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Let F be a local nonarchimedean field. We prove the uniqueness of the Whittaker model for irreducible admissible representations of $\overline{\mathrm{Sp}_{2n}(F)}$, the metaplectic double cover of $\mathrm{Sp}_{2n}(F)$. An ingredient of the proof is an explicit extension of Rao's normalized cocycle from $\mathrm{Sp}_{2n}(F)$ to $\mathrm{GSp}_{2n}(F)$.

1. Introduction

Let F be a p-adic field. Let $\overline{\mathrm{Sp}_{2n}(F)}$ be the metaplectic double cover of $\mathrm{Sp}_{2n}(F)$. The uniqueness of Whittaker models for irreducible admissible representations of $\overline{\mathrm{Sp}_{2n}(F)}$ is well known to experts. Although it has been used many times, there is no general written proof, although a uniqueness theorem for principal series representations can be found in [Bump et al. 1991]. In this paper, we correct the situation and prove this uniqueness as Theorem 3.4. It turns out that one may use similar methods to those used in [Shalika 1974; Gel'fand and Kazhdan 1975; Bernstein and Zelevinskii 1976] for quasisplit groups over F. A central role is played by $\overline{\tau}$, a certain involution on $\overline{\mathrm{Sp}_{2n}(F)}$ which is an extension of the involution $g \mapsto {}^{\tau}g = \sigma_0\, {}^{\mathrm{t}}g \sigma_0^{-1}$ defined on $\mathrm{Sp}_{2n}(F)$, where σ_0 is a certain Weyl element.

Here, we realize $\overline{\mathrm{Sp}_{2n}(F)}$ using Rao's cocycle [1993]. In Section 2, we extend it explicitly to $\mathrm{GSp}_{2n}(F) \times \mathrm{GSp}_{2n}(F)$ and then use this extension to define $\overline{\tau}$. The unexpected fact, at least to us, is that τ extends to $\overline{\mathrm{Sp}_{2n}(F)}$ in the simplest possible way, namely ${}^{\overline{\tau}}(g, \epsilon) = ({}^{\tau}g, \epsilon)$.

2. Rao's cocycle

Let F be a local nonarchimedean field. Let $X = F^{2n}$ be a vector space of even dimension over F equipped with $\langle \cdot, \cdot \rangle : X \times X \to F$, a non degenerate symplectic form, and let $\mathrm{Sp}(X) = \mathrm{Sp}_{2n}(F)$ be the subgroup of $\mathrm{GL}(X)$ of symplectomorphisms of X onto itself. We shall write the action of $GL(X)$ on X from the right. R. Rao [1993] constructs an explicit nontrivial 2-cocycle $c(\cdot, \cdot)$ on $\mathrm{Sp}(X)$. The

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set $\overline{\mathrm{Sp}(X)} = \mathrm{Sp}(X) \times \{\pm 1\}$ is then made a group, called the metaplectic group, via

$$(g_1, \epsilon_1)(g_2, \epsilon_2) = \big(g_1 g_2, \epsilon_1 \epsilon_2 c(g_1, g_2)\big). \tag{2-1}$$

In Section 2A, we describe this cocycle.

Let $\mathrm{GSp}(X)$ be the group of similitudes of the symplectic form above. It is the subgroup of $\mathrm{GL}(X)$ of elements g such that $\langle v_1 g, v_2 g\rangle = \lambda_g \langle v_1, v_2\rangle$ for all $v_1, v_2 \in X$, where $\lambda_g \in F^*$. The similitude factor λ_g defines a homomorphism from $\mathrm{GSp}(X)$ to F^* whose kernel is $\mathrm{Sp}(X)$. In Section 2B, we will extend $c(\cdot\,,\cdot)$ from $\mathrm{Sp}(X) \times \mathrm{Sp}(X)$ to $\mathrm{GSp}(X) \times \mathrm{GSp}(X)$. The result of this extension is that the set $\overline{\mathrm{GSp}(X)} = \mathrm{GSp}(X) \times \{\pm 1\}$ is given a group structure by (2-1).

2A. ***Description of Rao's cocycle.*** We start with recalling some of the notations and facts that appear in Rao's formula. Detailed proofs can be found in [Rao 1993].

Let $E = \{e_1, e_2, \dots, e_n, e_1^*, e_2^*, \dots, e_n^*\}$ be a symplectic basis of X: For $1 \le i, j \le n$, we have $\langle e_i, e_j\rangle = \langle e_i^*, e_j^*\rangle = 0$ and $\langle e_i, e_j^*\rangle = \delta_{i,j}$. In this basis, $\mathrm{Sp}(X)$ is realized as the set $\{A \in \mathrm{Mat}_{2n\times 2n}(F) \mid AJ\,{}^{\mathrm{t}}A = J\}$, where $J = ((0, I_n), (-I_n, 0))$. Define $V = \mathrm{span}\{e_1, e_2, \dots, e_n\}$, $V^* = \mathrm{span}\{e_1^*, e_2^*, \dots, e_n^*\}$. These are two transverse Lagrangian subspaces. Let P be the Siegel parabolic subgroup of $\mathrm{Sp}(X)$ consisting of the elements that preserve V^*. In coordinates,

$$P = \left\{ \begin{pmatrix} A & B \\ 0 & {}^{\mathrm{t}}A^{-1} \end{pmatrix} \;\middle|\; A \in \mathrm{GL}_n(F),\ B \in \mathrm{Mat}_{n\times n}(F),\ {}^{\mathrm{t}}B = A^{-1} B\,{}^{\mathrm{t}}A \right\}.$$

Let S be a subset of $\{1, 2, \dots, n\}$. We define τ_S, a_S to be the following elements of $\mathrm{Sp}(X)$:

$$e_i \cdot \tau_S = \begin{cases} -e_i^*, & i \in S, \\ \ \ e_i, & \text{otherwise,} \end{cases} \qquad e_i^* \cdot \tau_S = \begin{cases} e_i, & i \in S, \\ e_i^*, & \text{otherwise,} \end{cases}$$

$$e_i \cdot a_S = \begin{cases} -e_i, & i \in S, \\ \ \ e_i, & \text{otherwise,} \end{cases} \qquad e_i^* \cdot a_S = \begin{cases} -e_i^*, & i \in S, \\ \ \ e_i^*, & \text{otherwise.} \end{cases}$$

The elements τ_S, a_S, $\tau_{S'}$, $a_{S'}$ commute. Note that $a_S \in P$, $a_S^2 = I_{2n}$, and that $\tau_{S_1}\tau_{S_2} = \tau_{S_1 \triangle S_2} a_{S_1 \cap S_2}$, where $S_1 \triangle S_2 = S_1 \cup S_2 \backslash S_1 \cap S_2$. In particular, $\tau_S^2 = a_S$. For $S = \{1, 2, \dots, n\}$, we define $\tau = \tau_S$; in this case $a_S = -I_{2n}$.

Define now $\Omega_j = \{\sigma \in \mathrm{Sp}(X) \mid \dim(V^* \cap V^*\sigma) = n - j\}$. Note that $P = \Omega_0$, $\tau_S \in \Omega_{|S|}$, and, more generally, if $\alpha, \beta, \gamma, \delta \in \mathrm{Mat}_{n\times n}(F)$ and $\sigma = ((\alpha, \beta), (\gamma, \delta)) \in \mathrm{Sp}(X)$, then $\sigma \in \Omega_{\mathrm{rank}\,\gamma}$. The Bruhat decomposition states that each Ω_j is a single double coset in ${}_P\backslash^{\mathrm{Sp}(X)}/_P$, that $\Omega_j^{-1} = \Omega_j$, and that $\bigcup_{j=1}^n \Omega_j = \mathrm{Sp}(X)$. In particular, every element of $\mathrm{Sp}(X)$ has the form $p\tau_S p'$, where $p, p' \in P$, $S \subseteq \{1, 2, \dots, n\}$.

Let $p_1, p_2 \in P$. Rao defines

$$x(p_1 \tau_S p_2) \equiv \det(p_1 p_2 \,|_{V^*}) \pmod{(F^*)^2}$$

and proves that it is a well defined map from $\mathrm{Sp}(X)$ to $F^*/(F^*)^2$. Note that $x(a_S) \equiv (-1)^{|S|}$, and, more generally, if $p = ((A, B), (0, {}^{t}A^{-1})) \in P$, then $x(p) \equiv \det A$. Also note that $x(\tau_S) \equiv 1$ and that for $g \in \Omega_j$, and $p_1, p_2 \in P$,

$$x(g^{-1}) \equiv x(g)(-1)^j, \quad x(p_1 g p_2) \equiv x(p_1)x(g)x(p_2).$$

Theorem 5.3 in [Rao 1993] states that a nontrivial 2-cocycle on $\mathrm{Sp}(X)$ can be defined by

$$\begin{aligned} c(\sigma_1, \sigma_2) = \big(x(\sigma_1), x(\sigma_2)\big)_F \big(-x(\sigma_1)x(\sigma_2), x(\sigma_1\sigma_2)\big)_F \\ \cdot \big((-1)^l, d_F(\rho)\big)_F \big(-1, -1\big)_F^{l(l-1)/2} h_F(\rho), \end{aligned} \tag{2-2}$$

where ρ is the Leray invariant $-q(V^*, V^*\sigma_1, V^*\sigma_2^{-1})$, $d_F(\rho)$ and $h_F(\rho)$ are its discriminant and Hasse invariant, $(\,\cdot\,,\cdot\,)_F$ is the quadratic Hilbert symbol of F, and $2l = j_1 + j_2 - j - \dim\rho$, where $\sigma_1 \in \Omega_{j_1}$, $\sigma_2 \in \Omega_{j_2}$, $\sigma_1\sigma_2 \in \Omega_j$. We use Rao's normalization of the Hasse invariant. (Note that the cocycle formula just given differs slightly from the one appearing in Rao's paper, as there is a small mistake in its Theorem 5.3. A correction by Adams can be found in [Kudla 1994], Theorem 3.1). For future reference, we recall some of the properties of the Hilbert symbol:

$$(a, -a)_F = 1, \quad (aa', b)_F = (a, b)_F (a', b)_F, \quad (a, b)_F = (a, -ab)_F. \tag{2-3}$$

From (2-2) and from previous remarks we obtain the following properties of $c(\,\cdot\,,\cdot\,)$. For $\sigma \in \Omega_j$, $p \in P$ we have

$$c(\sigma, \sigma^{-1}) = \big(x(\sigma), (-1)^j x(\sigma)\big)_F (-1, -1)_F^{j(j-1)/2}, \tag{2-4}$$

$$c(p, \sigma) = c(\sigma, p) = \big(x(p), x(\sigma)\big)_F. \tag{2-5}$$

As a consequence of (2-5) and (2-3), we see that

$$(p, \epsilon_1)(\sigma, \epsilon)(p, \epsilon_1)^{-1} = (p\sigma p^{-1}, \epsilon), \tag{2-6}$$

for all $\sigma \in \mathrm{Sp}(X)$, $p \in P$, and $\epsilon_1, \epsilon \in \{\pm 1\}$.

Finally, we recall Corollary 5.6 in Rao's paper. For $S \subset \{1, 2, \dots, n\}$, define $X_S = \mathrm{span}\{e_i, e_i^* \mid i \in S\}$. We may now consider x_S and $c_{X_S}(\,\cdot\,,\cdot\,)$ defined by analogy with x and $c(\,\cdot\,,\cdot\,)$. Let S_1 and S_2 be a partition of $\{1, 2, \dots, n\}$. Suppose $\sigma_1, \sigma_1' \in \mathrm{Sp}(X_{S_1})$ and $\sigma_2, \sigma_2' \in \mathrm{Sp}(X_{S_2})$. Put $\sigma = \mathrm{diag}(\sigma_1, \sigma_2)$, $\sigma' = \mathrm{diag}(\sigma_1', \sigma_2')$. Rao proves that

$$\begin{aligned} c(\sigma, \sigma') = c_{S_1}(\sigma_1, \sigma_1') c_{S_2}(\sigma_2, \sigma_2') \big(x_{S_1}(\sigma_1), x_{S_2}(\sigma_2)\big)_F \\ \cdot \big(x_{S_1}(\sigma_1'), x_{S_2}(\sigma_2')\big)_F \big(x_{S_1}(\sigma_1\sigma_1'), x_{S_2}(\sigma_2\sigma_2')\big)_F. \end{aligned} \tag{2-7}$$

2B. ***Extension of Rao's cocycle to*** $\mathbf{GSp}(X)$**.** F^* is embedded in $\mathrm{GSp}(X)$ via

$$\lambda \mapsto i(\lambda) = \begin{pmatrix} I_n & 0 \\ 0 & \lambda I_n \end{pmatrix}.$$

Using this embedding we define an action of F^* on $\mathrm{Sp}(X)$: $(g, \lambda) \mapsto g^\lambda = i(\lambda^{-1}) g\, i(\lambda)$. Let $F^* \ltimes \mathrm{Sp}(X)$ be the semidirect product corresponding to this action. For $\alpha, \beta, \gamma, \delta \in \mathrm{Mat}_{n\times n}(F)$ and $g = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{GSp}(X)$, define

$$p(g) = \begin{pmatrix} \alpha & \beta \\ \lambda_g^{-1}\gamma & \lambda_g^{-1}\delta \end{pmatrix} \in \mathrm{Sp}(X),$$

and note that $g = i(\lambda_g) p(g)$. The map $g \mapsto \iota(g) = (\lambda_g, p(g))$ is an isomorphism between $\mathrm{GSp}(X)$ and $F^* \ltimes \mathrm{Sp}(X)$.

From the theory of the Weil representation (see [Mœglin et al. 1987, page 36]), we know that for any $\lambda \in F^*$, we can define a map $v_\lambda : \mathrm{Sp}(X) \to \{\pm 1\}$ such that $(g, \epsilon) \mapsto (g, \epsilon)^\lambda = \big(g^\lambda, \epsilon v_\lambda(g)\big)$ is an automorphism of $\overline{\mathrm{Sp}(X)}$. In 2B, we compute v_λ. We shall also show there that $\big(\lambda, (g, \epsilon)\big) \mapsto (g, \epsilon)^\lambda$ defines an action of F^* on $\overline{\mathrm{Sp}(X)}$. Here we just want to show how this computation enables us to extend $c(\,\cdot\,,\cdot\,)$ to a 2-cocycle $\tilde{c}(\,\cdot\,,\cdot\,)$ on $\mathrm{GSp}(X)$ and hence write an explicit multiplication formula of $\overline{\mathrm{GSp}(X)}$, the metaplectic double cover of $\mathrm{GSp}(X)$. We define the group $F^* \ltimes \overline{\mathrm{Sp}(X)}$ using the multiplication

$$\big(a, (g, \epsilon_1)\big)\big(b, (h, \epsilon_2)\big) = \big(ab, (g, \epsilon_1)^b (h, \epsilon_2)\big).$$

We also define a bijection from $\mathrm{GSp}(X) \times \{\pm 1\}$ to $F^* \times \overline{\mathrm{Sp}(X)}$ by $\bar{\iota}(g, \epsilon) = \big(\lambda_g, (p(g), \epsilon)\big)$, whose inverse is given by $\bar{\iota}^{-1}\big(\lambda, (h, \epsilon)\big) = \big(i(\lambda)h, \epsilon\big)$. We use $\bar{\iota}$ to define a group structure on $\mathrm{GSp}(X) \times \{\pm 1\}$. A straightforward computation will show that the multiplication in $\overline{\mathrm{GSp}(X)}$ is given by

$$(g, \epsilon_1)(h, \epsilon_2) = \big(gh, v_{\lambda_h}(p(g))\, c(p(g)^{\lambda_h}, p(h)\epsilon_1\epsilon_2)\big).$$

Thus,

$$\tilde{c}(g, h) = v_{\lambda_h}\big(p(g)\big)\, c\big(p(g)^{\lambda_h}, p(h)\big)$$

serves as a nontrivial 2-cocycle on $\mathrm{GSp}(X)$. We remark here that Kubota [1969] (see also [Gelbart 1976]) used a similar construction to extend a nontrivial double cover of $\mathrm{SL}_2(F)$ to a nontrivial double cover of $\mathrm{GL}_2(F)$. For $n = 1$, our construction agrees with Kubota's.

Computation of $v_\lambda(g)$. Barthel [1991] extended Rao's unnormalized cocycle to $\mathrm{GSp}(X)$. One may compute $v_\lambda(g)$ using Barthel's work and Rao's normalizing factors. Instead, we compute $v_\lambda(g)$ using Rao's (normalized) cocycle. Fix $\lambda \in F^*$.

Since $(g,\epsilon)\mapsto(g,\epsilon)^\lambda$ is an automorphism, v_λ satisfies

$$v_\lambda(g)\,v_\lambda(h)\,v_\lambda(gh)=\frac{c(g^\lambda,h^\lambda)}{c(g,h)}. \tag{2-8}$$

We shall show that this property determines v_λ.

We first note that

$$\begin{pmatrix}A & B\\ 0 & {}^{\mathrm{t}}A^{-1}\end{pmatrix}^\lambda=\begin{pmatrix}A & \lambda B\\ 0 & {}^{\mathrm{t}}A^{-1}\end{pmatrix},\quad x\left(\begin{pmatrix}A & B\\ 0 & {}^{\mathrm{t}}A^{-1}\end{pmatrix}^\lambda\right)\equiv x\left(\begin{pmatrix}A & B\\ 0 & {}^{\mathrm{t}}A^{-1}\end{pmatrix}\right). \tag{2-9}$$

For $S\subseteq\{1,2,\dots,n\}$ define $a_S(\lambda)\in P$ by

$$e_i\cdot a_S(\lambda)=\begin{cases}\lambda^{-1}e_i, & i\in S,\\ e_i, & \text{otherwise},\end{cases}\qquad e_i^*\cdot a_S(\lambda)=\begin{cases}\lambda e_i^*, & i\in S,\\ e_i^*, & \text{otherwise}.\end{cases}$$

Note that $a_S=a_S(-1)$. One can verify that

$$\tau_S^\lambda=a_S(\lambda)\tau_S=\tau_S a_S(\lambda^{-1}). \tag{2-10}$$

Since $x\big(a_S(\lambda)\big)\equiv\lambda^{|S|}$, we obtain, using (2-9) and (2-10), the Bruhat decomposition, and the properties of x presented earlier, that $\Omega_j^\lambda=\Omega_j$ and that, for $g\in\Omega_j$,

$$x(g^\lambda)\equiv\lambda^j x(g). \tag{2-11}$$

Lemma 2.1. *For $p\in P$, $g\in\Omega_j$ we have*

$$v_\lambda(p)\,v_\lambda(g)\,v_\lambda(pg)=\big(x(p),\lambda^j\big)_F \tag{2-12}$$

and

$$v_\lambda(g)\,v_\lambda(p)\,v_\lambda(gp)=\big(x(p),\lambda^j\big)_F. \tag{2-13}$$

Proof. We prove (2-12) only. (2-13) follows in the same way. We use (2-8), (2-5), (2-11) and (2-3):

$$\begin{aligned}v_\lambda(p)\,v_\lambda(g)\,v_\lambda(pg)&=\frac{c(p^\lambda,g^\lambda)}{c(p,g)}=\frac{\big(x(p^\lambda),x(g^\lambda)\big)_F}{\big(x(p),x(g)\big)_F}\\&=\frac{\big(x(p),x(g)\lambda^j\big)_F}{\big(x(p),x(g)\big)_F}=\big(x(p),\lambda^j\big)_F.\end{aligned}$$ □

Lemma 2.2. *There exists a unique $t_\lambda\in F^*/F^{*2}$ such that $v_\lambda(p)=\big(x(p),t_\lambda\big)_F$ for all $p\in P$.*

Proof. Substituting $p' \in P$ for g in (2-12) we see that $v_\lambda|_P$ is a quadratic character. Since N, the unipotent radical of P, is isomorphic to a vector space over F, it follows that $N^2 = N$. Thus, $v_\lambda|_N$ is trivial. We conclude that $v_\lambda|_P$ is a quadratic character of $\mathrm{GL}_n(F)$ extended to P. Every quadratic character of $\mathrm{GL}_n(F)$ is of the form $g \mapsto \chi(\det g)$, where χ is a quadratic character of F^*. The nondegeneracy of the Hilbert symbol implies every such character of F^* has the form $\chi(a) = (a, t_\chi)_F$, where $t_\chi \in F^*$ uniquely determined by χ up to multiplication by squares. □

Lemma 2.3. *For $\sigma \in \Omega_j$ we have $v_\lambda(\sigma) = \big(x(\sigma), t_\lambda \lambda^j\big)_F v_\lambda(\tau_S)$, where $S \subseteq \{1, 2, \dots, n\}$ is such that $|S| = j$. In particular, if $|S| = |S'|$ then $v_\lambda(\tau_S) = v_\lambda(\tau_{S'})$.*

Proof. An element $\sigma \in \Omega_j$ has the form $\sigma = p\tau_S p'$, where $p, p' \in P$ and $|S| = j$. Substituting $g = \tau_S p'$ in (2-12) yields $v_\lambda(p\tau_S p') = v_\lambda(p) v_\lambda(\tau_S p')(x(p), \lambda^j)_F$, while substituting $g = \tau_S$ and $p = p'$ in (2-13) yields $v_\lambda(\tau_S p') = v_\lambda(\tau_S) v_\lambda(p') \cdot (x(p'), \lambda^j)_F$. Using these two equalities together with (2-3) and Lemma 2.2, we obtain $v_\lambda(p\tau_S p') = \big(x(pp'), t_\lambda \lambda^j\big)_F v_\lambda(\tau_S)$. Since $|S| = |S'|$ implies $p\tau_S p^{-1} = \tau_{S'}$, for some $p \in P$, the last argument shows that $v_\lambda(\tau_S) = v_\lambda(\tau_{S'})$. □

It is clear now that, once we compute t_λ and $v_\lambda(\tau_S)$ for all $S \subseteq \{1, 2, \dots, n\}$, we will find the explicit formula for v_λ.

Lemma 2.4. $t_\lambda = \lambda$ *and* $v_\lambda(\tau_S) = (\lambda, \lambda)_F^{|S|(|S|-1)/2}$.

Proof. Let k be a symmetric matrix in $\mathrm{GL}_n(F)$. Put

$$p_k = \begin{pmatrix} k & -I_n \\ 0 & k^{-1} \end{pmatrix} \in P, \qquad n_k = \begin{pmatrix} I_n & k \\ 0 & I_n \end{pmatrix} \in N,$$

and note that $x(n_k) \equiv 1$, $x(p_k) \equiv \det k$, and that

$$\tau n_k \tau = n_{-k^{-1}} \tau p_k. \tag{2-14}$$

We are going to compute $v_\lambda(\tau) v_\lambda(n_k\tau) v_\lambda(\tau n_k \tau)$ in two ways: First, by Lemma 2.3 and by (2-14), we have $v_\lambda(\tau) v_\lambda(n_k\tau) v_\lambda(\tau n_k \tau) = v_\lambda(\tau) v_\lambda(\tau) v_\lambda(n_{-k^{-1}} \tau p_k) = v_\lambda(n_{-k^{-1}} \tau p_k)$. Since

$$x(\tau n_k \tau) \equiv x(n_{-k^{-1}} \tau p_k) \equiv \det(k), \tag{2-15}$$

we obtain, using Lemma 2.3 again,

$$v_\lambda(\tau) v_\lambda(n_k\tau) v_\lambda(\tau n_k \tau) = \big(\det k, t_\lambda \lambda^n\big)_F v_\lambda(\tau). \tag{2-16}$$

Second, by (2-8) we have

$$v_\lambda(\tau) v_\lambda(n_k\tau) v_\lambda(\tau n_k \tau) = \frac{c\big(\tau, n_k\tau\big)}{c\big(\tau^\lambda, (n_k\tau)^\lambda\big)}. \tag{2-17}$$

We shall compute the two terms on the right side of (2-17), starting with $c(\tau, n_k\tau)$. Let ρ and l be the factors in (2-2), where $\sigma_1 = \tau$, $\sigma_2 = n_k\tau$. Recall that the Leray invariant is stable under the action of $\mathrm{Sp}(X)$ on Lagrangian triplets, see [Rao 1993, Theorem 2.11]. Hence,

$$q\big(V^*, V^*\tau, V^*(n_k\tau)^{-1}\big) = q\big(V^*, V, V^*(-I_{2n}\tau n_{-k})\big) = q(V^*, V, Vn_{-k}).$$

We conclude that $\rho = k$, $l = 0$. Using (2-2), and (2-15) we observe that

$$c(\tau, n_k\tau) = \big(-1, \det k\big)_F h_F(k). \tag{2-18}$$

We now turn to $c\big(\tau^\lambda, (n_k\tau)^\lambda\big)$. Let ρ and l be the factors in (2-2), where $\sigma_1 = \tau^\lambda$ and $\sigma_2 = (n_k\tau)^\lambda$. Note that (2-9) and (2-10) imply $(n_k\tau)^\lambda = n_{\lambda k}\lambda I_{2n}\tau$, and hence

$$q\big((V^*, V^*\tau^\lambda, V^*((n_k\tau)^\lambda)^{-1}\big) = q(V^*, V, Vn_{-\lambda k}).$$

Therefore $\rho = mk$ and $l = 0$, and so we get

$$c\big(\tau^\lambda, (n_k\tau)^\lambda\big) = \big(x(\lambda I_{2n}), x(\lambda I_{2n})\big)_F \big(-1, x(\tau^\lambda(n_k\tau)^\lambda)\big)_F h_F(\lambda k).$$

We recall (2-14) and note now that

$$\tau^\lambda(n_k\tau)^\lambda = (\tau n_k\tau)^\lambda = (n_{-k^{-1}}\tau p_k)^\lambda = n_{-\lambda k^{-1}}\lambda I_{2n}\tau \begin{pmatrix} k & \lambda k \\ 0 & k^{-1} \end{pmatrix}.$$

Hence, $c\big(\tau^\lambda, (n_k\tau)^\lambda\big) = (\lambda^n, \lambda^n)_F \big(-1, \lambda^n \det k\big)_F h_F(\lambda k)$, or, using (2-3):

$$c\big(\tau^\lambda, (n_k\tau)^\lambda\big) = (-1, \det k)_F\, h_F(\lambda k). \tag{2-19}$$

Using (2-16), (2-17), (2-18) and (2-19) we finally get

$$\upsilon_\lambda(\tau)\,\big(\det k, t_\lambda\lambda^n\big)_F = \frac{h_F(\lambda k)}{h_F(k)}.$$

By substituting $k = I_n$ in the above, we get $\upsilon_\lambda(\tau) = (\lambda, \lambda)_F^{n(n-1)/2}$, and we can rewrite it as

$$\big(\det k, t_\lambda\lambda^n\big)_F = \frac{h_F(\lambda k)}{h_F(k)} (\lambda, \lambda)_F^{n(n-1)/2}. \tag{2-20}$$

To find t_λ, we note that for any $y \in F^*$ we can put $k_y = \mathrm{diag}(1, 1, \dots, y)$ in (2-20) and obtain $(y, t_\lambda\lambda^n)_F = (\lambda, \lambda)_F^{(n-1)(n-2)/2}(\lambda, \lambda y)_F^{n-1}(\lambda, \lambda)_F^{n(n-1)/2}$. For both even and odd n, this is equivalent to $(y, \lambda)_F = (y, t_\lambda)_F$. That the last equality holds for all $y \in F^*$ implies that $t_\lambda \equiv \lambda \pmod{(F^*)^2}$.

We are left to compute $\upsilon_\lambda(\tau_S)$ for $S \subsetneq \{1, 2, \dots, n\}$. For such S, define ${}_S\tau \in \mathrm{Sp}(X_S)$ by analogy with $\tau \in \mathrm{Sp}(X)$. We can embed $\mathrm{Sp}(X_S)$ in $\mathrm{Sp}(X)$ in a way that maps ${}_S\tau$ to τ_S. We may now use (2-7) and repeat the computation of $\upsilon_\lambda(\tau)$. □

Joining Lemma 2.3, and Lemma 2.4, we write the explicit formula for v_λ. For $g \in \Omega_j$ we have

$$v_\lambda(g) = \big(x(g), \lambda^{j+1}\big)_F (\lambda, \lambda)^{j(j-1)/2}. \tag{2-21}$$

One can easily check now that $v_\lambda(g)v_\eta(g^\lambda) = v_{\lambda\eta}(g)$ and conclude that the map $\big(\lambda, (g, \epsilon)\big) \mapsto (g, \epsilon)^\lambda$ defines an action of F^* on $\overline{\mathrm{Sp}(X)}$, namely $\big((g, \epsilon)^\lambda\big)^\eta = (g, \epsilon)^{\lambda\eta}$. Lastly, comparing (2-21) and (2-4), keeping (2-3) in mind, we note that

$$v_{-1}(g) = v_{-1}(g^{-1}) = c(g, g^{-1}). \tag{2-22}$$

This fact will play an important role in the proof of the uniqueness of Whittaker models for $\overline{\mathrm{Sp}(X)}$.

3. Uniqueness of the Whittaker model

3A. ***Statement of the main results.*** Let ψ be a nontrivial character of F. Let Z' be the group of upper triangular unipotent matrices in $\mathrm{GL}_n(F)$. Let Z be the subgroup of $\mathrm{Sp}_{2n}(F)$ that consists of elements of the form $((z, b), (0, {}^t z^{-1}))$ in which $z \in Z'$ and $b \in \mathrm{Mat}_{n\times n}(F)$ satisfy ${}^t b = z^{-1} b \, {}^t z$. We shall continue to denote by ψ the character of Z given by $\psi(z) = \psi(b_{(n,n)} + \sum_{i=1}^{n-1} z_{(i,i+1)})$. Let $\overline{Z}$ be the inverse image of Z in $\overline{\mathrm{Sp}_{2n}(F)}$. From (2-5) it follows that $\overline{Z} \simeq Z \times \{\pm 1\}$. We define a character of $\overline{Z}$ by $(z, \epsilon) \mapsto \epsilon\psi(z)$ and continue to denote it by ψ.

Let (π, V_π) be a smooth representation of $\overline{\mathrm{Sp}_{2n}(F)}$. By a Whittaker functional on π with respect to ψ, we mean a linear functional w on V_π satisfying $w\big(\pi(z)v\big) = \psi(z)w(v)$ for all $v \in V_\pi$, $z \in \overline{Z}$. Define $W_{\pi,\psi}$ to be the space of Whittaker functionals on π with respect to ψ. We define $\hat{\pi}$ as the (smooth) dual representation to π.

Theorem 3.1. *If (π, V_π) is an irreducible admissible representation of $\overline{\mathrm{Sp}_{2n}(F)}$, then*

$$\dim(W_{\pi,\psi}) \cdot \dim(W_{\hat{\pi},\psi^{-1}}) \le 1.$$

The proof of this theorem will show:

Theorem 3.2. *Suppose (π, V_π) is an irreducible admissible representation of $\overline{\mathrm{Sp}_{2n}(F)}$. If, from the existence of a nontrivial Whittaker functional on π with respect to ψ, one can deduce the existence of a nontrivial Whittaker functional on $\hat{\pi}$ with respect to ψ^{-1}, then* $\dim(W_{\pi,\psi}) \le 1$.

Corollary 3.3. *If (π, V_π) is an irreducible admissible unitary representation of $\overline{\mathrm{Sp}_{2n}(F)}$, then* $\dim(W_{\pi,\psi}) \le 1$.

Proof of the corollary. We show that the conditions of Theorem 3.2 hold in this case. Indeed, if (π, V_π) is an irreducible admissible unitary representation of $\overline{\mathrm{Sp}_{2n}(F)}$, one can realize the dual representation in the space $\widetilde{V_\pi}$, which is identical to V_π as a commutative group. The scalars act on $\widetilde{V_\pi}$ by $\lambda\cdot v=\bar{\lambda}v$. The action of $\hat{\pi}$ in this realization is given by $\hat{\pi}(g)=\pi(g)$. It is clear now that if L is a nontrivial Whittaker functional on π with respect to ψ then L, acting on $\widetilde{V_\pi}$, is a nontrivial Whittaker functional on $\hat{\pi}$ with respect to ψ^{-1}. □

Since every cuspidal representation π is unitary, it follows from Corollary 3.3 that $\dim(W_{\pi,\psi})\le 1$. Furthermore, assume now that π is an irreducible admissible representation of $\overline{\mathrm{Sp}_{2n}(F)}$. Then π is a subquotient of a representation induced from a cuspidal representation of a parabolic subgroup. Let H be a parabolic subgroup of $\mathrm{Sp}_{2n}(F)$. It is known [Mœglin et al. 1987, page 39] that if g and h commute in $\mathrm{Sp}_{2n}(F)$ then (g,ϵ_1) and (h,ϵ_2) commute in $\overline{\mathrm{Sp}_{2n}(F)}$. Therefore, $\overline{M}_H$, the inverse image of the Levy part of H, is isomorphic to

$$\mathrm{GL}_{n_1}(F)\times\mathrm{GL}_{n_2}(F)\times\cdots\times\mathrm{GL}_{n_r}(F)\times\overline{\mathrm{Sp}_{2k}(F)}.$$

Suppose that for $1\le i\le r$, the representation σ_i of $\mathrm{GL}_{n_i}(F)$ is cuspidal, and that π' is a cuspidal representation of $\overline{\mathrm{Sp}_{2k}(F)}$. Denote by ψ_i and ψ' the restrictions of ψ to the unipotent radicals of GL_{n_i} and $\overline{\mathrm{Sp}_{2k}(F)}$, respectively, embedded in $\overline{M}_H$. Note that $\dim(W_{\sigma_i,\psi_i})\le 1$ and $\dim(W_{\pi',\psi'})\le 1$. Let τ be the representation of $\overline{M}_H$ defined by

$$\big(\mathrm{diag}(g_1,g_2,\dots,g_r,h,\ {}^{\mathrm{t}}g_r^{-1},\ {}^{\mathrm{t}}g_{r-1}^{-1},\dots,\ {}^{\mathrm{t}}g_1^{-1}),\epsilon\big)\mapsto \bigotimes_{i=1}^{i=r}\sigma_i(g_i)\gamma_\psi(\det g_i)\otimes\pi(h,\epsilon),$$

where γ_ψ is the Weil index of ψ (for details on γ_ψ see [Rao 1993, appendix]). We extend τ from M_H to $\overline{H}$, letting the unipotent radical act trivially. Define $\mathrm{Ind}(\overline{H},\overline{\mathrm{Sp}_{2n}(F)},\tau)$ to be the corresponding induced representation. One may use the methods of [Rodier 1973], extended in [Banks 1998] to a nonalgebraic setting, and conclude that

$$\dim(W_{\mathrm{Ind}(\overline{H},\overline{\mathrm{Sp}_{2n}(F)},\tau),\psi})=\dim(W_{\pi',\psi'})\Pi_{i=1}^{i=r}\dim(W_{\sigma_i,\psi_i}).$$

Now, if $V_2\subseteq V_1\subseteq\mathrm{Ind}(\overline{H},\overline{\mathrm{Sp}_{2n}(F)},\tau)$, are two $\overline{\mathrm{Sp}_{2n}(F)}$ modules then clearly the dimension of the Whittaker functionals on V_1 and V_2 with respect to ψ is not greater then $\dim(W_{\mathrm{Ind}(\overline{H},\overline{\mathrm{Sp}_{2n}(F)},\tau),\psi})$. It follows now that $\dim(W_{\pi,\psi})\le 1$. Thus, we proved

Theorem 3.4. *Let π be an irreducible, admissible representation of $\overline{\mathrm{Sp}_{2n}(F)}$. Then $\dim(W_{\pi,\psi})\le 1$.*

Proof of Theorem 3.1. Define on $\mathrm{Sp}_{2n}(F)$ the map

$$g\mapsto {}^{\tau}g=\sigma_0({}^{\mathrm{t}}g)\sigma_0^{-1},$$

where

$$\sigma_0 = \begin{pmatrix} 0 & \epsilon_n \\ \epsilon_n & 0 \end{pmatrix}, \qquad \epsilon_n = \operatorname{diag}\bigl(1, -1, 1 \dots, (-1)^{n+1}\bigr) \in \mathrm{GL}_n(F).$$

We note that $\sigma_0^{-1} = {}^{t}\sigma_0 = \sigma_0$, and that $\sigma_0 \in \mathrm{GSp}_{2n}(F)$ with similitude factor -1. Hence, $g \mapsto {}^{\tau}g$ is an antiautomorphism of $\mathrm{Sp}_{2n}(F)$ of order 2 of. We now extend τ to $\overline{\mathrm{Sp}_{2n}(F)}$. A similar lifting was used in [Gelbart et al. 1979] for $\overline{\mathrm{GL}_2(F)}$.

Lemma 3.5. *The map $(g, \epsilon) \mapsto {}^{\overline{\tau}}(g, \epsilon) = ({}^{\tau}g, \epsilon)$ is an antiautomorphism of $\overline{\mathrm{Sp}_{2n}(F)}$ of order 2. It preserves both $\overline{Z}$ and ψ and satisfies ${}^{\overline{\tau}}\bigl(\mathrm{Sp}_{2n}(F), \epsilon\bigr) = \bigl(\mathrm{Sp}_{2n}(F), \epsilon\bigr)$.*

Proof. We note that if $g \in \mathrm{Sp}_{2n}(F)$ then ${}^{t}g = -Jg^{-1}J$. Hence,

$$^{\tau}g = \begin{pmatrix} \epsilon_n & 0 \\ 0 & \epsilon_n \end{pmatrix} \begin{pmatrix} -I_n & 0 \\ 0 & I_n \end{pmatrix} g^{-1} \begin{pmatrix} -I_n & 0 \\ 0 & I_n \end{pmatrix} \begin{pmatrix} \epsilon_n & 0 \\ 0 & \epsilon_n \end{pmatrix}^{-1}.$$

Thus, the map

$$(g, \epsilon) \mapsto \bigl({}^{\tau}g, \epsilon c(g, g^{-1}) v_{-1}(g^{-1}) c(p_\epsilon, \tilde{g}) c(p_\epsilon \tilde{g}, p_\epsilon^{-1}) c(p_\epsilon, p_\epsilon^{-1})\bigr),$$

where

$$p_\epsilon = \begin{pmatrix} \epsilon_n & 0 \\ 0 & \epsilon_n \end{pmatrix} \in P \quad \text{and} \quad \tilde{g} = \begin{pmatrix} -I_n & 0 \\ 0 & I_n \end{pmatrix} g^{-1} \begin{pmatrix} -I_n & 0 \\ 0 & I_n \end{pmatrix},$$

is an antiautomorphism of $\overline{\mathrm{Sp}_{2n}(F)}$. We now show that

$$c(g, g^{-1}) v_{-1}(g^{-1}) c(p_\epsilon, \tilde{g}) c(p_\epsilon \tilde{g}, p_\epsilon^{-1}) c(p_\epsilon, p_\epsilon^{-1}) = 1.$$

Indeed, $c(p_\epsilon, \tilde{g}) c(p_\epsilon \tilde{g}, p_\epsilon^{-1}) c(p_\epsilon, p_\epsilon^{-1}) = 1$ is a property of Rao's cocycle noted in (2-6). The fact that $c(g, g^{-1}) v_{-1}(g^{-1}) = 1$ is a consequence of the calculation of $v_\lambda(g)$ and is noted in (2-22). The remaining assertions of this lemma are clear. □

Let $S\bigl(\overline{\mathrm{Sp}_{2n}(F)}\bigr)$ be the space of Schwartz functions on $\overline{\mathrm{Sp}_{2n}(F)}$. For $h \in \overline{\mathrm{Sp}_{2n}(F)}$ and $\phi \in S\bigl(\overline{\mathrm{Sp}_{2n}(F)}\bigr)$, we define $\lambda(h)\phi$, $\rho(h)\phi$, and ${}^{\overline{\tau}}\phi$ by

$$\bigl(\rho(h)\phi\bigr)(g) = \phi(gh), \quad \bigl(\lambda(h)\phi\bigr)(g) = \phi(h^{-1}g), \quad {}^{\overline{\tau}}\phi(g) = \phi({}^{\overline{\tau}}g).$$

These maps are elements of $S(\overline{\mathrm{Sp}_{2n}(F)})$. We give $S(\overline{\mathrm{Sp}_{2n}(F)})$ an algebra structure, called the Hecke algebra, by the convolution. Given (π, V_π), a representation of $\overline{\mathrm{Sp}_{2n}(F)}$, we define as usual a representation of this algebra in the space V_π by

$$\pi(\phi) v = \int_{\overline{\mathrm{Sp}_{2n}(F)}} \phi(g) \pi(g) v \, dg.$$

The following theorem, known as the Gelfand–Kazhdan Theorem in the context of $\mathrm{GL}_n(F)$ (see [Gel'fand and Kazhdan 1975; Bernstein and Zelevinskii 1976]), will be used in the proof of Theorem 3.1.

Theorem 3.6. *Suppose D is a functional on $S(\overline{\mathrm{Sp}_{2n}(F)})$ with $D\big(\lambda(z_1)\rho(z_2)\phi\big) = \psi(z_2 z_1^{-1})D(\phi)$ for all $\phi \in S(\overline{\mathrm{Sp}_{2n}(F)})$ and $z_1, z_1 \in \overline{Z}$. Then D is $\overline{\tau}$ invariant, that is, $D({}^{\overline{\tau}}\phi) = D(\phi)$ for all $\phi \in S(\overline{\mathrm{Sp}_{2n}(F)})$.*

We will prove this theorem in Section 3B. Here we use it for:

Proof of Theorem 3.1. Since any irreducible admissible representation of $\overline{\mathrm{Sp}_{2n}(F)}$ may be realized as a dual representation, the proof amounts to showing that if $W_{\pi,\psi} \neq 0$ then $\dim W_{\hat{\pi},\psi^{-1}} \leq 1$. We shall use an argument similar to the one in [Soudry 1987, Theorem 2.1]. Let w be a nontrivial Whittaker functional on (π, V_π) with respect to ψ. Suppose $\widehat{w}_1$ and $\widehat{w}_2$ are two nontrivial Whittaker functionals on $\hat{\pi}$ with respect to ψ^{-1}. The proof will be achieved once we show that $\widehat{w}_1$ and $\widehat{w}_2$ are proportional.

For $\phi \in S(\overline{\mathrm{Sp}_{2n}(F)})$, let $\pi^*(\phi)\,w$ be a functional on V_π defined by

$$\big(\pi^*(\phi)\,w\big)\,v = \int_{\overline{\mathrm{Sp}_{2n}(F)}} \phi(g)\,w\big(\pi(g^{-1})\,v\big)\,dg.$$

Note that $\phi \in S(\overline{\mathrm{Sp}_{2n}(F)})$ implies $\pi^*(\phi)\,w$ is smooth even if w is not. Thus, $\pi^*(\phi)\,w$ lies in $\in V_{\hat{\pi}}$. By a change of variables we have

$$\hat{\pi}(h)\big(\pi^*(\phi)\,w\big) = \pi^*\big(\lambda(h)\phi\big)\,w. \tag{3-1}$$

Now define R_1 and R_2, two functionals on $S(\overline{\mathrm{Sp}_{2n}(F)})$, through

$$R_i(\phi) = \widehat{w}_i\big(\pi^*(\phi)\,w\big).$$

Using (3-1), the facts that w, $\widehat{w}_1$, $\widehat{w}_2$ are Whittaker functionals, and by changing variables, we observe for all $z \in \overline{Z}$ that

$$R_i\big(\lambda(z)\phi\big) = \psi^{-1}(z)R_i(\phi), \qquad R_i\big(\rho(z)\phi\big) = \psi(z)R_i(\phi).$$

From Theorem 3.6 it follows that $R_i(\phi) = R_i({}^{\overline{\tau}}\phi)$. Hence,

$$\widehat{w}_i\big(\hat{\pi}(h)\pi^*(\phi)\,w\big) = \widehat{w}_i\big(\pi^*({}^{\overline{\tau}}(\lambda(h)\phi)w)\big). \tag{3-2}$$

Using a change of variables again, we also have

$$\pi^*({}^{\overline{\tau}}(\lambda(h)\phi))\,w = \pi^*({}^{\overline{\tau}}\phi)\,\big(\pi^*({}^{\overline{\tau}}h)\,w\big)\,. \tag{3-3}$$

Joining (3-2) and (3-3) we obtain

$$\widehat{w}_i\,\big(\hat{\pi}(h)\pi^*(\phi)\,w\big) = \widehat{w}_i\big(\pi^*({}^{\overline{\tau}}\phi)\,\big(\pi^*({}^{\overline{\tau}}h)\,w\big)\big).$$

In particular, if $\pi^*(\phi)\,w$ is the zero functional for some $\phi \in S(\overline{\mathrm{Sp}_{2n}(F)})$, then, for all $h \in \overline{\mathrm{Sp}_{2n}(F)}$, $\widehat{w}_i\big(\pi^*({}^{\overline{\tau}}\phi)(\pi^*({}^{\overline{\tau}}h)\,w)\big) = 0$. Here, for all $f \in S(\overline{\mathrm{Sp}_{2n}(F)})$, we have

$$0=\widehat{w}_i\biggl(\int_{\overline{\mathrm{Sp}_{2n}(F)}} f(h)\widehat{w}_i\bigl(\pi^*({}^{\overline{\tau}}\phi)(\pi^*({}^{\overline{\tau}}h)\,w)\bigr)\,dh\biggr).$$

By the definition of $\pi^*(\phi)\,w$ and by changing the order of integration, we have, for all $v\in V_\pi$,

$$\int_{\overline{\mathrm{Sp}_{2n}(F)}} f(h)\widehat{w}_i\bigl(\pi^*({}^{\overline{\tau}}\phi)(\pi^*\bigl({}^{\overline{\tau}}h\bigr)\,w)\bigr)(v)dh=\hat{\pi}({}^{\overline{\tau}}\phi)(\pi^*(f)\,w)(v).$$

The last two equalities yield that if $\pi^*(\phi)\,w=0$, then, for all $v\in V_\pi$ and $f\in S(\overline{\mathrm{Sp}_{2n}(F)})$,

$$\widehat{w}_i\bigl(\hat{\pi}({}^{\overline{\tau}}\phi)(\pi^*(f)\,w)\bigr)(v)=0.$$

Because π is irreducible one can conclude that

$$\pi^*\bigl(S(\overline{\mathrm{Sp}_{2n}(F)})w\bigr)=V_{\hat{\pi}}.$$

Indeed, since $\pi^*\bigl(S(\overline{\mathrm{Sp}_{2n}(F)})w\bigr)$ is an $\overline{\mathrm{Sp}_{2n}(F)}$, invariant subspace we only have to show that $\pi^*\bigl(S(\overline{\mathrm{Sp}_{2n}(F)})w\bigr)\neq\{0\}$, which is clear.

Hence, changing variables once more, we see that for all $\xi\in V_{\hat{\pi}}$ we have

$$0=\widehat{w}_i\bigl(\hat{\pi}({}^{\overline{\tau}}\phi)\xi\bigr)=\int_{\overline{\mathrm{Sp}_{2n}(F)}} {}^{\overline{\tau}}\phi(g^{-1})\widehat{w}_i\bigl(\hat{\pi}(g^{-1})\xi\bigr)dg.$$

For $g\in\overline{\mathrm{Sp}_{2n}(F)}$, define ${}^{\omega}g={}^{\overline{\tau}}g^{-1}$, and for $\phi\in S(\overline{\mathrm{Sp}_{2n}(F)})$, define ${}^{\omega}\phi(g)=\phi({}^{\omega}g)$. We have just seen that if $\pi^*(\phi)\,w=0$ then

$$\bigl((\hat{\pi})^*({}^{\omega}\phi)\bigr)\widehat{w}_i=0.$$

This and the equality $\pi^*(S(\overline{\mathrm{Sp}_{2n}(F)})w)=V_{\hat{\pi}}$ show the following linear maps are well defined. For $i=1,2$, define $S_i:V_{\hat{\pi}}\to V_{\hat{\hat{\pi}}}$ by $S_i\bigl(\pi^*(\phi)\,w\bigr)=\bigl((\hat{\pi})^*({}^{\omega}\phi)\bigr)\widehat{w}_i$. One can easily check that S_1 and S_2 are two intertwining maps from $\hat{\pi}$ to $h\mapsto\hat{\hat{\pi}}(\sigma_0\,{}^{t}h^{-1}\sigma_0)$. The last representation is clearly isomorphic to $h\mapsto\pi(\sigma_0\,{}^{t}h^{-1}\sigma_0)$, which, due to the irreducibility of π, is itself irreducible. Schur's Lemma guarantees now the existence of a complex number c such that $S_2=cS_1$. Thus, for all $\phi\in S(\overline{\mathrm{Sp}_{2n}(F)})$ and for all $\xi\in V_{\hat{\pi}}$, we have

$$\int_{\overline{\mathrm{Sp}_{2n}(F)}}\phi(g)(\widehat{w}_2-c\widehat{w}_1)\bigl(\hat{\pi}(g^{-1})\xi\bigr)dg=0.$$

We can now conclude that $\widehat{w}_1$ and $\widehat{w}_2$ are proportional. □

3B. ***Proof of Theorem 3.6.*** Put $G=\overline{\mathrm{Sp}_{2n}(F)}$ and define $H=\overline{Z}\times\overline{Z}$. Let $\widetilde{\psi}$ be the character of H defined by $\widetilde{\psi}(n_1,n_2)=\psi(n_1^{-1}n_2)$. H acts on G by $(n_1,n_2)\cdot g=n_1gn_2^{-1}$. For $g\in G$, we denote by H_g the stabilizer of g in H. It is clearly a unimodular group. Let Y be an H orbit, that is, a subset of G of the form

$H \cdot g = ZgZ$, where g is a fixed element in G. Let $S(Y)$ be the space of Schwartz functions on Y. H acts on $S(Y)$ by

$$(h \cdot \phi)(k) = \phi(h^{-1} \cdot k)\widetilde{\psi}^{-1}(h).$$

With this notation, the proof of Theorem 3.6 goes almost word for word as [Soudry 1987, proof of Theorem 2.3]. The main ingredient of that proof was [Bernstein and Zelevinskii 1976, Theorem 6.10], which asserts that the following four conditions imply Theorem 3.6:

(i) The set $\{(g, h \cdot g) \mid g \in G, h \in H\}$ is a union of finitely many locally closed subsets of $G \times G$.

(ii) For each $h \in H$, there exists $h_{\overline{\tau}} \in H$ such that $h \cdot {}^{\overline{\tau}}g = {}^{\overline{\tau}}(h_{\overline{\tau}} \cdot g)$ for all $g \in G$.

(iii) $\overline{\tau}$ is of order 2.

(iv) Let Y be an H orbit. Suppose that there exists a non zero functional on $S(Y)$ satisfying $D(h \cdot \phi) = D(\phi)$ for all $\phi \in S(Y)$ and $h \in H$. Then ${}^{\overline{\tau}}Y = Y$ and $D({}^{\overline{\tau}}\phi) = D(\phi)$ for all $\phi \in S(Y)$.

Of these four conditions, only the forth requires some work. To make Soudry's proof work in our context, we have only to change [Soudry 1987, Theorem 2.2] to

Theorem 3.7. *Fix $g \in G$. If for all $h \in H_g$, we have $\widetilde{\psi}^{-1}(h) = 1$, then there exists an $h^g \in H$ such that $h^g \cdot g = {}^{\overline{\tau}}g$ and $\widetilde{\psi}^{-1}(h) = 1$.*

Before we prove this theorem, we state and prove its analog for $\mathrm{Sp}_{2n}(F)$. After the proof, we give a short argument which completes the proof of Theorem 3.7.

Lemma 3.8. *For a fixed $g \in \mathrm{Sp}_{2n}(F)$ one of the following holds*

A. *There exist $n_1, n_2 \in Z$ such that $n_1 g n_2 = g$ and $\psi(n_1 n_2) \neq 1$*

B. *There exist $n_1, n_2 \in Z$ such that $n_1 g n_2 = {}^{\tau}g$ and $\psi(n_1 n_2) = 1$.*

Proof. Because τ preserves both Z and ψ, it is enough to prove this Lemma only for a complete set of representatives of $Z \diagdown \mathrm{Sp}_{2n}(F) \diagup Z$. We recall the Bruhat decomposition: $\mathrm{Sp}_{2n}(F) = \bigcup_{w \in W} ZTwZ$, where W is the Weyl group of $\mathrm{Sp}_{2n}(F)$ and T is the group of diagonal elements in $\mathrm{Sp}_{2n}(F)$. We realize the set of Weyl elements as $\{\overline{w}_\sigma \tau_S \mid \sigma \in S_n,\ S \subseteq \{1, 2, \dots, n\}\}$, where, for $\sigma \in S_n$, we define $w_\sigma \in \mathrm{GL}_n(F)$ by $(w_\sigma)_{i,j} = \delta_{i,\sigma(j)}$ and $\overline{w}_\sigma \in \mathrm{Sp}_{2n}(F)$ by

$$\overline{w}_\sigma = \begin{pmatrix} w_\sigma & 0 \\ 0 & w_\sigma \end{pmatrix}.$$

Thus we may take $\{d^{-1}\overline{w}_\sigma^{-1}\varphi_S \mid d \in T,\ \sigma \in S_n,\ S \subseteq \{1, 2, \dots, n\}\}$ as a complete set of representatives of $Z \diagdown \mathrm{Sp}_{2n}(F) \diagup Z$, where

$$\varphi_S = \tau_{S^c} a_{S^c} = \begin{pmatrix} M_S & M_{S^c} \\ -M_{S^c} & M_S \end{pmatrix}$$

and where, for $S \subseteq \{1, 2, \ldots, n\}$, $M_S \in \mathrm{Mat}_{n\times n}(F)$ has $(M_S)_{i,j} = \delta_{i,j}\delta_{i\in S}$.

Denote by w_k the $k \times k$ invertible matrix with elements $(w_k)_{i,j} = \delta_{i+j,k+1}$. Suppose that $k_1, k_2, \ldots, k_p, k$ are non negative integers such that $k + \sum_{i=1}^{p} k_i = n$. Suppose also that $a_1, a_2, \ldots, a_p \in F^*$ and $\eta \in \{\pm 1\}$. For

$$\overline{w}_\sigma = \mathrm{diag}(w_{k_1}, w_{k_2}, \ldots, w_{k_p}, I_k, w_{k_1}, w_{k_2}, \ldots, w_{k_p}, I_k), \tag{3-4}$$

$$d = \mathrm{diag}(a_1\epsilon_{k_1}, a_2\epsilon_{k_2}, \ldots, a_p\epsilon_{k_p}, \eta I_k, a_1^{-1}\epsilon_{k_1}, a_2^{-1}\epsilon_{k_2}, \ldots, a_p^{-1}\epsilon_{k_p}, \eta I_k), \tag{3-5}$$

and $S = \{n-k+1, n-k, \ldots, n\}$, one checks that

$$^{\tau}(d^{-1}\overline{w}_\sigma^{-1}\varphi_S) = d^{-1}\overline{w}_\sigma^{-1}\varphi_S.$$

Thus $d^{-1}\overline{w}_\sigma^{-1}\varphi_S$ is of type B.

We shall show that all other representatives are of type A. We will prove that in all the other cases one can find $n_1, n_2 \in Z$ such that

$$\overline{w}_\sigma d n_1 d^{-1}\overline{w}_\sigma^{-1} = \varphi_S n_2^{-1}\varphi_S^{-1}, \tag{3-6}$$

$$\psi(n_1 n_2) \neq 1. \tag{3-7}$$

We shall use the following notations and facts: Denote by $E_{p,q}$ the $n \times n$ matrix defined by $(E_{p,q})_{i,j} = \delta_{p,i}\delta_{q,j}$. For $i, j \in \{1, 2, \ldots, n\}$, $i \neq j$ we define the root subgroups of $\mathrm{Sp}_{2n}(F)$ through

$$U_{i,j} = \left\{ u_{i,j}(x) = \begin{pmatrix} I_n + xE_{i,j} & 0 \\ 0 & I_n - xE_{j,i} \end{pmatrix} \mid x \in F \right\} \simeq F,$$

$$V_{i,j} = \left\{ v_{i,j}(x) = \begin{pmatrix} I_n & xE_{i,j} + xE_{j,i} \\ 0 & I_n \end{pmatrix} \mid x \in F \right\} \simeq F,$$

$$V_{i,i} = \left\{ v_{i,i}(x) = \begin{pmatrix} I_n & xE_{i,i} \\ 0 & I_n \end{pmatrix} \mid x \in F \right\} \simeq F.$$

If $i < j$, we call $U_{i,j}$ a positive root subgroup; if $j = i+1$, we call $U_{i,j}$ a simple root subgroup; and if $j > i+1$, we call $U_{i,j}$ a nonsimple root subgroup. We call $^{t}U_{i,j} = U_{j,i}$ the negative of $U_{i,j}$. The group S_n acts on the set $\{U_{i,j} \mid i, j \in \{1, 2, \ldots, n\},\ i \neq j\}$ by

$$\overline{w}_\sigma u_{i,j}(x)\overline{w}_\sigma^{-1} = u_{\sigma(i),\sigma(j)}(x) \tag{3-8}$$

and on the set $\{V_{i,j} \mid i, j \in \{1, 2, \ldots, n\}\}$ by

$$\overline{w}_\sigma v_{i,j}(x)\overline{w}_\sigma^{-1} = v_{\sigma(i),\sigma(j)}(x). \tag{3-9}$$

T acts on each root subgroup via rational characters:

$$du_{i,j}(x)d^{-1} = u_{i,j}(xd_i d_j^{-1}), \tag{3-10}$$

$$dv_{i,j}(x)d^{-1} = v_{i,j}(xd_i d_j), \tag{3-11}$$

where $d = \mathrm{diag}(d_1, d_2, \ldots, d_n, d_1^{-1}, d_2^{-1}, \ldots, d_n^{-1})$. We also note that

$$\varphi_S v_{i,i}(x)\varphi_S^{-1} = v_{i,i}(x) \qquad \text{if } i \in S, \tag{3-12}$$

$$\varphi_S v_{i,j}(x)\varphi_S^{-1} = u_{i,j}(x) \qquad \text{if } i \in S,\ j \notin S, \tag{3-13}$$

$$\varphi_S u_{i,j}(x)\varphi_S^{-1} = u_{i,j}(x) \qquad \text{if } i \in S,\ j \in S,\ i \neq j, \tag{3-14}$$

$$\varphi_S u_{i,j}(x)\varphi_S^{-1} = u_{j,i}(-x) = {}^{\mathrm{t}}u_{i,j}(x)^{-1} \qquad \text{if } i \notin S,\ j \notin S,\ i \neq j. \tag{3-15}$$

Consider the representative $d^{-1}\overline{w}_\sigma^{-1}\varphi_S$. Assume first that S is empty. If there exists a simple root subgroup $U_{k,k+1}$ taken by σ to the negative of a nonsimple root subgroup, we choose $n_1 = u_{k,k+1}(x)$ and $n_2 = {}^{\mathrm{t}}u_{\sigma(k),\sigma(k+1)}(d_k d_{k+1}^{-1}x)$. For such a choice, by (3-8), (3-10) and (3-15), Equation (3-6) holds. Also, since $\psi(n_1 n_2) = \psi(n_1) = \psi(x)$, it is possible, by choosing x properly, to satisfy (3-7). Next, if there exists a nonsimple positive root subgroup $U_{i,j}$ taken by σ to the negative of a simple root subgroup, we choose $n_1 = u_{i,j}(x)$ and $n_2 = {}^{\mathrm{t}}u_{\sigma(i),\sigma(j)}(d_i d_j^{-1}x)$ and repeat the argument. Thus, $d^{-1}\overline{w}_\sigma^{-1}\varphi_\varnothing$ is of type A unless σ has the following two properties: first, if σ takes a simple root to a negative root, then it is taken to the negative of a simple root; second, if σ takes a nonsimple positive root to a negative root, then it is taken to the negative of a nonsimple root. An easy argument shows that if σ has these two properties, $\overline{w}_\sigma$ must be as in (3-4) with $k = 0$. We assume now that $\overline{w}_\sigma$ has this form. To finish the case $S = \varnothing$, we show that unless d has the form (3-5), with $k = 0$ and $k_1, k_2, \ldots, k_p$ corresponding to $\overline{w}_\sigma$, then $d^{-1}\overline{w}_\sigma^{-1}\varphi_\varnothing$ is of type A. Indeed, suppose that there exist d_k and d_{k+1} that belong to the same block in $\overline{w}_\sigma$ such that $d_k \neq -d_{k+1}$. Then we choose $n_1 = u_{k,k+1}(x)$ and $n_2 = {}^{\mathrm{t}}u_{\sigma(k),\sigma(k+1)}(d_k d_{k+1}^{-1}x)$. For such a choice (3-6) holds as before, and $\psi(n_1 n_2) = \psi\big(x(1 + d_k d_{k+1}^{-1})\big)$. Hence it is possible to choose x so (3-7) holds.

We may now assume $|S| \geq 1$. We show that if $n \notin S$ then $d^{-1}\overline{w}_\sigma^{-1}\varphi_S$ is of type A. Indeed, if $\sigma(n) \in S$ (so $\sigma(n) \neq n$), then, for all $x \in F$, if we choose $n_1 = v_{n,n}(x)$ and $n_2 = v_{\sigma(n),\sigma(n)}(-xd_n^2)$, then (3-9), (3-10), and (3-12) imply Equation (3-6) holds. Clearly, there exists $x \in F$ such that $\psi(n_1 n_2) = \psi(n_1) = \psi(x) \neq 1$. Suppose now that $n \notin S \neq \varnothing$ and $\sigma(n) \notin S$. In this case, we can find $1 \leq k \leq n-1$ such that

$$\sigma(k) \in S \quad \text{and} \quad \sigma(k+1) \notin S. \tag{3-16}$$

We choose

$$n_1 = u_{k,k+1}(x) \quad \text{and} \quad n_2 = v_{\sigma(k),\sigma(k+1)}(-xd_k d_{k+1}^{-1}). \tag{3-17}$$

By (3-8), (3-10) and (3-13), Equation (3-6) holds, and since $\psi(n_1n_2) = \psi(n_1) = \psi(x)$, we can satisfy (3-7) by properly choosing x. We assume now $n \in S$. We also assume $\sigma(n) \in S$, and otherwise we use the last argument. Fix $n_1 = v_{n,n}(x)$ and $n_2 = v_{\sigma(n),\sigma(n)}(-xd_n^2)$. One can check using (3-9), (3-11), and (3-12) that (3-6) holds. Note that

$$\psi(n_1n_2) = \begin{cases} x, & \sigma(n) \neq n, \\ x(d_n^2 - 1), & \sigma(n) = n. \end{cases}$$

Hence, $d^{-1}\overline{w}_\sigma^{-1}\varphi_S$ is of type A unless $\sigma(n) = n$ and $d_n = \pm 1$, and so we now assume that of $\sigma(n)$ and d_n. If $S = \{n\}$, we use a similar argument to the one we used for $S = \varnothing$ but this time analyze the action of σ on $\{1, 2, 3, \dots, n-1\}$.

We are left with the case $\sigma(n) = n$, $d_n = \pm 1$, and $S \supsetneq \{n\}$. If $\sigma(n-1) \notin S$ we repeat an argument we used already. We choose $1 \leq k \leq n-2$, n_1, and n_2 as in (3-16) and (3-17). We now assume $\sigma(n-1) \in S$. We choose $n_1 = u_{n-1,n}(x)$ and $n_2 = u_{\sigma(n-1),n}(-xd_{n-1}d_n^{-1})$. Using (3-8), (3-10) and (3-14), we observe that (3-6) holds. Also

$$\psi(n_1n_2) = \begin{cases} x, & \sigma(n-1) \neq n-1, \\ x(d_{n-1}d_n^{-1} - 1), & \sigma(n-1) = n-1. \end{cases}$$

Thus, unless $\sigma(n-1) = n-1$ and $d_{n-1} = d_n = \pm 1$, $d^{-1}\overline{w}_\sigma^{-1}\varphi_S$ is of type A. Therefore, we should only consider the case $\sigma(n) = n$, $\sigma(n-1) = n-1$, $d_{n-1} = d_n = \pm 1$, and $\{n-1, n\} \subseteq S$. We continue in the same course. If $S = \{n-1, n\}$, we use similar argument we used for $S = \varnothing$, analyzing the action of σ on $\{1, 2, 3, \dots, n-2\}$. If $S \supsetneq \{n-1, n\}$, we show that unless $\sigma(n-2) = n-2 \in S$ and $d_{n-2} = d_{n-1} = d_n = \pm 1$ we are in type A etc. □

We now complete the proof of Theorem 3.7. We define types $\overline{A}$ and $\overline{B}$ for $\overline{\mathrm{Sp}_{2n}(F)}$ by analogy with the definitions given in Lemma 3.8 and show that each element of $\overline{Z}\backslash\overline{\mathrm{Sp}_{2n}(F)}/\overline{Z}$ is either of type $\overline{A}$ or of type $\overline{B}$. Given $\overline{g} = (g, \epsilon) \in \overline{\mathrm{Sp}_{2n}(F)}$, if g is of type A, then there are $n_1, n_2 \in Z$ such that $n_1gn_2 = g$ and $\psi(n_1n_2) \neq 1$. Let $\overline{n}_i = (n_i, 1)$. Clearly $\overline{n}_1\overline{g}\,\overline{n}_2 = \overline{g}$ and $\psi(\overline{n}_1\overline{n}_2) = \psi(n_1n_2) \neq 1$. If g is not of type A, then by Lemma 3.8 it is of type B. There are $n_1, n_2 \in Z$ such that $n_1gn_2 = {}^\tau g$ and $\psi(n_1n_2) = 1$. Define $\overline{n}_i$ as before. Note that $\psi(\overline{n}_1\overline{n}_2) = \psi(n_1n_2) = 1$. From Lemma 3.5 it follows that $\overline{n}_1\overline{g}\,\overline{n}_2 = {}^{\overline{\tau}}\,\overline{g}$. This proves Lemma 3.8 for $\overline{\mathrm{Sp}_{2n}(F)}$, which is Theorem 3.7. □

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