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We prove some rigidity results for compact manifolds with boundary. For a compact Riemannian manifold with nonnegative Ricci curvature and simply connected mean convex boundary, we show that if the sectional curvature vanishes on the boundary, the metric must be flat.

Schroeder and Strake [1989, Theorem 1] proved a rigidity theorem:

Theorem. *Let* (M, g) *be a compact Riemannian manifold with convex boundary and nonnegative Ricci curvature. Assume that the sectional curvature is identically zero in some neighborhood* U *of* ∂M *and that one of these conditions holds*:

- ∂M *is simply connected*;
- $\dim \partial M$ *is even and* ∂M *is strictly convex at some point* $p \in \partial M$.

Then M *is flat.*

As they remarked, the condition that the metric is flat in a whole neighborhood of ∂M is very strong. They conjectured that it suffices to only assume that the sectional curvature vanishes on ∂M and proved this in the special case of a convex metric ball. Xia [1997; 2002] studied the problem and confirmed the conjecture under various additional conditions, such as the boundary has constant mean curvature or constant scalar curvature, or the second fundamental form satisfies some pinching condition. We refer to Xia's papers for the precise statements. Here we present some results related to the conjecture.

Theorem 1. *Let* M *be a smooth compact connected Riemannian manifold with boundary and nonnegative Ricci curvature. If every component of* ∂M *is simply connected and has nonnegative mean curvature and the sectional curvature of* M *vanishes on* ∂M, *then* M *is flat and* ∂M *has only one component.*

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Therefore when ∂M is simply connected, the conjecture of Schroeder and Strake is true. Moreover, one only needs ∂M to be mean convex instead of convex. That ∂M has only one component follows from theorems in [Ichida 1981; Kasue 1983]. Below we present a different argument for it based on Reilly's formula [1977].

To continue, we need some notation. We will often write $\langle\ ,\ \rangle$ for the metric on M and denote its connection by D. For convenience we write $\Sigma = \partial M$ and denote the Levi-Civita connection, curvature tensor, and so on of the induced metric on Σ by adding the subscript Σ to the standard notations. Let ν be the unit outer normal vector. The shape operator is given by $A(X) = D_X \nu$ and the second fundamental form is given by $h(X, Y) = \langle A(X), Y\rangle = \langle D_X \nu, Y\rangle$, where $X, Y \in T\Sigma$. The mean curvature $H = \operatorname{tr} A$. Recall Reilly's formula [1977, formula (14)] for a smooth function u on M

$$\frac{1}{2}\int_M \left((\Delta u)^2 - \left|D^2 u\right|^2\right) d\mu = \frac{1}{2}\int_M Rc(\nabla u, \nabla u)\, d\mu + \int_\Sigma \Delta_\Sigma u \cdot \frac{\partial u}{\partial \nu} dS + \frac{1}{2}\int_\Sigma H\left(\frac{\partial u}{\partial \nu}\right)^2 dS + \frac{1}{2}\int_\Sigma \langle A(\nabla_\Sigma u), \nabla_\Sigma u\rangle\, dS.$$

A special case of theorems in [Ichida 1981; Kasue 1983] claims that if M^n is a compact connected Riemannian manifold with mean convex boundary Σ and nonnegative Ricci curvature, then Σ has at most two components; moreover if Σ has two components, then M is isometric to $\Gamma \times [0, a]$ for some connected compact Riemannian manifold Γ with nonnegative Ricci curvature and $a > 0$. For Theorem 1, it is clear that M cannot have the product metric; hence Σ has one component. Interestingly, one may give an argument for the above special case based on Reilly's formula. Indeed, assume Σ is not connected, and fix a component Σ_0 of Σ; then we may solve the Dirichlet problem

$$\Delta u = 0 \text{ on } M, \qquad u|_{\Sigma_0} = 0, \qquad u|_{\Sigma \setminus \Sigma_0} = 1.$$

Applying Reilly's formula to u, we get

$$-\int_M \left|D^2 u\right|^2 d\mu = \int_M Rc(\nabla u, \nabla u)\, d\mu + \int_\Sigma H\left(\frac{\partial u}{\partial \nu}\right)^2 dS.$$

Hence $D^2 u = 0$. This implies $|\nabla u| \equiv c > 0$. Since $\nabla u = -c\nu$ on Σ_0 and $\nabla u = c\nu$ on $\Sigma \setminus \Sigma_0$, we see $D_X \nu = 0$ for $X \in T\Sigma$, that is, Σ is totally geodesic. If we look at the flow generated by $\nabla u / c$, then it sends Σ_0 to $\Sigma \setminus \Sigma_0$ at time $1/c$ and hence Σ has exactly two components. Note that the flow lines are just geodesics. If we fix a coordinate on Σ_0, namely $\theta^1, \cdots, \theta^{n-1}$ and let $r = u/c$, then we have $g = dr \otimes dr + g_{ij}(r, \theta)\, d\theta^i \otimes d\theta^j$. Using $D^2 r = 0$, we see $\partial_r g_{ij}(r, \theta) = 0$. Hence M is isometric to $\Sigma_0 \times [0, 1/c]$.

Under the assumption of Theorem 1 that the sectional curvature of M vanishes on Σ, it follows from the Gauss and Codazzi equations that

$$\begin{aligned} R_\Sigma(X,Y,Z,W) &= h(X,Z)\,h(Y,W) - h(X,W)\,h(Y,Z), \\ (D_\Sigma)_X\, h(Y,Z) &= (D_\Sigma)_Y\, h(X,Z), \end{aligned}$$

where X, Y, Z, and W belong to $T\Sigma$. By the fundamental theorem for hypersurfaces [Spivak 1999, part (2) of Theorem 21 on p. 63] and that Σ is simply connected, we may find a smooth isometric immersion $\phi : \Sigma \to \mathbb{R}^n$ such that the second fundamental form of the immersion has $h_\phi = h$. If Σ is convex, then ϕ is an embedding by a Hadamard-type theorem of Sacksteder [1960]. With this immersion ϕ in hand, Theorem 1 follows from:

Proposition 2. *Let M^n be a smooth compact connected Riemannian manifold with connected boundary $\Sigma = \partial M$ and $Rc \geq 0$. If $\phi : \Sigma \to \mathbb{R}^l$ is an isometric immersion with $|H_\phi| \leq H$ on Σ, where H_ϕ is the mean curvature vector of the immersion ϕ, then M is flat. If ϕ is also an imbedding, M is isometric to a domain in $\mathbb{R}^n$.*

This generalizes [Ros 1988, Theorem 2], a congruence theorem for hypersurface in Euclidean space. Following the Ros's argument, we will show by Reilly's formula that the harmonic extension of the map ϕ is in fact an isometric immersion.

Proof. We can find a smooth function $F : M \to \mathbb{R}^l$ such that $\Delta F = 0$ in M and $F\big|_\Sigma = \phi$. Applying Reilly's formula to each component of F and summing yields

$$\begin{aligned} -\frac{1}{2}\sum_\alpha \int_M \left|D^2 F^\alpha\right|^2 d\mu \\ = \frac{1}{2}\int_M \sum_\alpha Rc\left(\nabla F^\alpha, \nabla F^\alpha\right) d\mu + \int_\Sigma \sum_\alpha \Delta_\Sigma \phi^\alpha \cdot \frac{\partial F^\alpha}{\partial \nu} dS \\ + \frac{1}{2}\int_\Sigma \sum_\alpha H\left(\frac{\partial F^\alpha}{\partial \nu}\right)^2 dS + \frac{1}{2}\int_\Sigma \sum_\alpha \left\langle A\left(\nabla_\Sigma \phi^\alpha\right), \nabla_\Sigma \phi^\alpha \right\rangle dS. \end{aligned}$$

Note that

$$\begin{aligned} \sum_\alpha \left\langle A\left(\nabla_\Sigma \phi^\alpha\right), \nabla_\Sigma \phi^\alpha\right\rangle &= \left\langle Ae_i, e_j\right\rangle e_i\phi^\alpha \cdot e_j\phi^\alpha = \left\langle Ae_i, e_j\right\rangle \phi_* e_i \cdot \phi_* e_j \\ &= \left\langle Ae_i, e_j\right\rangle \delta_{ij} = \operatorname{tr} A = H, \end{aligned}$$

where $e_1, \dots, e_{n-1}$ is a local orthonormal frame on Σ; hence

$$\begin{aligned} 0 = \frac{1}{2}\sum_\alpha \int_M \left|D^2 F^\alpha\right|^2 d\mu + \frac{1}{2}\int_M \sum_\alpha Rc\left(\nabla F^\alpha, \nabla F^\alpha\right) d\mu \\ + \int_\Sigma H_\phi \cdot F_*\nu\, dS + \frac{1}{2}\int_\Sigma H\left|F_*\nu\right|^2 dS + \frac{1}{2}\int_\Sigma H\, dS \end{aligned}$$

$$\geq \frac{1}{2} \sum_{\alpha} \int_M \left|D^2 F^\alpha\right|^2 d\mu + \frac{1}{2} \int_\Sigma H \left(|F_* \nu|^2 - 2\,|F_* \nu| + 1\right) dS.$$

Hence $D^2 F^\alpha = 0$ for all α. It follows that $F^* g_{\mathbb{R}^l}$ is parallel on M. We may find some $p \in \Sigma$ such that $|H_\phi| > 0$ at p; hence $H(p) > 0$. From the above argument, this implies $|F_* \nu| = 1$ at p and $F_* \nu$ is perpendicular to $\phi_* \Sigma_p$; hence $F^* g_{\mathbb{R}^l} = g_M$ at p. It follows that $F^* g_{\mathbb{R}^l} = g_M$ on M, that is, F is an isometric immersion and M is flat. Now assume ϕ is an imbedding. Let $\overline{D}$ be the connection on $\mathbb{R}^l$, then $\overline{D}_X F_* Y - F_* D_X Y = XYF - (D_X Y)\, F = 0$. It follows that $F : M \to \mathbb{R}^l$ is a totally geodesic submanifold, and hence the image lies in a n dimensional affine subspace. Without loss of generality, we may assume $l = n$ and Σ is a compact hypersurface in $\mathbb{R}^n$. Then there exists a bounded open domain Ω such that $\partial\Omega = \Sigma$. Since F is an immersion, we see $F(M) \setminus \overline{\Omega}$ is both open and closed in $\mathbb{R}^n \setminus \overline{\Omega}$; hence it must be empty. Based on this, we show $F : M \to \overline{\Omega}$ is a covering map, and hence it must be a diffeomorphism. □

If we assume that ∂M is convex, then it is clear from the above that M is isometric to a convex domain in $\mathbb{R}^n$. In fact, in this case one may replace the nonnegativity of the Ricci curvature by the much weaker nonnegativity of the scalar curvature, at least when M is spin.

Theorem 3. *Let M be a smooth compact connected Riemannian manifold with boundary and nonnegative scalar curvature. If M is spin, each component of ∂M is convex and simply connected, and the sectional curvature of M vanishes on ∂M, then M is isometric to a convex domain in $\mathbb{R}^n$.*

Proof. For every component Γ of ∂M, we have an isometric embedding $\phi : \Gamma \to \mathbb{R}^n$ that has h as the second fundamental form. Let Ω be the convex domain enclosed by $\phi(\Gamma)$. We glue M and $\mathbb{R}^n \setminus \Omega$ along Γ via the diffeomorphism ϕ for all the Γ's and obtain a complete Riemannian manifold N that has nonnegative scalar curvature and is flat outside a compact set. Note that the metric is C^1 along the gluing hypersurface. Since M is spin, we conclude by the generalized positive mass theorem [Shi and Tam 2002, Theorem 3.1] that N is isometric to $\mathbb{R}^n$. It follows that M is isometric to a convex domain in $\mathbb{R}^n$. □

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