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COMPLEX TANGENTIAL FLOWS AND COMPACTNESS OF THE $\bar{\partial}$-NEUMANN OPERATOR

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We provide geometric conditions on the set of boundary points of infinite type of a smooth bounded pseudoconvex domain in $\mathbb{C}^n$ implying that the $\bar{\partial}$-Neumann operator is compact. These conditions are formulated in terms of certain short time flows in suitable complex tangential directions. It is noteworthy that compactness is *not* established via the known potential theoretic sufficient conditions. Our results generalize to $\mathbb{C}^n$ the $\mathbb{C}^2$ results of the second author.

1. Introduction

In [Straube 2004], the second author provided geometric sufficient conditions for compactness of the $\bar{\partial}$-Neumann operator on the boundary points of infinite type of a bounded smooth pseudoconvex domain in $\mathbb{C}^2$. In this paper, we study the situation in higher dimensions and obtain suitable generalizations of the results in that reference.

Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. The $\bar{\partial}$-Neumann operator N_q on $(0, q)$-forms is the inverse of the complex Laplacian $\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ associated to the Dolbeault complex. For detailed information on the $\bar{\partial}$-Neumann problem and related questions, see, for example, [Folland and Kohn 1972; Boas and Straube 1999; Chen and Shaw 2001; Lieb and Michel 2002; McNeal 2006; Straube 2006]. The compactness of N_q is discussed in [Fu and Straube 2001; Straube 2006].

Whether or not N_q is compact is relevant in a number of situations. These include global regularity [Kohn and Nirenberg 1965], the Fredholm theory of Toeplitz operators [Henkin and Iordan 1997], and the (non)existence of solution operators to $\bar{\partial}$ with well-behaved solution kernels [Hefer and Lieb 2000]. There are also interesting connections to the theory of Schrödinger operators [Fu and Straube 2002; Christ and Fu 2005]. Catlin [1984] gave a sufficient condition, which he

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called property (P): near the boundary, there should exist plurisubharmonic functions bounded between 0 and 1 with arbitrarily large Hessians. (The smoothness assumptions on the boundary of the domain were considerably weakened in [Henkin and Iordan 1997] and were completely removed in [Straube 1997].) Property (P) was studied in detail (under the name B-regularity) by Sibony [1987]; see also [Sibony 1991]. On sufficiently regular domains, property (P) is equivalent to a quantitative version of Oka's lemma [Harrington 2007]. A version of property(P), called condition $(\widetilde{P})$, was introduced, and shown to still imply compactness, by McNeal [2002]. The uniform bound on the family of functions is replaced by a uniform bound on the gradient measured in the metric induced by the complex Hessian of the functions. (Both (P) and $(\widetilde{P})$ can also be formulated naturally at the level of $(0, q)$-forms; then (P_q) implies (P_{q+1}), $(\widetilde{P}_q)$ implies $(\widetilde{P}_{q+1})$, and (P_q) implies $(\widetilde{P}_q)$, for $1 \leq q \leq n$; see [Fu and Straube 2001; McNeal 2002; Straube 2006].) A sufficient condition that is intermediate, in a sense one can make precise (see [Straube 2006] for discussion), had appeared earlier in [Takegoshi 1991].

On locally convexifiable domains, (P_q) and $(\widetilde{P}_q)$ are equivalent and equivalent to the compactness of N_q for $1 \leq q \leq n$. Moreover, the three properties are equivalent to a simple geometric condition, the absence of (germs of) q-dimensional varieties from the boundary. For this, see [Fu and Straube 1998; 2001]. Thus the potential theory, the analysis, and the geometry mesh perfectly on locally convexifiable domains. It is also known that on smooth bounded Hartogs domains in $\mathbb{C}^2$, compactness of N_1 is equivalent to (P_1) and to $(\widetilde{P}_1)$; see [Fu and Straube 2002; Christ and Fu 2005]. However, it is well understood that the boundaries of convex (hence of locally convexifiable) domains do not exhibit some of the more intriguing aspects of the interaction with the ambient space that occur on general pseudoconvex boundaries [D'Angelo 1993]. For example, matters concerning orders of contact are always decided by orders of contact of *manifolds* (affine manifolds in the convex case) [McNeal 1992; Boas and Straube 1992; Yu 1992; Fu and Straube 1998]. A similar caveat applies in the case of domains in $\mathbb{C}^2$ [D'Angelo 1993]. As a result, these facts give no clear indication of how much (or how little) room there is, in the general pseudoconvex case, between $(P)/(\widetilde{P})$ and compactness of the $\overline{\partial}$-Neumann operator. (As far as we know, the exact relationship between (P) and $(\widetilde{P})$ is also unknown, but we do not address this question here.)

For obstructions, the situation is as follows. q-dimensional varieties in the boundary obstruct property (P_q); see [Sibony 1987; Fu and Straube 2001; Straube $\geq$ 2007]. Analytic discs in the boundary have been shown to obstruct condition $(\widetilde{P}_1)$ in [Straube 2006]. A q-dimensional complex manifold M in the boundary of a smooth bounded pseudoconvex domain is known to be an obstruction to compactness of N_q, provided M contains a point at which the domain is strictly pseudoconvex in the directions transverse to M [Şahutoğlu 2006; Şahutoğlu and

Straube 2006], but it is an open question whether the conclusion holds without assuming that there is such a point. One would expect that a flatter boundary is even more favorable for noncompactness, but the methods of [Şahutoğlu and Straube 2006] do not seem strong enough to yield noncompactness without some additional assumption on how M sits inside the boundary. On the other hand, both for (P)/$(\widetilde{P})$ and for compactness of the $\bar{\partial}$-Neumann operator, there are obstructions more subtle than varieties in the boundary [Sibony 1987; Matheos 1997; Fu and Straube 2001]. For a connection of some of these issues with properties of the Kobayashi metric, see [Kim 2004].

[Straube 2004] provided, for the first time, a method to prove compactness of the $\bar{\partial}$-Neumann operator that does not proceed by verifying property (P) or condition $(\widetilde{P})$. That the dimension is two was only used in applying so-called maximal estimates. Consequently, these results hold more generally on domains in $\mathbb{C}^n$, $n \geq 2$, where such estimates hold, or, equivalently, on domains where all the eigenvalues of the Levi form are comparable [Derridj 1978; 1991]. However, for the problem of compactness of the $\bar{\partial}$-Neumann operator, this assumption is too restrictive. It excludes, for example, the situation where the Levi form has at most one degenerate eigenvalue (see Remark 5 below). However, this assumption on the Levi form has been shown to be a useful generalization of the case of $\mathbb{C}^2$ in the context of compactness [Şahutoğlu 2006; Şahutoğlu and Straube 2006].

The obvious examples that satisfy the assumptions in Theorem 1 below also satisfy property $(\widetilde{P})$; we do not know whether the theorem can actually furnish domains where the $\bar{\partial}$-Neumann operator is compact but where $(\widetilde{P})$ fails. But just as in [Straube 2004], we obtain a simple geometric proof of compactness in these cases. Moreover, the assumptions are in some instances "minimal": they are necessary modulo the size of certain balls; see Remark 6 for details.

We will only consider the case $q = 1$ in the remainder of this paper. This is the main case in terms of understanding compactness. But note that compactness of N_1 implies compactness of N_q for $q > 1$. (See [McNeal 2003; Fu 2005, Proposition 2.2]; this fact is also implicit in [Kohn 2002, Proposition 5.2].)

2. Results

If Z is a (real) vector field defined in some open subset of the boundary (or of $\mathbb{C}^n$), we denote by $\mathcal{F}^t_Z$ the flow generated by Z. We use the D'Angelo's notion of finite or infinite type [1993]. For a boundary point ζ, we denote by $\lambda_0(\zeta)$ the smallest eigenvalue of the Levi form of the boundary at ζ. Recall that "the" Levi form is the restriction to the complex tangent space of the complex Hessian $H_\rho(w, \bar{w}) = \sum_{j,k=1}^{n} (\partial^2 \rho / \partial z_j \partial \overline{z_k}) w_j \overline{w_k}$ of a defining function ρ, and that the relevant properties of this restriction do not depend on the choice of ρ. In particular, the condition in Theorem 1 below is independent of this choice. Since the domains in question are pseudoconvex, $\lambda_0(\zeta) \geq 0$. For a set of real vector fields $T_1, \ldots, T_m$ on an open subset of $b\Omega$ (or of $\mathbb{C}^n$), we define $\mathrm{span}_{\mathbb{R}}(T_1, \ldots, T_m)$ to be the set of all linear combinations of $T_1, \ldots, T_m$ whose coefficients are smooth real-valued functions (not necessarily constants).

Theorem 1. *Let Ω be a C^∞-smooth bounded pseudoconvex domain in $\mathbb{C}^n$. Denote by K the set of boundary points of infinite type. Assume that there exist smooth complex tangential vector fields $X_1, \ldots X_m$ (of type $(1,0)$), defined on $b\Omega$ near K, so that $H_\rho(X_i(\zeta), \overline{X_i(\zeta)}) \leq C\lambda_0(\zeta)$, for some constant C, a sequence $\{\epsilon_j\}_{j=1}^\infty$ with $\lim_{j\to\infty} \epsilon_j = 0$, and constants C_1, $C_2 > 0$, C_3 with $1 \leq C_3 < (n+1)/n$, so that the following holds. For every $j \in \mathbb{N}$ and $p \in K$ there is a real vector field $Z_{p,j} \in \mathrm{span}_{\mathbb{R}}(\mathrm{Re}\, X_1, \mathrm{Im}\, X_1, \ldots, \mathrm{Re}\, X_m, \mathrm{Im}\, X_m)$ of unit length, defined in some neighborhood of p in $b\Omega$ with $\max|\mathrm{div}\, Z_{p,j}| \leq C_1$, such that $\mathcal{F}_{Z_{p,j}}^{\epsilon_j}\big(B(p, C_2(\epsilon_j)^{C_3}) \cap K\big) \subseteq b\Omega \setminus K$. Then the $\bar\partial$-Neumann operator N_1 on Ω is compact.*

Remark 1. In $\mathbb{C}^2$, take $m = 1$ and $X_1 = L$, where L is a smooth nonvanishing complex tangential vector field of type $(1,0)$ on $b\Omega$. Then the condition on the vector fields $Z_{p,j}$ becomes simply that they be complex tangential, as in [Straube 2004], and Theorem 1 therefore generalizes the main result there.

Remark 2. Compared to the main result in [Straube 2004], Theorem 1 assumes additionally that there is a family of vector fields $X_1 \ldots X_m$ satisfying

$$H_\rho(X_j(\zeta), \overline{X_j(\zeta)}) \leq C\lambda_0(\zeta)\,, \tag{1}$$

such that the vector fields $Z_{p,j}$ are contained in the linear span (over $\mathbb{R}$) of the real and imaginary parts of these fields (as opposed to just being complex tangential). Without some assumption on the fields $Z_{p,j}$ more restrictive than being complex tangential, such as the one made here, the result does not generalize to $\mathbb{C}^n$. To see this, consider a smooth bounded convex domain in $\mathbb{C}^3$ that is strictly convex except for an analytic (affine) disc in the boundary. Then one can flow along complex tangential directions from points of the disc into the set of strictly (pseudo) convex boundary points as required in the second part of the assumption in the theorem. Nonetheless, because there is a disc in the boundary, the $\bar\partial$-Neumann operator on $(0,1)$-forms is not compact on such a domain; see [Fu and Straube 1998].

Remark 3. In the previous example, the value of the Levi form in the direction parallel to the disc apparently goes to zero faster, upon approach to the disc, than the value in the direction transverse to the disc. Thus (1) cannot hold for a vector field transverse to the disc, and Theorem 1 does not apply. It would be very desirable to have a direct general proof of this, that is, a proof that the assumptions in Theorem 1 exclude discs from the boundary (or, less likely, a counterexample). This relates to the open question of whether discs in the boundary generally obstruct compactness (for what is known, see the introduction). Note that in $\mathbb{C}^2$, such discs are known to obstruct compactness [Fu and Straube 2001]; in this case, the theorem's assumptions obviously exclude discs from the boundary.

Remark 4. Because compactness of the $\bar{\partial}$-Neumann operator follows from a compactness estimate for forms supported in fixed, but possibly small neighborhoods of boundary points (see [Fu and Straube 2001], for example), there is a version of Theorem 1 where the assumptions are localized. See Example 3 below.

In Theorem 1, one would like to have a collection of vector fields $\{X_j\}_{j=1}^{m}$ such that, at each point p of K, $\mathrm{span}_{\mathbb{R}}\{\mathrm{Re}\, X_1(p), \mathrm{Im}\, X_1(p), \dots, \mathrm{Re}\, X_m(p), \mathrm{Im}\, X_m(p)\}$ is as big as possible, thus putting the least restrictions on the fields $Z_{p,j}$. On the other hand, this needs to be balanced with the requirement (1).

Example 1. When the eigenvalues of the Levi form are all comparable, any finite collection of complex tangential vector fields $X_1, \dots, X_m$ will satisfy condition (1). Taking a collection that spans $T^{\mathbb{C}}_{b\Omega}(p)$ at each point $p \in K$, we see that this part of the assumptions of Theorem 1 reduces to $Z_{p,j}$ being complex tangential (as in $\mathbb{C}^2$). Domains where all the eigenvalues of the Levi form are comparable play a special role in the theory of the $\bar{\partial}$-Neumann problem: certain estimates, called maximal estimates, hold. This class of domains was studied in detail by Derridj [1978; 1991].

Example 2. Assume there exists a smooth complex tangential vector field X_1 near K such that for $\zeta \in K$, $X_1(\zeta)$ is an eigenvector associated with the smallest eigenvalue of the Levi form at ζ. This vector field trivially satisfies the condition (1). Then the assumption in Theorem 1 requires that $Z_{p,j}(\zeta)$ is in the two real-dimensional plane spanned by $X_1(\zeta)$ for all ζ.

Example 3. Assume that the Levi form has at most one degenerate eigenvalue at each point of K (hence near K). Fix a point $p \in K$. Choose an $(n-2)$-dimensional subspace of $T^{\mathbb{C}}_{b\Omega}(p)$ on which the Levi form is strictly positive, and choose a basis. Extending the basis vectors to local sections of $T^{\mathbb{C}}_{b\Omega}$ gives vector fields (defined near p) $Y_2, \dots, Y_{n-1}$. In a neighborhood of p, the Levi form is strictly positive on the span of $Y_2, \dots, Y_{n-1}$. As a consequence, at each point there is a unique one-dimensional subspace of $T^{\mathbb{C}}_{b\Omega}$ that is orthogonal to this span with respect to the Levi form. Indeed, if Y_1 is such that $Y_1, \dots, Y_{n-1}$ is a basis for $T^{\mathbb{C}}_{b\Omega}$ (near p), then $X_1 = Y_1 + b_2 Y_2 \cdots + b_{n-1} Y_{n-1}$, where $b_j = -H_\rho(Y_1, \overline{Y}_j)/H_\rho(Y_j, \overline{Y}_j)$, $j = 2, \dots, n-1$, will span this subspace. This was observed already in [Machedon 1988, Lemma 2.1]. When $p \in K$, the eigenvector of the Levi form at p with zero eigenvalue is orthogonal to $T^{\mathbb{C}}_{b\Omega}(p)$ with respect to the Levi form (because the Levi form is positive definite). Therefore, by uniqueness, $X_1(p)$ is an eigenvector of the Levi form at p with eigenvalue zero. As a result, $H_\rho(X_1(\zeta), \overline{X_1(\zeta)}) \le C\lambda_0(\zeta)$ for

ζ close enough to p (by continuity), that is, (1) holds for the family consisting of the single field X_1. We note that when ζ is a strictly pseudoconvex point, $X_1(\zeta)$ need not be an eigenvector of the Levi form. By multiplying with a cutoff function that is identically equal to one near p, we may assume that $X_1(\zeta)$ is defined on all of $b\Omega$, with (1) still valid. Of course, this is at the expense of having a trivial span on a big set. We may proceed in two ways. We can cover K with finitely many open sets $U_1, \dots, U_s$ on which the cutoff functions multiplying these local fields are one, and then add finitely many of the fields. However, the resulting field may still vanish at some points of K. Alternatively, we can take advantage of the fact that compactness localizes: it suffices to prove a compactness estimate for forms supported in (small) neighborhoods of boundary points (as long as the neighborhood do not depend on the ϵ in the compactness estimate). The proof of Theorem 1, using the field X_j, gives a compactness estimate for forms whose support meets the boundary in one of the U_j's. Since the U_j's cover K, the result is a compactness estimate for all forms in $\mathrm{dom}(\bar{\partial}) \cap \mathrm{dom}(\bar{\partial}^*)$ (since away from K, we have subelliptic estimates).

Remark 5. On the domains discussed in Example 3, the Levi form is locally diagonalizable (use Gram–Schmidt to orthonormalize $Y_2, \dots, Y_{n-1}$ with respect to the Levi form). Derridj [1991, Theorem 7.1] showed that if maximal estimates hold at $p \in b\Omega$ and p is a weakly pseudoconvex point, then the Levi form of Ω cannot be diagonalizable near p when Ω is a domain in $\mathbb{C}^n$ with $n \geq 3$. Therefore, Examples 1 and 2 are mutually exclusive (when $n \geq 3$).

Remark 6. In Example 3, the assumptions in the theorem are “minimal”, that is, they are necessary modulo the size of the balls $B(p, C_2(\epsilon_j)^{C_3})$. The discussion is analogous to that of [Straube 2004, Remarks 2 and 3] but uses recent results from [Şahutoğlu and Straube 2006]. For $p \in K$, let X_1 be the complex tangential field from Example 3 above (defined near p, with $H_\rho(X_1(p), \overline{X_1(p)}) = 0$). Denote by T^θ the field $T^\theta = \cos(\theta)\,\mathrm{Re}\,X_1 + \sin(\theta)\,\mathrm{Im}\,X_1$. For ζ near p, set $M_{\zeta,\theta} = \{\mathcal{F}^t_{T^\theta}(\zeta) \mid 0 \leq \theta \leq 2\pi,\ 0 \leq t \leq t_0\}$. Then $M_{\zeta,\theta}$ is a smooth two-dimensional submanifold of the boundary. Because N_1 is compact, the boundary contains no analytic discs (since the Levi form has at most one degenerate eigenvalue; see [Şahutoğlu and Straube 2006, Theorem 1]). Therefore, [Şahutoğlu and Straube 2006, Lemma 3] implies that there exist points $\zeta \in M_{T^\theta}$ arbitrarily close to p with $H_\rho(X_1(\zeta), \overline{X_1(\zeta)}) > 0$. Because of the way X_1 was constructed, such a point ζ is a strictly pseudoconvex point. Consequently, for $\epsilon_j > 0$ and ζ near p, there exist real fields $Z_{\zeta,j} \in \mathrm{span}_{\mathbb{R}}(\mathrm{Re}\,X_1, \mathrm{Im}\,X_1)$ of unit length so that $\mathcal{F}^{\epsilon_j}_{Z_{\zeta,j}}(z) \notin K$ for z close enough to ζ. This yields balls as in the theorem, but without control of the radii from below in terms of ϵ_j. Because of the form of $Z_{\zeta,j}$, the uniform boundedness condition on the divergence of the fields $Z_{\zeta,j}$ is satisfied (near p).

We say that $b\Omega \setminus K$ satisfies a complex tangential cone condition if there exists a finite, open real cone Γ in $\mathbb{C}^n \approx \mathbb{R}^{2n}$ such that: For each $p \in K$, there exists a complex tangential direction such that when Γ is moved by a rigid motion to have vertex at p and axis in that direction (that is, in the two dimensional real affine subspace determined by that direction), the (open) cone obtained intersects $b\Omega$ in a set contained in $b\Omega \setminus K$.

Theorem 1 has the following corollaries.

Corollary 1. *Let Ω be a C^∞-smooth bounded pseudoconvex domain in $\mathbb{C}^n$. Denote by K the set of boundary points of infinite type. For all points ζ in a neighborhood of K in $b\Omega$, denote by $\lambda_0(\zeta)$ the smallest eigenvalue of the Levi form. Assume there exist smooth complex tangential vector fields $X_1, \dots, X_m$, defined on $b\Omega$ near K, such that $H_\rho(X_i(\zeta), \overline{X_i(\zeta)}) \leq C\,\lambda_0(\zeta)$ for some constant C and all ζ, and such that $b\Omega \setminus K$ satisfies a complex tangential cone condition with the axis of the cone at $p \in K$ in* $\operatorname{span}_{\mathbb{R}}(\operatorname{Re} X_1(p), \operatorname{Im} X_1(p), \dots, \operatorname{Re} X_m(p), \operatorname{Im} X_m(p))$ *for all $p \in K$. Then the $\bar\partial$-Neumann operator on Ω is compact.*

In $\mathbb{C}^2$, the assumption in Corollary 1 reduces to the simple requirement that K satisfy a complex tangential cone condition; this is Corollary 2 in [Straube 2004]. The example of Remark 2 shows that this is not sufficient in $\mathbb{C}^n$ when $n \geq 3$, not even when one assumes that the axis of the cone at $p \in K$ lies in the null space of the Levi form at p. So some complication in the statement of the corollary cannot be avoided. On the other hand, when the Levi form of $b\Omega$ has at most one degenerate eigenvalue, there is (locally) a complex tangential vector field X_1 and a constant such that $H_\rho(X_1(p), \overline{X_1}(p)) = 0$ and $H_\rho(X_1, \overline{X}_1) \leq C\lambda_0$ near $p \in K$ (see Example 3 on page 347). With this additional information, it suffices to require that the axis of the cone lie in the null space of the Levi form.

Corollary 2. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$. Assume that at each boundary point, the Levi form has at most one degenerate eigenvalue. If the set $b\Omega \setminus K$ satisfies a complex tangential cone condition with the axis of the cone at $p \in K$ in the null space of the Levi form at p, then the $\bar\partial$-Neumann operator on Ω is compact.*

In $\mathbb{C}^2$, this also reduces to [Straube 2004, Corollary 2].

3. Proof of Theorem 1

We will establish a compactness estimate for forms in $C^\infty_{(0,1)}(\overline{\Omega}) \cap \operatorname{dom}(\bar\partial^*)$: for all $\epsilon > 0$, there is a constant C_ϵ such that

$$\|u\|^2 \leq \epsilon(\|\bar\partial u\|^2 + \|\bar\partial^* u\|^2) + C_\epsilon \|u\|^2_{-1}.$$

This is equivalent to the compactness of N_1; see, for example, [Fu and Straube 2001, Lemma 1.1].

We may assume that the vector fields X_k, $1 \leq k \leq m$, are defined on all of $b\Omega$, by multiplying them with suitable cutoff functions that are identically equal to one near K. This preserves (1), which now holds on all of $b\Omega$. Then we can extend the vector fields $Z_{p,j}$ and X_k from $b\Omega$ to the inside of Ω by letting them be constant along the real normal, so that the extended fields, still denoted by $Z_{p,j}$ and X_k, are complex tangential to the level sets of the boundary distance. Finally, by multiplying by a suitable cutoff function that equals one near the boundary, we may assume that the fields X_k are defined and smooth on $\overline{\Omega}$.

There are two ideas in the proof. The first comes from [Straube 2004] and says that near a point $p \in K$, the values of a form u can be expressed by the values near $\mathcal{F}^{\epsilon_j}_{Z_{p,j}}(p)$ plus the integrals of $Z_{p,j}u$ along the integral curves of $Z_{p,j}$. If the points near $\mathcal{F}^{\epsilon_j}_{Z_{p,j}}(p)$ are of finite type, the contribution from there can easily be estimated by subelliptic estimates. Because the integrals of $Z_{p,j}u$ in the second contribution are over curves of length ϵ_j ($|Z_{p,j}| = 1$), a (small) factor ϵ_j appears (via the Cauchy–Schwarz inequality) when computing the relevant $\mathcal{L}^2$-norms. There are overlap issues, but these are handled by the uniformity built into the assumptions (for example the uniform bound on $\operatorname{div} Z_{p,j}$). The second idea concerns control of $\|Z_{p,j}u\|^2$, or ultimately $\|X_k u\|^2$, by $\|\bar\partial u\|^2 + \|\bar\partial^* u\|^2$. In $\mathbb{C}^2$, this can be done for any complex tangential field via maximal estimates; see [Derridj 1978; Straube 2004]. In higher dimensions, additional assumptions are needed; that (1) suffices is implicit in [Derridj 1978].

The first part of the proof follows [Straube 2004] verbatim, with only one obvious modification. Fix $\epsilon > 0$ and choose j big enough so that $\epsilon_j < \epsilon$. The arguments in [Straube 2004, pages 705–708] give estimate (2) below, the only modification being the exponent of ϵ_j, which depends on the dimension. This dimension dependence arises from a comparison of volumes argument used to resolve certain overlap issues; see the paragraph in [Straube 2004] that starts at the bottom of page 707. Note that when $n = 2$ we have $2n+2-2nC_3 = 6-4C_3$, as in equation (14) of [Straube 2004]. Combining (the analogues of) equations (6) and (14) in the same reference gives

$$
\begin{aligned}
(2)\quad \int_\Omega |u|^2
&\leq 2\epsilon(\|\bar\partial u\|_0^2 + \|\bar\partial^* u\|_0^2) + C_\epsilon \|u\|_{-1}^2 \\
&\quad + 4\epsilon_j \int_0^{\epsilon_j} \left(C(C_2)\epsilon_j^{2n-2nC_3} 2m \sum_{k=1}^m \int_\Omega \big(|\operatorname{Re} X_k u(y)|^2 + |\operatorname{Im} X_k u(y)|^2\big)\, dV(y) \right) dt \\
&\leq 2\epsilon(\|\bar\partial u\|_0^2 + \|\bar\partial^* u\|_0^2) + C_\epsilon \|u\|_{-1}^2 \\
&\qquad\qquad + 16mC(C_2)\epsilon_j^{2n+2-2nC_3} \sum_{k=1}^m \int_\Omega \big(|X_k u(y)|^2 + |\overline{X}_k u(y)|^2\big)\, dV(y).
\end{aligned}
$$

In $\mathbb{C}^2$, the last term can be estimated using maximal estimates [Derridj 1978]:

$$\|\bar{X}_k u\|^2 + \|X_k u\|^2 \le C_k \left(\|\bar{\partial} u\|_0^2 + \|\bar{\partial}^* u\|_0^2\right). \tag{3}$$

We next show that (3) also holds under the hypothesis (1) on the vector fields X_k. The estimates below owe much to [Derridj 1978]. For convenience, fix k and write X for X_k. First, the estimate on $\|\bar{X} u\|^2$ follows directly from the Kohn–Morrey formula; see [Chen and Shaw 2001, Proposition 4.3.1], with weight $e^{-\phi} \equiv 1$. For $\|Xu\|^2$, we integrate by parts to obtain

$$\begin{aligned} \|Xu\|^2 &= -\int_\Omega u X\overline{Xu} + O(\|u\|\,\|Xu\|) \\ &= -\int_\Omega u \left[X, \bar{X}\right] \bar{u} - \int_\Omega u \bar{X} X \bar{u} + O(\|u\|\,\|Xu\|) \\ &= -\int_\Omega u \left[X, \bar{X}\right] \bar{u} + \|\bar{X} u\|^2 + O(\|u\|\,\|Xu\| + \|u\|\,\|\bar{X} u\|). \end{aligned} \tag{4}$$

Using that $\|u\|^2 \lesssim (\|\bar{\partial} u\|^2 + \|\bar{\partial}^* u\|^2)$, $\|u\|\,\|Xu\| \le c_l \|u\|^2 + c_s \|Xu\|^2$ [Chen and Shaw 2001, 4.4.6, page 79], and again the Kohn–Morrey formula for $\|\bar{X} u\|^2$, we see that it suffices to estimate the first term on the last line of (4). Here c_s and c_l denote small and large constants; the term $\|Xu\|^2$ on the right-hand side in (4) can then be absorbed into the left-hand side.

Near $b\Omega$ (say, on the support of the cutoff functions used at the beginning of this section to extend X to all of Ω), we write

$$\left[X, \bar{X}\right] \bar{u} = H_\rho(X, \bar{X})(L_n - \bar{L}_n)\bar{u} + A\bar{u} + \bar{B}\bar{u},$$

where A and B are smooth complex tangential fields and L_n is the complex normal $\sum_{j=1}^n (\partial\rho/\partial\bar{z}_j)(\partial/\partial z_j)$, appropriately normalized. The contributions from $A\bar{u}$ and $L_n\bar{u}$ (or $\bar{A}u$ and $\bar{L}_n u$) are estimated as above. In the contribution from $\bar{B}\bar{u}$, we integrate $\bar{B}$ by parts and proceed as before. We are left with estimating the term involving $H_\rho(X, \bar{X})\bar{L}_n\bar{u}$. We integrate $\bar{L}_n$ by parts (with a boundary term) to obtain

$$\left|\int_\Omega u H_\rho(X, \bar{X})\bar{L}_n\bar{u}\right| \lesssim \|\bar{L}_n u\|\,\|u\| + \int_{b\Omega} H_\rho(X, \bar{X})|u|^2 + O(\|u\|^2). \tag{5}$$

The first and third term on the right are estimated as above, while for the middle term we have

$$\int_{b\Omega} H_\rho(X, \bar{X})|u|^2 \lesssim \int_{b\Omega} \lambda_0 |u|^2 \lesssim \int_{b\Omega} H_\rho(u, \bar{u}) \lesssim \|\bar{\partial} u\|^2 + \|\bar{\partial}^* u\|^2. \tag{6}$$

The first inequality comes from (1), the last one from the Kohn–Morrey formula. We have slightly abused notation in the third term: u is a (0, 1)-form, not a vector field, but it is identified with a vector field in the usual way via its coefficients.

Combining Equations (4), (5), and (6) shows that (3) holds under the assumptions of Theorem 1 — that is, under (1). Inserting (3) into (2) gives

$$\int_\Omega |u|^2 \le C(\epsilon + \epsilon^{2n+2-2nC_3})\big(\|\bar\partial u\|^2 + \|\bar\partial^* u\|^2\big) + C_\epsilon \|u\|_{-1}^2\,, \tag{7}$$

with C independent of ϵ. Since $2n+2-2nC_3 > 0$ (because $C_3 < (n+1)/n$), the limit $\lim_{\epsilon\to 0^+} C(\epsilon + \epsilon^{2n+2-2nC_3})$ vanishes, and so (7) implies the compactness estimate we set out to prove, concluding the demonstration of Theorem 1. □

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