\mathbb{T}^n, g_0)^{n+1}.$$

Corollary 1.6′. *Assume that the topological restrictions of a closed n-manifold V are as Corollary 1.6, W is a $(n+1)$-manifold with boundary $\partial W = V$. Given a Riemannian metric g_0 on V, for any Riemannian metric g on W satisfying $d_g|_{\partial W} \ge d_{g_0}$, we have*

$$\mathrm{Vol}(W, g) \ge n^{-(n+1)/2} c_n \cdot \mathrm{stsys}_1(V, g_0)^{n+1}.$$

The constants in these corollaries are not sharp. We do not look for anything like the optimal constants in all estimate inequalities of this paper; but these inequalities ensure the upper bounds of dilatation invariants, or the lower bounds of filling volume, in terms of packing radius, stable 1-systole, etc.

2. Dilatation estimates with upper bounds of sectional curvature

The proof of Theorem A is a direct extension of the arguments used to prove Theorem 1.1 and [Liu 2005, Propositions 3.2 and 3.3]. For completeness, we will give a detailed proof of Theorem A in this section.

Let $f : X \to Y$ be a map between metric spaces. The local dilatation at $x \in X$ is the number

$$\operatorname{dil}_x f = \lim_{\varepsilon \to 0} (\operatorname{dil}(f|_{B(x,\varepsilon)})),$$

where $B(x, \varepsilon)$ is an open ball in X of radius ε centered at x. If (X, d) is a path metric space (for example, a Riemannian manifold), then $\operatorname{dil} f = \sup_{x \in X} \operatorname{dil}_x f$. If X and Y are Riemannian manifolds, and f is differentiable, then $\operatorname{dil}_x f = \|df_x\|$, where $df_x : T_x X \to T_{f(x)} Y$ is the differential of f at x.

Proof of Theorem A. (1) Assume $K = \sup \sec(V) > 0$. Let $B(x_1, r), \dots, B(x_q, r)$ be disjoint r-balls of V centered at the points $x_1, \dots, x_q$, where

$$r = \operatorname{pack}_q V < \min\{\pi/\sqrt{K}, \operatorname{Inj} V\}.$$

Denote by $B_T(x_i, r)$ the ball of radius r *in the tangent space* $T_{x_i} V$, centered at the origin. Since $r < \pi/\sqrt{K}$, the map $\exp_{x_i} |B_T(x_i, r)$ is nonsingular for each i; see the remark after 1.29 in [Cheeger and Ebin 1975]. Using the geometric Rauch theorem [Chavel 1993, Theorem 7.3 and subsequent Remark 7.1], we have

$$\frac{|d_v(\exp_{x_i})(X)|}{|X|} \geq \frac{S_K(|v|)}{|v|} = \frac{\sin(\sqrt{K}|v|)}{\sqrt{K}|v|}$$

for any $v \in B_T(x_i, r)$ and any $X \in T_{x_i} V$, where $d_v(\exp_{x_i}) : T_v(B_T(x_i, r)) \to T_y V$, with $y = \exp_{x_i}(v)$, denotes the differential of $\exp_{x_i}$ at v and the tangent space $T_v(B_T(x_i, r))$ is naturally identified with $T_{x_i} V$. Since $\exp_{x_i} : B_T(x_i, r) \to B(x_i, r)$ is a diffeomorphism, we have $d_y(\exp_{x_i}^{-1}) = (d_v(\exp_{x_i}))^{-1}$ for $v \in B_T(x_i, r)$ and $y = \exp_{x_i}(v) \in B(x_i, r)$. Let $d_v(\exp_{x_i})(X) = Y$. We have

$$\frac{|d_y(\exp_{x_i}^{-1})(Y)|}{|Y|} \leq \frac{\sqrt{K}|v|}{\sin(\sqrt{K}|v|)}, \quad \operatorname{dil}_y \exp_{x_i}^{-1} \leq \frac{\sqrt{K}|v|}{\sin(\sqrt{K}|v|)}.$$

Note that $\sqrt{K}|v|/\sin(\sqrt{K}|v|)$ is an increasing function of $|v| \in (0, r]$. Thus

$$\operatorname{dil} \exp_{x_i}^{-1} \leq \sup_{v \in B_T(x_i, r)} \frac{\sqrt{K}|v|}{\sin(\sqrt{K}|v|)} = \frac{\sqrt{K} \cdot r}{\sin(\sqrt{K} \cdot r)}.$$

Fix a point $p \in S^n$ and consider the composite $f_i : B(x_i, r) \to S^n$ defined by

$$B(x_i, r) \xrightarrow{\exp_{x_i}^{-1}} B_T(x_i, r) \xrightarrow{\varphi_1} B_T(x_i, \pi) \xrightarrow{\varphi_2} B_T(p, \pi) \xrightarrow{\varphi_3} S^n, \tag{1}$$

where φ_1 is the obvious diffeomorphism with dilatation π/r, φ_2 is the obvious isometry onto a ball $B_T(p, \pi)$ of radius π centered at the origin in the tangent space to S^n at p, and φ_3 has degree 1, dilatation 1, and maps $\partial B_{T,S^n}(p, \pi)$ to the antipode p' of p in S^n. The map f_i sends $\partial B_V(x_i, r)$ to p'.

It follows that $\deg f_i = 1$ and

$$\operatorname{dil} f_i \le \operatorname{dil}\varphi_3 \ \operatorname{dil}\varphi_2 \ \operatorname{dil}\varphi_1 \ \operatorname{dil}\exp_{x_i}^{-1} \le \frac{r\sqrt{K}}{\sin(r\sqrt{K})}.$$

Finally, define $f : V \to S^n$ by

$$f(x) = \begin{cases} f_i(x) & \text{for } x \in B(x_i, r), \\ p' & \text{elsewhere.} \end{cases}$$

It is clear that $\deg f = q$ and $\operatorname{dil} f \le r\sqrt{K}/\sin(r\sqrt{K})$.

(2) Now suppose $K = \sup \sec(V) \le 0$. Take q disjoint open balls $B(x_i, r)$ with $r = \operatorname{pack}_q V$. In this case, since V has no conjugate points, we require only $r = \operatorname{pack}_q V \le \operatorname{Inj} V$. In analogy with the above argument, we also have $\exp_{x_i}^{-1} : B(x_i, r) \to B_T(x_i, r)$, the composite map $f_i : B(x_i, r) \to S^n$, and $f : V \to S^n$. Since V is nonpositively curved, $\exp_{x_i}^{-1}$ is distance-decreasing. Then $\operatorname{dil}\exp_{x_i}^{-1} \le 1$ and $\operatorname{dil} f_i \le \pi/r$, hence $\operatorname{dil} f \le \pi/r$. This completes the proof of Theorem A. □

In proving Theorem A we have shown Corollary 1.2. Indeed, take a point $x_0 \in V$ such that $\operatorname{Inj}_{x_0} V = \operatorname{Inj}_{\max} V =: r$, we have a map $B_V(x_0, r) \to S^n$, whose extension $\varphi_{x_0} : V \to S^n$ is of degree one. We then estimate $\operatorname{dil}\varphi_{x_0}$. For details see [Liu 2005].

3. Abel–Jacobi map, stable 1-systoles and the proof of Theorem B

Let V be a compact, oriented n-manifold with first Betti number $b_1(V) = n$ and let $i_* : H_1(V, \mathbb{Z}) \to H_1(V, \mathbb{R})$ be the map induced by chain inclusion. We denote by $H_1(V, \mathbb{Z})_{\mathbb{R}}$ the image $\Im(i_*(H_1(V, \mathbb{Z})))$, which is a lattice in $H_1(V, \mathbb{R}) \simeq \mathbb{R}^n$. Let $J_1(V) = H_1(V, \mathbb{R})/H_1(V, \mathbb{Z})_{\mathbb{R}} \simeq \mathbb{T}^n$ be the Jacobi torus of V. Up to homotopy, we have the Abel–Jacobi map

$$\mathcal{A}_V : V \to J_1(V)$$

induced by the harmonic one-forms on V, originally introduced by A. Lichnerowicz [1969] (see also [Gromov 1999b, p. 249; Bangert and Katz 2004; Katz and Lescop 2005). The covering $\tilde{V} \to V$ is the pull-back along $\mathcal{A}_V$ of the universal covering $\mathbb{R}^n \to J_1(V)$, with the group of deck transformations $H_1(V, \mathbb{Z})_{\mathbb{R}} \simeq \mathbb{Z}^n$, as depicted in the commutative diagram

$$\begin{array}{ccc} \tilde{V} & \xrightarrow{\tilde{\mathcal{A}}_V} & H_1(V, \mathbb{R}) \simeq \mathbb{R}^n \\ \downarrow & & \downarrow \\ V & \xrightarrow[\mathcal{A}_V]{} & J_1(V) \simeq \mathbb{T}^n \end{array}$$

The key point in the following topological lemma is that $H^1(\mathcal{A}_V) : H^1(\mathbb{T}^n, \mathbb{R}) \to H^1(V, \mathbb{R})$ is an isomorphism and $H^*(\mathcal{A}_V)$ preserves cup products.

Lemma 3.1 [Gromov 1983, Section 7.5]. *Let V be a compact, oriented n-manifold with first Betti number $b_1(V) = n$. If the real cuplength is also n, then $\mathcal{A}_V$ has nonzero degree.* □

If V is equipped with a Riemannian metric g, there is a corresponding stable norm $\|\cdot\|_{\mathrm{st}}$ on $H_1(V, \mathbb{R})$ (see [Gromov 1999b, pp. 245–247; Bangert and Katz 2004], for example). We define the *stable 1-systole* of (V, g) as

$$\mathrm{stsys}_1(V) = \min\{\|h\|_{\mathrm{st}} : h \in H_1(V, \mathbb{Z})_{\mathbb{R}} \setminus \{0\}\}.$$

The construction of f in Theorem B is a combination $h \circ \psi$ where $\psi : V \to \mathbb{T}^n$ (a flat torus) comes from [Ivanov and Katz 2004, Section 7], based on the techniques of [Burago and Ivanov 1995].

From the stable norm $\|\cdot\|_{\mathrm{st}}$ on $H_1(V, \mathbb{R})$ one derives a Euclidean norm $|\cdot|_e$ such that $|\cdot|_e \geq \|\cdot\|_{\mathrm{st}}$ and

$$|\cdot|_e^2 = \sum_{i=1}^{N} a_i L_i^2 \quad \text{with} \quad N \leq \frac{n(n+1)}{2}, \quad a_i > 0 \text{ for } i = 1, \ldots, N, \quad \sum_{i=1}^{N} a_i = n,$$

where the $L_i : H_1(V, \mathbb{R}) \to \mathbb{R}$ $(i = 1, \ldots, N)$ are linear functions with $\|L_i\|_{\mathrm{st}} = |L_i|_e = 1$ (see [Burago and Ivanov 1995, Theorem 1.3]). The linear map $L : H_1(V, \mathbb{R}) \to \mathbb{R}^N$ defined by

$$L(x) = \left(\sqrt{a_1} L_1(x), \ldots, \sqrt{a_N} L_N(x)\right)$$

is an isometry from $(H_1(V, \mathbb{R}), |\cdot|_e)$ onto a linear subspace $L(H_1(V, \mathbb{R}))$ of $\mathbb{R}^N$, equipped with the restriction of the standard coordinate metric $|\cdot|_E$ of $\mathbb{R}^N$.

Since the stable norm $\|\cdot\|_{\mathrm{st}}$ satisfies

$$\|\gamma\|_{\mathrm{st}} \leq d_g(x, x + \gamma)$$

for all $\gamma \in H_1(V, \mathbb{Z})_{\mathbb{R}}$ and all $x \in \tilde{V}$, where $+\gamma$ denotes the action on $\tilde{V}$ of an element γ of the deck transformation group $H_1(V, \mathbb{Z})_{\mathbb{R}}$, we have:

Lemma 3.2 [Ivanov and Katz 2004, Lemma 7.3]. *For each linear function $L_i : H_1(V, \mathbb{R}) \to \mathbb{R}$ with $\|L_i\|_{\mathrm{st}} = 1$ there is a Lipschitz map $\varphi_i : \tilde{V} \to \mathbb{R}$ such that* $\operatorname{dil} \varphi_i \leq 1$ *and*

$$\varphi_i(x + \gamma) = \varphi_i(x) + L_i(\gamma) \qquad \text{for } x \in \tilde{V},\ \gamma \in H_1(V, \mathbb{Z})_{\mathbb{R}}.$$ □

Proposition 3.3 [Ivanov and Katz 2004, Section 7]. *Let (V, g) be a compact, oriented Riemannian manifold with* $\dim(V) = b_1(V) =$ *real cuplength*$(V) = n$. *There exists a Lipschitz map $\psi : (V, d_g) \to (J_1(V), |\cdot|_e)$ of nonzero degree such that* $\operatorname{dil} \psi \leq \sqrt{n}$, *where $|\cdot|_e$ on $J_1(V)$ is the flat Riemannian metric induced from $H_1(V, \mathbb{R}), |\cdot|_e)$.*

Proof. From Lemma 3.2, define $F : \tilde{V} \to \mathbb{R}^N$ by

$$F(x) = \big(\sqrt{a_1}\varphi_1(x), \sqrt{a_2}\varphi_2(x), \dots, \sqrt{a_N}\varphi_N(x)\big).$$

It is easy to check that L and F are $H_1(V,\mathbb{Z})_{\mathbb{R}}$-equivariant with respect to the following action of $H_1(V,\mathbb{Z})_{\mathbb{R}}$ on $\mathbb{R}^N$

$$\mathbb{R}^N \times H_1(V,\mathbb{Z})_{\mathbb{R}} \to \mathbb{R}^N \quad (u,\gamma) \to u + L(\gamma)$$

where L is as the beginning of this section.

Let $P : \mathbb{R}^N \to L(H_1(V,\mathbb{R}))$ be the orthogonal projection. Since the composite map

$$L^{-1} \circ P \circ F : \tilde{V} \to \mathbb{R}^N \to L(H_1(V,\mathbb{R})) \to H_1(V,\mathbb{R})$$

is $H_1(V,\mathbb{Z})_{\mathbb{R}}$-equivariant, we can lower it to a map between base spaces:

$$\psi : V \to H_1(V,\mathbb{R})/H_1(V,\mathbb{Z})_{\mathbb{R}} = J_1(V).$$

From $\operatorname{dil}\varphi_i \le 1$ we have

$$\operatorname{dil} F \le \Big(\sum_{i=1}^{N} \operatorname{dil}(\sqrt{a_i}\varphi_i)^2\Big)^{1/2} \le \Big(\sum_{i=1}^{N} a_i\Big)^{1/2} = \sqrt{n},$$

$$\operatorname{dil}(L^{-1} \circ P \circ F) \le \operatorname{dil} L^{-1} \ \operatorname{dil} P \ \operatorname{dil} F \le \sqrt{n},$$

with respect to d_g on $\tilde{V}$ and $|\cdot|_e$ on $H_1(V,\mathbb{R})$. This implies that $\operatorname{dil}\psi \le \sqrt{n}$.

Finally, there is a linear homotopy $\{G_t\}$ between $L^{-1} \circ P \circ F$ and $\tilde{\mathscr{A}}_V$, i.e., $G_t(x) = t(L^{-1} \circ P \circ F(x)) + (1-t)(\tilde{\mathscr{A}}_V(x))$. It is easy to see that each G_t is $H_1(V,\mathbb{Z}_{\mathbb{R}})$-equivariant; hence it may lower to a homotopy between ψ and $\mathscr{A}_V$. Therefore $\deg\psi = \deg\mathscr{A}_V$. By Lemma 3.1, this completes the proof of the proposition. □

Remark 3.4. By estimating the Jacobian of ψ, Ivanov and Katz [2004] proved that ψ is volume-decreasing.

Proof of Theorem B. Let $(J_1(V), |\cdot|_e)$ be as above. For $x \in J_1(V)$, let $B(x,r)$ be the ball in $J_1(V)$ of radius $r = \operatorname{Inj}(J_1(V), |\cdot|_e)$ centered at x, and $B_T(x,r)$ the ball of radius r centered at the origin of the tangent space $T_x J_1(V)$. As in the proof of Theorem A, we fix $p \in S^n$ and define a map $h : B(x,r) \to S^n$ through the composition

$$B(x,r) \xrightarrow{\exp_x^{-1}} B_T(x,r) \xrightarrow{\varphi_1} B_T(x,\pi) \xrightarrow{\varphi_2} B_T(p,\pi) \xrightarrow{\varphi_3} S^n,$$

where the notation is just as in diagram (1) (page 450). This map h sends $\partial B(x,r)$ to p'. Since $(J_1(V), |\cdot|_e)$ is flat, we have $\operatorname{dil}\exp_x^{-1} \le 1$. Extend h to $J_1(V) \to S^n$

by setting $h(J_1(V)\backslash B(x,r)) = p'$. As in the proof of Theorem A, h has degree 1 and its dilatation satisfies $\operatorname{dil} h \le \operatorname{dil}\varphi_3 \operatorname{dil}\varphi_2 \operatorname{dil}\varphi_1 \operatorname{dil}\exp_x^{-1} \le \pi/r = \pi/\operatorname{Inj} J_1(V)$.

Next, a closed geodesic C in the flat torus $(J_1(V), |\cdot|_e)$ is the projection of a segment $[x, x+\gamma]$ joining x and $x+\gamma$ in the Euclidean space $(H_1(V,\mathbb{R}), |\cdot|_e)$, for $\gamma \in H_1(V,\mathbb{Z})_{\mathbb{R}}$. Moreover, C is not homologous to zero (hence not homotopic to zero since $\pi_1(J_1(V))$ is abelian) if and only if $\gamma \neq 0$. The length of C equals the displacement $d_{|\cdot|_e}(x, x+\gamma)$, which does not depend on x; see [Gromov 1996, 1.A]. Hence

$$\begin{aligned}\operatorname{sys}_1(J_1(V), |\cdot|_e) &= \inf\{|\gamma|_e : \gamma \in H_1(V,\mathbb{Z}) \setminus 0\} \\ &\ge \inf\{\|\gamma\|_{\mathrm{st}} : \gamma \in H_1(V,\mathbb{Z}) \setminus 0\} = \operatorname{stsys}_1 V.\end{aligned}$$

Thus $\operatorname{Inj}(J_1(V), |\cdot|_e) = \frac{1}{2}\operatorname{sys}_1(J_1(V), |\cdot|_e) \ge \frac{1}{2}\operatorname{stsys}_1 V$, and $\operatorname{dil} h \le 2\pi/(\operatorname{stsys}_1 V)$.

Composing h with the map ψ of Proposition 3.3 yields a Lipschitz map $f = h \circ \psi : (V, d_g) \to (S^n, \mathrm{can})$ satisfying $\deg f = \deg h \, \deg\psi = \deg \mathcal{A}_V \neq 0$ and

$$\operatorname{dil} f \le \operatorname{dil} h \, \operatorname{dil}\psi \le \frac{2\pi\sqrt{n}}{\operatorname{stsys}_1 V)}.$$

This completes the proof of the theorem. □

4. Filling volume and its mapping property

We recall from [Gromov 1983] the definition of filling volume. Let (X, d) be a metric space and $\sigma : \triangle^{n+1} \to X$ a singular simplex. Define

$$\operatorname{Vol}(\sigma) = \inf_g\{\operatorname{Vol}_g(\triangle^{n+1}\},$$

where the infimum is taken over all Riemannian metrics g on $\triangle^{n+1}$ such that $d_X(\sigma(x), \sigma(y)) \le d_g(x, y)$ for all $x, y \in \triangle^{n+1}$. For a singular chain $c = \sum_i r_i\sigma_i$, we can define

$$\operatorname{Vol}(c) = \sum_i |r_i| \operatorname{Vol}(\sigma_i),$$

where the coefficients r_i may be real numbers, integers, or integers mod 2. When (X, g) is a Riemannian manifold, $\operatorname{Vol}(c)$, with respect to d_g, is just the usual Riemannian volume of a singular chain.

Let z be an n-dimensional singular $\mathbb{G}$-cycle in X, and $\mathbb{G} = \mathbb{R}$, $\mathbb{Z}$ or $\mathbb{Z}_2$. We define

$$\operatorname{FillVol}(z \hookrightarrow X; \mathbb{G}) = \inf\{\operatorname{Vol}(c) : c \text{ are } (n+1)\text{-chains in } X, \partial c = z\},$$

where the coefficients of the chains c lie in $\mathbb{G}$.

Let V be a compact submanifold in X and $[V]$ the fundamental class of V (if V is not oriented, $[V]$ denotes the fundamental $\mathbb{Z}_2$-class in the group $H_{\dim(V)}(V, \mathbb{Z}_2) \simeq \mathbb{Z}_2$). We define the filling volume relative to the imbedding $i : V \hookrightarrow X$ by

$$\mathrm{FillVol}(V \hookrightarrow X; \mathbb{G}) = \inf\big\{\mathrm{FillVol}(z) : z \text{ represents } i_*[V]\big\}.$$

Let V be a compact manifold equipped with a metric d and the isometric imbedding $V \hookrightarrow L^\infty(V)$ is as Section 1. We define the absolute filling volume of V as

$$\mathrm{FillVol}(V; \mathbb{G}) =: \mathrm{FillVol}(V \hookrightarrow L^\infty(V); \mathbb{G}).$$

To simplify the notation, we will generally write $\mathrm{FillVol}(V)$ as $\mathrm{FillVol}(V, \mathbb{G})$, understanding the coefficient group $\mathbb{G}$ to be $\mathbb{Z}$ or $\mathbb{Z}_2$, depending on whether or not V is oriented.

The following proposition shows a mapping property of filling volume

Proposition 4.1. *Let $f : V \to W$ be a map between compact, connected, n-dimensional manifolds V, W with metrics d_V, d_W respectively.*

(i) *If V, W are oriented and* $|\deg f| = 1$, *then*

$$\mathrm{FillVol}(V; \mathbb{Z}) \ge \frac{1}{(\mathrm{dil}\, f)^{n+1}} \mathrm{FillVol}(W; \mathbb{Z}).$$

(ii) *If V, W are oriented and f is of nonzero degree, then*

$$\mathrm{FillVol}(V; \mathbb{R}) \ge \frac{|\deg f|}{(\mathrm{dil}\, f)^{n+1}} \mathrm{FillVol}(W; \mathbb{R}).$$

(iii) *If V, W are not oriented and* $\deg f \neq 0\, mod 2$, *then*

$$\mathrm{FillVol}(V; \mathbb{Z}_2) \ge \frac{1}{(\mathrm{dil}\, f)^{n+1}} \mathrm{FillVol}(W; \mathbb{Z}_2).$$

Lipschitz Extension Lemma 4.2 [Gromov 1983, p. 8]. *Let V be a compact submanifold of a metric space X, and let W be a metric space. Every Lipschitz map $\varphi : V \to L^\infty(W)$ has a Lipschitz extension $\tilde\varphi : X \to L^\infty(W)$ with* $\mathrm{dil}\,\tilde\varphi = \mathrm{dil}\,\varphi$.

For a detailed proof, see [Liu 2005].

Proof of Proposition 4.1. We may suppose that f is a Lipschitz map (if not, the desired inequalities are obvious). By the Lipschitz Extension Lemma 4.2, $f : V \to W$ has a Lipschitz extension $\tilde f : L^\infty(V) \to L^\infty(W)$ with $\mathrm{dil}\,\tilde f = \mathrm{dil}\, f$. Let the isometric imbedding $i_V : V \hookrightarrow L^\infty(V)$ and $i_W : W \hookrightarrow L^\infty(W)$ be as above. Let $c = \sum r_i\sigma_i$ be any $(n+1)$-chain in $L^\infty(V)$ with $[\partial c] = i_*[V]$, where the $\sigma_i : \triangle^{n+1} \to L^\infty(V)$ are singular simplexes. Then $\tilde f(c) = \sum r_i \tilde f(\sigma_i)$ is an $(n+1)$-chain in $L^\infty(W)$. Let g be any Riemannian metric on $\triangle^{n+1}$ with $d_g(x, y) \ge d_{L^\infty(V)}(\sigma_i(x), \sigma_i(y))$ for any $x, y \in \triangle^{n+1}$. Set $\alpha := \mathrm{dil}\, f$ and consider the new Riemannian metric $\alpha^2 g$ on $\triangle^{n+1}$. For any $x, y \in \triangle^{n+1}$ we have

$$d_{L^\infty(W)}(\tilde f\sigma_i(x), \tilde f\sigma_i(y)) \le \alpha d_{L^\infty(V)}(\sigma_i(x), \sigma_i(y)) \le \alpha d_g(x, y) = d_{\alpha^2 g}(x, y).$$

By the definition of the volume of a singular simplex,

$$\mathrm{Vol}(\tilde{f}\sigma_i) \le \mathrm{Vol}(\Delta^{n+1}, \alpha^2 g) = \alpha^{n+1}\,\mathrm{Vol}(\Delta^{n+1}, g).$$

In view of the freedom of g, we have proved that $\mathrm{Vol}(\tilde{f}\sigma_i) \le \alpha^{n+1}\,\mathrm{Vol}(\sigma_i)$; hence $\mathrm{Vol}(\tilde{f}(c)) \le \alpha^{n+1}\,\mathrm{Vol}(c)$.

In cases (i) and (iii), $\partial\tilde{f}(c)$ represents $\pm(i_W)_*[W]$. Indeed,

$$[\partial\tilde{f}(c)] = \tilde{f}_*[\partial c] = \tilde{f}_* \circ (i_V)_*[V] = (i_W)_* \circ f_*[V] = \deg f\,(i_W)_*[W].$$

In case (ii), the real singular chain $|\deg f|^{-1}\tilde{f}(c)$ represents $\pm(i_W)_*[W]$ and

$$\mathrm{Vol}\Big(\frac{1}{|\deg f|}\tilde{f}(c)\Big) = \frac{1}{|\deg f|}\mathrm{Vol}(\tilde{f}(c)) \le \frac{1}{|\deg f|}\alpha^{n+1}\,\mathrm{Vol}(c).$$

This completes the proof of the proposition. □

Proposition 4.3. *Let V^n be a compact manifold with a metric d and let X be a metric space. For any isometric imbedding $j : V \hookrightarrow X$, we have*

$$\mathrm{FillVol}(V \hookrightarrow X) \ge \mathrm{FillVol}(V \hookrightarrow L^\infty(V)) = \mathrm{FillVol}(V).$$

Proof. Again using Lipschitz Extension Lemma 4.2, $i : V \hookrightarrow L^\infty(V)$ has a Lipschitz extension $\varphi : X \to L^\infty(V)$ with $\mathrm{dil}\,\varphi \le 1$. Let z be any n-cycle in X which represent $j_*[V]$ and c be any $(n+1)-$*chain* in X with $\partial c = z$. Then $[\partial\varphi(c)] = \varphi_*[\partial c] = \varphi_*[z] = \varphi_* \circ j_*[V] = i_*[V]$, namely $\partial\varphi(c)$ represent $i_*[V]$. From the argument of Proposition 4.1, we know that $\mathrm{Vol}(\varphi(c)) \le \mathrm{Vol}(c)$. The arbitrariness of z and c implies that $\mathrm{FillVol}(V \hookrightarrow L^\infty(V)) \le \mathrm{FillVol}(V \hookrightarrow X)$. □

Let W be a compact manifold with boundary ∂W, with a Riemannian metric g. The *chordal metric* on ∂W is the (non-Riemannian) metric $d_g|_{\partial W}$ on ∂W defined by g-shortest paths in W. (See [Croke and Katz 2003] for details.)

Corollary 4.4 [Gromov 1983, p. 12]. $\mathrm{FillVol}(\partial W, d_g|_{\partial W}) \le \mathrm{Vol}(W, g)$.

Proof. For the isometric imbedding $(\partial W, d_g|_{\partial W}) \hookrightarrow (W, d_g)$, applying Proposition 4.3 we have

$$\mathrm{Vol}(W, g) = \mathrm{FillVol}(\partial W \hookrightarrow W) \ge \mathrm{FillVol}(\partial W, d_g|_{\partial W}).$$ □

Thanks to Proposition 4.1 and Corollary 4.4, we obtain Corollaries 1.4′ and 1.6′ from Corollaries 1.4 and 1.6.

5. Filling volume estimates by cube Besicovitch inequality

In this section we adopt the notation $I = [-1, 1]$. Then I^n is a topological cube in $\mathbb{R}^n$, with boundary $\partial I^n = \bigcup_{i=1}^n (F_i^{-1} \cup F_i^1)$, where F_i^{-1} is the set of points in I^n whose i-th coordinate equals -1, and likewise for F_i^1. The sets F_i^{-1} and F_i^1 are called $(n-1)$-faces of the cube; the notion of opposite faces is obvious.

Theorem 5.1 [Gromov 1983, pp. 85–86]. *Let V be a compact, oriented n-manifold with a metric d and let $f : V \to \partial I^{n+1} (\simeq S^n)$ be a continuous map with nonzero degree. Then*

$$\mathrm{FillVol}(V) > |\deg f| \prod_{i=1}^{n+1} d\big(f^{-1}(F_i^{-1}),\, f^{-1}(F_i^{1})\big).$$

We regard this inequality still as a Besicovitch inequality, and we will use it in deducing Theorem C.

Proof of Theorem C. We establish a homeomorphism $\varphi : S^n \to \partial I^{n+1}$ by central projection:

$$\varphi(u) = \frac{u}{\|u\|_\infty} \quad \text{for } u \in S^n, \qquad \varphi^{-1}(x) = \frac{x}{|x|_E} \quad \text{for } x \in \partial I^{n+1},$$

where $\|\cdot\|_\infty$ is the l^∞-norm (whose unit sphere is ∂I^{n+1}) and $|\cdot|_E$ is the Euclidean norm. Next we determine $d_S\big(\varphi^{-1}(F_1^{-1}), \varphi^{-1}(F_1^{1})\big)$, where d_S is the spherical distance on S^n. Take $x = (1, x_2, \dots, x_{n+1}) \in F_1^1$, $y = (-1, y_2, \dots, y_{n+1}) \in F_1^{-1}$, and assume without loss of generality that $|x|_E \geq |y|_E$. Then

$$\left|\frac{x}{|x|_E} - \frac{y}{|y|_E}\right|_E = \frac{1}{|x|_E}\left|x - \frac{|x|_E}{|y|_E} y\right|_E \geq \frac{2}{|x|_E} \geq \frac{2}{\max_{x \in \partial I^{n+1}} |x|_E} = \frac{2}{\sqrt{n+1}}.$$

For any

$$u = \frac{x}{|x|_E} \in \varphi^{-1}(F_1^1) \subset S^n \quad \text{and} \quad v = \frac{y}{|y|_E} \in \varphi^{-1}(F_1^{-1}) \subset S^n,$$

the cosine theorem implies

$$|u - v|_E^2 = 2 - 2\cos d_S(u, v).$$

Therefore

$$\begin{aligned} d_S(u, v) &= \arccos\big(1 - \tfrac{1}{2}|u - v|_E^2\big) \\ &= \arccos\left(1 - \frac{1}{2}\left|\frac{x}{|x|_E} - \frac{y}{|y|_E}\right|_E^2\right) \geq \arccos\Big(1 - \frac{2}{n+1}\Big) = \arccos\frac{n-1}{n+1}. \end{aligned}$$

Taking

$$x_0 = (1, 1, \dots, 1) \in F_1^1, \qquad y_0 = (-1, 1, \dots, 1) \in F_1^{-1},$$

$$u_0 = \frac{x_0}{|x_0|_E} = \frac{x_0}{\sqrt{n+1}} \in f^{-1}(F_1^1), \qquad v_0 = \frac{y_0}{|y_0|_E} = \frac{y_0}{\sqrt{n+1}} \in f^{-1}(F_1^{-1}),$$

we have

$$d_S(u_0, v_0) = \arccos\left(1 - \frac{1}{2}\left|\frac{x_0}{|x_0|_E} - \frac{y_0}{|y_0|_E}\right|_E^2\right) = \arccos\frac{n-1}{n+1}.$$

Hence $d_S\big(\varphi^{-1}(F_1^1), \varphi^{-1}(F_1^{-1})\big) = \arccos\dfrac{n-1}{n+1}$, and more generally

$$d_S\big(\varphi^{-1}(F_i^1), \varphi^{-1}(F_i^{-1})\big) = \arccos\frac{n-1}{n+1} \quad \text{for } i = 1, \dots, n+1.$$

Next, we estimate $d_S\big((\varphi\circ f)^{-1}(F_i^1), (\varphi\circ f)^{-1}(F_i^{-1})\big)$ for the composite map $\varphi\circ f : V \to \partial I^{n+1}$. For any $x \in (\varphi\circ f)^{-1}(F_i^1)$ and $y \in (\varphi\circ f)^{-1}(F_i^{-1})$, we have $f(x) \in \varphi^{-1}(F_i^1)$, $f(y) \in \varphi^{-1}(F_i^{-1})$. Therefore

$$\begin{aligned} d_V(x, y) &\geq \frac{1}{\operatorname{dil} f} d_S(f(x), f(y)) \\ &\geq \frac{1}{\operatorname{dil} f} d_S(\varphi^{-1}(F_i^1), \varphi^{-1}(F_i^{-1})) = \frac{1}{\operatorname{dil} f}\arccos\frac{n-1}{n+1}. \end{aligned}$$

Since $\deg(\varphi\circ f) = \deg f$, applying the Besicovitch inequality (Theorem 5.1) to $\varphi\circ f$, we get

$$\operatorname{FillVol}(V) \geq \frac{|\deg f|}{(\operatorname{dil} f)^{n+1}} \cdot \left(\arccos\frac{n-1}{n+1}\right)^{n+1}. \qquad \square$$

Remark 5.2. As is well-known, it is difficult to compute Gromov invariants. There is still not a single Riemannian manifold whose filling volume is known. Gromov's filling volume conjecture [1983, p. 13], which is still open in all dimensions, says that $\operatorname{FillVol}(S^n, \text{can}) = \frac{1}{2}\operatorname{Vol}(S^{n+1}, \text{can})$. The case $n = 1$ can be broken into separate problems depending on the genus of the filling [Gromov 1983, pp. 59–60; Bangert et al. 2005; Croke and Katz 2003; Katz and Lescop 2005]. Taking $W =$the canonical positive hemisphere (S_+^{n+1}, g), obviously, $\partial S_+^{n+1} = S^n$ and $d_g|_{S^n} =$the canonical distance of S^n. From Corollary 4.4, we have

$$\operatorname{FillVol}(S^n) \leq \operatorname{Vol}(S_+^{n+1}) = \tfrac{1}{2}\operatorname{Vol}(S^{n+1}).$$

On the other hand, as an intermediate result of the argument of Theorem C, we have

$$\operatorname{FillVol}(S^n) > \left(\arccos\frac{n-1}{n+1}\right)^{n+1}.$$

Unfortunately, this rough lower bound is far from the exact value conjectured by Gromov. It is also inferior to the easy estimate obtained as follows. Since the Euclidean unit ball B^{n+1} is flat and simply connected, we know from [Gromov 1983, 2.1] that

$$\operatorname{FillVol}(S^n, \text{Euclid}) = \operatorname{FillVol}(\partial B^{n+1}, \text{Euclid}) = \operatorname{Vol}(B^{n+1}) = \frac{\pi^{(n+1)/2}}{\Gamma\big(\frac{n+1}{2}+1\big)}.$$

The canonical Riemannian metric on S^n (of diameter π) dominates the Euclidean metric of diameter 2, so $\operatorname{FillVol}(S^n, \text{can}) \geq \operatorname{FillVol}(S^n, \text{Euclid})$ (Proposition 4.1 is a generalization of this fact).

Acknowledgements

This article is part of a Ph.D. thesis written under the direction of Fuquan Fang at the Nankai Institute of Mathematics. The author thanks Professor Fang for his guidance. Special thanks go to Mikhail Katz for constructive criticism of the first version of the manuscript, which prompted an overall improvement in the work.

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Received December 20, 2005.

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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department
Berkeley, CA 94720-3840 is published monthly except July and August. Periodical rate postage paid at B
mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163

PACIFIC JOURNAL OF MATHEMATICS

Volume 231 No. 2 June 2007