Pacific Journal of Mathematics

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Volume 232 No. 2 October 2007

PACIFIC JOURNAL OF MATHEMATICS
Vol. 232, No. 2, 2007

ON THE ESTIMATE OF THE FIRST EIGENVALUE OF A SUBLAPLACIAN ON A PSEUDOHERMITIAN 3-MANIFOLD

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In this paper, we study a lower bound estimate of the first positive eigenvalue of the sublaplacian on a three-dimensional pseudohermitian manifold. S.-Y. Li and H.-S. Luk derived the lower bound estimate under certain conditions for curvature tensors bounded below by a positive constant. By using the Li–Yau gradient estimate, we are able to get an effective lower bound estimate under a general curvature condition. The key is the discovery of a new CR version of the Bochner formula which involves the CR Paneitz operator.

1. Introduction

Let M be a closed 3-manifold with an oriented contact structure ξ. There always exists a global contact form θ obtained by patching together local ones with a partition of unity. The characteristic vector field of θ is the unique vector field T such that $\theta(T)=1$ and $\mathscr{L}_T\theta=0$ or $d\theta(T,\cdot)=0$. A CR structure compatible with ξ is a smooth endomorphism $J:\xi\to\xi$ such that $J^2=-\mathrm{Id}$. A pseudohermitian structure compatible with ξ is a CR-structure J compatible with ξ together with a global contact form θ. The CR structure J can extend to $\mathbf{C}\otimes\xi$ and decomposes $\mathbf{C}\otimes\xi$ into the direct sum of $T_{1,0}$ and $T_{0,1}$, which are eigenspaces of J with respect to i and $-i$, respectively.

Let $\{T, Z_1, Z_{\bar{1}}\}$ be a frame of $TM\otimes\mathbf{C}$, where T is the characteristic vector field, Z_1 is any local frame of $T_{1,0}$, and $Z_{\bar{1}}=\overline{Z_1}\in T_{0,1}$. Then $\{\theta,\theta^1,\theta^{\bar{1}}\}$, the coframe dual to $\{T, Z_1, Z_{\bar{1}}\}$, satisfies

$$d\theta = i h_{1\bar{1}}\,\theta^1\wedge\theta^{\bar{1}},$$

for some positive function $h_{1\bar{1}}$. Actually we can always choose Z_1 such that $h_{1\bar{1}}=1$; hence, throughout this paper, we assume $h_{1\bar{1}}=1$.

MSC2000: primary 32V05, 32V20; secondary 53C56.

Keywords: eigenvalue, gradient estimate, pseudohermitian manifold, Tanaka–Webster curvature, pseudohermitian torsion, CR Paneitz operator, sublaplacian, Carnot–Carathéodory distance, diameter.

Research supported in part by the NSC of Taiwan.

The Levi form $\langle\ ,\ \rangle_{L_\theta}$ is the Hermitian form on $T_{1,0}$ defined by

$$\langle Z, W\rangle_{L_\theta} = -i\left\langle d\theta, Z\wedge\overline{W}\right\rangle.$$

We can extend $\langle\ ,\ \rangle_{L_\theta}$ to $T_{0,1}$ by defining $\left\langle\overline{Z}, \overline{W}\right\rangle_{L_\theta} = \overline{\langle Z, W\rangle}_{L_\theta}$ for all $Z, W\in T_{1,0}$. The Levi form induces naturally a Hermitian form on the dual bundle $\langle\ ,\ \rangle_{L_\theta^*}$ of $T_{1,0}$ and hence on all the induced tensor bundles. Integrating the Hermitian form (when acting on sections) over M with respect to the volume form $dV=\theta\wedge d\theta$, we get an inner product on the space of sections of each tensor bundle. We denote the inner product by $\langle\ ,\ \rangle$. For example,

$$\langle\varphi, \psi\rangle = \int_M \varphi\bar{\psi}\, dV,$$

for functions φ and ψ.

The pseudohermitian connection of (J,θ) is the connection ∇ on $TM\otimes\mathbf{C}$ (and extended to tensors) given in terms of a local frame $Z_1\in T_{1,0}$ by

$$\nabla Z_1 = {\theta_1}^1\otimes Z_1, \quad \nabla Z_{\bar 1} = {\theta_{\bar 1}}^{\bar 1}\otimes Z_{\bar 1}, \quad \nabla T = 0,$$

where ${\theta_1}^1$ is the 1-form uniquely determined by

$$\begin{aligned} d\theta^1 &= \theta^1\wedge{\theta_1}^1 + \theta\wedge\tau^1,\\ \tau^1 &\equiv 0 \mod \theta^{\bar 1},\\ 0 &= {\theta_1}^1 + {\theta_{\bar 1}}^{\bar 1}, \end{aligned}$$

where τ^1 is the pseudohermitian torsion. Put $\tau^1 = {A^1}_{\bar 1}\,\theta^{\bar 1}$. The structure equation for the pseudohermitian connection is

$$d{\theta_1}^1 = R\theta^1\wedge\theta^{\bar 1} + 2i\,\mathrm{Im}({A^{\bar 1}}_{1,\bar 1}\,\theta^1\wedge\theta),$$

where R is the Tanaka–Webster curvature.

We will denote components of covariant derivatives with indices preceded by comma; thus we write ${A^{\bar 1}}_{1,\bar 1}\,\theta^1\wedge\theta$. The indices $\{0, 1, \bar 1\}$ indicate derivatives with respect to $\{T, Z_1, Z_{\bar 1}\}$. For derivatives of a scalar function, we will often omit the comma, for instance, $\varphi_1 = Z_1\varphi$, $\varphi_{1\bar 1} = Z_{\bar 1}Z_1\varphi - \theta_1^1(Z_{\bar 1})Z_1\varphi$, $\varphi_0 = T\varphi$ for a (smooth) function.

For a real function φ, the subgradient ∇_b is defined by $\nabla_b\varphi\in\xi$, and $\langle Z, \nabla_b\varphi\rangle_{L_\theta} = d\varphi(Z)$ for all vector fields Z tangent to contact plane. Locally, $\nabla_b\varphi = \varphi_{\bar 1}Z_1 + \varphi_1 Z_{\bar 1}$. We can use the connection to define the subhessian as the complex linear map

$$(\nabla^H)^2\varphi : T_{1,0}\oplus T_{0,1}\to T_{1,0}\oplus T_{0,1},$$

and

$$(\nabla^H)^2\varphi(Z) = \nabla_Z\nabla_b\varphi.$$

The sublaplacian Δ_b is defined as -1 times the trace of the subhessian, that is, $\Delta_b\varphi = -\operatorname{Tr}\big((\nabla^H)^2\varphi\big) = -(\varphi_{1\bar{1}} + \varphi_{\bar{1}1})$. For all $Z = x^1 Z_1 \in T_{1,0}$, define

$$\operatorname{Ric}(Z, Z) = R x^1 x^{\bar{1}} = R|Z|^2_{L_\theta},$$
$$\operatorname{Tor}(Z, Z) = 2\Re\, i A_{\bar{1}\bar{1}}\, x^{\bar{1}} x^{\bar{1}}.$$

Greenleaf [1985] proved the lower bound $(n/n+1)k_0$ of the first positive eigenvalue λ_1 of the sublaplacian for a pseudohermitian manifold M^{2n+1} with $n \geq 3$ under a condition on the Webster curvature and the torsion. Li and Luk [2004] proved the same result for $n = 1$ and $n = 2$. However, for $n = 1$, they needed a condition depending not only on the Webster curvature and the torsion, but also on a covariant derivative of the torsion.

The same result was proved in [Chiu 2006] under a more geometric condition which involved the positivity of the CR Paneitz operator P_0 (see Section 2 for a definition) with respect to (J, θ).

Proposition 1.1 [Chiu 2006]. *Let (M, J, θ) be a closed three-dimensional pseudohermitian manifold with nonnegative Paneitz operator P_0. Suppose that*

$$\operatorname{Ric}_m(Z, Z) - \operatorname{Tor}_m(Z, Z) \geq k_0 \langle Z, Z\rangle_{L_\theta},$$

for all $m \in M$, $Z \in T_{1,0}$, and for some positive constant k_0. Let λ_1 be the first positive eigenvalue of Δ_b. Then

$$\lambda_1 \geq \frac{k_0}{2} > 0.$$

Let (S^3, J, θ) be a 3-sphere with the induced CR structure from $\mathbf{C}^2$ and the standard contact form θ. One can show that [Chang et al. 2005; Chiu 2006]

$$\lambda_1 = \frac{k_0}{2}.$$

Here k_0 is the positive, constant Webster curvature of S^3. Thus we get a sharp estimate of λ_1 on the standard sphere (S^3, J, θ).

Conjecture 1.2. Let (M, J, θ) be a closed three-dimensional pseudohermitian manifold. Suppose that

$$\lambda_1 = \frac{k_0}{2}.$$

We conjecture that (M, J, θ) is the standard CR 3-sphere due to the theorems of Lichnérowicz [1958] and Obata [1962] in the Riemannian case. In fact, here we have (see the proof of Theorem 1.5)

(i) $\operatorname{Ric}_m(Z, Z) - \operatorname{Tor}_m(Z, Z) = k_0 \langle Z, Z\rangle_{L_\theta}$,

(ii) $\ker(\Delta_b - \lambda_1 I) \subset \ker P_0$,

(iii) $\varphi_{11} = 0$ for $\varphi \in \ker(\Delta_b - \lambda_1 I)$.

In this paper, we will try to place a good lower bound on the first positive eigenvalue when the curvature satisfies

$$\mathrm{Ric}_m(Z, Z) - \mathrm{Tor}_m(Z, Z) \geq -k_0 \langle Z, Z \rangle_{L_\theta}$$

for some nonnegative constant k_0.

Definition 1.3. On a closed pseudohermitian 3-manifold (M, J, θ), we call the Paneitz operator P_0 with respect to (J, θ) essentially positive if there exists a constant $\Lambda > 0$ such that

$$\int_M P\varphi \cdot \varphi \, d\mu \geq \Lambda \int_M \varphi^2 \, d\mu$$

for all real C^∞ smooth functions $\varphi \in (\ker P_0)^\perp$ (that is, those perpendicular to the kernel of P_0 in the L^2 norm with respect to the volume form $d\mu = \theta \wedge d\theta$).

Remark 1.4. The essential positivity of P_0 is a CR invariant in the sense that it is independent of the choice of the contact form θ. Actually, if $\widetilde{\theta} = e^{2\lambda}\theta$ is another contact form, then we have $d\widetilde{V} = \widetilde{\theta} \wedge d\widetilde{\theta} = e^{4\lambda}\theta \wedge d\theta$ and the transformation law $\widetilde{P}_0 = e^{-4\lambda} P_0$ of the CR Paneitz operator [Hirachi 1993]. Therefore, we have $\int \widetilde{P}_0\varphi \cdot \phi \, d\widetilde{V} = \int P_0\varphi \cdot \phi \, dV$.

Firstly, by using the same method as in [Chiu 2006], we are able to prove:

Theorem 1.5. *Let (M, J, θ) be a closed three-dimensional pseudohermitian manifold with essentially positive Paneitz operator P_0. Suppose*:

(i) *For some nonnegative constant k_0,*

$$\mathrm{Ric}_m(Z, Z) - \mathrm{Tor}_m(Z, Z) \geq -k_0 \langle Z, Z \rangle_{L_\theta} .$$

(ii) $\ker(\Delta_b - \lambda_1 I) \cap (\ker P_0)^\perp \neq \varnothing$.

Then

$$\lambda_1 \geq \frac{-k_0 + \sqrt{k_0^2 + 6\Lambda}}{4} > 0.$$

However if the torsion is zero, then the corresponding Paneitz operator is essentially positive [Chang et al. 2005]. Therefore,

Corollary 1.6. *Let (M, J, θ) be a closed three-dimensional pseudohermitian manifold. Suppose*:

(i) *For some nonnegative constant k_0,*

$$R \geq -k_0 \quad \textit{and} \quad A_{11} = 0.$$

(ii) $\ker(\Delta_b - \lambda_1 I) \cap (\ker P_0)^\perp \neq \phi$.

Then

$$\lambda_1 \geq \frac{-k_0 + \sqrt{k_0^2 + 6\Lambda}}{4} > 0.$$

Definition 1.7. We say that (M, J) has a transversal symmetry if M admits a one-parameter group of CR automorphisms transverse to the holomorphic tangent bundle.

For example, (M, J, θ) has a transversal symmetry if $A_{11} = 0$. For details, we refer to [Graham and Lee 1988] and [Hirachi 1993].

Definition 1.8. A piecewise smooth curve $\gamma : [0, 1] \to M$ is said to be horizontal if $\gamma'(t) \in \xi$ whenever $\gamma'(t)$ exists. The length of γ is then defined by

$$l(\gamma) = \int_0^1 dt\, \sqrt{\langle \gamma'(t), \gamma'(t)\rangle_{L_\theta}}.$$

The Carnot–Carathéodory distance between two points $p,\ q \in M$ is

$$d(p, q) = \inf \left\{ l(\gamma) |\ \gamma \in C_{p,q} \right\},$$

where $C_{p,q}$ is the set of all horizontal curves joining p and q. By the Chow connectivity theorem [1939], there always exists a horizontal curve joining p and q, so the distance is finite. The diameter d is defined by

$$d = \sup \left\{ d(p, q) |\ p, q \in M \right\}.$$

Note that there is a minimizing geodesic joining p and q so that its length is equal to the distance $d(p, q)$.

Next, by using the Li–Yau gradient estimates [Yau 1975; Li and Yau 1980], we have:

Theorem 1.9. *Let (M, J, θ) be a closed three-dimensional pseudohermitian manifold that has a transversal symmetry. Suppose*:

(i) *For some nonnegative constant k_0,*

$$\mathrm{Ric}_m(Z, Z) - \mathrm{Tor}_m(Z, Z) \geq -k_0 \langle Z, Z\rangle_{L_\theta}.$$

(ii) $\ker(\Delta_b - \lambda_1 I) \cap \ker P_0 \neq \phi$.

Then

$$\lambda_1 \geq \frac{\left(1 + \sqrt{1 + 2(k_0 + \tau_0)d^2}\right)}{6d^2} e^{-\left(1+\sqrt{1+2(k_0+\tau_0)d^2}\right)}.$$

Here $\tau_0 = \max |A_{11}|$ and d is the diameter of M.

Corollary 1.10. *Let (M, J, θ) be a closed three-dimensional pseudohermitian manifold. Suppose*:

(i) *For some nonnegative constant* k_0,

$$R \geq -k_0 \quad \textit{and} \quad A_{11} = 0.$$

(ii) $\ker(\Delta_b - \lambda_1 I) \cap \ker P_0 \neq \phi$.

Then

$$\lambda_1 \geq \frac{\left(1+\sqrt{1+2k_0 d^2}\right)}{6d^2} e^{-\left(1+\sqrt{1+2k_0 d^2}\right)}.$$

Combining Theorem 1.5 and Theorem 1.9, we can prove:

Theorem 1.11. *Let* (M, J, θ) *be a closed three-dimensional pseudohermitian manifold which has a transversal symmetry. Suppose*:

(i) *For some nonnegative constant* k_0,

$$\mathrm{Ric}_m(Z, Z) - \mathrm{Tor}_m(Z, Z) \geq -k_0 \langle Z, Z\rangle_{L_\theta}.$$

(1) $\Delta_b \ker P_0 \subset \ker P_0$.

Then

$$\lambda_1 \geq \max\left\{ \frac{\left(1+\sqrt{1+2(k_0+\tau_0)d^2}\right)}{6d^2} e^{-\left(1+\sqrt{1+2(k_0+\tau_0)d^2}\right)};\ \frac{-k_0+\sqrt{k_0^2+6\Lambda}}{4} \right\}.$$

Here $\tau_0 = \max|A_{11}|$ *and* d *is the diameter of* M.

In particular, if $A_{11} = 0$, then (M, J, θ) has a transversal symmetry and we also have $\Delta_b \ker P_0 \subset \ker P_0$. Therefore, as a consequence of Theorem 1.5 and Theorem 1.11,

Corollary 1.12. *Let* (M, J, θ) *be a closed three-dimensional pseudohermitian manifold with* $A_{11} = 0$. *Suppose*

$$R \geq -k_0,$$

for some nonnegative constant k_0. *Then*

$$\lambda_1 \geq \max\left\{ \frac{\left(1+\sqrt{1+2k_0 d^2}\right)}{6d^2} e^{-\left(1+\sqrt{1+2k_0 d^2}\right)};\ \frac{-k_0+\sqrt{k_0^2+6\Lambda}}{4} \right\}.$$

That is, there is a positive constant $C(k_0, d, \Lambda)$ *such that*

$$\lambda_1 \geq C(k_0, d, \Lambda).$$

We briefly describe the methods used in our proofs. In Section 2, we first derive the CR version of Bochner formula which involves the CR Paneitz operator. This formula, involving a term that has no analogue in the Riemannian case, is hard to control. A key step is that we relate this extra term to a third-order operator P that characterizes CR-pluriharmonic functions [Lee 1988]. After integrating by parts, we get the CR Paneitz operator.

Section 3 contains the second crucial step. By using the Li–Yau gradient estimate [Yau 1975; Li and Yau 1980], we are able to prove the main Theorem 1.11.

2. The Bochner formula and CR Paneitz operator

We define an operator through

$$P\varphi = (\varphi_{\bar{1}}{}^{\bar{1}}{}_{1} + iA_{11}\varphi^{1})\theta^{1} = P\varphi = (P_1\varphi)\theta^{1},$$

which characterizes the CR-pluriharmonic functions. Here $P_1\varphi = \varphi_{\bar{1}}{}^{\bar{1}}{}_{1} + iA_{11}\varphi^{1}$, and $\overline{P}\varphi = (\overline{P}_1)\theta^{\bar{1}}$ is the conjugate of P. Now define δ_b as the divergence operator that takes $(1,0)$-forms to functions by $\delta_b(\sigma_1\theta^1) = \sigma_{1,}{}^{1}$, and similarly define $\bar{\delta}_b$ through $\bar{\delta}_b(\sigma_{\bar{1}}\theta^{\bar{1}}) = \sigma_{\bar{1},}{}^{\bar{1}}$. The CR Paneitz operator P_0 is then defined through

$$P_0\varphi = 4\left(\delta_b(P\varphi) + \bar{\delta}_b(\overline{P}\varphi)\right).$$

We observe that

$$\int \langle P\varphi + \overline{P}\varphi, d_b\varphi\rangle_{L_\theta^*}\, dV = -\frac{1}{4}\int P_0\varphi\cdot\varphi\, dV. \tag{1}$$

One can check that P_0 is self-adjoint, that is, $\langle P_0\varphi, \psi\rangle = \langle\varphi, P_0\psi\rangle$ for all smooth functions φ and ψ. For more details about these operators, read [Lee 1988; Graham and Lee 1988; Hirachi 1993; Gover and Graham 2003; Fefferman and Hirachi 2003].

We first derive the following new CR version of the Bochner formula:

Lemma 2.1. *For a real function φ,*

$$\begin{aligned}\frac{1}{2}\Delta_b|\nabla_b\varphi|^2 = &-|(\nabla^H)^2\varphi|^2 + 3\langle\nabla_b\varphi, \nabla_b\Delta_b\varphi\rangle_{L_\theta}\\ &-(2\,\mathrm{Ric} - 3\,\mathrm{Tor})((\nabla_b\varphi)_{\mathbf{C}}, (\nabla_b\varphi)_{\mathbf{C}})\\ &+4\langle P\varphi + \overline{P}\varphi, d_b\varphi\rangle_{L_\theta^*}.\end{aligned}$$

Here $(\nabla_b\varphi)_{\mathbf{C}} = \varphi_{\bar{1}}Z_1$ is the corresponding complex $(1,0)$-vector field of $\nabla_b\varphi$ and $d_b\varphi = \varphi_1\theta^1 + \varphi_{\bar{1}}\theta^{\bar{1}}$.

Proof. From [Greenleaf 1985], we have for a real function φ

$$\Delta_b|\nabla_b\varphi|^2 = -2|(\nabla^H)^2\varphi|^2 + 2\langle\nabla_b\varphi, \nabla_b\Delta_b\varphi\rangle_{L_\theta} - (4\,\mathrm{Ric} + 2\,\mathrm{Tor})((\nabla_b\varphi)_{\mathbf{C}}, (\nabla_b\varphi)_{\mathbf{C}}) - 4\langle J\nabla_b\varphi, \nabla_b\varphi_0\rangle_{L_\theta}. \tag{2}$$

Lemma 2.1 follows from this and

$$\langle J\nabla_b\varphi, \nabla_b\varphi_0\rangle_{L_\theta} = -\langle\nabla_b\varphi, \nabla_b\Delta_b\varphi\rangle_{L_\theta} - 2\,\mathrm{Tor}((\nabla_b\varphi)_{\mathbf{C}}, (\nabla_b\varphi)_{\mathbf{C}}) - 2\langle P\varphi + \overline{P}\varphi, d_b\varphi\rangle_{L_\theta^*}, \tag{3}$$

which we next prove. The commutation relation $i\varphi_0 = \varphi_{1\bar{1}} - \varphi_{\bar{1}1}$ [Lee 1988] gives $\varphi_{1\bar{1}1} - \varphi_{\bar{1}11} = i\varphi_{01}$. Thus

$$\begin{aligned}\langle J\nabla_b\varphi, \nabla_b\varphi_0\rangle_{L_\theta} &= i(\varphi_{\bar{1}}\varphi_{01} - \varphi_1\varphi_{0\bar{1}}) \\ &= \varphi_{\bar{1}}(\varphi_{1\bar{1}1} - \varphi_{\bar{1}11}) + \varphi_1(\varphi_{\bar{1}1\bar{1}} - \varphi_{1\bar{1}\bar{1}}).\end{aligned} \tag{4}$$

On the other hand,

$$\begin{aligned}\langle\nabla_b\varphi, \nabla_b\Delta_b\varphi\rangle_{L_\theta} &= \varphi_{\bar{1}}(\Delta_b\varphi)_1 + \varphi_1(\Delta_b\varphi)_{\bar{1}} \\ &= -\varphi_{\bar{1}}(\varphi_{1\bar{1}1} + \varphi_{\bar{1}11}) - \varphi_1(\varphi_{\bar{1}1\bar{1}} + \varphi_{1\bar{1}\bar{1}}).\end{aligned} \tag{5}$$

It follows from (4) and (5) that

$$\begin{aligned}\langle J\nabla_b\varphi, \nabla_b\varphi_0\rangle_{L_\theta} + \langle\nabla_b\varphi, \nabla_b\Delta_b\varphi\rangle_{L_\theta} &= -2\varphi_{\bar{1}}\varphi_{\bar{1}11} - 2\varphi_1\varphi_{1\bar{1}\bar{1}} \\ &= -2\varphi_{\bar{1}}\left(P_1\varphi - iA_{11}\varphi_{\bar{1}}\right) - 2\varphi_1\left(\overline{P}_1\varphi + iA_{\bar{1}\bar{1}}\varphi_1\right) \\ &= -2\ \mathrm{Tor}((\nabla_b\varphi)_{\mathbb{C}}, (\nabla_b\varphi)_{\mathbb{C}}) - 2\langle P\varphi + \overline{P}\varphi, d_b\varphi\rangle_{L_\theta^*}. \quad \square\end{aligned}$$

Proof of Theorem 1.5. Let $\varphi \in (\ker P_0)^\perp$ be an eigenfunction of Δ_b having the first positive eigenvalue λ_1. By definition,

$$\int_M \varphi P_0\varphi dV \geq \Lambda \int_M \varphi^2 dV. \tag{6}$$

By integrating (2), (3) and using (1), we have

$$\int |(\nabla^H)^2\varphi|^2_{L_\theta} dV = \int (\Delta_b\varphi)^2 dV + 2\int \varphi_0^2\, dV - \int (2\,\mathrm{Ric} + \mathrm{Tor})((\nabla_b\varphi)_{\mathbb{C}}, (\nabla_b\varphi)_{\mathbb{C}}) dV, \tag{7}$$

and

$$\int \varphi_0^2 dV = \int (\Delta_b\varphi)^2 dV + 2\int \mathrm{Tor}((\nabla_b\varphi)_{\mathbb{C}}, (\nabla_b\varphi)_{\mathbb{C}}) dV - \frac{1}{2}\int P_0\varphi \cdot \varphi dV. \tag{8}$$

On the other hand, it is easy to verify that

$$\left|(\nabla^H)^2\varphi\right|^2_{L_\theta} = 2\,|\varphi_{11}|^2 + \tfrac{1}{2}(\Delta_b\varphi)^2 + \tfrac{1}{2}\varphi_0^2.$$

Substituting this into the left hand of (7) and combining with (8), we get

$$2\int |\varphi_{11}|^2\ dV = 2\int (\Delta_b\varphi)^2 dV - 2\int (\mathrm{Ric}-\mathrm{Tor})((\nabla_b\varphi)_{\mathbb{C}}, (\nabla_b\varphi)_{\mathbb{C}})\ dV \\ -\frac{3}{4}\int P_0\varphi\cdot\varphi\ dV.$$

By combining with (6), we get

$$\begin{aligned} 0 &\geq -2\int (\Delta_b\varphi)^2 dV + 2\int (\mathrm{Ric}-\mathrm{Tor})((\nabla_b\varphi)_{\mathbb{C}}, (\nabla_b\varphi)_{\mathbb{C}})\ dV + \frac{3}{4}\int P_0\varphi\cdot\varphi\ dV \\ &\geq \int -2\lambda_1|\nabla_b\varphi|^2_{L_\theta} dV - \int k_0|\nabla_b\varphi|^2_{L_\theta} dV + \int \frac{3\Lambda}{4}\varphi^2 dV \\ &= \int \left(-2\lambda_1 - k_0 + \frac{3\Lambda}{4\lambda_1}\right)|\nabla_b\varphi|^2_{L_\theta} dV. \end{aligned}$$

This holds if and only if

$$-2\lambda_1 - k_0 + \frac{3\Lambda}{4\lambda_1} \leq 0,$$

and Theorem 1.5 follows immediately. □

3. The Li–Yau gradient estimate

Let (M, J, θ) be a closed three-dimensional pseudohermitian manifold. In the case that $\ker(\Delta_b - \lambda_1 I) \cap (\ker P_0) \neq \phi$, then, by using the so-called Li–Yau gradient estimate [Yau 1975; Li and Yau 1980], one can place a lower bound on the positive first eigenvalue of a sublaplacian Δ_b.

Lemma 3.1. *Let $\varphi = \ln f$ for $f > 0$. Then*

$$4\left\langle P\varphi + \bar{P}\varphi, d_b\varphi\right\rangle_{L_\theta^*} = 4\frac{\left\langle Pf + \bar{P}f, d_bf\right\rangle_{L_\theta^*}}{f^2} - 4\langle\nabla_b\varphi, \nabla_b|\nabla_b\varphi|^2\rangle_{L_\theta} \\ + 2\frac{\Delta_b f}{f}|\nabla_b\varphi|^2.$$

Proof. Let $Q(x) = |\nabla_b\varphi|^2(x)$. We compute

$$\begin{aligned} \nabla_b Q &= Q_{\bar{1}}Z_1 + Q_1 Z_{\bar{1}} = 2\nabla_b(\varphi_1\varphi_{\bar{1}}) \\ &= \frac{f^2 f_1 f_{\bar{1}\bar{1}} + f^2 f_{\bar{1}} f_{1\bar{1}} - 2ff_{\bar{1}}^2 f_1}{f^4} Z_1 + \text{complex conjugate}. \end{aligned}$$

It follows that

$$P_1\varphi = \varphi_{\bar{1}11} + iA_{11}\varphi_{\bar{1}} = \frac{f^3 f_{\bar{1}11} - f^2 f_{\bar{1}} f_{11} - 2f^2 f_1 f_{\bar{1}1} + 2f f_1{}^2 f_{\bar{1}}}{f^4} + iA_{11}\frac{f_{\bar{1}}}{f}$$
$$= \frac{P_1 f}{f} - Q_1 - \frac{f_1 f_{\bar{1}1}}{f^2} = \frac{P_1 f}{f} - Q_1 - \varphi_1 \frac{f_{\bar{1}1}}{f}.$$

Thus

$$4\left\langle P\varphi + \bar{P}\varphi, d_b\varphi\right\rangle_{L_\theta^*} = 4\left\langle (P_1\varphi)\,\theta^1 + \left(\bar{P}_1\varphi\right)\theta^{\bar{1}}, \varphi_1\theta^1 + \varphi_{\bar{1}}\theta^{\bar{1}}\right\rangle_{L_\theta^*}$$
$$= 4\left[(P_1\varphi)\,\varphi_{\bar{1}} + \left(\bar{P}_1\varphi\right)\varphi_1\right] = 4\left(\frac{P_1 f}{f} - Q_1 - \varphi_1\frac{f_{\bar{1}1}}{f}\right)\varphi_{\bar{1}} + \text{complex conjugate}$$
$$= 4\frac{\left\langle Pf + \bar{P}f, d_b f\right\rangle_{L_\theta^*}}{f^2} - 4\left\langle \nabla_b\varphi, \nabla_b\left|\nabla_b\varphi\right|^2\right\rangle + 2\left(\frac{\Delta_b f}{f}\left|\nabla_b\varphi\right|^2\right).$$

This implies the lemma. □

The next lemma will ready us to show Theorem 1.9.

Lemma 3.2 [Graham and Lee 1988; Hirachi 1993]. *Let (M, J, θ) be a closed three-dimensional pseudohermitian manifold with a transversal symmetry, and let θ be any pseudohermitian structure on M. Then a smooth real-valued function f satisfies $P_0 f = 0$ on M if and only if $P_1 f = 0$ on M, that is, f is CR-pluriharmonic.*

Proof of Theorem 1.9. Let f be an eigenfunction of Δ_b with eigenvalue λ_1, that is, $\Delta_b f = \lambda_1 f$. Also suppose $P_0 f = 0$. Since

$$\lambda_1 \int_M f = \int_M \Delta_b f = 0,$$

f must change sign. We may normalize f to satisfy $\min f = -1$ and $\max f \le 1$. Let us consider the function $\varphi = \ln(a + f)$, for some constant $a > 1$. Then the function φ satisfies

$$\Delta_b\varphi = \frac{\Delta_b f}{a+f} - \left\langle \nabla_b\left(\frac{1}{a+f}\right), \nabla_b(a+f)\right\rangle_{L_\theta}$$
$$= \frac{\Delta_b f}{a+f} + \frac{|\nabla_b f|^2}{(a+f)^2} = \frac{\lambda_1 f}{a+f} + |\nabla_b\varphi|^2.$$

Since $\left|(\nabla^\xi)^2\varphi\right|_{L_\theta}^2 = 2\,|\varphi_{11}|^2 + \frac{1}{2}(\Delta_b\varphi)^2 + \frac{1}{2}\varphi_0^2$, we have

$$-\left|(\nabla^\xi)^2\varphi\right|_{L_\theta}^2 \le -\frac{1}{2}(\Delta_b\varphi)^2 \le -\frac{1}{2}|\nabla_b\varphi|^4 - \frac{\lambda_1 f}{a+f}|\nabla_b\varphi|^2.$$

On the other hand, we have

$$\langle \nabla_b\varphi, \nabla_b\Delta_b\varphi\rangle_{L_\theta} = \left\langle \nabla_b\varphi, \nabla_b\left(\frac{\lambda_1 f}{a+f} + |\nabla_b\varphi|^2\right)\right\rangle_{L_\theta}$$
$$= \left\langle \nabla_b\varphi, \nabla_b\,|\nabla_b\varphi|^2\right\rangle_{L_\theta} + \frac{\lambda_1 a}{a+f}\,|\nabla_b\varphi|^2 .$$

Because $\mathrm{Ric}_m(Z, Z) - \mathrm{Tor}_m(Z, Z) \geq -k_0\,\langle Z, Z\rangle_{L_\theta}$, we have

(9) $$2\,\mathrm{Ric}_m(Z, Z) - 2\,\mathrm{Tor}_m(Z, Z) \geq -2k_0\,\langle Z, Z\rangle_{L_\theta} .$$

On the other hand, put $\tau_0 = \max|A_{11}|$. Then from

$$-2|A_{11}|\,\langle Z, Z\rangle_{L_\theta} \leq -\mathrm{Tor}(Z, Z) \leq 2|A_{11}|\,\langle Z, Z\rangle_{L_\theta} ,$$

we have

(10) $$-2\tau_0\,\langle Z, Z\rangle_{L_\theta} \leq -\mathrm{Tor}(Z, Z) \leq 2\tau_0\,\langle Z, Z\rangle_{L_\theta} .$$

Combining (9) and (10), one has

$$2\,\mathrm{Ric}_m(Z, Z) - 3\,\mathrm{Tor}_m(Z, Z) \geq -2(k_0 + \tau_0)\,\langle Z, Z\rangle_{L_\theta} .$$

Now we define $Q(x) = |\nabla_b\varphi|^2$. Then, by Lemma 2.1 and Lemma 3.1, we see that the sublaplacian satisfies

$$\frac{1}{2}\Delta_b Q + \langle\nabla_b\varphi, \nabla_b Q\rangle \leq -\frac{1}{2}Q^2 - \left(\lambda_1 - \frac{2(k_0+\tau_0)}{2} - \frac{4\lambda_1 a}{a+f} - \frac{2\lambda_1 f}{a+f}\right) Q$$
$$\leq -\frac{1}{2}Q^2 - \left(\lambda_1 - \frac{2(k_0+\tau_0)}{2} - \frac{6\lambda_1 a}{a-1}\right) Q.$$

If $x_0 \in M$ is a point where Q achieves its maximum, we have

$$0 \leq \frac{1}{2}\Delta_b Q(x_0) + \langle\nabla_b\varphi, \nabla_b Q\rangle\,(x_0).$$

Hence

$$\frac{1}{2}Q^2(x_0) + \left(\lambda_1 - \frac{2(k_0+\tau_0)}{2} - \frac{6\lambda_1 a}{a-1}\right) Q(x_0) \leq 0$$

which implies that

$$Q(x) \leq Q(x_0) \leq -2\left(\lambda_1 - \frac{2(k_0+\tau_0)}{2} - \frac{6\lambda_1 a}{a-1}\right) \leq \frac{12a}{a-1}\lambda_1 + 2(k_0+\tau_0),$$

for all $x \in M$. Integrating $Q^{\frac{1}{2}} = |\nabla_b \varphi| = |\nabla_b \ln(a+f)|$ along a minimal horizontal geodesic γ joining the points at which $f=-1$ and $f=\max f$, it follows that

$$\begin{aligned}\ln\frac{a}{a-1} &\le \ln\left(\frac{a+\max f}{a-1}\right) = \ln(a+\max f) - \ln(a-1)\\ &\le \int_\gamma |\nabla_b \ln(a+f)| \le d\sqrt{\frac{12a}{a-1}\lambda_1 + 2(k_0+\tau_0)},\end{aligned}$$

for all $a>1$. Setting $t=(a-1)/a$, we have

$$12\lambda_1 \ge \left(\frac{1}{d^2}\left(\ln\frac{1}{t}\right)^2 - 2(k_0+\tau_0)\right)t$$

for all $0<t<1$. Maximizing the right hand side as a function of t by setting $t=\exp(-1-\sqrt{1+2(k_0+\tau_0)d^2})$, we obtain the estimate

$$\begin{aligned}\lambda_1 &\ge \frac{1}{12}\left(\frac{(1+\sqrt{1+2(k_0+\tau_0)d^2})^2}{d^2} - 2(k_0+\tau_0)\right)e^{\left(-1-\sqrt{1+2(k_0+\tau_0)d^2}\right)}\\ &= \frac{\left(1+\sqrt{1+2(k_0+\tau_0)d^2}\right)}{6d^2}e^{\left(-1-\sqrt{1+2(k_0+\tau_0)d^2}\right)}.\end{aligned}$$ □

Proof of Theorem 1.11. If (M, J, θ) is a closed three-dimensional pseudohermitian manifold that has a transversal symmetry, then there exists a torsion free pseudohermitian contact structure $\widetilde{\theta} = e^{2f}\theta$ for some real smooth function f. Therefore $\widetilde{P}_0$, the CR Paneitz operator with respect to $\widetilde{\theta}$, is essentially positive. But $\widetilde{P}_0 = e^{-4f} P_0$. It follows that P_0 is essentially positive.

On the other hand, suppose that the CR Paneitz operator P_0 and the sublaplacian Δ_b satisfy $\Delta_b(\ker P_0) \subset \ker P_0$. Hence we have the following decomposition (see [Chang et al. 2005, Section 5] for details):

$$\ker(\Delta_b - \lambda_1 I) = E_K \oplus_{P_0} {E_K}^{\perp},$$

where $E_K \subset \ker P_0$ and ${E_K}^{\perp} \subset (\ker P_0)^{\perp}$.

Let f be an eigenfunction of Δ_b with respect to the first positive eigenvalue λ_1. P_0 decomposes f as

$$f = f^{\perp} \oplus f_{\ker},$$

whence

$$\Delta_b f^{\perp} = \lambda_1 f^{\perp} \quad \text{and} \quad \Delta_b f_{\ker} = \lambda_1 f_{\ker}.$$

Theorem 1.11 then follows directly from Theorem 1.5 and Theorem 1.9. □

Acknowledgments

The first author would like to express his thanks to Professor S.-T. Yau for constant encouragement and Professor J.-P. Wang for valuable discussions during his visit at NCTS, Hsinchu, Taiwan.

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Received April 25, 2006.

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