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# TRANSVERSE POISSON STRUCTURES TO ADJOINT ORBITS IN SEMISIMPLE LIE ALGEBRAS

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**We study the transverse Poisson structure to adjoint orbits in a complex semisimple Lie algebra. The problem is first reduced to the case of nilpotent orbits. We prove then that in suitably chosen quasihomogeneous coordinates, the quasidegree of the transverse Poisson structure is −2. For *s*ubregular nilpotent orbits, we show that the structure may be computed using a simple determinantal formula that involves the restriction of the Chevalley invariants on the slice. In addition, using results of Brieskorn and Slodowy, the Poisson structure is reduced to a three dimensional Poisson bracket, which is intimately related to the simple rational singularity that corresponds to the subregular orbit.**

## 1. Introduction

The transverse Poisson structure was introduced by A. Weinstein [1983], stating in his famous splitting theorem that every (real smooth or complex holomorphic) Poisson manifold $M$ is, in the neighborhood of each point $m$, the product of a symplectic manifold and a Poisson manifold of rank 0 at $m$. The two factors of this product can be geometrically realized as follows. Let $S$ be the symplectic leaf through $m$, and let $N$ be any submanifold of $M$ containing $m$ such that

$$T_m(M) = T_m(S) \oplus T_m(N).$$

There exists a neighborhood $V$ of $m$ in $N$, endowed with a Poisson structure, and a neighborhood $U$ of $m$ in $S$ such that, near $m$, $M$ is isomorphic to the product Poisson manifold $U \times V$. The submanifold $N$ is called a *transverse slice* at $m$ to the symplectic leaf $S$. The Poisson structure on $V \subset N$ is called the *transverse Poisson structure* to $S$; up to Poisson isomorphism, it is independent of the point $m \in S$ and the chosen transverse slice $N$ at $m$: given two points $m, m' \in S$ with

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transverse slices $N$, $N'$ to $S$, there exist neighborhoods $V$ of $m$ in $N$ and $V'$ of $m'$ in $N'$ such that $(V, m)$ and $(V', m')$ are Poisson diffeomorphic.

When $M$ is the dual $\mathfrak{g}^*$ of a complex Lie algebra $\mathfrak{g}$ and is equipped with its standard Lie–Poisson structure, we know that the symplectic leaf through $\mu \in \mathfrak{g}^*$ is the coadjoint orbit $\mathrm{G} \cdot \mu$ of the adjoint Lie group G of $\mathfrak{g}$. In this case, a natural transverse slice to $\mathrm{G} \cdot \mu$ is obtained in the following way. We choose any complement $\mathfrak{n}$ to the centralizer $\mathfrak{g}(\mu)$ of $\mu$ in $\mathfrak{g}$, and we take $N$ to be the affine subspace $\mu + \mathfrak{n}^\perp$ of $\mathfrak{g}^*$. Since $\mathfrak{g}(\mu)^\perp = \mathrm{ad}^*_{\mathfrak{g}} \mu$, we have

$$T_\mu(\mathfrak{g}^*) = T_\mu(\mathrm{G} \cdot \mu) \oplus T_\mu(N),$$

so that $N$ is indeed a transverse slice to $\mathrm{G} \cdot \mu$ at $\mu$. Furthermore, defining on $\mathfrak{n}^\perp$ any system of linear coordinates $(q_1, \dots, q_k)$ and using the explicit formula for Dirac reduction (see formula (4) below), one can write down explicit formulas for the Poisson matrix $\Lambda_N := \left(\{q_i, q_j\}_N\right)$, $1 \le i, j \le k$ of the transverse Poisson structure, from which it follows easily that the coefficients of $\Lambda_N$ are actually rational functions in $(q_1, \dots, q_k)$. As a corollary, in the Lie–Poisson case, the transverse Poisson structure is always rational [Saint-Germain 1999]. One immediately wonders, for which cases — more precisely, for which Lie algebras $\mathfrak{g}$, coadjoint orbits, and complements $\mathfrak{n}$ — is the Poisson structure on $N$ polynomial?

Partial answers have been given in the literature for (co)adjoint orbits in a semisimple Lie algebra. P. Damianou [1996] computed explicitly how the transverse Poisson structure to nilpotent orbits of $\mathfrak{gl}_n$ for $n \le 7$ correspond to a particular complement $\mathfrak{n}$; in this case the transverse Poisson structure is polynomial. Cushman and Roberts [2002] proved that there exists for any nilpotent adjoint orbit of a semisimple Lie algebra a special choice of a complement $\mathfrak{n}$ such that the corresponding transverse Poisson structure is polynomial. For the latter case, H. Sabourin [2005] gave a more general class of complements having a polynomial transverse structure, using essentially the machinery of semisimple Lie algebras; he also showed that the choice of complement $\mathfrak{n}$ is relevant for the polynomial character of the transverse Poisson structure by giving an example where the structure is rational for a generic choice of complement.

When the transverse Poisson structure is polynomial, one is tempted to define its degree as the maximal degree of the coefficients $\{q_i, q_j\}_N$ of its Poisson matrix, as was done in [Damianou 1996] and [Cushman and Roberts 2002], where several conjectures about this degree are formulated. Unfortunately, as shown in [Sabourin 2005], this degree depends strongly on the choice of the complement $\mathfrak{n}$, and hence it is not intrinsically attached to the transverse Poisson structure. We show in Section 3 that the right approach is to use the more general notion of quasidegree; that is, we assign natural quasidegrees $\varpi(q_i)$ to the variables $q_i$ $(i = 1, \dots, k)$ and we show that, in the above mentioned class of complements, the quasidegree of the transverse Poisson structure is always $-2$, irrespective of the simple Lie algebra, the chosen adjoint orbit, and the chosen transverse slice $N$! In fact, the weights $\varpi(q_i)$ have a Lie-theoretic origin and are also independent of the particular complement. It follows that $\{q_i, q_j\}_N$ for $1 \le i, j \le k$ is a quasihomogeneous polynomial of quasidegree $\varpi(q_i) + \varpi(q_j) - 2$.

Another result, established in this article, is that the study of the transverse Poisson structure to any adjoint orbit $G\cdot x$ can be reduced, via the Jordan–Chevalley decomposition of $x \in \mathfrak{g}$, to the case of an adjoint *nilpotent* orbit. Thereby we explain why we are merely interested in the case of nilpotent orbits.

The transverse structure to the regular nilpotent orbit $\mathcal{O}_{reg}$ of $\mathfrak{g}$ is always trivial. So, the next step is to consider the case of the *subregular* nilpotent orbit $\mathcal{O}_{sr}$ of $\mathfrak{g}$. Then $N \cong \mathbb{C}^{\ell+2}$, where $\ell$ is the rank of $\mathfrak{g}$. The dimension of $\mathcal{O}_{sr}$ is two less than the dimension of the regular orbit, so that the transverse Poisson structure has rank 2. It has $\ell$ independent polynomial Casimir functions $\chi_1, \dots, \chi_\ell$, where $\chi_i$ is the restriction of the $i$-th Chevalley invariant $G_i$ to the slice $N$. In this case, the transverse Poisson structure may be obtained by a simple determinantal formula instead of the usual, rather complicated Dirac constraints. That formula is as follows. In linear coordinates $q_1, q_2, \dots, q_{\ell+2}$ on $N$,

$$\{f, g\}_{\det} := \frac{df \wedge dg \wedge d\chi_1 \wedge \cdots \wedge d\chi_\ell}{dq_1 \wedge dq_2 \wedge \ldots \wedge dq_{\ell+2}} \tag{1}$$

defines a Poisson bracket on $N$ that coincides (up to a nonzero constant) with the transverse Poisson structure on $N$.

As an application of formula (1), we show in Theorem 5.6 that the Poisson matrix of the transverse Poisson on $N$ takes, in suitable coordinates, the block form

$$\widetilde{\Lambda}_N = \begin{pmatrix} 0 & 0 \\ 0 & \Omega \end{pmatrix}, \quad \text{where} \quad \Omega = \begin{pmatrix} 0 & \dfrac{\partial F}{\partial q_{\ell+2}} & -\dfrac{\partial F}{\partial q_{\ell+1}} \\ -\dfrac{\partial F}{\partial q_{\ell+2}} & 0 & \dfrac{\partial F}{\partial q_\ell} \\ \dfrac{\partial F}{\partial q_{\ell+1}} & -\dfrac{\partial F}{\partial q_\ell} & 0 \end{pmatrix}.$$

The polynomial $F = F(u_1, \dots, u_{\ell-1}, q_\ell, q_{\ell+1}, q_{\ell+2})$ is precisely the one that describes the universal deformation of the (homogeneous or inhomogeneous) simple singularity of the singular surface $N \cap \mathcal{N}$, where $\mathcal{N}$ is the nilpotent cone of $\mathfrak{g}$. The $u_1, \dots, u_{\ell-1}$ are the deformation parameters, which are also Casimirs for the Poisson structure on $N$. In particular, the restriction of this Poisson structure to

$N \cap \mathcal{N}$ is given by

$$\{x, y\} = \frac{\partial F_0}{\partial z}, \qquad \{y, z\} = \frac{\partial F_0}{\partial x}, \qquad \{z, x\} = \frac{\partial F_0}{\partial y},$$

where $F_0(x, y, z) := F(0, \dots, 0, x, y, z)$ is the polynomial that defines $N \cap \mathcal{N}$ as a surface in $\mathbb{C}^3$. As we will recall in Section 5, Brieskorn [1971] showed that, in the ADE case, the so-called adjoint quotient $G = (G_1, \dots, G_\ell) : \mathfrak{g} \to \mathbb{C}^\ell$ is, when restricted to the slice $N$, a semiuniversal deformation of the singular surface $N \cap \mathcal{N}$; this result was generalized by Slodowy [1980a] to the other simple Lie algebras. Our Theorem 5.6 adds a Poisson dimension to this result.

The article is organized as follows. In Section 2, we recall a few basic facts concerning transverse Poisson structures, and we show that a general orbit in a semisimple Lie algebra can be reduced to the case of a nilpotent orbit. In Section 3, we recall the notion of quasihomogeneity, and we show that, for a natural class of slices, the transverse Poisson structure is quasihomogeneous of quasidegree $-2$. In Section 4 and the end of Section 5, we show, in the Lie algebras $\mathfrak{g}_2$, $\mathfrak{so}_8$, and $\mathfrak{sl}_4$, how the transverse Poisson structure can be computed explicitly, and we use these examples to illustrate our results. In Section 5, we prove that, in the case of the subregular orbit, the transverse Poisson structure is given by a determinantal formula; we also show that this Poisson structure is entirely determined by the singular variety of nilpotent elements of the slice.

## 2. Transverse Poisson structures in semisimple Lie algebras

In this section, we recall the main setup for studying the transverse Poisson structure to a (co)adjoint orbit of a complex semisimple Lie algebra $\mathfrak{g}$, and we show how the general orbit is related to the case of a nilpotent orbit. We use the Killing form $\langle \cdot \mid \cdot \rangle$ of $\mathfrak{g}$ to identify $\mathfrak{g}$ with its dual $\mathfrak{g}^*$. This leads to a Poisson structure on $\mathfrak{g}$ that is given for functions $F, G$ on $\mathfrak{g}$ at $x \in \mathfrak{g}$ by

$$\{F, G\}(x) := \langle x \mid [dF(x), dG(x)] \rangle, \tag{2}$$

where we think of $dF(x)$ and $dG(x)$ as elements of $\mathfrak{g} \cong \mathfrak{g}^* \cong T_x^*\mathfrak{g}$. Since the Killing form is Ad-invariant, the isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$ identifies the adjoint orbits $\mathrm{G} \cdot x$ of G with the coadjoint orbits $\mathrm{G} \cdot \mu$, and so the symplectic leaf of $\{\cdot, \cdot\}$ that passes through $x$ is the adjoint orbit $\mathrm{G} \cdot x$. Also, as a transverse slice at $x$ to $\mathrm{G} \cdot x$, we can take an affine subspace $N := x + \mathfrak{n}^\perp$, where $\mathfrak{n}$ is any complementary subspace to the centralizer $\mathfrak{g}(x) := \{y \in \mathfrak{g} \mid [x, y] = 0\}$ of $x$ in $\mathfrak{g}$ and $\perp$ is the orthogonal complement with respect to the Killing form. To give an explicit formula for the Poisson structure $\{\cdot, \cdot\}_N$ transverse to $\mathrm{G} \cdot x$, let $(Z_1, \dots, Z_k)$ be a basis for $\mathfrak{g}(x)$, and let $(X_1, \dots, X_{2r})$ be a basis for $\mathfrak{n}$, where $2r = \dim(\mathrm{G} \cdot x)$ is the rank of the Poisson structure (2) at $x$. These bases lead to linear coordinates $q_1, \dots, q_{k+2r}$

on $\mathfrak{g}$, centered at $x$, defined by $q_i(y) := \langle y - x \mid Z_i \rangle$, for $i = 1, \ldots, k$ and $q_{k+i}(y) := \langle y - x \mid X_i \rangle$, for $i = 1, \ldots, 2r$. Since $dq_i(y) = Z_i$ for $i = 1, \ldots, k$ and $dq_{k+i}(y) = X_i$ for $i = 1, \ldots, 2r$, it follows from (2) that the Poisson matrix of $\{\cdot, \cdot\}$ at $y \in \mathfrak{g}$ is given by

$$\left(\left\{q_i, q_j\right\}(y)\right)_{1 \leq i, j \leq k+2r} = \begin{pmatrix} A(y) & B(y) \\ -B(y)^\top & C(y) \end{pmatrix}, \tag{3}$$

where

$$\begin{aligned} A_{i,j}(y) &= \left\langle y \mid [Z_i, Z_j] \right\rangle, && \text{for } 1 \leq i, j \leq k; \\ B_{i,m}(y) &= \langle y \mid [Z_i, X_m] \rangle, && \text{for } 1 \leq i \leq k, \quad 1 \leq m \leq 2r; \\ C_{l,m}(y) &= \langle y \mid [X_l, X_m] \rangle, && \text{for } 1 \leq l, m \leq 2r. \end{aligned}$$

It is easy to see that the skew-symmetric matrix $C(x)$ is invertible, and so $C(y)$ is invertible for $y$ in a neighborhood of $x$ in $\mathfrak{g}$, and hence for $y$ in a neighborhood $V$ of $x$ in $N$. By Dirac reduction, the Poisson matrix of $\{\cdot, \cdot\}_N$ at $n \in V$ in the coordinates $q_1, \ldots, q_k$ (restricted to $V$), is given by

$$\Lambda_N(n) = A(n) + B(n)C(n)^{-1}B(n)^\top. \tag{4}$$

According to the Jordan–Chevalley decomposition theorem, we can write $x = s + e$, where $s$ is semisimple, $e$ is nilpotent, and $[s, e] = 0$. Moreover, the respective centralizers of $x$, $s$ and $e$ are related as follows:

$$\mathfrak{g}(x) = \mathfrak{g}(s) \cap \mathfrak{g}(e). \tag{5}$$

This leads to a natural class of complements $\mathfrak{n}$ to $\mathfrak{g}(x)$. Since the restriction of $\langle \cdot \mid \cdot \rangle$ to $\mathfrak{g}(s)$ is nondegenerate [Dixmier 1996, Prop. 1.7.7.], we have a vector space decomposition of $\mathfrak{g}$ as

$$\mathfrak{g} = \mathfrak{g}(s) \oplus \mathfrak{n}_s,$$

where $\mathfrak{n}_s = \mathfrak{g}(s)^\perp$. Notice that $\mathfrak{n}_s$ is $\mathfrak{g}(s)$-invariant, that is, $[\mathfrak{g}(s), \mathfrak{n}_s] \subset \mathfrak{n}_s$, since

$$\langle \mathfrak{g}(s) \mid [\mathfrak{g}(s), \mathfrak{n}_s] \rangle = \langle [\mathfrak{g}(s), \mathfrak{g}(s)] \mid \mathfrak{n}_s \rangle \subset \langle \mathfrak{g}(s) \mid \mathfrak{n}_s \rangle = \{0\}.$$

Choosing any complement $\mathfrak{n}_e$ of $\mathfrak{g}(x)$ in $\mathfrak{g}(s)$, we get the following decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{g}(x) \oplus \mathfrak{n}_e \oplus \mathfrak{n}_s.$$

We take then $\mathfrak{n} := \mathfrak{n}_e \oplus \mathfrak{n}_s$, and we denote $N_x := x + \mathfrak{n}^\perp$. It follows that, if $n \in N_x$ such that $n \in \mathfrak{g}(s)$, then $\langle n \mid [\mathfrak{g}(s), \mathfrak{n}_s] \rangle \subset \langle \mathfrak{g}(s) \mid \mathfrak{n}_s \rangle = \{0\}$. In particular,

$$\langle n \mid [\mathfrak{g}(x), \mathfrak{n}_s] \rangle = \{0\} \quad \text{and} \quad \langle n \mid [\mathfrak{n}_e, \mathfrak{n}_s] \rangle = \{0\}. \tag{6}$$

Let us assume that the basis vectors $X_1, \ldots, X_{2r}$ of $\mathfrak{n}$ have been chosen such that $X_1, \ldots, X_{2p} \in \mathfrak{n}_e$ and $X_{2p+1}, \ldots, X_{2r} \in \mathfrak{n}_s$. Then the formulas (6) imply that the

Poisson matrix (3) takes at $n \in N_x$ the form

$$\Lambda(n) = \begin{pmatrix} A(n) & B_e(n) & 0 \\ -B_e(n)^\top & C_e(n) & 0 \\ 0 & 0 & C_s(n) \end{pmatrix},$$

where

$$\begin{aligned} A_{i,j}(n) &= \langle n \mid [Z_i, Z_j] \rangle, && \text{for } 1 \le i, j \le k; \\ B_{e;i,m}(n) &= \langle n \mid [Z_i, X_m] \rangle, && \text{for } 1 \le i \le k,\ 1 \le m \le 2p; \\ C_{e;l,m}(n) &= \langle n \mid [X_l, X_m] \rangle, && \text{for } 1 \le l,\ m \le 2p; \\ C_{s;l,m}(n) &= \langle n \mid [X_l, X_m] \rangle, && \text{for } 2p+1 < l,\ m \le 2r. \end{aligned}$$

It follows from (4) that the Poisson matrix of the transverse Poisson structure on $N_x$ is given by

$$\Lambda_{N_x}(n) = A(n) + B_e(n) C_e(n)^{-1} B_e(n)^\top. \tag{7}$$

Let us now restrict our attention to the Lie algebra $\mathfrak{g}(s)$, which, being reductive, decomposes as

$$\mathfrak{g}(s) = \mathfrak{z}(s) \oplus \mathfrak{g}_{ss}(s),$$

where $\mathfrak{z}(s)$ is the center of $\mathfrak{g}(s)$ and $\mathfrak{g}_{ss}(s) = [\mathfrak{g}(s), \mathfrak{g}(s)]$ is the semisimple part of $\mathfrak{g}(s)$. At the group level we have a similar decomposition of $\mathrm{G}(s)$, the centralizer of $s$ in G whose Lie algebra is $\mathfrak{g}(s)$, namely,

$$\mathrm{G}(s) = \boldsymbol{Z}(s)\mathrm{G}_{ss}(s),$$

where $\boldsymbol{Z}(s)$ is a central subgroup of $\mathrm{G}(s)$ and $\mathrm{G}_{ss}(s)$ is the semisimple part of $\mathrm{G}(s)$ with Lie algebra $\mathfrak{g}_{ss}(s)$. Since $e \in \mathfrak{g}(s)$, we can consider $\mathrm{G}(s) \cdot e$ as an adjoint orbit of the reductive Lie algebra $\mathfrak{g}(s)$. We may think of it as an adjoint orbit of a *semisimple* Lie algebra, since $\mathrm{G}(s) \cdot e = \mathrm{G}_{ss}(s) \cdot e$; similarly we may think of a transverse slice to the adjoint orbit $\mathrm{G}(s) \cdot e$ as a transverse slice to $\mathrm{G}_{ss}(s) \cdot e$ up to a summand with trivial Lie bracket. Denoting by $\perp_s$ the $\langle \cdot \mid \cdot \rangle$ orthogonal complement restricted to $\mathfrak{g}(s)$, we have that $N := e + \mathfrak{n}_e^{\perp_s}$ is a transverse slice to $\mathrm{G}(s) \cdot e$, since

$$\mathfrak{g}(s) = \mathfrak{g}(x) \oplus \mathfrak{n}_e = \mathfrak{z}(s) \oplus \mathfrak{g}_{ss}(s)(e) \oplus \mathfrak{n}_e.$$

We have used that $\mathfrak{g}(x) = \mathfrak{g}(s)(e)$ is the centralizer of $e$ in $\mathfrak{g}(s)$, which follows from (5). In the chosen bases $(Z_1, \ldots, Z_k)$ of $\mathfrak{g}(x)$ and $(X_1, \ldots, X_{2p})$ of $\mathfrak{n}_e$, the Poisson matrix at $n \in N$ takes the form

$$\begin{pmatrix} A(n) & B_e(n) \\ -B_e(n)^\top & C_e(n) \end{pmatrix},$$

which leads by Dirac reduction to the transverse Poisson structure $\Lambda_N$ on $N$:

$$\Lambda_N(n) = A(n) + B_e(n)C_e(n)^{-1}B_e(n)^\top,$$

where $n \in N$. This yields formally the same formula as (7), except that it is evaluated at points $n$ of $N$ rather than at points of $N_x$. However, since $\mathfrak{n}_e^{\perp_s} = \mathfrak{g}(s) \cap \mathfrak{n}_e^\perp = \mathfrak{n}_s^\perp \cap \mathfrak{n}_e^\perp = (\mathfrak{n}_s + \mathfrak{n}_e)^\perp = \mathfrak{n}^\perp$, the affine subspaces $N_x$ and $N$ only differ by a translation, $N_x = s + e + \mathfrak{n}^\perp = s + N$. Thus they, and their Poisson matrices with respect to the coordinates $q_1, \ldots, q_k$, can be identified, leading to:

**Proposition 2.1.** *Let $x \in \mathfrak{g}$ be any element, $\mathrm{G} \cdot x$ its adjoint orbit, and $x = s + e$ its Jordan–Chevalley decomposition. Given any complement $\mathfrak{n}_e$ of $\mathfrak{g}(x)$ in $\mathfrak{g}(s)$ and putting $\mathfrak{n} := \mathfrak{n}_s \oplus \mathfrak{n}_e$, where $\mathfrak{n}_s = \mathfrak{g}(s)^\perp$, the parallel affine spaces $N_x := x + \mathfrak{n}^\perp$ and $N := e + \mathfrak{n}^\perp$ are respectively transverse slices to the adjoint orbit $\mathrm{G} \cdot x$ in $\mathfrak{g}$ and to the nilpotent orbit $\mathrm{G}(s) \cdot e$ in $\mathfrak{g}(s)$. The Poisson structure on both transverse slices has the same Poisson matrix, namely that of (7), in the same affine coordinates restricted to the corresponding transverse slice.*

In short, the transverse Poisson structure to any adjoint orbit $\mathrm{G} \cdot x$ of a semisimple (or reductive) Lie algebra $\mathfrak{g}$ is essentially determined by the transverse Poisson structure of the underlying nilpotent orbit $\mathrm{G}(s) \cdot e$ defined by the Jordan–Chevalley decomposition $x = s + e$. A refinement of this proposition will be given in Corollary 3.5.

## 3. The polynomial and the quasihomogeneous character of the tranverse Poisson structure

In this section we show that, for a natural class of transverse slices to a nilpotent orbit $\mathbb{O}$ which we equip with an adapted set of linear coordinates centered at a nilpotent element $e \in \mathbb{O}$, the transverse Poisson structure is quasihomogeneous (of quasidegree $-2$) in the following sense.

**Definition 3.1.** Let $\nu = (\nu_1, \ldots, \nu_d)$ be nonnegative integers. A polynomial $P$ in $\mathbb{C}[x_1, \ldots, x_d]$ is said to be *quasihomogeneous* (relative to $\nu$) if, for some integer $\kappa$,

$$P(t^{\nu_1} x_1, \ldots, t^{\nu_d} x_d) = t^\kappa P(x_1, \ldots, x_d) \quad \text{for all } t \in \mathbb{C},$$

and $\kappa$ is then called the *quasidegree* (relative to $\nu$) of $P$, denoted $\varpi(P)$. Similarly, a polynomial Poisson structure $\{\cdot\,,\cdot\}$ on $\mathbb{C}[x_1, \ldots, x_d]$ is said to be *quasihomogeneous* (relative to $\nu$) if there exists $\kappa \in \mathbb{Z}$ such that, for any quasihomogeneous polynomials $F$ and $G$, their Poisson bracket $\{F, G\}$ is quasihomogeneous of degree

$$\varpi(\{F, G\}) = \varpi(F) + \varpi(G) + \kappa;$$

equivalently, for any $i$, $j$ the polynomial $\left\{x_i, x_j\right\}$ is quasihomogeneous of quasidegree $\nu_i + \nu_j + \kappa$. Then $\kappa$ is called the *quasidegree* of $\{\cdot\,,\cdot\}$.

We first show that, given $\mathbb{O}$, we can choose a system of linear coordinates on $\mathfrak{g}$, centered at some nilpotent element $e \in \mathbb{O}$, such that the Lie–Poisson structure on $\mathfrak{g}$ is quasihomogeneous relative to some vector $\nu$ that has a natural Lie-theoretic interpretation. To describe how this happens, we need to recall some facts from the theory of semisimple Lie algebras, which will be used throughout this paper. First, one chooses a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, with corresponding root system $\Delta(\mathfrak{h})$, from which a basis $\Pi(\mathfrak{h})$ of simple roots is selected. The *rank* of $\mathfrak{g}$, which is the dimension of $\mathfrak{h}$, is denoted by $\ell$. According to the Jacobson–Morosov–Kostant correspondence (see [Tauvel and Yu 2005, paragraphs 32.1 and 32.4]), there is a canonical triple $(h, e, f) \in \mathfrak{g}$ associated with $\mathbb{O}$ and completely determined, up to conjugation by $G(h)$, by the following properties:

- $(h, e, f)$ is a $\mathfrak{sl}_2$-triple, that is, $[h, e] = 2e$, $[h, f] = -2f$, and $[e, f] = h$;
- $h$ is the characteristic of $\mathbb{O}$, that is, $h \in \mathfrak{h}$ and $\alpha(h) \in \{0, 1, 2\}$ for any simple root $\alpha \in \Pi(\mathfrak{h})$.
- $\mathbb{O} = \mathrm{G} \cdot e$.

The triple $(h, e, f)$ leads to two decompositions of $\mathfrak{g}$.

First, $\mathfrak{g}$ decomposes into eigenspaces relative to $\mathrm{ad}_h$. Since each eigenvalue is an integer, we have

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i),$$

where $\mathfrak{g}(i)$ is the eigenspace of $\mathrm{ad}_h$ with eigenvalue $i$. For example, $e \in \mathfrak{g}(2)$ and $f \in \mathfrak{g}(-2)$.

Second, let $\mathfrak{s}$ be the Lie subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2$ that is generated by $h$, $e$ and $f$. The Lie algebra $\mathfrak{g}$ is an $\mathfrak{s}$-module, hence it decomposes as

$$\mathfrak{g} = \bigoplus_{j=1}^{k} V_{n_j},$$

where each $V_{n_j}$ is a simple $\mathfrak{s}$-module, with $n_j + 1 = \dim V_{n_j}$ and $\mathrm{ad}_h$-weights $n_j, n_j - 2, n_j - 4, \ldots, -n_j$. Moreover, $k = \dim \mathfrak{g}(e)$, since the centralizer $\mathfrak{g}(e)$ is generated by the highest weight vectors of each $V_{n_j}$. It follows that

$$\sum_{j=1}^{k} n_j = \dim \mathfrak{g} - k = \dim(\mathrm{G} \cdot e) = 2r. \tag{8}$$

We center at $e$ a system of linear coordinates on $\mathfrak{g}$ by using the action of Slodowy [1980b]: First, he considers the one-parameter subgroup of G,

$$\begin{array}{rcl} \lambda : \mathbb{C}^* & \to & \mathrm{G} \\ t & \mapsto & \exp(\lambda_t h), \end{array}$$

where $\lambda_t$ is a complex number such that $e^{-\lambda_t} = t$. The restriction of Ad to this subgroup leaves every eigenspace $\mathfrak{g}(i)$ invariant and acts for each $t$ as a homothecy with ratio $t^{-i}$ on $\mathfrak{g}(i)$:

$$\mathrm{Ad}_{\lambda(t)}\, x = t^{-i} x \quad \text{for all } x \in \mathfrak{g}(i). \tag{9}$$

Since $e \in \mathfrak{g}(2)$, the action $\rho$ of $\mathbb{C}^*$ on $\mathfrak{g}$ — defined for $t \in \mathbb{C}^*$ and for $y \in \mathfrak{g}$ by $\rho_t \cdot y := t^2\, \mathrm{Ad}_{\lambda(t)}\, y$ — fixes $e$. We refer to $\rho$ as *Slodowy's action*. To see how it leads to quasihomogeneous coordinates, let us define for $x \in \mathfrak{g}$ the function $\mathcal{F}_x(y) := \langle y - e \mid x\rangle$ for $y \in \mathfrak{g}$. Then (9) and the Ad-invariance of the Killing form imply that if $x \in \mathfrak{g}(i)$ then

$$\begin{aligned}\left(\rho_t^* \mathcal{F}_x\right)(y) &= \left\langle \rho_{t^{-1}} \cdot y - e \mid x\right\rangle = t^{-2}\left\langle \mathrm{Ad}_{\lambda(t^{-1})}(y-e) \mid x\right\rangle \\ &= t^{-2}\left\langle y - e \mid \mathrm{Ad}_{\lambda(t)}\, x\right\rangle = t^{-2}\left\langle y - e \mid t^{-i} x\right\rangle = t^{-i-2}\mathcal{F}_x(y).\end{aligned}$$

It follows that the quasidegree $\varpi(\mathcal{F}_x)$ of $\mathcal{F}_x$ is $i+2$ for $x \in \mathfrak{g}(i)$. According to (2), one has, for any $x,\ y,\ z \in \mathfrak{g}$,

$$\left\{\mathcal{F}_x, \mathcal{F}_y\right\}(z) = \langle z \mid [x, y]\rangle = \mathcal{F}_{[x,y]}(z) + \langle e \mid [x, y]\rangle\,. \tag{10}$$

If $x \in \mathfrak{g}(i)$ and $y \in \mathfrak{g}(j)$ with $i + j \neq -2$, then $\langle e \mid [x, y]\rangle = 0$ and so

$$\begin{aligned}\varpi(\left\{\mathcal{F}_x, \mathcal{F}_y\right\}) - \varpi(\mathcal{F}_x) - \varpi(\mathcal{F}_y) &= \varpi(\mathcal{F}_{[x,y]}) - \varpi(\mathcal{F}_x) - \varpi(\mathcal{F}_y) \\ &= i + j + 2 - (i+2) - (j+2) = -2.\end{aligned}$$

This result extends to the case $i + j = -2$, since then $\varpi(\mathcal{F}_{[x,y]}) = i + j + 2 = 0$, which is the quasidegree of the constant function $\langle e \mid [x, y]\rangle$. This proves:

**Proposition 3.2.** *Let $\mathfrak{g}$ be a semisimple Lie algebra identified with its dual using its Killing form. Let $\mathcal{O}$ be a nilpotent adjoint orbit of $\mathfrak{g}$ with canonical triple $(h, e, f)$. Let $x_1, \ldots, x_d$ be any basis in $\mathfrak{g}$, where each $x_k$ belongs to some eigenspace $\mathfrak{g}(i_k)$ of $\mathrm{ad}_h$, and let $\mathcal{F}_k$ be the dual coordinates on $\mathfrak{g}$ centered at $e$ as $\mathcal{F}_k(y) := \langle y - e \mid x_k\rangle$. Then the Lie–Poisson structure $\{\cdot, \cdot\}$ on $\mathfrak{g}$ is quasihomogeneous of degree $-2$ with respect to $(\varpi(\mathcal{F}_1), \ldots, \varpi(\mathcal{F}_d)) = (i_1 + 2, \ldots, i_d + 2)$.* □

We now wish to show that, upon picking a suitable transverse slice $N$ to $\mathcal{O}$ at $e$, the transverse Poisson structure on $N$ is also quasihomogeneous (of degree $-2$). Following [Sabourin 2005], we consider the set $\mathcal{N}_h$ of all subspaces $\mathfrak{n}$ of $\mathfrak{g}$ that are complementary to $\mathfrak{g}(e)$ in $\mathfrak{g}$ and are $\mathrm{ad}_h$-invariant. For $\mathfrak{n} \in \mathcal{N}_h$ we let $N := e + \mathfrak{n}^\perp$, which is a transverse slice to $\mathrm{G}\cdot e$. The $\mathrm{ad}_h$-invariance of $\mathfrak{n}$ implies on the one hand that $\rho$ leaves $N$ invariant: if $y \in e + \mathfrak{n}^\perp$ then

$$0 = \left\langle y - e \mid \mathrm{Ad}_{\lambda(t^{-1})}\, \mathfrak{n}\right\rangle = \left\langle \mathrm{Ad}_{\lambda(t)}(y-e) \mid \mathfrak{n}\right\rangle = t^{-2}\left\langle \rho_t \cdot y - e \mid \mathfrak{n}\right\rangle,$$

so that indeed $\rho_t \cdot y \in e + \mathfrak{n}^\perp$. On the other hand, it implies that $\mathfrak{n}$ admits a basis consisting of eigenvectors of $\mathfrak{h}$. Thus we can adapt the above basis $x_1, \ldots, x_d$ to $\mathfrak{n}$.

We can choose a basis $(Z_1,\dots,Z_k)$ for $\mathfrak{g}(e)$ and a basis $(X_1,\dots,X_{2r})$ for $\mathfrak{n}$ so that:

- each $Z_i$ for $1\le i\le k$ is a highest weight vector of weight $n_i$;
- each $X_i$ for $1\le i\le 2r$ is a weight vector of weight $\nu_i$.

The linear coordinates (centered at $e$) $\mathcal{F}_{Z_1},\dots,\mathcal{F}_{Z_k}$, when restricted to $N$, will be denoted $q_1,\dots,q_k$. By the above, their quasidegrees are defined as $\varpi(q_i):=n_i+2$. That the transverse Poisson structure is polynomial in these coordinates was first shown in [Sabourin 2005, Thm 2.3]. We now refine this statement.

**Proposition 3.3.** *In the notation of Proposition 3.2, the transverse Poisson structure on $N:=e+\mathfrak{n}^\perp$, where $\mathfrak{n}\in\mathcal{N}$, is a polynomial Poisson structure that is quasihomogeneous of degree $-2$ with respect to the quasidegrees $n_1+2,\dots,n_k+2$, where $n_1,\dots,n_k$ denote the highest weights of $\mathfrak{g}$ as an $\mathfrak{s}$-module.*

*Proof.* According to (4), we need to show that for any $1\le i,\ j\le k$ the functions $A_{ij}$ and $(BC^{-1}B^\top)_{ij}$ are quasihomogeneous of degree $\varpi(q_i)+\varpi(q_j)-2=n_i+n_j+2$. For $A_{ij}$ this is clear, since $A$ is part of the Poisson matrix of the Lie–Poisson structure on $\mathfrak{g}$, which we know is quasihomogeneous of degree $-2$. Similarly, we have $\varpi(B_{ip})=n_i+\nu_p+2$. Since

$$\varpi(B_{ip}C^{-1}_{ps}B_{js})=n_i+n_j+\nu_p+\nu_s+4+\varpi(C^{-1}_{ps}),$$

we must show that

$$\varpi(C^{-1}_{ps})=-\nu_p-\nu_s-2. \tag{11}$$

This follows from $\sum_{i=1}^{2r}(\nu_i+1)=0$, which is itself a consequence of (8). Indeed, consider a term of the form $C'_{ij}=C_{i_1j_1}\dots C_{i_{2r-1}j_{2r-1}}$, where

$$\{i_1,i_2,\dots,i_{2r-1}\}=\{1,2,\dots,2r\}\setminus\{s\},$$
$$\{j_1,i_2,\dots,j_{2r-1}\}=\{1,2,\dots,2r\}\setminus\{p\}.$$

Then

$$\varpi(C'_{ij})=\sum_{k=1}^{2r-1}(\nu_{i_k}+\nu_{j_k}+2)=-\nu_s-\nu_p-2,$$

A typical term of $C^{-1}_{ps}$ is of the form $C'_{ij}/\Delta(C)$, where $\Delta(C)$ is the determinant of $C$. As $C$ is of quasidegree zero, $\Delta(C)$ is constant by the previous argument. This observation was made in [Sabourin 2005, Theorem 2.3]. This gives us (11). □

**Remark 3.4.** Our referee pointed out that the quasihomogeneity of the transverse Poisson structure is implicit in [Gan and Ginzburg 2002] and [Premet 2002] for the special transversal $\mathfrak{n}=\operatorname{Ker}\operatorname{ad}_f$. Using the eigenspaces of $\operatorname{ad}_h$, these authors consider a filtration on the universal enveloping algebra $\mathcal{U}\mathfrak{g}$ of $\mathfrak{g}$, which yields a grading on the transversal Poisson algebra for this $\mathfrak{n}$. With the quasidegrees that we use, the filtration's graded algebra is $\mathrm{Sym}(\mathfrak{g}^*)$.

Let us consider now any adjoint orbit $\mathrm{G}\cdot x$ and $x = s + e$, the Jordan–Chevalley decomposition of $x$. We already considered this case in Proposition 2.1. A well-known result [Tauvel and Yu 2005, par. 32.1.7.] says that there exists an $\mathfrak{sl}_2$-triple $(h, e, f)$ such that $[s, h] = [s, f] = 0$. Consequently, $(h, e, f)$ is an $\mathfrak{sl}_2$-triple of the reductive Lie algebra $\mathfrak{g}(s)$, and we can also suppose that, up to conjugation by elements of $\mathrm{G}(s)$, $h$ is the characteristic of $\mathrm{G}(s)\cdot e$. Let $\mathcal{N}_{s,h}$ be the set of all complementary subspaces to $\mathfrak{g}(x)$ in $\mathfrak{g}(s)$ that are $\mathrm{ad}_h$-invariant. Then, by applying Proposition 3.3, we get:

**Corollary 3.5.** *As in Proposition 2.1, let $\mathfrak{n}_s = \mathfrak{g}(s)^\perp$, $\mathfrak{n}_e \in \mathcal{N}_{s,h}$ and $\mathfrak{n} = \mathfrak{n}_s \oplus \mathfrak{n}_e$. Let $N_x := x + \mathfrak{n}^\perp$, which is a transverse slice to $\mathrm{G}\cdot x$. Then the transverse Poisson structure on $N_x$ is polynomial and is quasihomogeneous of quasidegree $-2$.*

From now on, a transverse Poisson structure given by Proposition 3.3 will be called an *adjoint transverse Poisson structure* or ATP-structure.

## 4. Examples

We want to show in two examples how to compute the ATP-structure. In the first example, we consider the subregular orbit of $\mathfrak{g}_2$, and we compute it without choosing a representation of $\mathfrak{g}_2$. In the second example, the subregular orbit of $\mathfrak{so}_8$, we use a concrete representation rather than referring to tables of the Lie brackets in a Chevalley basis. These two examples will also serve later to illustrate the results we will prove on the nature of the ATP-structure. Both examples correspond to subregular orbits and lead to two of the simplest nontrivial ATP-structures in the following sense. If $\mathbb{O}$ is an adjoint orbit in $\mathfrak{g}$, then the ATP-structure to $\mathbb{O}$ has rank $\dim\mathfrak{g} - \ell - \dim\mathbb{O}$ at a generic point of any slice transverse to $\mathbb{O}$, since the Lie–Poisson structure on $\mathfrak{g}$ has rank[1] $\dim\mathfrak{g} - \ell$ at a generic point of $\mathfrak{g}$. For the regular nilpotent orbit $\mathbb{O}_{reg}$, the ATP-structure is trivial because $\dim\mathbb{O}_{reg} = \dim\mathfrak{g} - \ell$. So, the first interesting nilpotent orbit to consider is the subregular orbit, denoted by $\mathbb{O}_{sr}$. We recall two well-known facts [Collingwood and McGovern 1993]:

(1) the subregular orbit $\mathbb{O}_{sr}$ is the unique nilpotent orbit that is open and dense in the complement of $\mathbb{O}_{reg}$ in the nilpotent cone;

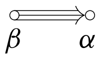
(2) $\dim\mathbb{O}_{sr} = \dim\mathfrak{g} - \ell - 2$.

It follows that the ATP-structure of the subregular orbit had dimension $\ell + 2$ and generic rank 2. In both of the following examples, we give the characteristic triplet

[1]Recall that $\ell$ denotes the rank of $\mathfrak{g}$.

$(h, e, f)$ that corresponds to the orbit; we derive from it a basis of the $\mathrm{ad}_h$-weight spaces, which leads to basis vectors $Z_i$ of $\mathfrak{g}(e)$ and $X_j$ of an $\mathrm{ad}_h$-invariant complement to $\mathfrak{g}(e)$ in $\mathfrak{g}$. The Lie brackets of these elements then lead to the matrices $A$, $B$ and $C$ in (3), which, by Dirac's formula (4), yields the matrix $\Lambda_N$ of the transverse Poisson structure.

***The subregular orbit of type $G_2$.*** We first consider the case of the subregular orbit of the Lie algebra $\mathfrak{g} := \mathfrak{g}_2$. Denoting the basis of simple roots by $\Pi = \{\alpha, \beta\}$, where $\beta$ is the longer root, its Dynkin diagram is given by




and it has the positive roots

$$\Delta_+ = \{\alpha,\ \beta,\ \alpha+\beta,\ 2\alpha+\beta,\ 3\alpha+\beta,\ 3\alpha+2\beta\}.$$

The vectors in the Chevalley basis[2] of $\mathfrak{g}$ are denoted by $H_\alpha$, $H_\beta$ for the Cartan subalgebra, $X_\gamma$ for the six positive roots $\gamma \in \Delta_+$, and $Y_\gamma$ for the six negative roots $-\gamma$, where $\gamma \in \Delta_+$. According to [Collingwood and McGovern 1993, Chapter 8.4], the characteristic $h$ of the subregular orbit $\mathbb{O}_{sr}$ is given by the sequence of weights $(0, 2)$, which means that $\langle \alpha, h\rangle = 0$ and $\langle \beta, h\rangle = 2$ and yields $h = 2H_\alpha + 4H_\beta$. The decomposition of $\mathfrak{g}$ into $\mathrm{ad}_h$-weight spaces $\mathfrak{g}(i)$ consists of five subspaces:

$$\begin{aligned}
\mathfrak{g}(4) &= \langle X_{3\alpha+2\beta}\rangle,\\
\mathfrak{g}(2) &= \langle X_\beta,\ X_{\alpha+\beta},\ X_{2\alpha+\beta},\ X_{3\alpha+\beta}\rangle,\\
\mathfrak{g}(0) &= \langle H_\alpha,\ H_\beta,\ X_\alpha,\ Y_\alpha\rangle,\\
\mathfrak{g}(-2) &= \langle Y_\beta,\ Y_{\alpha+\beta},\ Y_{2\alpha+\beta},\ Y_{3\alpha+\beta}\rangle,\\
\mathfrak{g}(-4) &= \langle Y_{3\alpha+2\beta}\rangle.
\end{aligned} \tag{12}$$

Taking for $e$ and $f$ an arbitrary linear combination of the above basis elements of $\mathfrak{g}(2)$ and $\mathfrak{g}(-2)$, respectively, and using $[e, f] = h$, one easily finds that the $\mathfrak{sl}_2$-triple corresponding to $\mathbb{O}_{sr}$ is

$$e = X_\beta + X_{3\alpha+\beta},\ \ h = 2H_\alpha + 4H_\beta,\ \ f = 2Y_\beta + 2Y_{3\alpha+\beta}.$$

Picking the vectors in the positive subspaces $\mathfrak{g}(i)$ that commute with $e$ leads to basis vectors of $\mathfrak{g}(e)$:

$$\begin{aligned}
Z_1 &= X_\beta + X_{3\alpha+\beta}, & Z_2 &= X_{2\alpha+\beta},\\
Z_3 &= X_{\alpha+\beta}, & Z_4 &= X_{3\alpha+2\beta}.
\end{aligned} \tag{13}$$

[2] The Chevalley basis that we use is explicitly described in [Tauvel 1998, Chapter VII.4].

We obtain an $\mathrm{ad}_h$-invariant complementary subspace $\mathfrak{n}$ of $\mathfrak{g}(e)$ by completing these vectors with additional vectors taken from the bases (12) of the subspaces $\mathfrak{g}(i)$. Our choice of basis vectors for $\mathfrak{n}$, ordered by weight, is

$$\begin{aligned} X_1 &= X_\beta, & X_6 &= Y_\beta, \\ X_2 &= X_\alpha, & X_7 &= Y_{\alpha+\beta}, \\ X_3 &= H_\alpha, & X_8 &= Y_{2\alpha+\beta}, \\ X_4 &= H_\beta, & X_9 &= Y_{3\alpha+\beta}, \\ X_5 &= Y_\alpha, & X_{10} &= Y_{3\alpha+2\beta}. \end{aligned}$$

The Lie brackets of these basis vectors for $\mathfrak{g}$, which are listed in [Tauvel 1998, Chapter VII.4], yield the Poisson matrix $((A, B), (-B^\top, C))$ of the Lie–Poisson structure on $\mathfrak{g}$ in the coordinates $\mathscr{F}_{Z_1}, \dots, \mathscr{F}_{Z_4}, \mathscr{F}_{X_1}, \dots, \mathscr{F}_{X_{10}}$ on $\mathfrak{g}$, as

$$A_{ij} = \left\{\mathscr{F}_{Z_i}, \mathscr{F}_{Z_j}\right\} = \mathscr{F}_{[Z_i, Z_j]} + \left\langle e \mid [Z_i, Z_j]\right\rangle$$

(see (10)), and similarly for the other elements of the Poisson matrix. We give the restriction of the matrices $A$, $B$ and $C$ to the transverse slice $N := e + \mathfrak{n}^\perp$ only, which amounts to keeping in the Lie brackets only the vectors $Z_1, \dots, Z_4$, as $\mathscr{F}_X(n) = \langle e - n \mid X\rangle = 0$ for $X \in \mathfrak{n}$ and $n \in N = e + \mathfrak{n}^\perp$. In the coordinates $q_1, \dots, q_4$ on $N$, where $q_i$ is the restriction of $\mathscr{F}_{Z_i}$ to $N$, we get

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -3q_4 & 0 \\ 0 & 3q_4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 & 0 & -q_4 & 0 & q_1 & -q_2 & q_3 & 0 & 0 \\ 0 & 3q_1 & -q_2 & 0 & 2q_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2q_2 & q_3 & -q_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ q_4 & q_3 & -3q_1 & q_1 & q_2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$C = \frac{1}{3}\begin{pmatrix} 0 & 3q_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -3q_3 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & -3 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Substituted in (4), this yields the Poisson matrix for the ATP-structure:

$$\Lambda_N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -3q_4 & 2q_1q_2-2q_3^2 \\ 0 & 3q_4 & 0 & 2q_2^2-2q_1q_3 \\ 0 & -2q_1q_2+2q_3^2 & -2q_2^2+2q_1q_3 & 0 \end{pmatrix}. \tag{14}$$

It follows from (12) and (13) that the quasidegree of $q_1$, $q_2$ and $q_3$ is 4, while the quasidegree of $q_4$ is 6. One easily reads off from (14) that, with respect to these quasidegrees, the ATP-structure is quasihomogeneous of quasidegree $-2$.

***The subregular orbit of type $D_4$.*** We now take $\mathfrak{g} = \mathfrak{so}_8$ and we realize $\mathfrak{g}$ as the following set of matrices:

$$\left\{ \begin{pmatrix} Z_1 & Z_2 \\ Z_3 & -Z_1^\top \end{pmatrix} \mid Z_i \in \mathrm{Mat}_4(\mathbb{C}),\ \text{with } Z_2, Z_3 \text{ skew-symmetric} \right\}.$$

Let $\mathfrak{h}$ denote the Cartan subalgebra of $\mathfrak{g}$ consisting of all diagonal matrices in $\mathfrak{g}$. Clearly, $\mathfrak{h}$ is spanned by the four matrices $H_i := E_{i,i} - E_{4+i,4+i}$, $1 \le i \le 4$. Define for $i = 1, \dots, 4$ the linear map $e_i \in \mathfrak{h}^*$ by

$$e_i\left(\sum a_k H_k\right) = a_i.$$

Then the root system of $\mathfrak{g}$ is

$$\Delta := \{\pm e_i \pm e_j \mid 1 \le i, j \le 4, i \ne j\},$$

and a basis of simple roots is $\Pi := \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, where

$$\alpha_1 = e_1 - e_2,\ \alpha_2 = e_2 - e_3,\ \alpha_3 = e_3 - e_4,\ \alpha_4 = e_3 + e_4.$$

It leads to the following Chevalley basis of $\mathfrak{g}$:

$$\begin{aligned} X_{e_i-e_j} &= E_{i,j} - E_{4+j,4+i}, \\ X_{e_i+e_j} &= E_{i,4+j} - E_{j,4+i}, && i < j, \\ X_{-e_i-e_j} &= -E_{4+i,j} + E_{4+j,i}, && i < j, \\ H_{e_i-e_j} &= H_i - H_j, \\ H_{e_i+e_j} &= H_i + H_j. \end{aligned}$$

According to [Collingwood and McGovern 1993, Chapter 5.4], the characteristic $h$ of the subregular orbit is given by the sequence of weights $(2, 0, 2, 2)$. It follows that

$$h = 4H_{\alpha_1} + 6H_{\alpha_2} + 4H_{\alpha_3} + 4H_{\alpha_4}.$$

The positive $\mathrm{ad}_h$-weight spaces are

$$
(15)\qquad
\begin{aligned}
\mathfrak{g}(0) &= \mathfrak{h} \oplus \langle X_{\alpha_2}, X_{-\alpha_2} \rangle, \\
\mathfrak{g}(2) &= \langle X_{\alpha_1}, X_{\alpha_3}, X_{\alpha_4}, X_{\alpha_1+\alpha_2}, X_{\alpha_2+\alpha_3}, X_{\alpha_2+\alpha_4} \rangle, \\
\mathfrak{g}(4) &= \langle X_{\alpha_1+\alpha_2+\alpha_3}, X_{\alpha_2+\alpha_3+\alpha_4}, X_{\alpha_1+\alpha_2+\alpha_4} \rangle, \\
\mathfrak{g}(6) &= \langle X_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}, X_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4} \rangle.
\end{aligned}
$$

As in the first example, it follows that the canonical $\mathfrak{sl}_2$-triple associated to $\mathbb{O}_{sr}$ is

$$
\begin{aligned}
e &= X_{\alpha_1} + X_{\alpha_1+\alpha_2} - X_{\alpha_2+\alpha_4} + 2X_{\alpha_3} - X_{\alpha_4}, \\
h &= 4H_{\alpha_1} + 6H_{\alpha_2} + 4H_{\alpha_3} + 4H_{\alpha_4}, \\
f &= X_{-\alpha_1} + 3X_{-\alpha_1-\alpha_2} - 3X_{-\alpha_2-\alpha_4} + 2X_{-\alpha_3} - X_{-\alpha_4}.
\end{aligned}
$$

We can now define the basis vectors $Z_i$ of $\mathfrak{g}(e)$ and $X_j$ of an $\mathrm{ad}_h$-invariant complementary subspace $\mathfrak{n}$ to $\mathfrak{g}(e)$ in the Chevalley basis:

$$
(16)\qquad
\begin{aligned}
Z_1 &= X_{\alpha_1+\alpha_2} - X_{\alpha_2+\alpha_4} + 2X_{\alpha_3}, \\
Z_2 &= X_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}, \\
Z_3 &= X_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}, \\
Z_4 &= X_{\alpha_1} - X_{\alpha_4}, \\
Z_5 &= X_{\alpha_2+\alpha_3} + X_{\alpha_2+\alpha_4} - X_{\alpha_3} - X_{\alpha_4}, \\
Z_6 &= X_{\alpha_1+\alpha_2+\alpha_3} + X_{\alpha_1+\alpha_2+\alpha_4} - X_{\alpha_2+\alpha_3+\alpha_4},
\end{aligned}
$$

$$
\begin{aligned}
X_1 &= X_{\alpha_1+\alpha_2+\alpha_3}, & X_8 &= H_{\alpha_3}, & X_{15} &= X_{-\alpha_1-\alpha_2}, \\
X_2 &= X_{\alpha_2+\alpha_3+\alpha_4}, & X_9 &= H_{\alpha_4}, & X_{16} &= X_{-\alpha_2-\alpha_3}, \\
X_3 &= X_{\alpha_4}, & X_{10} &= X_{\alpha_2}, & X_{17} &= -X_{-\alpha_2-\alpha_4}, \\
X_4 &= X_{\alpha_3}, & X_{11} &= X_{-\alpha_2}, & X_{18} &= -X_{-\alpha_1-\alpha_2-\alpha_3}, \\
X_5 &= X_{\alpha_2+\alpha_4}, & X_{12} &= X_{-\alpha_1}, & X_{19} &= -X_{-\alpha_1-\alpha_2-\alpha_4}, \\
X_6 &= H_{\alpha_1}, & X_{13} &= X_{-\alpha_3}, & X_{20} &= -X_{-\alpha_2-\alpha_3-\alpha_4}, \\
X_7 &= H_{\alpha_2}, & X_{14}, &= -X_{-\alpha_4}, & X_{21} &= -X_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4}, \\
& & & & X_{22} &= -X_{-\alpha_1-2\alpha_2-\alpha_3-\alpha_4}.
\end{aligned}
$$

If we denote by $\bar{Z}_1, \dots, \bar{Z}_6$ the dual basis (with respect to $\langle X \mid Y \rangle = \frac{1}{2}\,\mathrm{Trace}(XY)$) of the basis $Z_1, \dots, Z_6$ of $\mathfrak{g}(e)$, then a typical element of the transverse slice $N =$

$e+\mathfrak{n}^\perp$ is $e+\sum_{i=1}^{6} q_i \bar{Z}_i$, that is,

$$Q=\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0\\ q_4 & 0 & 0 & 0 & 0 & 0 & 0 & -1\\ q_1 & 0 & 0 & 2 & 0 & 0 & 0 & -1\\ 0 & q_5 & 0 & 0 & 0 & 1 & 1 & 0\\ 0 & -q_3 & -q_2 & 0 & 0 & -q_4 & -q_1 & 0\\ q_3 & 0 & q_6 & 0 & -1 & 0 & 0 & -q_5\\ q_2 & -q_6 & 0 & 0 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \end{pmatrix}, \tag{17}$$

and we can compute the matrix $A$ restricted to $N$ by $A_{ij}=\langle Q \mid [Z_i, Z_j]\rangle$, and similarly for the matrices $B$ and $C$. A direct substitution in (4) leads to the Poisson matrix for the ATP-structure:

$$\Lambda_N=\frac{1}{2}\begin{pmatrix} 0 & q_4q_6 & -q_4q_6 & 0 & -2q_6 & 2q_{16}\\ -q_4q_6 & 0 & 0 & q_4q_6 & -q_5q_6 & -2q_{36}\\ q_4q_6 & 0 & 0 & -q_4q_6 & q_5q_6 & 2q_{36}\\ 0 & -q_4q_6 & q_4q_6 & 0 & 2q_6 & -2q_{16}\\ 2q_6 & q_5q_6 & -q_5q_6 & -2q_6 & 0 & 2q_{56}\\ -2q_{16} & 2q_{36} & -2q_{36} & 2q_{16} & -2q_{56} & 0 \end{pmatrix}, \tag{18}$$

where

$$\begin{aligned} q_{16}&=2q_2-q_1q_4-q_4q_5+q_4^2,\\ q_{36}&=q_3q_4-q_2q_4-q_2q_5,\\ q_{56}&=2q_3-2q_2-q_5^2+q_4q_5-q_1q_5. \end{aligned} \tag{19}$$

It follows from (15) and (16) that the quasidegrees of the variables $q_i$ are $\varpi(q_1)=\varpi(q_4)=\varpi(q_5)=4$, $\varpi(q_2)=\varpi(q_3)=8$, and $\varpi(q_6)=6$. That the ATP-structure is quasihomogeneous of quasidegree $-2$ can again be easily read off from (18).

## 5. The subregular case

In this section we will explicitly describe the ATP-structure of the subregular orbit $\mathbb{O}_{sr}\subset\mathfrak{g}$, where $\mathfrak{g}$ is a semisimple Lie algebra. Since in the subregular orbit the generic rank of the ATP-structure on the transverse slice $N$ is two, and since we know $\dim(N)-2$ independent Casimirs, namely the basic Ad-invariant functions on $\mathfrak{g}$ restricted to $N$, we will easily derive that the ATP-structure is the determinantal structure (also called Nambu structure) determined by these Casimirs, up to multiplication by a *function.* What is much less trivial to show is that this function is only a *constant.* For this we will use Brieskorn's theory of simple singularities, which is recalled in Section 5 below. First we recall the basic facts on Ad-invariant functions on $\mathfrak{g}$ and link them to the ATP-structure.

***Invariant functions and Casimirs.*** Let $\mathbb{O}_{sr} = \mathrm{G} \cdot e$, be a subregular orbit in the semisimple Lie algebra $\mathfrak{g}$. Let $(h, e, f)$ be the corresponding canonical $\mathfrak{sl}_2$-triple, and consider the transverse slice $N := e + \mathfrak{n}^{\perp}$ to $\mathrm{G} \cdot e$, where $\mathfrak{n}$ is an $\mathrm{ad}_h$-invariant complement to $\mathfrak{g}(e)$. We know from Section 3 that the ATP-structure on $N$, when equipped with the linear coordinates $q_1, \dots, q_k$, is a quasihomogeneous polynomial Poisson structure of generic rank 2. Let $S(\mathfrak{g}^*)^{\mathrm{G}}$ be the algebra of Ad-invariant polynomial functions on $\mathfrak{g}$. By a classical theorem due to Chevalley, $S(\mathfrak{g}^*)^{\mathrm{G}}$ is a polynomial algebra generated by $\ell$ homogeneous polynomials $(G_1, \dots, G_\ell)$ whose degree $d_i := \deg(G_i) = m_i + 1$, where $m_1, \dots, m_\ell$ are the exponents of $\mathfrak{g}$. These functions are Casimirs of the Lie–Poisson structure on $\mathfrak{g}$, since Ad-invariance of $G_i$ implies that $[x, dG_i(x)] = 0$, and hence the Lie–Poisson bracket (2) is

$$\{F, G_i\}(x) = \langle x \mid [dF(x), dG_i(x)]\rangle = -\langle [x, dG_i(x)] \mid dF(x)\rangle = 0$$

for any function $F$ on $\mathfrak{g}$. If we denote by $\chi_i$ the restriction of $G_i$ to the transverse slice $N$ then, it follows that these functions are Casimirs of the ATP-structure. The polynomials $\chi_i$ are not homogeneous, but they are quasihomogeneous.

**Lemma 5.1.** *Each $\chi_i$ is a quasihomogeneous polynomial of quasidegree $2d_i$ relative to the quasidegrees $(2+n_1, \dots, 2+n_k)$.*

*Proof.* Since $\chi_i$ is of degree $d_i$ and $\chi_i$ is Ad-invariant, we get

$$\rho_t^*(\chi_i) = \chi_i \circ \rho_{t^{-1}} = \chi_i \circ (t^{-2}\,\mathrm{Ad}_{\lambda^{-1}(t)}) = t^{-2d_i}\chi_i \circ \mathrm{Ad}_{\lambda^{-1}(t)} = t^{-2d_i}\chi_i,$$

so that $\chi_i$ has quasidegree $2d_i$. □

***Simple singularities.*** Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. The Weyl group $\mathcal{W}$ acts on $\mathfrak{h}$, and the algebra $S(\mathfrak{g}^*)^{\mathrm{G}}$ of Ad-invariant polynomial functions on $\mathfrak{g}$ is isomorphic to $S(\mathfrak{h}^*)^{\mathcal{W}}$, the algebra of $\mathcal{W}$-invariant polynomial functions on $\mathfrak{h}^*$. The inclusion homomorphism $S(\mathfrak{g}^*)^{\mathrm{G}} \hookrightarrow S(\mathfrak{g}^*)$, is dual to a morphism $\mathfrak{g} \to \mathfrak{h}/\mathcal{W}$, called the *adjoint quotient*. Concretely, the adjoint quotient is given by

$$\begin{aligned} G : \mathfrak{g} &\to \mathbb{C}^\ell \\ x &\mapsto (G_1(x),\ G_2(x), \dots, G_\ell(x)). \end{aligned} \tag{20}$$

The zero-fiber $G^{-1}(0)$ of $G$ is exactly the nilpotent variety $\mathcal{N}$ of $\mathfrak{g}$. As we are interested in $N \cap \mathcal{N} = N \cap G^{-1}(0) = \chi^{-1}(0)$ — which is an affine surface with an isolated, simple singularity — let us recall the notion of simple singularity (see [Slodowy 1980a] for details). Up to conjugacy, there are five types of finite subgroups of $\mathrm{SL}_2 = \mathrm{SL}_2(\mathbb{C})$, which are denoted by $\mathcal{C}_p$, $\mathcal{D}_p$, $\mathcal{T}$, $\mathbb{O}$, and $\mathcal{I}$. Given such a subgroup $\boldsymbol{F}$, one looks at the corresponding ring of invariant polynomials $\mathbb{C}[u, v]^{\boldsymbol{F}}$.

In each of the five cases, $\mathbb{C}[u, v]^{\boldsymbol{F}}$ is generated by three fundamental polynomials $X, Y, Z$, subject to only one relation $R(X, Y, Z) = 0$; hence the quotient space $\mathbb{C}^2/\boldsymbol{F}$ can be identified, as an affine surface, with the singular surface in $\mathbb{C}^3$ defined by $R = 0$. The origin is its only singular point; it is called a *(homogeneous) simple singularity*. The exceptional divisor of the minimal resolution of $\mathbb{C}^2/\boldsymbol{F}$ is a finite set of projective lines. If two of these lines meet, then they meet in a single point, and transversally. Moreover, the intersection pattern of these lines forms a graph that coincides with one of the simply laced Dynkin diagrams of type $A_\ell$, $D_\ell$, $E_6$, $E_7$, or $E_8$. This type is called the type of the singularity. Moreover, every such Dynkin diagram (that is, of type ADE) appears in this way; see Table 1.

For the other simple Lie algebras (of type $B_\ell$, $C_\ell$, $F_4$ or $G_2$), there exists a similar correspondence. By definition, an *(inhomogeneous) simple singularity* of type $\Delta$ is a couple $(V, \Gamma)$ consisting of a homogeneous simple singularity $V = \mathbb{C}^2/\boldsymbol{F}$ and a group $\Gamma = \boldsymbol{F}'/\boldsymbol{F}$ of automorphisms of $V$, according to Table 2.

The connection between the diagram of $(V, \Gamma)$ and that of $V$ can be described as follows. The action of $\Gamma$ on $V$ lifts to an action on a minimal resolution of $V$ that permutes the components of the exceptional set. Then, we obtain the diagram of $(V, \Gamma)$ as a $\Gamma$-quotient of that of $V$. It leads to Table 3, which is the nonsimply-laced analog of Table 1.

We can now state an extension of a theorem of Brieskorn.

**Proposition 5.2** [Slodowy 1980a, Theorems 1 and 2]. *Let $\mathfrak{g}$ be a simple complex Lie algebra with Dynkin diagram of type $\Delta$. Let $\mathbb{O}_{sr} = \mathrm{G}\cdot e$ be the subregular orbit, and let $N = e + \mathfrak{n}^\perp$ be a transverse slice to $\mathrm{G}\cdot e$. The surface $N \cap \mathcal{N} = \chi^{-1}(0)$ has a* (*homogeneous or inhomogeneous*) *simple singularity of type $\Delta$.*

To finish this section we illustrate the results above for the examples of Section 4. In both cases we give the invariants restricted to the slice $N$ and their zero locus,

| Group $\boldsymbol{F}$ | Singularity $R(X, Y, Z) = 0$ | Type $\Delta$ |
|---|---|---|
| $\mathscr{C}_{\ell+1}$ | $X^{\ell+1} + YZ = 0$ | $A_\ell$ |
| $\mathscr{D}_{\ell-2}$ | $X^{\ell-1} + XY^2 + Z^2 = 0$ | $D_\ell$ |
| $\mathscr{T}$ | $X^4 + Y^3 + Z^2 = 0$ | $E_6$ |
| $\mathscr{O}$ | $X^3Y + Y^3 + Z^2 = 0$ | $E_7$ |
| $\mathscr{I}$ | $X^5 + Y^3 + Z^2 = 0$ | $E_8$ |

**Table 1.** The basic correspondence between finite subgroups $\boldsymbol{F}$ of $\mathrm{SL}_2$, homogeneous simple singularities defined by an equation $R(X, Y, Z) = 0$, and simply laced simple Lie algebras of type $\Delta$.

the surface $\chi^{-1}(0)$.

First, for the subregular orbit of $\mathfrak{g}_2$, the invariant functions restricted to the slice $N$ are

$$\begin{aligned} \chi_1 &= q_1, \\ \chi_2 &= 12q_1q_2q_3 - 4q_2^3 - 4q_3^3 + 9q_4^2, \end{aligned} \tag{21}$$

which leads to an affine surface $\chi^{-1}(0)$ in $\mathbb{C}^4$ that is isomorphic to the surface in $\mathbb{C}^3$ defined by

$$4q_2^3 + 4q_3^3 - 9q_4^2 = 0.$$

Up to a rescaling, this is the polynomial $R$ that was given in Table 3.

Second, for the subregular orbit of $\mathfrak{so}_8$, the invariant functions restricted to the slice $N$ are found as the (nonconstant) coefficients of the characteristic polynomial

| Type $\Delta$ | $V$ | $\boldsymbol{F}$ | $\boldsymbol{F}'$ | $\Gamma = \boldsymbol{F}'/\boldsymbol{F}$ |
|---|---|---|---|---|
| $B_\ell$ | $A_{2\ell-1}$ | $\mathscr{C}_{2\ell}$ | $\mathscr{D}_\ell$ | $\mathbb{Z}/2\mathbb{Z}$ |
| $C_\ell$ | $D_{\ell+1}$ | $\mathscr{D}_{\ell-1}$ | $\mathscr{D}_{2\ell-2}$ | $\mathbb{Z}/2\mathbb{Z}$ |
| $F_4$ | $E_6$ | $\mathscr{T}$ | $\mathbb{O}$ | $\mathbb{Z}/2\mathbb{Z}$ |
| $G_2$ | $D_4$ | $\mathscr{D}_2$ | $\mathbb{O}$ | $\mathbb{Z}/3\mathbb{Z}$ |

**Table 2.** List of all possible inhomogeneous singularities of type $\Delta = (V, \Gamma)$, where $V$ is one of the homogeneous simple singularities and $\Gamma = \boldsymbol{F}'/\boldsymbol{F}$ is a group of automorphisms of $V$. The labels $B_\ell$, $C_\ell$, $F_4$ and $G_2$ for these types will become clear in Proposition 5.2.

| Type $\Delta$ | Singularity $R(X, Y, Z) = 0$ | $\Gamma$-action |
|---|---|---|
| $B_\ell$ | $X^{2\ell} + YZ = 0$ | $(X, Y, Z) \longrightarrow (-X, Z, Y)$ |
| $C_\ell$ | $X^\ell + XY^2 + Z^2 = 0$ | $(X, Y, Z) \longrightarrow (X, -Y, -Z)$ |
| $F_4$ | $X^4 + Y^3 + Z^2 = 0$ | $(X, Y, Z) \longrightarrow (-X, Y, -Z)$ |
| $G_2$ | $X^3 + Y^3 + Z^2 = 0$ | $(X, Y, Z) \longrightarrow (\alpha X, \alpha^2 Y, Z)$ |

**Table 3.** For each of the inhomogeneous simple singularities of type $\Delta$ (see Table 2), the corresponding homogeneous simple singularity $V = \mathbb{C}^2/\boldsymbol{F}$ is given by its equation $R(X, Y, Z) = 0$ together with the action of $\Gamma = \boldsymbol{F}'/\boldsymbol{F}$ on $V$. In the last line, $\alpha$ is a nontrivial cubic root of unity.

of the matrix $Q$ (see (17)):

$$
\begin{aligned}
\chi_1 &= -2q_1 - 2q_4,\\
\chi_2 &= -12q_2 - 4q_3 - 4q_4q_5 + (q_1+q_4)^2,\\
\chi_3 &= -q_2 + q_3 - q_4q_5,\\
\chi_4 &= -4q_1q_2 - 16q_2q_5 - 12q_3q_4 + 12q_2q_4 + 4q_1q_3 + 4q_4^2q_5 + 4q_1q_4q_5 - 4q_6^2.
\end{aligned} \tag{22}
$$

By linearly eliminating the variables $q_1$, $q_2$ and $q_3$ from the equations $\chi_i = 0$ for $i = 1, 2, 3$, we find that $\chi^{-1}(0)$ is isomorphic to the affine surface in $\mathbb{C}^3$ defined by

$$4q_4^2q_5 - 2q_4q_5^2 + q_6^2 = 0.$$

Its defining polynomial corresponds to the polynomial $R$ in Table 1, after putting $X = i\gamma q_4$, $Y = \gamma(q_5 - q_4)$, and $Z = q_6$, where $\gamma$ is any cubic root of $2i$.

***The determinantal Poisson structure.*** We prove here the announced result that the ATP-structure in the subregular case is a determinantal Poisson structure determined by the Casimirs. Let us first point out how such a structure is defined. Let $C_1, \dots, C_{d-2}$ be $d-2$ (algebraically) independent polynomials in $d > 2$ variables $x_1, \dots, x_d$. For a polynomial $F$ in the variables $x_1, \dots, x_d$, let us denote by $\nabla F$ its differential $\mathrm{d}F$, expressed in the natural basis $\mathrm{d}x_i$, that is, $\nabla F$ is a column vector with elements $(\nabla F)_i = \partial F/\partial x_i$. Then a polynomial Poisson structure is defined on $\mathbb{C}^d$ by

$$\{F, G\}_{\det} := \det(\nabla F,\ \nabla G,\ \nabla C_1,\ \dots,\ \nabla C_{d-2}), \tag{23}$$

where $F$ and $G$ are arbitrary polynomials. It is clear that each of the $C_i$ is a Casimir of $\{\cdot\,,\cdot\}_{\det}$, so that in particular the generic rank of $\{\cdot\,,\cdot\}_{\det}$ is two. Notice also that if the Casimirs $C_i$ are quasihomogeneous with respect to the weights $\varpi_i := \varpi(x_i)$, then for any quasihomogeneous elements $F$ and $G$ we have

$$\varpi(\{F, G\}_{\det}) = \varpi(F) + \varpi(G) + \sum_{i=1}^{d-2} \varpi(C_i) - \sum_{i=1}^{d} \varpi_i.$$

This follows easily from the definition of a determinant and that if $F$ is any quasihomogeneous polynomial, then $\partial F/\partial x_i$ is quasihomogeneous and $\varpi(\partial F/\partial x_i) = \varpi(F) - \varpi_i$. It follows that $\{\cdot\,,\cdot\}_{\det}$ is quasihomogeneous of quasidegree $\kappa$, where

$$\kappa = \sum_{i=1}^{d-2} \varpi(C_i) - \sum_{i=1}^{d} \varpi_i. \tag{24}$$

Applied to our case, it means that we have two polynomial Poisson structures on the transverse slice $N$ that have $\chi_1, \dots, \chi_\ell$ as Casimirs on $N \cong \mathbb{C}^{\ell+2}$, namely, the ATP-structure and the determinantal structure constructed by using these Casimirs.

**Remark 5.3.** The determinantal Poisson structure first appears (without proof that it is a Poisson structure) in [Damianou 1989], who attributes the formula to H. Flaschka and T. Ratiu. The first explicit proof appears in [Grabowski et al. 1993]. A more conceptual proof appears in [Takhtajan 1994, Remark 1 and Theorem 4].

In our two examples (see Section 4), these structures are easily compared by explicit computation. For the subregular orbit of $\mathfrak{g}_2$, we have, according to (23),

$$(\Lambda_{\text{det}})_{ij} = \det\left(\nabla q_i \ \nabla q_j \ \nabla \chi_1 \ \nabla \chi_2\right),$$

where $\chi_1$ and $\chi_2$ are the Casimirs (21). This leads to

$$\Lambda_{\text{det}} = -6 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -3q_4 & 2q_1q_2 - 2q_3^2 \\ 0 & 3q_4 & 0 & 2q_2^2 - 2q_1q_3 \\ 0 & -2q_1q_2 + 2q_3^2 & -2q_2^2 + 2q_1q_3 & 0 \end{pmatrix}.$$

In view of (14), it follows that $\Lambda_{\text{det}} = -6\Lambda_N$, so that both Poisson structures coincide. For $\mathfrak{so}_8$, one finds similarly, using the Casimirs $\chi_1, \dots, \chi_4$ in (22),

$$\Lambda_{\text{det}} = -128 \begin{pmatrix} 0 & q_4q_6 & -q_4q_6 & 0 & -2q_6 & 2q_{16} \\ -q_4q_6 & 0 & 0 & q_4q_6 & -q_5q_6 & -2q_{36} \\ q_4q_6 & 0 & 0 & -q_4q_6 & q_5q_6 & 2q_{36} \\ 0 & -q_4q_6 & q_4q_6 & 0 & 2q_6 & -2q_{16} \\ 2q_6 & q_5q_6 & -q_5q_6 & -2q_6 & 0 & 2q_{56} \\ -2q_{16} & 2q_{36} & -2q_{36} & 2q_{16} & -2q_{56} & 0 \end{pmatrix},$$

where $q_{16}$, $q_{36}$ and $q_{56}$ are given by (19). In view of (18), both Poisson structures again coincide, $\Lambda_{\text{det}} = -256\Lambda_N$.

To show that, in the subregular case, the ATP-structure and the determinantal structure always coincide, that is, they differ only by a constant factor, we first show that both structures coincide up to a rational function.

**Proposition 5.4.** *Suppose* $\{\cdot\,,\cdot\}$ *and* $\{\cdot\,,\cdot\}'$ *are two nontrivial polynomial Poisson structures on* $\mathbb{C}^d$ *that have* $d-2$ *common independent polynomial Casimirs* $C_1, \dots, C_{d-2}$. *Then there exists a rational function* $R \in \mathbb{C}(x_1, \dots, x_d)$ *such that* $\{\cdot\,,\cdot\} = R\,\{\cdot\,,\cdot\}'$.

*Proof.* Let $M$ and $M'$ denote the Poisson matrices of $\{\cdot\,,\cdot\}$ and $\{\cdot\,,\cdot\}'$ in the coordinates $x_1, \dots, x_d$. If we denote $\mathcal{R} := \mathbb{C}(x_1, \dots, x_d)$, then $M$ and $M'$ both act naturally as skew-symmetric endomorphisms on the $\mathcal{R}$-vector space $\mathcal{R}^d$. The subspace $H$ of $\mathcal{R}^d$ spanned by $\nabla C_1, \dots, \nabla C_{d-2}$ is the kernel of both maps; hence we have two induced skew-symmetric endomorphisms $\varphi$ and $\varphi'$ of the quotient space $\mathcal{R}^d/H$. Since the latter is two-dimensional, $\varphi'$ and $\varphi$ are proportional, that is, $\varphi' = R\varphi$ with $R \in \mathcal{R}$. Since $M$ and $M'$ have the same kernel, $M' = RM$. □

Applied to our two Poisson structures $\{\cdot\,,\cdot\}_N$ and $\{\cdot\,,\cdot\}_{\det}$, the proposition yields that $\{\cdot\,,\cdot\}_N = R\{\cdot\,,\cdot\}_{\det}$, where $R = P/Q \in \mathcal{R}$. We show next that $R$ is actually a (nonzero) constant and thereby characterize completely the ATP-structure in the subregular case.

**Theorem 5.5.** *Let $\mathcal{O}_{sr}$ be the subregular nilpotent adjoint orbit of a complex semisimple Lie algebra $\mathfrak{g}$, and let $(h, e, f)$ be the canonical triple associated to $\mathcal{O}_{sr}$. Let $N = e + \mathfrak{n}^\perp$ be a slice transverse to $\mathcal{O}_{sr}$, where $\mathfrak{n}$ is an $\mathrm{ad}_h$-invariant complementary subspace to $\mathfrak{g}(e)$. Let $\{\cdot\,,\cdot\}_N$ and $\{\cdot\,,\cdot\}_{det}$ denote respectively the ATP-structure and the determinantal structure on $N$. Then $\{\cdot\,,\cdot\}_N = c\{\cdot\,,\cdot\}_{det}$ for some $c \in \mathbb{C}^*$.*

*Proof.* By the above, $\{\cdot\,,\cdot\}_N = R\{\cdot\,,\cdot\}_{\det}$, where $R \in \mathcal{R}$. If $R$ has a nontrivial denominator $Q$, then all elements of the Poisson matrix of $\{\cdot\,,\cdot\}_{\det}$ must be divisible by $Q$, since both Poisson structures are polynomial. Then along the hypersurface $Q = 0$, the rank of $(\nabla\chi_1, \dots, \nabla\chi_\ell)$ is smaller than $\ell$; hence $\chi^{-1}(0)$ is singular along the curve $\chi^{-1}(0) \cap (Q = 0)$. However, by Proposition 5.2, we know that $\chi^{-1}(0)$ has an isolated singularity, which leads to a contradiction. This shows that $Q$ is a constant and hence that $R$ is a polynomial.

To show that the polynomial $R$ is constant, it suffices to show that the quasidegrees of $\{\cdot\,,\cdot\}_N$ and $\{\cdot\,,\cdot\}_{\det}$ are the same, which amounts (in view of Proposition 3.2) to showing that the quasidegree of $\{\cdot\,,\cdot\}_{\det}$ is $-2$. This follows from the following formula due to Kostant [1963, Thm 7], which expresses the dimension of the regular orbit in terms of the exponents $m_i$ of $\mathfrak{g}$:

$$2\sum_{i=1}^{\ell} m_i = \dim \mathcal{O}_{reg} = \dim \mathfrak{g} - \ell.$$

Indeed, if we apply this formula, Lemma 5.1, and (8) to the formula (24) for the quasidegree of $\{\cdot\,,\cdot\}_{\det}$, then we find

$$\begin{aligned}\kappa &= \sum_{i=1}^{\ell} \varpi(\chi_i) - \sum_{i=1}^{\ell+2} \varpi(q_i) = 2\sum_{i=1}^{\ell} d_i - \sum_{i=1}^{\ell+2}(n_i + 2)\\ &= 2\sum_{i=1}^{\ell} m_i - \sum_{i=1}^{\ell+2} n_i - 4\\ &= \dim \mathfrak{g} - \ell - (\dim \mathfrak{g} - \ell - 2) - 4 = -2.\end{aligned}$$

□

***Reduction to a* $3\times 3$ *Poisson matrix.*** Let $\mathbb{O}_{sr}$ be the subregular nilpotent adjoint orbit of a complex semisimple Lie algebra $\mathfrak{g}$ of rank $\ell$. Let $(h, e, f)$ be its associated canonical $\mathfrak{sl}_2$-triple, and let $N := e + \mathfrak{n}^{\perp}$ be a transverse slice to $\mathbb{O}_{sr}$, where $\mathfrak{n}$ is an $\mathrm{ad}_h$-invariant complementary subspace to $\mathfrak{g}(e)$. Let $\{\cdot,\cdot\}_N$ be the ATP-structure defined on $N$. Recall that $N$ is equipped with linear coordinates $q_1, \dots, q_{\ell+2}$ defined in Section 2, and that $\{\cdot,\cdot\}_N$ has independent Casimirs $\chi_1, \dots, \chi_\ell$, which are the restrictions to $N$ of the basic homogeneous invariant polynomial functions on $\mathfrak{g}$.

Our goal now is to show that, in well-chosen coordinates, the ATP-structure $\{\cdot,\cdot\}_N$ on $N$ is essentially given by a $3\times 3$ skew-symmetric matrix which is closely related to the polynomial that defines the singularity. More precisely:

**Theorem 5.6.** *After possibly relabeling the coordinates $q_i$ and the Casimirs $\chi_i$, the $\ell+2$ functions*

$$\chi_i,\ 1 \le i \le \ell-1, \quad \textit{and} \quad q_\ell,\ q_{\ell+1},\ q_{\ell+2}$$

*form a system of (global) coordinates on the affine space $N$. The Poisson matrix of the ATP-structure on $N$ in these coordinates is*

$$\widetilde{\Lambda}_N = \begin{pmatrix} 0 & 0 \\ 0 & \Omega \end{pmatrix}, \quad \textit{where} \quad \Omega = c' \begin{pmatrix} 0 & \dfrac{\partial \chi_\ell}{\partial q_{\ell+2}} & -\dfrac{\partial \chi_\ell}{\partial q_{\ell+1}} \\ -\dfrac{\partial \chi_\ell}{\partial q_{\ell+2}} & 0 & \dfrac{\partial \chi_\ell}{\partial q_\ell} \\ \dfrac{\partial \chi_\ell}{\partial q_{\ell+1}} & -\dfrac{\partial \chi_\ell}{\partial q_\ell} & 0 \end{pmatrix}, \tag{25}$$

*for some nonzero constant $c'$. It has the polynomial $\chi_\ell$ as Casimir, which reduces to the polynomial that defines the singularity if we set $\chi_j = 0$ for $j = 1, 2, \dots, \ell-1$.*

*Proof.* The non-Poisson part of this theorem is due to Brieskorn and Slodowy. Before proving the Poisson part of the theorem, namely, that the Poisson matrix takes the form (25), we explain for the reader's convenience the basics of singularity theory used in their proof, but see [Slodowy 1980a] for details. Let $(X_0, x)$ be the germ of an analytic variety $X_0$ at the point $x$. A deformation of $(X_0, x)$ is a pair $(\Phi, \iota)$ where $\Phi : X \to U$ is a flat morphism of varieties with $\Phi(x) = u$ and where the map $\iota : X_0 \to \Phi^{-1}(u)$ is an isomorphism. Such a deformation is called *semiuniversal* if any other deformation of $(X_0, x)$ is isomorphic to a deformation induced from $(\Phi, \iota)$ by a local change of variables in a neighborhood of $x$. The semiuniversal deformation of $(X_0, x)$ is unique up to isomorphism. It can be explicitly described in the following case. Let $(X_0, 0)$ be a germ of a hypersurface of $\mathbb{C}^d$ that is singular at 0, and say $X_0$ is locally given by $f(z) = 0$. Then the semiuniversal deformation of $(X_0, 0)$ is the (germ at the origin of the) map

$$\begin{array}{rcl} \Phi : \mathbb{C}^k \times \mathbb{C}^d & \to & \mathbb{C}^k \times \mathbb{C} \\ (u, z) & \mapsto & (u, F(u, z)), \end{array}$$

where

$$F(u, z) = f(z) + \sum_{i=1}^{k} g_i(z) u_i$$

and where the polynomials $1, g_1, g_2, \dots, g_k$ represent a vector space basis of the Milnor (or Tjurina) algebra

$$\mathcal{M}(f) := \frac{\mathbb{C}[z_1, \dots, z_d]}{\left(f, \frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_d}\right)} = \frac{\mathbb{C}[z_1, \dots, z_d]}{\left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_d}\right)}. \tag{26}$$

The last equality is valid whenever $f$ is quasihomogeneous, which is true in this case. The dimension $\dim \mathcal{M}(f) = k + 1$ is called the *Milnor number* of $f$.

We can now formulate Brieskorn's result. It says that the map $\chi : N \to \mathbb{C}^\ell$, which is the restriction of the adjoint quotient (20) to the slice $N$, is a semiuniversal deformation of the singular surface $N \cap \mathcal{N}$. More precisely, when the Lie algebra is of the type ADE, then the map

$$\begin{array}{rcl} \Phi : \mathbb{C}^{\ell-1} \times \mathbb{C}^3 & \to & \mathbb{C}^{\ell-1} \times \mathbb{C} \\ ((\chi_1, \dots, \chi_{\ell-1}), (q_\ell, q_{\ell+1}, q_{\ell+2})) & \mapsto & ((\chi_1, \dots, \chi_{\ell-1}), \chi_\ell) \end{array}$$

is the semiuniversal deformation of the singular surface $N \cap \mathcal{N}$; for the other types one has to consider $\Gamma$-invariant semiuniversal deformations, as was shown by Slodowy [1980a], see Table 2. It is implicit in Brieskorn's statement that $(\chi_1, \dots, \chi_{\ell-1}, q_\ell, q_{\ell+1}, q_{\ell+2})$ form a system of coordinates on $N$, which comes from the fact that one can solve the $\ell - 1$ equations $\chi_i = \chi_i(q)$ *linearly* for $\ell - 1$ of the variables $q_i$. That is, the Casimirs have the form

$$\begin{pmatrix} \chi_1 \\ \vdots \\ \chi_{\ell-1} \end{pmatrix} = A \begin{pmatrix} q_1 \\ \vdots \\ q_{\ell-1} \end{pmatrix} + \begin{pmatrix} F_1(q_\ell, q_{\ell+1}, q_{\ell+2}) \\ \vdots \\ F_{\ell-1}(q_\ell, q_{\ell+1}, q_{\ell+2}) \end{pmatrix}, \tag{27}$$

where $A$ is a constant matrix with $\det A \neq 0$; this will be illustrated in the examples below.

We now get to the Poisson part of the proof. Since the coordinate functions $\chi_1, \dots, \chi_{\ell-1}$ are Casimirs, the Poisson matrix $\widetilde{\Lambda}_N$ has with respect to these coordinates the block form

$$\widetilde{\Lambda}_N = \begin{pmatrix} 0 & 0 \\ 0 & \Omega \end{pmatrix},$$

where $\Omega$ is a $3 \times 3$ skew-symmetric matrix. We know from Theorem 5.5 that the ATP-structure is a constant multiple of the determinantal structure. Since $\det A$

lies in $\mathbb{C}^*$, it follows from (27) that, for $\ell \leq i, j \leq \ell+2$,

$$\widetilde{\Lambda}_{ij} := c \det(\nabla q_i \ \nabla q_j \ \nabla \chi_1 \ \ldots \ \nabla \chi_\ell) = c' \det(\nabla' q_i \ \nabla' q_j \ \nabla' \chi_\ell),$$

where $c$ and $c'$ are nonzero constants and $\nabla'$ denotes the restriction of $\nabla$ to $\mathbb{C}^3$, namely,

$$\nabla' F = \left( \frac{\partial F}{\partial q_\ell} \ \frac{\partial F}{\partial q_{\ell+1}} \ \frac{\partial F}{\partial q_{\ell+2}} \right)^\top .$$

The explicit formula (25) for $\Omega$ follows at once. □

## 6. Examples

***The subregular orbit of*** $\mathfrak{g}_2$. For this we have, according to (21), that $\chi_1 = q_1$. Then $\chi_2$ expressed in terms of $q_2$, $q_3$, $q_4$, and $\chi_1$ is

$$\chi_2 = 9q_4^2 - 4q_2^3 - 4q_3^3 + 12\chi_1 q_2 q_3.$$

The Poisson matrix (14) of the ATP-structure is already in the form (25), with $c' = -1/6$ (and $\chi_1 = q_1$). Since the Milnor algebra (26) is given in this case by $\mathcal{M}(9q_4^2 - 4q_2^3 - 4q_3^3) = \mathbb{C}[q_2, q_3, q_4]/\left(q_2^2, q_3^2, q_4\right)$, one easily sees that 1 and the coefficient $q_2 q_3$ of $u_1$ indeed form a vector space basis for the $\Gamma$-invariant elements of the Milnor algebra (see Table 3); compare [Slodowy 1980a, page 136].

***The subregular orbit of*** $\mathfrak{so}_8$. Recall from (22) that its ATP structure has Casimirs $\chi_1, \ldots, \chi_4$. As stated in the proof of Theorem 5.6, we can solve three of them linearly for $q_1$, $q_2$, $q_3$ in terms of $\chi_1$, $\chi_2$, $\chi_3$ and the last three variables $q_4$, $q_5$, and $q_6$. We obtain

$$\begin{aligned} q_1 &= -q_4 - \tfrac{1}{2}\chi_1, \\ q_2 &= \tfrac{1}{64}\left(\chi_1^2 - 16\chi_3 - 4\chi_2 - 32q_4q_5\right), \\ q_3 &= \tfrac{1}{64}\left(\chi_1^2 + 48\chi_3 - 4\chi_2 + 32q_4q_5\right). \end{aligned}$$

Substituted in $\chi_4$, this yields

$$\chi_4 = 8q_4q_5^2 - 16q_4^2q_5 - 4q_6^2 - 4\chi_1q_4q_5 + (\chi_2 - \tfrac{1}{4}\chi_1^2 + 4\chi_3)q_5 - 16\chi_3q_4 - 2\chi_1\chi_3,$$

so that

$$\hat{\chi}_4 = 8q_4q_5^2 - 16q_4^2q_5 - 4q_6^2 - 4\chi_1q_4q_5 + \hat{\chi}_2q_5 - 16\chi_3q_4,$$

where $\hat{\chi}_2 := \chi_2 - \frac{1}{4}\chi_1^2 + 4\chi_3$ and $\hat{\chi}_4 := \chi_4 + 2\chi_1\chi_3$ can be used instead of $\chi_2$ and $\chi_4$ as basic Ad-invariant polynomials restricted to $N$. Using (18), expressed in the coordinates $\chi_1, \hat{\chi}_2, \chi_3, q_4, q_5$ and $q_6$, we find that the matrix $\Omega$ is indeed of the form (25) with $c' = -\frac{1}{8}$, since

$$\{q_4, q_5\} = q_6 = -\frac{1}{8}\frac{\partial \hat{\chi}_4}{\partial q_6}, \quad \{q_4, q_6\} = 2q_4q_5 - 2q_4^2 - \frac{1}{2}\chi_1 q_4 + \frac{1}{8}\hat{\chi}_2 = \frac{1}{8}\frac{\partial \hat{\chi}_4}{\partial q_5},$$
$$\{q_5, q_6\} = -q_5^2 + 4q_4q_5 + \frac{1}{2}\chi_1 q_5 + 2\chi_3 = -\frac{1}{8}\frac{\partial \hat{\chi}_4}{\partial q_4}.$$

It follows easily that the Milnor algebra is given by

$$\mathcal{M}(8q_4q_5^2 - 16q_4^2q_5 - 4q_6^2) = \mathbb{C}[q_4, q_5, q_6]/(q_6, q_4(q_5 - q_4), q_5(q_5 - 4q_4)),$$

so that 1 and the coefficients $q_4$, $q_5$ and $q_4q_5$ of $\hat{\chi}_4$ indeed form a vector space basis for it.

***The subregular orbit $\mathbb{O}_{sr}$ in $\mathfrak{sl}_4$.*** This example is from [Damianou 1996]. It was also examined by Sabourin [2005], who showed that the slice, originally due to Arnold [1971], belongs to the set $\mathcal{N}_h$. It is the orbit of the nilpotent element

$$e = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The transverse slice in Arnold's coordinates consists of matrices of the form

$$Q = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ q_1 & q_2 & q_3 & q_4 \\ q_5 & 0 & 0 & -q_3 \end{pmatrix}.$$

The basic Casimirs of the ATP-structure, as computed from the characteristic polynomial of $Q$, are

$$\chi_1 = q_2 + q_3^2, \quad \chi_2 = q_1 + q_2q_3, \quad \chi_3 = q_1q_3 + q_4q_5.$$

If we solve the first two equations for the variables $q_1$, $q_2$ in terms of $\chi_1$, $\chi_2$ and $q_3$, $q_4$, $q_5$, and substitute the result in $\chi_3$, then we find that

$$\chi_3 = q_3^4 + q_4q_5 - \chi_1 q_3^2 + \chi_2 q_3.$$

Using the explicit formulas for the ATP-structure given in [Damianou 1996], expressed in the coordinates $\chi_1, \chi_2, q_3, q_4$ and $q_5$, we find that the matrix $\Omega$ is indeed of the form (25) with $c' = 1$, since

$$\{q_3, q_4\} = q_4 = \frac{\partial \chi_3}{\partial q_5}, \quad \{q_3, q_5\} = -q_5 = -\frac{\partial \chi_3}{\partial q_4},$$

$$\{q_4, q_5\} = 4q_3^3 - 2\chi_1 q_3 + \chi_2 = \frac{\partial \chi_3}{\partial q_3}.$$

It can be read from these formulas that the Milnor algebra is given by

$$\mathcal{M}(q_3^4 + q_4 q_5) = \mathbb{C}[q_3, q_4, q_5]/(q_4, q_5, q_3^3),$$

so that the coefficients 1, $q_3$ and $q_3^2$ of $\chi_3$ indeed span its vector space.

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