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# THE RESIDUAL SPECTRUM OF INNER FORMS OF Sp(2)

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**Let $G$ be the unitary group of the hyperbolic hermitian space with rank two over a quaternion division algebra over a totally real number field. We determine the irreducible decomposition of the residual discrete spectrum of $G$. Finally we give expected description of Arthur parameters associated to these representations.**

## 1. Introduction

Let $k$ be a number field and $\boldsymbol{A}$ its adele ring. Let $G$ be a semisimple group defined over $k$. We write $L^2(G(k)\backslash G(\boldsymbol{A}))$ for the Hilbert space of square-integrable functions on $G(k)\backslash G(\boldsymbol{A})$. The space $L^2_0(G(k)\backslash G(\boldsymbol{A}))$ of cuspidal elements of $L^2(G(k)\backslash G(\boldsymbol{A}))$ is contained in the discrete spectrum of $L^2(G(k)\backslash G(\boldsymbol{A}))$, and the orthogonal complement of $L^2_0(G(k)\backslash G(\boldsymbol{A}))$ in the discrete spectrum is called the *residual spectrum*. In this paper we take as $G$ the unitary group of the two-dimensional hyperbolic hermitian space over a quaternion division algebra $D$ over $k$. It is an inner form of Sp(2). We determine the irreducible decomposition of its residual spectrum, as a first example of a nonquasisplit group.

For $G = \mathrm{Sp}(2)$, the irreducible decomposition of its residual spectrum has been determined by Kim [1995]. Kon-no [1994] described it using theta correspondence in the case that $k$ is totally real. We have a decomposition:

$$L^2(\mathrm{Sp}(2,k)\backslash \mathrm{Sp}(2,\boldsymbol{A})) = L^2(\mathrm{Sp}(2)) \oplus L^2(P_0) \oplus L^2(P_1) \oplus L^2(P_2),$$

along constant terms [Mœglin and Waldspurger 1995, Prop. II.2.4]. Here $P_0$ is a Borel subgroup, $P_1$ and $P_2$ are Siegel and non-Siegel maximal parabolic subgroups, respectively. Then the residual spectrum can be described as

$$L^2_d(P_0) \oplus L^2_d(P_1) \oplus L^2_d(P_2).$$

Here $L^2_d(\,\cdot\,)$ denotes the discrete spectrum of $L^2(\,\cdot\,)$. Similarly, the residual spectrum of our $G$ coincides with $L^2_d(P)$, where $P$ is a proper parabolic subgroup of $G$. $P$ corresponds to $P_1$ via the inner twist. Therefore we can make considerable use of the technique of decomposition of $L^2_d(P_1)$. Generally, the residual spectrum

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is spanned by residues of the Eisenstein series associated with the cuspidal representations of the Levi factors of parabolic subgroups. A calculation of the poles of such Eisenstein series reduces to a calculation of normalization factors of certain intertwining operators. In the case of Sp(2) the Langlands–Shahidi normalization was used, where its normalization factors are written by automorphic $L$-functions. Therefore we can define the normalization factors for $P$ as an analogy of the case of $P_1$.

The residual spectrum for $G$ has a further decomposition,

$$L^2_d(P)_\infty \oplus L^2_d(P)_1,$$

where $L^2_d(P)_\infty$ (resp. $L^2_d(P)_1$) is the space spanned by the residues of the Eisenstein series of infinite dimensional (resp. one-dimensional) cuspidal representations of a Levi factor $M$ of $P$. The irreducible decomposition of $L^2_d(P)_\infty$ is obtained by the Langlands classification (Theorem 4.1(2)); it is written in the form of the unique irreducible quotient of $\mathrm{Ind}_{P(\boldsymbol{A})}^{G(\boldsymbol{A})}\pi$ with an infinite dimensional cuspidal representation $\pi$ of $M$ satisfying some conditions. Then the representation replacing $\pi$ with the Jacquet–Langlands correspondence of $\pi$ appears in $L^2_d(P_1)$ [Kim 1995, Th. 3.3]. $L^2_d(P)_1$ contains the trivial representation (Theorem 4.1(1)). Every other irreducible constituent is described by the theta lift of the trivial representation of the unitary group of a $(-1)$-hermitian space over $D$ (Theorem 4.1(3)). This representation has a local component contained in $\mathrm{Ind}_{P(k_v)}^{G(k_v)}(\chi_v|\cdot|^{-1/2})\circ\nu$ for any place $v$ of $k$, where $\chi_v$ is a quadratic character of $k_v^\times$ and $\nu$ is the reduced norm of $D$. On the other hand, any irreducible constituent of $L^2_d(P_0)$ except for the trivial representation is the theta lift of the trivial representation of an orthogonal group; see [Kon-no 1994]. And this representation has a local component contained in $\mathrm{Ind}_{P_1(k_v)}^{\mathrm{Sp}(2,k_v)}(\chi_v|\cdot|^{-1/2})\circ\det$ for any $v$. It is interesting that the representations of Theorem 4.1(3) do not have a multiplicity of one, unlike the case of Sp(2).

By Arthur's conjecture, an irreducible constituent of the residual spectrum should give the corresponding Arthur parameter. I expect that the Arthur parameters for the residual spectrum of $G$ are contained in those for the residual spectrum of Sp(2). It seems that the Arthur parameters for $L^2_d(P)_\infty$ appear for $L^2_d(P_1)$, and those for $L^2_d(P)_1$ appear for $L^2_d(P_0)$. I give the expected description of the Arthur parameters in Section 5.

## 2. Preliminaries

Let $k$ be a number field with adele ring $\boldsymbol{A}=\boldsymbol{A}_k$. We write $\boldsymbol{A}_\infty$, $\boldsymbol{A}_f$ for the infinite and finite components of $\boldsymbol{A}$, while $|\ |_{\boldsymbol{A}}$ denotes the idele norm of $\boldsymbol{A}^\times$. For any place $v$ of $k$ we write $k_v$ for the completion of $k$ at $v$, and $|\ |_v$ for the $v$-adic norm. If $v$ is finite, $\mathcal{O}_v$ denotes the maximal compact subring of $k_v$. If $\psi$ is a nontrivial character of $\boldsymbol{A}$, trivial on $k$, and $v$ is a place of $k$, $\psi_v$ denotes the $v$-component of $\psi$.

Let $D$ be a quaternion division algebra over $k$. We write $\nu$, $\tau$, and $\iota$ for the reduced norm, the reduced trace, and the main involution of $D$, respectively. We write $D_- = \{x \in D \mid \tau(x) = 0\}$. Also we write $S_D$ for the set of places $v$ of $k$ at which $D$ is ramified, and $s_D$ for the number of its elements, which is finite and even. We denote by $\mathbb{M}(n, R)$ the algebra of all $n \times n$-matrices over a ring $R$. Let $W = D^{\oplus 2}$ be the free left module over $D$ with rank two, and we equip it with the hermitian form $\langle\ ,\ \rangle$ given by

$$\langle (x_1, y_1), (x_2, y_2)\rangle = x_1\, {}^{\iota}y_2 + y_1\, {}^{\iota}x_2 \qquad (x_1, x_2, y_1, y_2 \in D).$$

Writing $X := \{(*, 0) \in D^{\oplus 2}\}$, $Y := \{(0, *) \in D^{\oplus 2}\}$ we obtain a polarization of $W$. Let $G$ be the unitary group of this form, so that

$$G(R) = \left\{ g \in \mathrm{GL}(2, D \otimes_k R) \;\middle|\; g \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} {}^{*}g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

for any commutative $k$-algebra $R$. Here we write ${}^{*}(a_{i,j}) = ({}^{\iota}a_{j,i})$ for $(a_{i,j}) \in \mathbb{M}(2, D \otimes_k R)$. $G$ is nonquasisplit and is an inner form of Sp(2) with respect to a quadratic extension $k'$ of $k$ such that all $v \in S_D$ do not split fully in $k'/k$. Fix a $k$-parabolic subgroup $P$ and its Levi factor $M$ as

$$P = \left\{ \begin{pmatrix} * & * \\ & * \end{pmatrix} \in G \right\}, \quad M = \left\{ m(x) := \begin{pmatrix} x & 0 \\ 0 & ({}^{\iota}x)^{-1} \end{pmatrix} \;\middle|\; x \in D^{\times} \right\},$$

where $D^{\times}$ is regarded as an algebraic group over $k$ in the natural way. $P$ is the unique proper parabolic subgroup of $G$ up to $G(k)$-conjugate. We write again $\nu$ for the character of $M$ corresponding to the reduced norm via $m$. $U$ denotes the unipotent radical of $P$, so that

$$U = \left\{ u(y) := \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \;\middle|\; y \in D_- \right\}.$$

Here $D_-$ is also regarded as an algebraic group over $k$ in the natural way. The $k$-split component of the center of $M$ is

$$A = \left\{ a(t) := \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \;\middle|\; t \in \mathbf{G}_m \right\}.$$

The element $\alpha$ of the character group $X^*(A)$ of $A$ is defined by $\alpha(a(t)) = t^2$. It is the unique positive root of $A$ with respect to $P$, and $\alpha^{\vee} = a$ is the attached coroot. The Weyl group $W_G$ of $A$ in $G$ is equal to $\{1, w_0\}$, where

$$w_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We fix a maximal compact subgroup $\boldsymbol{K} = \prod_v \boldsymbol{K}_v$ of $G(\boldsymbol{A})$ such that

$$\boldsymbol{K}_v = \begin{cases} G(\mathbb{O}_v) & \text{if } v \notin S_D \text{ and } v \text{ is nonarchimedean,} \\ O(4) \cap G(\mathbb{R}) & \text{if } v \notin S_D \text{ and } v \text{ is real,} \\ U(4) \cap G(\mathbb{C}) & \text{if } v \text{ is complex,} \\ O(\nu^{\oplus 2}) \cap G(\mathbb{R}) & \text{if } v \in S_D \text{ and } v \text{ is real,} \end{cases}$$

and $\boldsymbol{K}_v$ is an $A$-good maximal compact subgroup if $v \in S_D$ and $v$ is nonarchimedean. Here $O(\nu^{\oplus 2})$ is the orthogonal group of a quadratic form $W \otimes \mathbb{R} \ni (x_1, x_2) \mapsto \nu(x_1) + \nu(x_2)$. Then we have the Iwasawa decomposition $G(\boldsymbol{A}) = P(\boldsymbol{A})\boldsymbol{K}$. We write $\boldsymbol{K}_\infty$ for $\prod_{v|\infty} \boldsymbol{K}_v \subset G(\boldsymbol{A}_\infty)$.

## 3. Decomposition of the $L^2$-inner product

We will make use of the results of [Mœglin and Waldspurger 1995], henceforth abbreviated [MW]. Let $(M, \mathfrak{P}_1)$ and $(M, \mathfrak{P}_2)$ be two cuspidal pairs [MW, II.1.1]. For their Paley–Wiener sections $\phi_1 \in P_{(M,\mathfrak{P}_1)}$ and $\phi_2 \in P_{(M,\mathfrak{P}_2)}$ [MW, II.1.2], $\theta_{\phi_1}$ and $\theta_{\phi_2}$ denote the pseudo-Eisenstein series, respectively [MW, II.1.10]. They are elements of $L^2(G(k)\backslash G(\boldsymbol{A}))$ and if $(M, \mathfrak{P}_1)$ and $(M, \mathfrak{P}_2)$ are not $G(k)$-conjugate, they are orthogonal. If $(M, \mathfrak{P}_1)$ and $(M, \mathfrak{P}_2)$ are $G(k)$-conjugate then the $L^2$-inner product between them is described as follows [MW, Theorem II.2.1].

$$\langle \theta_{\phi_1}, \theta_{\phi_2} \rangle = \int_{\substack{\pi \in \mathfrak{P}_1 \\ \mathrm{Re}(\pi)=\lambda_0}} A(\phi_1, \phi_2)(\pi)\, d\pi, \tag{3-1}$$

where

$$A(\phi_1, \phi_2)(\pi) = \sum_{w \in W((M,\mathfrak{P}_1),(M,\mathfrak{P}_2))} \langle M(w, \pi)\phi_1(\pi), \phi_2(-w(\overline{\pi})) \rangle.$$

Here all notation follows [MW]. $W((M, \mathfrak{P}_1), (M, \mathfrak{P}_2))$ is a subset of $W_G$ and $M(1, \pi)$ is the identity, and

$$(M(w_0, \pi)\phi_1(\pi))(g) = \int_{U(\boldsymbol{A})} \phi_1(\pi)(w_0^{-1} u g)\, du \qquad (g \in G(\boldsymbol{A})).$$

This integral converges absolutely at $\pi \in \mathfrak{P}_1$ such that $\langle \mathrm{Re}(\pi), \alpha^\vee \rangle \gg 0$ and $\lambda_0$ is chosen in this area. We have to extend $M(w_0, \pi)$ to $\langle \mathrm{Re}(\pi), \alpha^\vee \rangle \geq 0$ analytically and rewrite Equation (3-1).

For a cuspidal pair $(M, \mathfrak{P})$, $\pi \in \mathfrak{P}$, and $\phi \in P_{(M,\mathfrak{P})}$, $\phi(\pi)$ belongs to the space $A(U(\boldsymbol{A})M(k)\backslash G(\boldsymbol{A}))_\pi$ (defined in [MW, II.1.1]). Here $\pi$ is decomposed into a restricted tensor product $\bigotimes_v \pi_v$ as a (Lie $M(\boldsymbol{A}_\infty) \otimes_{\mathbb{R}} \mathbb{C}$, $M(\boldsymbol{A}_\infty) \cap \mathbf{K}_\infty) \times M(\boldsymbol{A}_f)$-module. If $\pi$ is infinite dimensional then it has a multiplicity of one in the space $A_0(M(k)\backslash M(\boldsymbol{A}))$ of cuspidal forms on $M(k)\backslash M(\boldsymbol{A})$ by Jacquet–Langlands theory.

Also if $\pi$ is one-dimensional it has a multiplicity of one clearly. Thus we have

$$A(U(\boldsymbol{A})M(k)\backslash G(\boldsymbol{A}))_\pi = \mathrm{Ind}_{\boldsymbol{K}\cap M(\boldsymbol{A})}^{\boldsymbol{K}}\pi = \bigotimes_v \mathrm{Ind}_{P(k_v)}^{G(k_v)}\pi_v$$

as $(\mathrm{Lie}\, G(\boldsymbol{A}_\infty)\otimes_{\mathbb{R}}\mathbb{C}, \boldsymbol{K}_\infty)\times G(\boldsymbol{A}_f)$-modules. We have a decomposition:

$$M(w_0,\pi) = \otimes_v M(w_0,\pi_v),$$

where $M(w_0,\pi_v)$ is defined by

$$[M(w_0,\pi_v)\phi_v](g) = \int_{U(k_v)} \phi_v(w_0^{-1}ug)\,du \quad (g\in G(k_v),\ \phi_v\in \mathrm{Ind}_{P(k_v)}^{G(k_v)}\pi_v).$$

Their elementary properties are found in [Knapp and Stein 1980] for the archimedean case and [Silberger 1979] for the nonarchimedean case. This allows us to deduce analytic properties of $M(w_0,\pi)$ from those of the local components $M(w_0,\pi_v)$. We define the normalization factor for $M(w_0,\pi_v)$ by

(3-2)
$$r(w_0,\pi_v,\psi_v) = \frac{L_{w_0}(0,\pi_v)}{L_{w_0}(1,\pi_v)\varepsilon_{w_0}(0,\pi_v,\psi_v)},$$
$$L_{w_0}(s,\pi_v) = L(s,\pi_v)L_{k_v}(2s,\omega_{\pi_v}),$$
$$\varepsilon_{w_0}(s,\pi_v,\psi_v) = \varepsilon(s,\pi_v,\psi_v)\varepsilon_{k_v}(2s,\omega_{\pi_v},\psi_v),$$

where $L(s,\pi_v)$, $\varepsilon(s,\pi_v,\psi_v)$ denote the $L$ and $\varepsilon$-factor defined by Godement and Jacquet [1972], and $L_{k_v}(s,\omega_{\pi_v})$ and $\varepsilon_{k_v}(s,\omega_{\pi_v},\psi_v)$ denote the Hecke $L$ and $\varepsilon$-factor. For any $v\notin S_D$, this normalization factor coincides with the Langlands–Shahidi normalization factor which is considered in [Shahidi 1990]. Let $\mathrm{Re}(\pi_v)$ denote the local analogue of $\mathrm{Re}(\pi)$.

**Lemma 3.1.** *For any $v$, the normalized intertwining operator*

$$N(w_0,\pi_v,\psi_v) = r(w_0,\pi_v,\psi_v)^{-1}M(w_0,\pi_v)$$

*is holomorphic for $\langle \mathrm{Re}(\pi_v),\alpha^\vee\rangle \geq 0$.*

*Proof.* If $v\notin S_D$ and $\pi_v$ is infinite dimensional then the lemma has already been shown by Kim [1995, p. 133–134]. If $v\in S_D$, then $\pi_v$ is square-integrable modulo the center, and therefore $M(w_0,\pi_v)$ is holomorphic and nonzero for $\langle \mathrm{Re}(\pi_v),\alpha^\vee\rangle \geq 0$. Since $r(w_0,\pi_v,\psi_v)$ does not vanish for $\langle \mathrm{Re}(\pi_v),\alpha^\vee\rangle \geq 0$, the lemma follows. Assume that $v\notin S_D$ and $\pi_v = \chi_v\circ\nu$, where $\chi_v$ is a quasi-character of $k_v^\times$. We write $B$ for the Borel subgroup of GL(2) consisting of upper triangular matrices. Identifying $M(k_v)$ with $\mathrm{GL}(2,k_v)$, $\pi_v$ is the unique irreducible subrepresentation of $\mathrm{Ind}_{B(k_v)}^{M(k_v)}(\chi_v|\cdot|_v^{-1/2}\otimes\chi_v|\cdot|_v^{1/2})$ [Jacquet and Langlands 1970]. We fix a set of positive roots of the root data for Sp(2). $r_1$ and $r_2$ denote the reflection attached to the short and long simple roots, respectively. We may assume that $w_0 = r_2 r_1 r_2$. By the functional equation we have

$$M(w_0, \rho[s]) = M(r_2, r_1 r_2 \rho[s]) \circ M(r_1, r_2\rho[s]) \circ M(r_2, \rho[s]),$$

where $\rho[s] := \chi_v|\cdot|_v^{-1/2+s} \otimes \chi_v|\cdot|_v^{1/2+s}$. It is known that

$$\frac{L_{k_v}(s+1, \chi_v|\cdot|_v^{-1/2})\varepsilon_{k_v}(s, \chi_v|\cdot|_v^{-1/2}, \psi_v)}{L_{k_v}(s, \chi_v|\cdot|_v^{-1/2})} M(r_2, r_1 r_2 \rho[s]),$$

$$\frac{L_{k_v}(2s+1, \chi_v^2)\varepsilon_{k_v}(2s, \chi_v^2, \psi_v)}{L_{k_v}(2s, \chi_v^2)} M(r_1, r_2\rho[s]),$$

and

$$\frac{L_{k_v}(s+1, \chi_v|\cdot|_v^{1/2})\varepsilon_{k_v}(s, \chi_v|\cdot|_v^{1/2}, \psi_v)}{L_{k_v}(s, \chi_v|\cdot|_v^{1/2})} M(r_2, \rho[s]),$$

are holomorphic in the region $\mathrm{Re}(s) \geq 0$. Since $N(w_0, \rho[s], \psi_v)$ is the composition of these three maps we obtain the lemma. □

Take a $\phi(\pi) = \bigotimes_v \phi(\pi)_v \in P_{(M,\mathfrak{P})}$. Let $S$ be a finite set of places of $k$ including all the archimedean places such that at every $v \notin S$, $D, \pi$, and $\psi$ are unramified and $\phi(\pi)_v$ is the $K_v$-fixed vector with $\phi(\pi)_v(1) = 1$. For any $v \notin S$ we have the Gindikin–Karpelevich formula [Langlands 1971, p.45]:

$$M(w, \pi_v)\phi(\pi)_v = r(w, \pi_v, \psi_v)\phi(\pi)_v.$$

Therefore

$$\begin{aligned} M(w_0, \pi)\phi(\pi) &= \bigotimes_{v\in S} r(w_0, \pi_v, \psi_v) N(w_0, \pi_v, \psi_v)\phi(\pi)_v \otimes \bigotimes_{v\notin S} r(w_0, \pi_v, \psi_v)\phi(\pi)_v \\ &= r(w_0, \pi) N(w_0, \pi)\phi(\pi), \end{aligned}$$

where

$$r(w_0, \pi) := \prod_v r(w_0, \pi_v, \psi_v),$$

$$N(w_0, \pi)\phi(\pi) := \bigotimes_{v\in S} N(w_0, \pi_v, \psi_v)\phi(\pi)_v \otimes \bigotimes_{v\notin S} \phi(\pi)_v.$$

From Lemma 3.1, $M(w_0, \pi)$ is continued for $\langle \mathrm{Re}(\pi), \alpha^\vee \rangle \geq 0$ and the poles of $M(w_0, \pi)$ coincide with those of $r(w_0, \pi)$ in this region.

**Proposition 3.2.** *Suppose that $M(w_0, \pi)$ has a pole $\pi'$ for $\langle \mathrm{Re}(\pi), \alpha^\vee \rangle \geq 0$.*

(i) *If $\mathfrak{P}$ consists of infinite dimensional representations, $\pi'$ is equal to $\pi_0|\nu|_{\boldsymbol{A}}^{1/2}$, where $\pi_0$ is an irreducible self-dual cuspidal representation of $M(\boldsymbol{A})$ whose standard $L$-function $L(s, \pi_0)$ does not vanish at $s = 1/2$. It is simple.*

(ii) *If $\mathfrak{P}$ consists of one-dimensional representations then $\pi'$ is equal to $|\nu|_{\boldsymbol{A}}^{3/2}$ or $(\omega_{k'/k}|\cdot|_{\boldsymbol{A}}^{1/2})\circ\nu$ where $k'/k$ is a quadratic extension such that all $v\in S_D$ do not split fully in $k'/k$. It is simple.*

*Proof.* (i) From Shahidi's nonvanishing theorem [1981, Theorem 5.1] and the Jacquet–Langlands theory, the only pole of $r(w_0,\pi)$ for $\langle \mathrm{Re}(\pi),\alpha^\vee\rangle\geq 0$ is $\pi_0|\nu|_{\boldsymbol{A}}^{1/2}$ where $\omega_{\pi_0}$ is trivial and $L(s,\pi_0)$ does not vanish at $s=1/2$. Note that $\omega_{\pi_0}=\mathbf{1}_{D^\times(\boldsymbol{A})}$ implies $\pi_0$ is self-dual.

(ii) Let $\pi=\chi\circ\nu$. $L(s,\pi_v)$ is described as follows [Jacquet and Langlands 1970].

$$L(s,\pi_v)=\begin{cases} L_{k_v}(s+1/2,\chi_v)L_{k_v}(s-1/2,\chi_v) & \text{if } v\notin S_D,\\ L_{k_v}(s+1/2,\chi_v) & \text{if } v\in S_D \text{ and } v \text{ is finite},\\ 2(2\pi)^{-(s+1/2)}\Gamma(s+1/2) & \text{if } v\in S_D \text{ and } v \text{ is real}.\end{cases}$$

Thus a factor of $r(w_0,\pi)$ related to the poles is

$$\Big(\prod_{v\in S_D} L_{k_v}(-1/2,\chi_v)\Big)^{-1}\cdot L_k(1/2,\chi)\cdot L_k(0,\chi^2)\cdot L_k(1,\chi^2)^{-1}.$$

Here $L_k(\,\cdot\,,\cdot)$ denotes the Hecke $L$-function. We can calculate its poles easily. □

Write $\mathfrak{S}(\pi_0)=\pi_0|\nu|_{\boldsymbol{A}}^{1/2}$ for $\pi_0$ satisfying the condition of Proposition 3.2(i). Similarly write $\mathfrak{S}_{\mathbf{1}}=|\nu|_{\boldsymbol{A}}^{3/2}$ and $\mathfrak{S}_\Theta(k')=(\omega_{k'/k}|\cdot|_{\boldsymbol{A}}^{1/2})\circ\nu$ for $\omega_{k'/k}$ satisfying the condition of Proposition 3.2(ii). From [Harish-Chandra 1968, Lemma 101], $M(w_0,\pi)$ is bounded on any region of the form

$$\{\pi\in\mathfrak{P}\mid 0\leq\langle\mathrm{Re}(\pi),\alpha^\vee\rangle\leq R\},\quad 0<R\in\mathbb{R}.$$

Thus we can apply the residue theorem to (3-1). From Proposition 3.2 we have the following.

**Theorem 3.3.** *Let $(M,\mathfrak{P}_1)$ and $(M,\mathfrak{P}_2)$ be cuspidal pairs, and $\theta_{\phi_1}$ ($\phi_1\in P_{(M,\mathfrak{P}_1)}$) and $\theta_{\phi_2}$ ($\phi_2\in P_{(M,\mathfrak{P}_2)}$) pseudo-Eisenstein series.*

(i) *If $\mathfrak{P}_2=w_0(\mathfrak{P}_1)$ and one of $\mathfrak{S}(\pi_0)$, $\mathfrak{S}_{\mathbf{1}}$ and $\mathfrak{S}_\Theta(k')$ is contained in $\mathfrak{P}_1$, which is denoted by $\mathfrak{S}$, then*

$$\langle\theta_{\phi_1},\theta_{\phi_2}\rangle=\int_{\pi\in\mathfrak{P}_1|\mathrm{Re}(\pi)=0}A(\phi_1,\phi_2)(\pi)\,d\pi+c\langle N(w_0,\mathfrak{S})\phi_1(\mathfrak{S}),\phi_2(\mathfrak{S})\rangle \tag{3-3}$$

*for some nonzero constant $c$.*

(ii) *Otherwise,*

$$\langle\theta_{\phi_1},\theta_{\phi_2}\rangle=\int_{\pi\in\mathfrak{P}_1|\mathrm{Re}(\pi)=0}A(\phi_1,\phi_2)(\pi)\,d\pi.$$

## 4. The residual spectrum of $G$

We are now ready to determine the irreducible decomposition of the residual discrete spectrum of $G$.

For a $(-1)$-hermitian right $D$-space $(V, h)$, $G(V)$ denotes the unitary group of $(V, h)$ and $G(V_{\boldsymbol{A}})$ the group of its adelic points. Set $\mathbb{W} = V \otimes_D W$ and $\langle\!\langle\ ,\ \rangle\!\rangle = \frac{1}{2}\tau(h_v \otimes {}^{\iota}\langle\ ,\ \rangle)$. Then $(\mathbb{W}, \langle\!\langle\ ,\ \rangle\!\rangle)$ is a symplectic space over $k$. We will define the Weil representation $\omega_{\psi,V}$ of $G(V_{\boldsymbol{A}}) \times G(\boldsymbol{A})$ in Section 4.1.

**Theorem 4.1.** *Let $k$ be a totally real number field. The irreducible constituents of the residual spectrum of $G$ consist of the following representations.*

(i) *The trivial representation $\mathbf{1}_{G(\boldsymbol{A})}$.*

(ii) *The unique irreducible quotient $J_P^G(\pi)$ of $\mathrm{Ind}_{P(\boldsymbol{A})}^{G(\boldsymbol{A})}(\pi|\nu|_{\boldsymbol{A}}^{1/2})$. Here $\pi$ runs over infinite dimensional irreducible self-dual cuspidal representations of $M(\boldsymbol{A})$ whose standard $L$-functions $L(s,\pi)$ do not vanish at $s = 1/2$.*

(iii) *The theta lift $R(V)$ from the trivial representation of $G(V)$ under the Weil representation $\omega_{\psi,V}$. Here $V$ runs over local isometry classes of $(-1)$-hermitian right $D$-spaces with dimension one.*

*In the cases* (1) *and* (2), *the multiplicity of each representation is one. In the case* (3), *the multiplicity of each representation is* $2^{s_D-2}$.

The proof of the theorem occupies the rest of this section. It is known that the discrete term, which is the second term on the right hand of (3-3), expresses the inner product for the residual spectrum [Mœglin and Waldspurger 1995, VI]. In other words, the residual spectrum of $G$ is isomorphic to the direct sum of the images of intertwining operators for all $\mathfrak{S} = \mathfrak{S}_{\mathbf{1}}$, $\mathfrak{S}(\pi_0)$ and $\mathfrak{S}_\Theta(k')$ appearing in Proposition 3.2.

First assume that $\mathfrak{S} = \mathfrak{S}_{\mathbf{1}}$. We have that at each $v$, $\mathrm{Im}N(w_0, \mathfrak{S}_v)$ is isomorphic to $\mathbf{1}_{G(k_v)}$ by the Langlands classification. Thus the representation of Theorem 4.1(1) is obtained. Next assume that $\mathfrak{S} = \mathfrak{S}(\pi_0)$. For $v \notin S_D$, the proof of Proposition 3.2 in [Kim 1995] shows that $\mathrm{Ind}_{P(k_v)}^{G(k_v)}\mathfrak{S}_v$ has a unique irreducible quotient and $\mathrm{Im}N(w_0, \mathfrak{S}_v)$ is isomorphic to this quotient. If $v \in S_D$, $\mathfrak{S}_v$ is tempered so that $\mathrm{Im}N(w_0, \mathfrak{S}_v)$ is also isomorphic to the unique irreducible quotient of $\mathrm{Ind}_{P(k_v)}^{G(k_v)}\mathfrak{S}_v$. Therefore, all images of $N(w_0, \mathfrak{S})$ are given by the representations of Theorem 4.1(2). We remark that the above results for $\mathfrak{S} = \mathfrak{S}_{\mathbf{1}}$ and $\mathfrak{S}(\pi_0)$ do not need the fact that $k$ is totally real.

**4.1.** ***Construction by theta correspondence.*** Finally assume that $\mathfrak{S} = \mathfrak{S}_\Theta(k')$. At each place $v$ write $I(s, \chi_v) = \mathrm{Ind}_{P(k_v)}^{G(k_v)}((\chi_v|\cdot|_v^s) \circ \nu)$, where $\chi_v$ is a character of $k_v^\times$ and $s \in \mathbb{C}$. Writing $\omega_{k'_v/k_v}$ for the $v$-component of $\omega_{k'/k}$, $\mathrm{Im}N(w_0, \mathfrak{S}_\Theta(k')_v, \psi_v)$ is a subrepresentation of $I(-1/2, \omega_{k'_v/k_v})$. We will describe the irreducible constituents of $I(-1/2, \omega_{k'_v/k_v})$ by using the theta correspondence.

*(-1)-Hermitian spaces over quaternion algebras.* We will review some basic facts about $(-1)$-hermitian spaces. The main involution $\iota$ of $D$ is of the first kind and of symplectic type [Scharlau 1985, p. 304]. For any place $v$ we again write $\iota$ for the involution of $D_v := D \otimes_k K_v$ induced by the main involution of $D$, which also becomes of the first kind and of symplectic type. Therefore, from Remarks (iii) in the same location we can identify the local involution $\iota$ with the local main involution.

Let $F$ be a local field and $R$ be a quaternion algebra over $F$ with the main involution $\iota$. For $\Gamma = (\gamma_{i,j}) \in \mathbb{M}(m, R)$ such that $({}^\iota\gamma_{j,i}) = -\Gamma$ (resp. $({}^\iota\gamma_{j,i}) = \Gamma$), a $(-1)$-hermitian (resp. hermitian) form on a right (resp. left) $R$-module $R^{\oplus m}$ (the set of column (resp. row) vectors) is defined by $((v_i), (v'_j)) \mapsto {}^t({}^\iota v_i)\Gamma(v'_j)$ (resp. $((w_i), (w'_j)) \mapsto (w_i)\Gamma\, {}^t({}^\iota w'_j)$). We denote this form by $\langle\Gamma\rangle$. Similarly for $B \in \mathbb{M}(m, F)$ such that ${}^tA = A$ (resp. ${}^tA = -A$) we can define a quadratic (resp. symplectic) form $\langle B\rangle$ on $F^{\oplus m}$ (the set of column (resp. row) vectors).

- The case $R = \mathbb{M}(2, F)$. To observe $(-1)$-hermitian modules over $\mathbb{M}(2, F)$ we make use of an available theory which is called hermitian Morita theory. This implies an equivalence between the category of $(-1)$-hermitian (right) $R$-modules and the category of quadratic $F$-spaces. We will describe the correspondence in this theory ([Knus 1991] § I.9, [Scharlau 1985] p.361,362).

Let $(V, h)$ be a $(-1)$-hermitian right $R$-module. Set

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathbb{M}(2, F) = R,$$

and $T_V = T_{(V,h)} := Ve$. A bilinear form $b_V = b_{(V,h)}$ on $T_V$ over $F$ is defined by

$$h(ve, v'e) = \begin{pmatrix} 0 & 0 \\ b_{(V,h)}(ve, v'e) & 0 \end{pmatrix} \in \mathbb{M}(2, F) \qquad \text{for } v, v' \in V.$$

Then $(T_V, b_V)$ is the quadratic space corresponding to $(V, h)$ in hermitian Morita theory. We define $\det V$ as the determinant of $(T_V, b_V)$. For two $(-1)$-hermitian modules $(V_1, h_1)$, $(V_2, h_2)$ and an isometry $\sigma : (V_1, h_1) \to (V_2, h_2)$, the restriction $\sigma|_{T_{V_1}} : T_{V_1} \to T_{V_2}$ becomes an isometry from $(T_{V_1}, b_{V_1})$ to $(T_{V_2}, b_{V_2})$ as quadratic $F$-spaces. Therefore, the unitary group of $(V, h)$ is isomorphic to the orthogonal group of $(T_V, b_V)$.

In particular, consider the case of $(-1)$-hermitian free module with rank one. For a

$$\Gamma = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \in \mathbb{M}(2, F)_-,$$

we define $\mathcal{M}(\Gamma) \in \mathrm{Sym}_2(F) = \{X \in \mathbb{M}(2, F) \mid {}^tX = X\}$ as

$$\mathcal{M}(\Gamma) = \begin{pmatrix} \gamma & -\alpha \\ -\alpha & -\beta \end{pmatrix} = J\Gamma.$$

For a $(-1)$-hermitian right $R$-module, $(V, h) = (R, \langle\Gamma\rangle)$, one has that a quadratic space $(F^{\oplus 2}, \langle\mathcal{M}(\Gamma)\rangle)$ is isometric to $(T_V, b_V)$. Note that $\det V = \det \mathcal{M}(\Gamma) = \det \Gamma$.

Hermitian Morita theory also implies the equivalence between the category of hermitian left $R$-modules and the category of symplectic $F$-spaces. Writing $(W_F, h_F)$ for the hermitian left $R$-module

$$(R^{\oplus 2}, \langle\begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}\rangle),$$

the Morita correspondence of $(W_F, h_F)$ is isometric to $(eW_F, s_{W_F})$ where $s_{W_F}$ is defined by

$$h_F(ew, ew') = \begin{pmatrix} 0 & s_{W_F}(ew, ew') \\ 0 & 0 \end{pmatrix} \in \mathbb{M}(2, F) \qquad \text{for } w, w' \in W_F.$$

Also, we have an isomorphism between the unitary group $G(W_F)$ of $(W_F, h_F)$ and $\mathrm{Sp}(eW_F)$ by restriction to $eW_F$.

- The case that $R$ is the quaternion division algebra over $F$. For a $(-1)$-hermitian right $R$-space $(V, h)$, the determinant $\det V \in F^\times/(F^\times)^2$ is defined as the reduced norm of a matrix expression of $h$.

**Proposition 4.2** [Scharlau 1985, Theorem 3.6, 3.7].

(a) *Let $F$ be nonarchimedean.*
(i) *The isometry class of any regular $(-1)$-hermitian right $R$-space is determined by its dimension and determinant.*
(ii) *There exists a regular $(-1)$-hermitian right $R$-space with any dimension and determinant except for dimension* 1 *and determinant* $-1$.

(b) *Let $F$ be real and archimedean.*
(i) *The isometry class of any regular $(-1)$-hermitian right $R$-space is determined by its dimension.*
(ii) *There exists a regular $(-1)$-hermitian right $R$-space with any dimension.*

The next proposition is a statement about the local-global property with regard to $(-1)$-hermitian spaces. For a $(-1)$-hermitian right $D$-space $(V, h)$, the determinant $\det V \in k^\times/(k^\times)^2$ is defined as the reduced norm of a matrix expression of $h$. For $\gamma \in D_-\backslash\{0\}$, let $k^\times_{D,\gamma} = \{c \in k^\times \mid (\gamma^2, c)_v = 1 \text{ for all } v \notin S_D\}$, where $(\ ,\ )_v$ is the Hilbert symbol at $v$. A group homomorphism $\lambda$ is defined by

$$k^\times_{D,\gamma} \ni c \mapsto \{(\gamma^2, c)_v\}_{v \in S_D} \in \{\pm 1\}^{s_D}.$$

Let $\{\pm 1\}$ be regarded as the subgroup of $\{\pm 1\}^{s_D}$ via the diagonal embedding. Note that the number of elements of $k_{D,\gamma}^{\times}/\lambda^{-1}(\{\pm 1\})$ is $2^{s_D-2}$.

**Proposition 4.3** [Scharlau 1985, Theorem 10.4.6, Remark 10.4.6]. *Let $(V, h)$ be a $(-1)$-hermitian right $D$-space.*

(i) *If $\dim_D V = 1$ and $h = \langle\gamma\rangle$ for some $\gamma \in D_-\backslash\{0\}$, then for any $c \in k_{D,\gamma}^{\times}$, $\langle c\gamma\rangle$ is locally isometric to $\langle\gamma\rangle$. For any $a \in \lambda^{-1}(\{\pm 1\})$, $\langle a\gamma\rangle$ is globally isometric to $\langle\gamma\rangle$. Moreover*

$$\{\langle a\gamma\rangle \mid a \in k_{D,\gamma}^{\times}/\lambda^{-1}(\{\pm 1\})\}$$

*is the set of classes locally isometric to $\langle\gamma\rangle$, so this set contains $2^{s_D-2}$ elements.*

(ii) *In general, for every dimension there exists exactly $2^{s_D-2}$ classes locally isometric to $(V, h)$.*

*Local theta correspondence.* Let $(V, h)$ be a $(-1)$-hermitian right $D$-module with dimension $m$. We denote by $G(V)$ the unitary group of $V$ and, if $R$ is a $k$-algebra, by $G(V_R)$ the group of $R$-valued points of $G(V)$. For a place $v$, let $(V_v, h_v)$ be a $(-1)$-hermitian space over $D_v$. We define $G(V_v)$ similarly, but we often use $G(V_v)$ for the group of $k_v$-valued points of $G(V_v)$ by an abuse of notation. Let $(W_v, \langle\ ,\ \rangle_v)$ denote the completion of the hermitian space $(W, \langle\ ,\ \rangle)$ over $D$ at $v$. We define the Weil representation of $G(V_v) \times G(k_v)$ as follows.

Let $\mathbb{W}_v := V_v \otimes_{D_v} W_v$ and $\langle\langle\ ,\ \rangle\rangle_v := \frac{1}{2}\tau(h_v \otimes {}^{\iota}\langle\ ,\ \rangle_v)$. Then $(\mathbb{W}_v, \langle\langle\ ,\ \rangle\rangle_v)$ is a symplectic space over $k_v$ of dimension $8m$.

- The case $v \notin S_D$. A homomorphism

$$\mathbb{W}_v = V_v \otimes W_v \ni v \otimes w \mapsto ve'e \otimes ee'w + ve \otimes ew \in V_v e \otimes_{k_v} eW_v$$

becomes an isometry between $(\mathbb{W}_v, \langle\langle\ ,\ \rangle\rangle_v)$ and $(V_v e \otimes_{k_v} eW_v, \frac{1}{2}b_{V_v} \otimes (-s_{W_v}))$. On the other hand, a homomorphism

$$O(V_2 e, \tfrac{1}{2}b_{V_v}) \times \mathrm{Sp}(eW_v, -s_{W_v}) \to \mathrm{Mp}(V_v e \otimes eW_v)$$

is given in [Kudla 1994], where $\mathrm{Mp}(V_v e \otimes eW_v)$ denotes the metaplectic group. From this and Section 4.1 we have a homomorphism $G(V_v) \times G(k_v) \to \mathrm{Mp}(\mathbb{W}_v)$. Therefore the Weil representation $\omega_{\psi_v}$ of $\mathrm{Mp}(\mathbb{W}_v)$ induces a representation $\omega_{V_v} = \omega_{\psi_v, V_v}$ of $G(V_v) \times G(k_v)$, which is realized on the space $\mathcal{S}(V_v)$ of Schwartz–Bruhat functions on $V_v = V_v \otimes X_v$.

- The case $v \in S_D$. A homomorphism $G(V_v) \times G(k_v) \to \mathrm{Mp}(\mathbb{W}_v)$ is given in [Kudla 1994]. The Weil representation $\omega_{\psi_v}$ of $\mathrm{Mp}(\mathbb{W}_v)$ induces a representation $\omega_{V_v} = \omega_{\psi_v, V_v}$ of $G(V_v) \times G(k_v)$, which is realized on $\mathcal{S}(V_v) = \mathcal{S}(V_v \otimes X_v)$.

For all places $v$ we have defined the Weil representation $\omega_{V_v}$ of $G(V_v) \times G(k_v)$. Some explicit formulae involving $\omega_{V_v}$ are as follows. Let $\phi \in \mathcal{S}(V_v)$ and $v \in V_v$.

- $\omega_{V_v}(\begin{pmatrix} a & 0 \\ 0 & {}^{\iota}a^{-1} \end{pmatrix})\phi(v) = ((-1)^m \det V_v, \nu(a))_v |a|_{k_v}^m \phi(va) \quad (a \in D^\times(k_v))$
- $\omega_{V_v}(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix})\phi(v) = \psi_v(\frac{1}{4}\tau(bh_v(v)))\phi(v) \quad (b \in D_{v,-})$
- $\omega_{V_v}(h)\phi(v) = \phi(h^{-1}v) \quad (h \in G(V_v))$

We will describe constituents of $I(-1/2, \chi_v)$. Let $(V_v, h_v)$ be a $(-1)$-hermitian (free) right module over $D_v$ with rank $m$. $\chi_{V_v}$ denotes the quadratic character of $k_v^\times$ defined by $\chi_{V_v}(x) = ((-1)^m \det V_v, x)_v$. For $v \notin S_D$ the description of the irreducible constituents of $I(-1/2, \chi_v)$ has been obtained by Kudla, Rallis, and Soudry [Kudla et al. 1992]. Therefore we will restrict to $v \in S_D$.

- The case of nonarchimedean $v \in S_D$. Consider the reducible points of $I(s, \chi_v)$, where $\chi_v$ is a quadratic character of $k_v^\times$.

**Lemma 4.4.** (i) *If $\chi_v = 1$ then $I(\pm 3/2, \chi_v)$ is reducible. $I(s, \chi_v)$ is irreducible for real $s \neq \pm 3/2$.*

(ii) *If $\chi_v \neq 1$ then $I(\pm 1/2, \chi_v)$ is reducible. $I(s, \chi_v)$ is irreducible for real $s \neq \pm 1/2$.*

*Proof.* The local Jacquet–Langlands correspondence of $\chi_v \circ \nu$ is $\sigma_0 = \chi_v \circ \det \otimes \delta_{St}$, where $\delta_{St}$ denotes the Steinberg representation of $\mathrm{GL}(2, k_v)$. By Proposition 2.1 in [Muić and Savin 2000] the Plancherel measure of $\chi_v$ coincides with the Plancherel measure of $\chi_v \circ \det \otimes \delta_{St}$. Therefore, the poles and zeros of $\mu(s, \chi_v \circ \nu)$ coincide with the poles and zeros of

$$\frac{L(1-s, \sigma_0^\vee) L_k(1-2s, \omega_{\sigma_0}^{-1}) L(1+s, \sigma_0) L_k(1+2s, \omega_{\sigma_0})}{L(-s, \sigma_0^\vee) L_k(-2s, \omega_{\sigma_0}^{-1}) L(s, \sigma_0) L_k(2s, \omega_{\sigma_0})},$$

respectively. Since $\mu(0, \chi_v \circ \nu) = 0$, the reducible points of $I(s, \chi_v)$ coincide with the poles of $\mu(s, \chi_v \circ \nu)$ [Silberger 1979]. All the poles are $s = \pm 3/2$ if $\chi_v = 1$ and $s = \pm 1/2$ if $\chi_v \neq 1$. □

In Proposition 3.2(ii), the local component $\omega_{k_v'/k_v}$ at $v$ of $\omega_{k'/k}$ is not trivial. Therefore we want the description of constituents of $I(-1/2, \chi_v)$ with a nontrivial $chi_v$, using theta correspondence. We write $\mathcal{S}(V_v)_{G(V_v)}$ for the $G(V_v)$-coinvariant space of $S(V_v)$. Let $R(V_v)$ denote the image of the map

$$\mathcal{S}(V_v) \ni \phi \mapsto [G(k_v) \ni g \mapsto \omega_{V_v}(g)\phi(0)] \in I(m - 3/2, \chi_{V_v}).$$

This map induces an isomorphism $\mathcal{S}(V_v)_{G(V_v)} \simeq R(V_v)$; see [Mœglin et al. 1987, Chap. 3 IV Th. 7]. Let $(V_v', h_v')$ be a 1-dimensional $(-1)$-hermitian space with det $V'_v \neq -1$. $(V''_v, h''_v)$ denotes the 2-dimensional $(-1)$-hermitian space given by det $V''_v = -$det $V'_v$. Proposition 4.2 guarantees the existence and uniqueness of $(V'_v, h'_v)$ and $(V''_v, h''_v)$.

**Proposition 4.5.** (i) *$R(V'_v)$ and $R(V''_v)$ are the unique irreducible subrepresentations of $I(-1/2, \chi_{V'_v})$ and $I(1/2, \chi_{V'_v})$, respectively.*

(ii) *$N(w_0, (\chi_{V'_v}|\cdot|_v^{1/2}) \circ \nu, \psi_v)$ induces an isomorphism $I(1/2, \chi_{V'_v})/R(V''_v) \simeq R(V'_v)$.*

*Proof.* The Jacquet module $I(-1/2, \chi_{V'_v})_P$ of $I(-1/2, \chi_{V'_v})$ along $P$ is

$$(\chi_{V'_v}|\cdot|_v^{1/2}) \circ \nu + (\chi_{V'_v}|\cdot|_v^{-1/2}) \circ \nu$$

in the Grothendieck group. On the other hand, $R(V'_v)_P \simeq (\chi_{V'_v}|\cdot|_v^{-1/2}) \circ \nu$. Since $I(-1/2, \chi_{V'_v})$ has at most one proper $G(k_v)$-invariant space, the former of (i) follows. Similarly the latter of (i) is proved. From the Langlands classification we obtain (ii). □

• The case of real archimedean $v \in S_D$. We have $G(k_v) \simeq \mathrm{Sp}(1,1)$. The reducible points of $I(s, \mathbf{1})$ have been obtained, as follows.

**Lemma 4.6** [Johnson 1990, Corollary of Lemma 5.4]. *If $s = \pm(2n+1)/2$ $(n \in \mathbb{N})$ then $I(s, \mathbf{1})$ is reducible, otherwise $I(s, \mathbf{1})$ is irreducible. In particular $I(\pm 1/2, \mathbf{1})$ are irreducible.*

Since $\nu(D_v^\times) = \mathbb{R}_+^\times$, we need not take the signed character in this lemma. Let $\mathcal{S}^0(V_v)$ denote the subspace of $\mathcal{S}(V_v)$ of functions of the form $e^{-\pi\nu(x)}P(x)$ where $P$ is a polynomial on $V_v (\cong k_v^{\oplus 4m})$. This space is a $(\mathfrak{g}_v, \boldsymbol{K}_v)$-module where $\mathfrak{g}_v = \mathrm{Lie}\, G(k_v) \otimes_{\mathbb{R}} \mathbb{C}$. Fix an isomorphism $V_v \cong D_v^{\oplus m}$. We set $\mathfrak{h}_v = \mathrm{Lie}\, O^*(2m) \otimes_{\mathbb{R}} \mathbb{C}$ and a maximal compact subgroup $L_v = O(\nu^{\oplus m}) \cap O^*(2m)$ of $O^*(2m)$ where $O(\nu^{\oplus m})$ is the orthogonal group of $\nu^{\oplus m}$ ($m$ times direct sum of the norm form on $D_v$). We write $\mathcal{S}^0(V_v)_{(\mathfrak{h}_v, L_v)}$ for the $(\mathfrak{h}_v, L_v)$-coinvariant space of $\mathcal{S}^0(V_v)$. $R(V_v)$ denotes the image of the map

$$\mathcal{S}^0(V_v) \ni \phi \mapsto [G(k_v) \ni g \mapsto \omega_{V_v}(g)\phi(0)] \in I(m - 3/2, \chi_{V_v}).$$

This map induces an isomorphism $\mathcal{S}^0(V_v)_{(\mathfrak{h}_v, L_v)} \simeq R(V_v)$ as $(\mathfrak{g}_v, \boldsymbol{K}_v)$-modules [Zhu 1992, Theorem II]. Let $(V'_v, h'_v)$ be a $(-1)$-hermitian space with dimension one. By Proposition 4.2 it is determined uniquely. From the above lemma and the Langlands classification we have the following.

**Proposition 4.7.** *$N(w_0, |\cdot|_v^{1/2}, \psi_v)$ induces an isomorphism $I(1/2, \mathbf{1}) \simeq R(V'_v)(= I(-1/2, \mathbf{1}))$.*

We go back to a general place $v$. Let $(V_v, h_v)$ be a $(-1)$-hermitian right $D_v$-module with rank $m$. For $\beta \in D_{v,-}$, let $\Omega_\beta = \{v \in V_v \mid h_v(v, v) = \beta\}$. Also we write $\psi_{v,\beta}$ for the character of $U(k_v)$ given by $\psi_{v,\beta}(u(b)) = \psi_v(\frac{1}{4}\tau(b\beta))$.

Suppose $k_v$ is nonarchimedean. Set

$$\mathcal{S}(V_v)_\beta = \mathcal{S}(V_v)/\mathrm{Span}\{\omega_{V_v}(u)\varphi - \psi_{v,\beta}(u)\varphi \mid \varphi \in \mathcal{S}(V_v),\ u \in U(k_v)\}.$$

The next two lemmas are shown by an argument similar to the proof of [Rallis 1987, Lemma 4.2].

**Lemma 4.8.** *If* $\Omega_\beta = \phi$ *then* $R(V_v)_\beta = 0$. *If* $\nu(\beta) \neq 0$ *and* $\Omega_\beta \neq \phi$ *then* $\dim R(V_v)_\beta = 1$.

Suppose that $k_v$ is real. $I^\infty(s, \chi_{V_v})$ denotes the smooth induced representation including $I(s, \chi_{V_v})$ with its Fréchet topology. Let the topology of $\mathcal{S}(V_v)$ be given by the usual one. The map $i : \mathcal{S}^0(V_v) \to R(V_v) \subset I((m-3)/2, \chi_{V_v})$ extends to a continuous map

$$i : \mathcal{S}(V_v) \to R^\infty(V_v) \subset I^\infty((m-3)/2, \chi_{V_v}),$$

where $R^\infty(V_v)$ is the image of $\mathcal{S}(V_v)$. For $\beta \in D_{v,-}$, let $R^\infty(V_v)'_\beta$ be the space of all continuous linear functionals $\mu$ on $R^\infty(V_v)$ such that $\mu(r(X)\Phi) = d\psi_{v,\beta}(X)\mu(\Phi)$ for all $X \in \mathfrak{u} = \mathrm{Lie}\, U(\mathbb{R})$ and all $\Phi \in R^\infty(V_v)$. Here $d\psi_{v,\beta} : \mathfrak{u} \to \mathbb{C}$ is the differential of $\psi_{v,\beta}$ and $r$ denotes the action on $R^\infty(V_v)$.

**Lemma 4.9.** *If* $\Omega_\beta = \phi$ *then* $R^\infty(V_v)'_\beta = 0$. *If* $\nu(\beta) \neq 0$ *and* $\Omega_\beta \neq \phi$ *then* $\dim R^\infty(V_v)'_\beta = 1$.

*Global theta correspondence.* For a $(-1)$-hermitian right $D$-module $(V, h)$ we write $\omega_V = \omega_{\psi,V}$ for the Weil representation of $G(V_{\boldsymbol{A}}) \times G(\boldsymbol{A})$ on the space $\mathcal{S}(V_{\boldsymbol{A}})$ of Schwartz–Bruhat functions on $V_{\boldsymbol{A}}$. For any $\beta \in D_-$, we define a character $\psi_\beta$ of $U(\boldsymbol{A})$ by

$$\psi_\beta(u(b)) = \prod_v \psi_{v,\beta}(u(b_v)) \qquad (u(b) = (u(b_v)) \in U(\boldsymbol{A}))\,.$$

Let $\{(V_v, h_v)\}$ be a collection of $(-1)$-hermitian right $D_v$-modules with rank one for all $v$, and let $\Pi = \bigotimes_v R(V_v)$ (if it can be defined). $\Pi$ is regarded as a representation of $(\mathrm{Lie}\, G(\boldsymbol{A}_\infty) \otimes_{\mathbb{R}} \mathbb{C}, \boldsymbol{K}_\infty) \times G(\boldsymbol{A}_f)$, whose action is written by $r$. We write $\mathcal{W}_\beta(\Pi)$ for the space of linear functionals $\mu$ on $\Pi$ which satisfy

$$\text{(4-1)} \qquad \begin{aligned} \mu(r(u)f) &= \psi_\beta(u)\mu(f) & &\big(f \in \Pi, u \in U(\boldsymbol{A}_f)\big)\,,\\ \mu(r(X)f) &= d\psi_\beta(X)\mu(f) & &(X \in \mathrm{Lie}\, U(\boldsymbol{A}_\infty))\,, \end{aligned}$$

where $d\psi_\beta$ is the differential of the restriction of the character $\psi_\beta$ to $U(\boldsymbol{A}_\infty)$. For each archimedean place $v$, $R^\infty(V_v)$ denotes the closure of $R(V_v)$ in $I^\infty(1/2, \chi_{V_v})$

as in the previous subsection. Let

$$\Pi^\infty = \Big( \bigotimes_{v:\text{arch}} R^\infty(V_v) \Big) \otimes \Big( \bigotimes_{v:\text{nonarch}} R(V_v) \Big),$$

which is a representation of $G(\boldsymbol{A})$. Let $\mathcal{W}_\beta^\infty(\Pi^\infty)$ be the space of all linear functionals $\mu$ on $\Pi^\infty$ which satisfy (4-1) and whose restrictions to $R^\infty(V_v)$ lie in $R^\infty(V_v)'_\beta$. Let $\mathcal{W}_\beta^\infty(\Pi)$ be the subspace of $\mathcal{W}_\beta(\Pi)$ spanned by the restrictions of functionals in $\mathcal{W}_\beta^\infty(\Pi^\infty)$. We have the following proposition which is the same as Proposition 2.1 in [Kudla et al. 1992].

**Proposition 4.10.** *Suppose that* $\{(V_v, h_v)\}$ *is given by the completions at all* $v$ *of some* $(-1)$*-hermitian right* $D$*-module* $(V, h)$ *with rank one and* $\det V \neq 0$. *We write* $\Pi(V) = \bigotimes_v R(V_v)$. *Let* $\mathcal{O}_V = \{\beta \in D_-\backslash\{0\} \mid h(x,x) = c\beta$ *for some* $x \in V$ *and some* $c \in k_{D,\beta}^\times\}$.

(i) *If* $\beta \notin \mathcal{O}_V$ *and* $\beta \neq 0$ *then* $\mathcal{W}_\beta^\infty(\Pi(V)) = 0$.

(ii) *If* $\beta \in \mathcal{O}_V$ *then* $\dim \mathcal{W}_\beta^\infty(\Pi(V)) = 1$.

$A(G(k)\backslash G(\boldsymbol{A}))$ denotes the space of automorphic forms on $G(k)\backslash G(\boldsymbol{A})$. For $f \in A(G(k)\backslash G(\boldsymbol{A}))$, the $\beta$-th Fourier coefficient $W_\beta(f)$ is defined by

$$W_\beta(f)(g) = \int_{U(k)\backslash U(\boldsymbol{A})} f(ug)\psi_\beta(u^{-1})\,du.$$

Denote by $\overline{W}_\beta$ the linear functional on $A(G(k)\backslash G(\boldsymbol{A}))$ defined by $f \mapsto W_\beta(f)(1)$. It satisfies

$$\overline{W}_\beta(r(u)f) = \psi_\beta(u)\overline{W}_\beta(f) \qquad (u \in U(\boldsymbol{A})).$$

Let $(V, h)$ be a $(-1)$-hermitian space over $D$. We write $A^\infty(G(k)\backslash G(\boldsymbol{A}))$ for the space of smooth automorphic forms without the $\boldsymbol{K}$-finiteness condition. Let $r$ again denote the right regular action on $A(G(k)\backslash G(\boldsymbol{A}))$ or on $A^\infty(G(k)\backslash G(\boldsymbol{A}))$.

By a parallel argument to the first part of the proof of Theorem 2.2 in [Kudla et al. 1992], we have that if $A, B : \Pi(V) \to A(G(k)\backslash G(\boldsymbol{A}))$ are two intertwining operators then $A$ and $B$ both extend to continuous $G(\boldsymbol{A})$-intertwining operators $A^\infty, B^\infty : \Pi(V)^\infty \to A^\infty(G(k)\backslash G(\boldsymbol{A}))$. Since $\overline{W}_\beta$ can extend to a continuous linear functional on $A^\infty(G(k)\backslash G(\boldsymbol{A}))$, we conclude that both $A_\beta = \overline{W}_\beta \circ A$ and $B_\beta = \overline{W}_\beta \circ B$ lie in $\mathcal{W}_\beta^\infty(\Pi(V))$. As in the observation just before Lemma 2.5 in [Kudla et al. 1992], we have the following results. If $\beta \notin \mathcal{O}_V$ and $\beta \neq 0$, then $A_\beta = B_\beta = 0$ by Proposition 4.10(i). If $\beta \in \mathcal{O}_V$, there is a $c_\beta \in \mathbb{C}$ such that $A_\beta = c_\beta B_\beta$ by Proposition 4.10(ii). Moreover, if both $\beta_1$ and $\beta_2$ lie in the same orbit in $\mathcal{O}_V$ then $c_{\beta_1} = c_{\beta_2}$.

**Proposition 4.11.** *Suppose that* $\{(V_v, h_v)\}$ *is as in the assumption of Proposition 4.10. Then* $\dim \operatorname{Hom}(\Pi(V), A(G(k)\backslash G(\boldsymbol{A}))) \leq 2^{s_D - 2}$.

*Proof.* From the definition of $\mathbb{O}_V$ and Proposition 4.3(i), $\mathbb{O}_V$ has $2^{s_D-2}$ orbits. By the preceding observation and the next lemma, the proposition is proved. □

**Lemma 4.12.** (*It is not necessary to impose the assumption of Proposition 4.10.*) *If* $E:\Pi\to A(G(k)\backslash G(\boldsymbol{A}))$ *is an intertwining operator satisfying* $\overline{W}_\beta\circ E=0$ *for all* $\beta\in D_-\backslash\{0\}$ *then* $E=0$.

This is shown by the same argument as [Kudla et al. 1992, Lemma 2.5].

Next consider the equality of the relation in Proposition 4.11. For $\phi\in\mathcal{S}(V_{\boldsymbol{A}})$ and $g\in G(\boldsymbol{A})$, we let

$$I_\phi(g)=\int_{G(V_k)\backslash G(V_{\boldsymbol{A}})}\theta(g,l;\phi)\,dl,$$

with the usual theta kernel

$$\theta(g,l;\phi)=\sum_{x\in V(k)}\omega_V(g)\phi(l^{-1}x).$$

Since $G(V)$ is anisotropic, $I_\phi(g)$ is well-defined and $I_\phi\in A(G(k)\backslash G(\boldsymbol{A}))$. This defines an intertwining operator $I_V$ from $\mathcal{S}(V_{\boldsymbol{A}})$ to $A(G(k)\backslash G(\boldsymbol{A}))$. It is $G(V_v)$-invariant for all nonarchimedean places and $(\mathfrak{h}_v,L_v)$-invariant for archimedean places. Therefore the image $\mathrm{Im}I_V$ of the intertwining operator is isomorphic to a quotient of $\Pi(V)$. Since $\Pi(V)$ is irreducible, $\mathrm{Im}I_V$ is isomorphic to $\Pi(V)$. Applying the square integrability criterion, and by Proposition 6.9 in [Kudla et al. 1992], we have that $\mathrm{Im}I_V\subset L^2(G(k)\backslash G(\boldsymbol{A}))$. For $\beta\in D_-\backslash\{0\}$,

$$\begin{aligned}W_\beta(I_\phi)(g)&=\int_{U(k)\backslash U(\boldsymbol{A})}I_\phi(ug)\psi_\beta(u^{-1})\,du\\&=\int_{U(k)\backslash U(\boldsymbol{A})}du\int_{G(V_k)\backslash G(V_{\boldsymbol{A}})}\sum_{x\in V(k)}\omega_V(ug)\phi(l^{-1}x)\psi_\beta(u^{-1})\,dl\\&=\int_{G(V_k)\backslash G(V_{\boldsymbol{A}})}\sum_{h(x,x)=\beta}\omega_V(g)\phi(l^{-1}x)\,dl.\end{aligned}$$

In particular, if $\beta$ is not represented by $h$ then $W_\beta(I_\phi)=0$. If $\beta=h(x_0,x_0)$ for some $x_0\in V\backslash\{0\}$ then

$$W_\beta(I_\phi)(g)=\int_{G(V_{\boldsymbol{A}})}\omega_V(g)\phi(l^{-1}x_0)\,dl.$$

Choosing a suitable $\phi$ we have $W_\beta(I_\phi)\neq 0$. In particular, if $V_1,\dots,V_n$ are not isometric to each other then $I_{V_1},\dots,I_{V_n}$ are linearly independent. Thus we have obtained the following theorem.

**Theorem 4.13.** *Suppose that* $\{(V_v,h_v)\}$ *is as in the assumption of Proposition 4.10. Then* $\dim\mathrm{Hom}(\Pi(V),A(G(k)\backslash G(\boldsymbol{A})))=2^{s_D-2}$.

**Proposition 4.14.** *Suppose that there are no global $(-1)$-hermitian space $(V, h)$ such that $\{(V_v, h_v)\}_v$ is given by the completions of $(V, h)$. Then*

$$\mathrm{Hom}(\Pi, A(G(k)\backslash G(\boldsymbol{A}))) = 0.$$

*Proof.* Let $E : \Pi \to A(G(k)\backslash G(\boldsymbol{A}))$ be a nonzero intertwining operator. There exists $\beta \in D_-\backslash\{0\}$ such that $E_\beta = \overline{W}_\beta \circ E$ is nonzero by Lemma 4.12. Therefore for any $v$, the restriction of $E_\beta$ to $R(V_v)$ is nonzero. By Lemma 4.8 and 4.9 $\langle\beta\rangle \simeq V_v$. This contradicts the assumption for $\{(V_v, h_v)\}$. □

Let $\{(V_v, h_v)\}$ be given by the completions at all $v$ for some $(-1)$-hermitian right $D$-module $(V, h)$ with rank one and $\det V \neq 0$. We define the quadratic character $\chi_V$ of $\boldsymbol{A}^\times/k^\times$ by $\chi_V = \prod_v \chi_{V_v}$. Since

$$-\nu(D_-\backslash\{0\}) = \{\alpha \in k^\times \mid \alpha \notin (k_v^\times)^2 \text{ for all } v \in S_D\},$$

any $\omega_{k'/k}$ appearing in Proposition 3.2(ii) can be written by the form $\chi_V$ for some $V$. By the definition of $R(V_v)$ we have

$$\bigotimes_v R(V_v) \subset \mathrm{Ind}_{P(\boldsymbol{A})}^{G(\boldsymbol{A})}((\chi_V|\cdot|^{-1/2}) \circ \nu).$$

On the other hand, by Proposition 1.1, 1.2 in [Kudla et al. 1992] and Proposition 4.5, 4.7, any irreducible constituent of $\mathrm{Ind}_{P(\boldsymbol{A})}^{G(\boldsymbol{A})}((\chi_V|\cdot|^{-1/2}) \circ \nu)$ is written by the form $\bigotimes_v R(\widetilde{V}_v)$ where $\{\widetilde{V}_v\}_v$ is a collection of $(-1)$-hermitian right $D_v$-modules with rank one and $\det \widetilde{V}_v = \det V_v$ for all $v$. Therefore Proposition 4.14 and Theorem 4.13 conclude that the representations obtained by the images of $N(w_0, (\chi_V|\cdot|_{\boldsymbol{A}}^{1/2}) \circ \nu)$ are exhausted by the representations given by Theorem 4.1(3).

## 5. Arthur parameters for residual spectrum

Here we give an expectation about the Arthur parameters for the residual spectrum of $G$. The dual group $\widehat{G}$ of $G$ equals $SO(5, \mathbb{C})$, which we realize as

$$\widehat{G} = \left\{ g \in \mathrm{SL}(5, \mathbb{C}) \;\middle|\; g \begin{pmatrix} & & & & 1 \\ & & & -1 & \\ & & 1 & & \\ & -1 & & & \\ 1 & & & & \end{pmatrix} {}^t g = \begin{pmatrix} & & & & 1 \\ & & & -1 & \\ & & 1 & & \\ & -1 & & & \\ 1 & & & & \end{pmatrix} \right\}.$$

For each $v$ we denote by $W_{k_v}$ the Weil group of $k_v$ and write ${}^L G_v$ for the $L$-group $\widehat{G} \times W_{k_v}$ of $G_v = G \otimes_k k_v$. We write $\mathscr{L}_v$ for the Langlands group introduced in

[Kottwitz 1984]:

$$\mathcal{L}_v := \begin{cases} W_{k_v} & \text{if } k_v \text{ is archimedean,} \\ W_{k_v} \times SU(2,\mathbb{R}) & \text{if } k_v \text{ is nonarchimedean.} \end{cases}$$

A local Arthur parameter for $G$ is a continuous homomorphism

$$\psi : \mathcal{L}_v \times \mathrm{SL}(2,\mathbb{C}) \to {}^L G_v$$

such that (i) $\psi|_{W_{k_v}}$ is semisimple and has bounded image, (ii) the composition

$$W_{k_v} \xrightarrow{\psi} {}^L G_v \xrightarrow{\mathrm{pr}_2} W_{k_v}$$

is the identity, and (iii) $\psi$ restricted to $\mathrm{SL}(2,\mathbb{C})$ or $SU(2) \times \mathrm{SL}(2,\mathbb{C})$ is analytic.

**Expectation 5.1.** Let $\pi = \bigotimes_v \pi_v$ be a residual discrete representation appearing in Theorem 4.1. The Arthur parameter $\psi_{\pi_v}$ associated to $\pi_v$ is given by the following.

(1) For $\mathbf{1}_{G(\boldsymbol{A})}$ we have $\psi_{\mathbf{1}_{G(\boldsymbol{A})}}|_{\mathcal{L}_v} = 1_5 \times \mathrm{pr}_{W_{k_v}}$ and

$$\psi_{\mathbf{1}_{G(\boldsymbol{A})}}\left(\begin{pmatrix} a & x \\ y & -a \end{pmatrix}\right) = \begin{pmatrix} 4a & x & & & \\ 2y & 2a & x & & \\ & 2y & 0 & x & \\ & & 2y & -2a & x \\ & & & 2y & -4a \end{pmatrix} \in \mathrm{Lie}\widehat{G}.$$

(2) For $J_P^G(\pi)$ we have

$$\psi_{J_P^G(\pi)}|_{\mathcal{L}_v} = \left(\begin{array}{c|c|c} \varphi_{\pi_v} & & \\ \hline & 1 & \\ \hline & & \mathrm{Ad}(J)^t\varphi_{\pi_v}^{-1} \end{array}\right) \times \mathrm{pr}_{W_{k_v}}$$

and

$$\psi_{J_P^G(\pi)}\left(\begin{pmatrix} a & x \\ y & -a \end{pmatrix}\right) = \left(\begin{array}{cc|c|cc} a & & & x & \\ & a & & & x \\ \hline & & 0 & & \\ \hline y & & & -a & \\ & y & & & -a \end{array}\right) \in \mathrm{Lie}\widehat{G}.$$

Here $\varphi_{\pi_v}$ is the Langlands parameter for $\pi_v$. Since $\pi_v$ is self-dual the image of $\varphi_{\pi_v}$ is contained in $\mathrm{SL}(2,\mathbb{C}) \times W_{k_v}$.

(3) For $R(V)$ we have

$$\psi_{R(V_v)}|_{\mathcal{L}_v} = \begin{pmatrix} \chi_{V_v} & & & & \\ & c_1 & & c_2 & \\ & & \chi_{V_v} & & \\ & c_3 & & c_4 & \\ & & & & \chi_{V_v} \end{pmatrix} \times \mathrm{pr}_{W_{k_v}}$$

and

$$\psi_{R(V_v)}\left(\begin{pmatrix} a & x \\ y & -a \end{pmatrix}\right) = \left(\begin{array}{cc|c|cc} 2a & 0 & x & & \\ 0 & 0 & 0 & & \\ \hline 2y & 0 & 0 & 0 & -x \\ \hline & & 0 & 0 & 0 \\ & & -2y & 0 & -2a \end{array}\right) \in \mathrm{Lie}\widehat{G},$$

where

$$\begin{pmatrix} c_1(w) & c_2(w) \\ c_3(w) & c_4(w) \end{pmatrix} = \begin{cases} 1_2 & \text{if } w \in W_{k'_v}, \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \text{if } w \in W_{k_v} \backslash W_{k'_v}, \end{cases}$$

for the quadratic extension $k'$ of $k$ attached to $\chi_V$.

**Remark 5.2.** The (global) Arthur parameter of $\mathbf{1}_{G(\boldsymbol{A})}$ should coincide with that of $\mathbf{1}_{\mathrm{Sp}(2,\boldsymbol{A})}$. For $J_P^G(\pi)$ the unique irreducible quotient $J_{P_1}^{\mathrm{Sp}(2)}(JL(\pi))$ of

$$\mathrm{Ind}_{P_1(\boldsymbol{A})}^{\mathrm{Sp}(2,\boldsymbol{A})}(JL(\pi)|\det|_{\boldsymbol{A}}^{1/2})$$

occurs in the residual spectrum of Sp(2) where $P_1$ is the Siegel parabolic subgroup of Sp(2) and $JL(\pi)$ is the Jacquet–Langlands correspondence of $\pi$ [Kim 1995]. The Arthur parameters of $J_P^G(\pi)$ and $J_{P_1}^{\mathrm{Sp}(2)}(JL(\pi))$ should coincide. For $R(V)$ the theta lift of the trivial representation of the orthogonal group of a 2-dimensional quadratic space with determinant $\det V$ occurs in the residual spectrum of Sp(2) [Kon-no 1994] and its Arthur parameter should coincide with that of $R(V)$.

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