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We show that the notion of asymptotic lift generalizes naturally to normal positive maps $\phi : M \rightarrow M$ acting on von Neumann algebras M . We focus on cases in which the domain of the asymptotic lift can be embedded as an operator subsystem $M_\infty \subseteq M$ and characterize when M_∞ is a Jordan subalgebra of M in terms of the asymptotic multiplicative properties of ϕ .

1. Introduction

Let $\phi : M \rightarrow M$ be a normal unit-preserving positive linear map acting on a dual operator system M ; we refer to such a pair (M, ϕ) as a *UP map*. While we are primarily interested in UP maps that act on von Neumann algebras M , it is useful to broaden the context as above. The powers of ϕ form an irreversible dynamical semigroup $\{\phi^n : n = 0, 1, 2, \dots\}$ acting on M . In this paper we generalize work begun in [Arveson 2004; 2007], together with complementary results in [Størmer 2007], to further develop the asymptotic theory of such semigroups.

One may view UP maps as the objects of a category, in which a homomorphism from $\phi_1 : M_1 \rightarrow M_1$ to $\phi_2 : M_2 \rightarrow M_2$ is a UP map $E : M_1 \rightarrow M_2$ such that $E \circ \phi_1 = \phi_2 \circ E$. There is a natural notion of isomorphism in this category. Our first general result is that every normal unit-preserving positive linear map acting on a dual operator system has an asymptotic lift which is unique up to isomorphism. This generalizes one of the main results of [Arveson 2007], which dealt with the subcategory in which the objects are unital normal *completely* positive maps (UCP maps) on dual operator systems, with UCP maps as morphisms.

We are primarily concerned with UP maps that act on von Neumann algebras M . In [Arveson 2007], it was shown that the asymptotic lift (N, α, E) of a UCP map $\phi : M \rightarrow M$ acting on a von Neumann algebra M also acts on a von Neumann algebra N . Moreover, it was shown that the W^* -dynamical system (N, α) can be identified with the tail flow of the minimal dilation of ϕ to a $*$ -endomorphism of a larger von Neumann algebra in most cases — namely, those in which the dilation endomorphism has trivial kernel, which includes all UCP maps acting on factors M . Since the minimal dilation of a UCP map on a von Neumann algebra can be

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constructed explicitly in principle, that provided a concrete identification of the asymptotic lift.

It is significant that the asymptotic lift (N, α, E) of a UP map acting on a von Neumann algebra need not act on a von Neumann algebra N . In this paper we show that, in general, N is order-isomorphic to a unique JW^* -algebra — namely, a dual operator system that is closed under the Jordan product $x \circ y = (xy + yx)/2$ — so that α becomes a Jordan automorphism of N . Thus, *the asymptotic behavior of a UP map on a von Neumann algebra is always associated with a JW^* -dynamical system (N, α) .*

Naturally, one would like to identify (N, α) more concretely in terms of the asymptotic properties of the semigroup $\phi, \phi^2, \phi^3, \dots$. But since there is no dilation theory for semigroups of UP maps — and perhaps an effective dilation theory is impossible — there seems to be no candidate to replace the “tail flow” identification described above for UCP maps. Thus, this identification problem becomes a significant issue for UP maps.

In many cases, N can be embedded as an operator system $N \subseteq M$ so that $\alpha = \phi \upharpoonright_N$. While the operator system N is order-isomorphic to a JW^* -algebra in general, it need not be closed under the ambient Jordan multiplication of M , and we address that issue in [Section 4](#). We identify N concretely in terms of the action of ϕ on M , and when N is a Jordan subalgebra of M , we are able to go farther by identifying N with the *multiplicative core* of ϕ introduced in [[Størmer 2007](#)].

The problem of finding a satisfactory concrete description of the JW^* -dynamical system (N, α) when N does *not* embed in M remains untouched.

2. Asymptotic lifts of positive maps

In this section we describe how the notion of asymptotic lift (of a completely positive map) can be generalized to normal unit-preserving maps that are merely positive. We summarize the basic properties of asymptotic lifts, indicating briefly how proofs of [[Arveson 2007](#)] should be modified.

Remark 2.1 (Dual operator systems, norm, and order). Every von Neumann algebra is the dual of a unique Banach space, and hence it carries a natural weak* topology (also known as the ultraweak or σ -weak topology). A *dual operator system* is a linear space of operators $M \subseteq \mathcal{B}(H)$ that contains the identity $\mathbf{1}$, is self-adjoint $M^* = M$, and is closed in the weak*-topology of $\mathcal{B}(H)$. Such an M also has a unique predual M_* , and its intrinsic M_* -topology coincides with the relative weak*-topology of $\mathcal{B}(H)$. A map of dual operator systems $\phi : M \rightarrow N$ is called *normal* when it is weak*-continuous. There is an intrinsic characterization of dual operator systems, which we do not require.

Let us recall the basic properties of unit-preserving normal positive linear maps $\phi : M \rightarrow N$ (UP maps) between dual operator systems. A UP map $\phi : M \rightarrow N$ defined on a C^* -algebra M must satisfy $\|\phi\| = 1$. That is most easily seen by making use of the Russo–Dye theorem [1966] which implies that $\|\phi\| = \sup \|\phi(u)\|$, where u ranges over the unitary group of M , together with the fact that for every unitary u , the restriction of ϕ to the commutative C^* -algebra $C^*(u)$ is completely positive and therefore satisfies the strong Schwarz inequality $\phi(u)^* \phi(u) \leq \phi(u^*u) = \phi(\mathbf{1}_M) = \mathbf{1}_N$. More generally, if M is merely an operator system, then one has $1 \leq \|\phi\| \leq 2$ in general, but the C^* -algebraic upper bound $\|\phi\| = 1$ often fails.

On the other hand, since the norm of a self-adjoint operator x is the smallest $\alpha \geq 0$ such that $-\alpha \mathbf{1} \leq x \leq \alpha \mathbf{1}$, the restriction of ϕ to the real Banach space M^{sa} of self-adjoint elements of M has norm 1. Conversely, if a linear map $\phi : M \rightarrow N$ carries self-adjoint elements to self-adjoint elements, maps $\mathbf{1}_M$ to $\mathbf{1}_N$, and satisfies $\|\phi \upharpoonright_{M^{\text{sa}}}\| = 1$, then ϕ must also preserve positivity. In particular, *an order isomorphism in the category of UP maps is characterized as a $*$ -preserving linear map $\phi : M \rightarrow N$ of operator systems that carries $\mathbf{1}_M$ to $\mathbf{1}_N$ and restricts to an isometry of M^{sa} onto N^{sa} .*

Definition 2.2. A *reversible lift* of a UP map $\phi : M \rightarrow M$ is a triple (N, α, E) consisting of a UP automorphism $\alpha : N \rightarrow N$ of another dual operator system N and a UP map $E : N \rightarrow M$ satisfying $E \circ \alpha = \phi \circ E$.

A reversible lift (N, α, E) of ϕ is said to be *nondegenerate* if

$$(2-1) \quad E(\alpha^{-n}(y)) = 0 \text{ for } n = 0, 1, 2, \dots \quad \text{implies } y = 0.$$

Significantly, if (2-1) fails, one can replace (N, α, E) with another reversible lift $(\tilde{N}, \tilde{\alpha}, \tilde{E})$ that is *nondegenerate*, as in [Arveson 2007, Remark 2.3].

Definition 2.3. Let $\phi : M \rightarrow M$ be a UP map on a dual operator system. An *asymptotic lift* of ϕ is a reversible lift (N, α, E) of ϕ that satisfies nondegeneracy (2-1) together with

$$(2-2) \quad \|\rho \circ E \upharpoonright_{N^{\text{sa}}}\| = \lim_{k \rightarrow \infty} \|\rho \circ \phi^k \upharpoonright_{M^{\text{sa}}}\| \quad \text{for } \rho \in M_*.$$

Remark 2.4. We shall make use of the dual formulation of (2-2), and we record that now for later reference: For every nondegenerate reversible lift (N, α, E) of ϕ , (2-2) is equivalent to the assertion

$$(2-3) \quad E(\text{ball}_r N^{\text{sa}}) = \bigcap_{n=1}^{\infty} \phi^n(\text{ball}_r M^{\text{sa}}) \quad \text{for } r > 0,$$

where $\text{ball}_r X$ denotes the closed ball of radius r in a real or complex Banach space X . The proof that (2-2) and (2-3) are equivalent follows along the lines of the proof of the corresponding result in [Arveson 2007].

There are two fundamental results on asymptotic lifts of UP maps. The first concerns existence and uniqueness:

Theorem 2.5. *Every UP map $\phi : M \rightarrow M$ of a dual operator system has an asymptotic lift. If (N_1, α_1, E_1) and (N_2, α_2, E_2) are two asymptotic lifts for ϕ , then there is a unique UP-isomorphism of dual operator systems $\theta : N_1 \rightarrow N_2$ such that $\theta \circ \alpha_1 = \alpha_2 \circ \theta$ and $E_2 \circ \theta = E_1$.*

As in the case of completely positive maps, the existence issue is settled by a direct construction based on *inverse sequences*, namely bounded bilateral sequences (x_n) of elements of M that satisfy $x_n = \phi(x_{n+1})$ for $n \in \mathbb{Z}$. The space of all inverse sequences is a dual operator system N , the bilateral shift $\alpha : (x_n) \mapsto (x_{n-1})$ is an automorphism of N , and the connecting map E carries (x_n) to x_0 . The proof that (N, α, E) is an asymptotic lift is a minor (and somewhat simpler) variation of the proof of the corresponding result in [Arveson 2007], and, similarly, the proof of uniqueness is a straightforward variation of arguments therein; we omit the details.

If a UP map $\phi : M \rightarrow M$ on a von Neumann algebra M is completely positive, then its asymptotic lift (N, α, E) gives rise to a W^* -dynamical system (N, α) [Arveson 2007]. While this need not be true for asymptotic lifts of UP maps, one can assert in general:

Theorem 2.6. *Let $\phi : M \rightarrow M$ be a UP map on a von Neumann algebra M and let (N, α, E) be its asymptotic lift. Then N is order isomorphic to a unique JW^* -algebra such that α is a Jordan automorphism of N .*

Again, the proof follows along the lines of arguments in [Arveson 2007], by introducing a Jordan multiplication on the range of a positive idempotent map as in [Effros and Størmer 1979, Corollary 1.6]. The key step shows that for the constructed asymptotic lift (N, α, E) in which N is the space of inverse sequences, there is a projection $Q : \ell^\infty(M) \rightarrow N$ of norm one. The existence of such a projection follows from the argument of [Arveson 2007]. Once one has a positive projection (which in the current setting is typically not *completely* positive) with these properties, one can introduce a Jordan product in N by way of

$$x \circ y = Q\left(\frac{1}{2}(xy + yx)\right) \quad \text{for } x, y \in N,$$

and at this point one can show that N is order isomorphic to a JW^* -algebra by imitating arguments in [Arveson 2007].

3. Embeddable asymptotic lifts

Let (N, α, E) be the asymptotic lift of a UP map $\phi : M \rightarrow M$ acting on a dual operator system. In this section we fix attention on those cases in which the asymptotic lift can be embedded as a subsystem of M in the following particular way. We say

that (N, α, E) embeds in M if E restricts to an isometry on the self-adjoint part of N :

$$(3-4) \quad \|E(y)\| = \|y\| \quad \text{for } y = y^* \in N.$$

This is equivalent to the claim that E implements an order isomorphism of N onto $E(N)$. Since all asymptotic lifts of $\phi : M \rightarrow M$ are isomorphic, this definition does not depend on the particular choice of (N, α, E) . In such cases, the range $E(N)$ of E is an operator subsystem of M with the property that ϕ restricts to an automorphism of $E(N)$, α is identified with $\phi \upharpoonright_{E(N)}$, and E is identified with the inclusion map $\iota : E(N) \subseteq M$.

The purpose of this section is to precisely summarize those facts and, especially, to explicitly describe $E(N)$ in terms of the action of ϕ on M . We begin by introducing an operator system $M_\infty \subseteq M$:

$$M_\infty = \bigcup_{r>0} \bigcap_{n=1}^{\infty} \phi^n(\text{ball}_r M).$$

M_∞ consists of all $y \in M$ for which there is a *bounded* sequence $x_n \in M$ with $y = \phi^n(x_n)$ for $n = 1, 2, \dots$. In general, M_∞ is a self-adjoint linear subspace of M containing the identity operator, it is invariant under ϕ , and in fact $\phi(M_\infty) = M_\infty$. Moreover, from (2-3) we may infer that

$$(3-5) \quad E(N^{\text{sa}}) = \bigcup_{r>0} E(\text{ball}_r N^{\text{sa}}) = \bigcup_{r>0} \bigcap_{n=1}^{\infty} \phi^n(\text{ball}_r M^{\text{sa}}) = M_\infty^{\text{sa}};$$

hence $M_\infty = E(N)$ is precisely the range of E in all cases. We sometimes refer to M_∞ as the *tail operator system* of M .

Remark 3.1. It is clear that $M_\infty \subseteq \bigcap_n \phi^n(M)$, but the inclusion is typically proper. As a simple example, let $M = M_2(\mathbb{C})$ and consider the completely positive map $\phi : M \rightarrow M$ defined by

$$\phi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & \lambda b \\ \lambda c & d \end{pmatrix},$$

where λ is a constant satisfying $0 < \lambda < 1$. One has $\bigcap_n \phi^n(M) = M$, but in this case, M_∞ is the two-dimensional subalgebra of diagonal matrices.

Proposition 3.2. *Let (N, α, E) be an asymptotic lift of $\phi : M \rightarrow M$. Then $\ker E = \{0\}$ if and only if the restriction of ϕ to M_∞ is both injective and surjective.*

Proof. In general, one has $\phi(M_\infty) = M_\infty$ by definition of M_∞ . Assuming that $\ker E = \{0\}$, choose $a \in M_\infty$ such that $\phi(a) = 0$. Since $E(N) = M_\infty$, there is a $y \in N$ such that $a = E(y)$; hence $0 = \phi(a) = \phi(E(y)) = E(\alpha(y))$, and therefore

$\alpha(y) = 0$ because E is injective. $y = 0$ follows because α is an automorphism of N ; hence $a = E(y) = 0$.

Conversely, if the restriction of ϕ to M_∞ is injective, choose a nonzero element $y \in N$. Since the norms $\|E(\alpha^{-n}(y))\|$ increase to $\|y\|$ as $n \uparrow \infty$ (see [Arveson 2007, Lemma 3.7]), we must have $E(\alpha^{-n}(y)) \neq 0$ for sufficiently large $n \geq 1$. Since each power ϕ^n restricts to an injective map on $M_\infty = E(N)$, it follows that

$$E(y) = \phi^n(E(\alpha^{-n}(y))) \neq 0$$

for large n ; hence E is injective. \square

We conclude that whenever $\ker E = \{0\}$, the restriction of ϕ to M_∞ defines an order-preserving linear bijection on M_∞ . However, M_∞ itself need not be closed in any topology in general, and the restriction of ϕ to M_∞ need not be an order automorphism. The following result implies that when E restricts to an isometry on N^{sa} , such anomalies cannot occur.

Theorem 3.3. *For every UP map $\phi : M \rightarrow M$ on a dual operator system, these are equivalent:*

- (i) *The asymptotic lift of $\phi : M \rightarrow M$ embeds in M .*
- (ii) *M_∞ is weak*-closed, and ϕ restricts to an order automorphism of it.*
- (iii) *The asymptotic lift of $\phi : M \rightarrow M$ is isomorphic to the triple $(M_\infty, \phi \upharpoonright_{M_\infty}, \iota)$, where $\iota : M_\infty \subseteq M$ is the inclusion map.*

Proof. Let (N, α, E) be an asymptotic lift of $\phi : M \rightarrow M$.

(i) \implies (ii): By hypothesis, E restricts to an isometry of N^{sa} onto the self-adjoint part of $E(N) = M_\infty$. Since M_∞ is a self-adjoint linear space of operators and the adjoint operation is weak*-continuous, we will know M_∞ is weak*-closed if we can show that its self-adjoint part M_∞^{sa} is weak*-closed. By a standard result on the weak*-closure of convex sets in dual Banach spaces, this will follow if we show that for every $r > 0$, the intersection $M_\infty^{\text{sa}} \cap \text{ball}_r M$ of M_∞^{sa} with the r -ball of M is weak*-closed. Now the connecting map $E : N \rightarrow M$ restricts to a weak*-continuous isometry of N^{sa} onto M_∞^{sa} , and by (2-3), it carries $\text{ball}_r N^{\text{sa}}$ onto $M_\infty^{\text{sa}} \cap \text{ball}_r M$. Since the unit ball of N is weak*-compact, it follows that $M_\infty^{\text{sa}} \cap \text{ball}_r M = E(\text{ball}_r N^{\text{sa}})$ is weak*-compact and hence weak*-closed. We conclude that M_∞^{sa} is weak*-closed.

Making use of Proposition 3.2, let ψ be the linear automorphism of M_∞ inverse to $\phi \upharpoonright_{M_\infty}$. Then since $E \circ \alpha = \phi \circ E$, we have $E \circ \alpha^{-1} = \psi \circ E$. Since both E and α^{-1} restrict to isometries on N^{sa} , it follows that ψ restricts to an isometry on $M_\infty^{\text{sa}} = E(N^{\text{sa}})$. Since $\psi(\mathbf{1}) = \mathbf{1}$, ψ must also be order preserving, and we conclude that $\phi \upharpoonright_{M_\infty}$ is an order automorphism.

(ii) \implies (iii): Under the hypothesis (ii), $(M_\infty, \phi \upharpoonright_{M_\infty}, \iota)$ becomes a reversible lift of $\phi : M \rightarrow M$ which is obviously nondegenerate. We show that it is the asymptotic lift of $\phi : M \rightarrow M$ by establishing (2-2). For that, consider the decreasing sequence of weak*-compact sets $\phi^n(\text{ball } M^{\text{sa}})$ for $n = 1, 2, \dots$, and choose a normal linear functional $\rho \in M_*$. By (2-3), we have

$$\bigcap_{n=1}^{\infty} \phi^n(\text{ball } M^{\text{sa}}) = E(\text{ball } N^{\text{sa}}).$$

Since ρ is weak*-continuous, it follows from Lemma 3.5 of [Arveson 2007]) that

$$\begin{aligned} \|\rho \circ \phi^n \upharpoonright_{M_\infty^{\text{sa}}}\| &= \sup\{\rho(x) : x \in \phi^n(\text{ball } M^{\text{sa}})\} \downarrow \sup\{\rho(x) : x \in E(\text{ball } N^{\text{sa}})\} \\ &= \|\rho \circ E \upharpoonright_{N^{\text{sa}}}\| \end{aligned}$$

as $n \uparrow \infty$, and (iii) follows.

The implication (iii) \implies (i) is obvious. \square

Taken together, Theorems 3.3 and 2.6 imply:

Corollary 3.4. *Let $\phi : M \rightarrow M$ be a UP map on a von Neumann algebra M whose asymptotic lift embeds in M . Then the tail operator system M_∞ is order-isomorphic to a JW^* -algebra such that the restriction of ϕ to M_∞ becomes a Jordan automorphism, and the asymptotic lift of ϕ is the triple $(M_\infty, \phi \upharpoonright_{M_\infty}, \iota)$, where ι is the inclusion map $\iota : M_\infty \subseteq M$.*

The asymptotic lifts of many UP maps are embeddable. That is true for all examples covered by the hypotheses of [Arveson 2004], and in Section 5 below, we elaborate on the special case of maps on finite-dimensional algebras. Not all infinite-dimensional UP maps are embeddable, as the next example illustrates.

Example 3.5. Let $L^2 = L^2(\mathbb{T}, d\theta/(2\pi))$, let $H^2 \subseteq L^2$ be the usual Hardy space, and let $s \in \mathcal{B}(H^2)$ be the unilateral shift. Consider the UCP map ϕ defined on $M = \mathcal{B}(H^2)$ by

$$\phi(x) = s^* x s \quad \text{for } x \in \mathcal{B}(H^2).$$

Let $u \in \mathcal{B}(L^2)$ be the bilateral shift and let p be the projection of L^2 on H^2 . Then the asymptotic lift of ϕ is the triple $(\mathcal{B}(L^2), \alpha, E)$, where $\alpha(y) = u^* y u$, and E is the compression map $E(y) = p y \upharpoonright_{H^2}$. Since $\ker E \neq \{0\}$, the asymptotic lift of ϕ is not embeddable. We omit the calculations.

4. Embedding and the multiplicative core

If the asymptotic lift of a UP map $\phi : M \rightarrow M$ acting on a von Neumann algebra embeds in M , then by Corollary 3.4, M_∞ is order isomorphic to a JW^* -algebra, and the restriction of ϕ to M_∞ becomes a Jordan automorphism of M_∞ . Notice

that this does not imply that M_∞ itself is closed under the Jordan product $x \circ y = (xy + yx)/2$ inherited from M . [Section 5](#) contains a simple example that illustrates the phenomenon.

In the present section, we study the Jordan structure of M_∞ in more detail for the case when the asymptotic lift embeds in M . In particular, we show that if M_∞ is itself a JW^* -algebra, then it coincides with the *multiplicative core* C_ϕ of ϕ introduced in [[Størmer 2007](#)]. We also give necessary and sufficient conditions for $(C_\phi, \phi \upharpoonright_{C_\phi}, \iota)$ to be the asymptotic lift.

Following [Størmer](#), the *definite set* of a UP map $\phi : M \rightarrow M$ is defined by

$$(4-6) \quad M_\phi = \{x \in M : \phi(x^* \circ x) = \phi(x)^* \circ \phi(x)\};$$

it is a JW^* -algebra, and we have

$$\phi(x \circ y) = \phi(x) \circ \phi(y) \quad \text{for } x \in M_\phi \text{ and } y \in M.$$

Continuing to follow [Størmer](#), one can show that

$$B_\phi = \{x \in M : \phi^n(x) \in M_\phi, \quad n = 0, 1, 2, \dots\}$$

is a JW^* -algebra on which ϕ restricts to a Jordan endomorphism, and one forces surjectivity on ϕ by restricting it to the “tail” subalgebra

$$C_\phi = \bigcap_{n=1}^{\infty} \phi^n(B_\phi).$$

C_ϕ is called the *multiplicative core* of ϕ . We require the following variation of [[Størmer 2007](#), Lemma 6].

Lemma 4.1. *The multiplicative core is the largest JW^* -subalgebra $N \subseteq M$ with*

- (i) $\phi(N) = N$, and
- (ii) $\phi(x \circ y) = \phi(x) \circ \phi(y)$ for all $x, y \in N$.

Proof. Let N be a JW^* -subalgebra of M with properties (i) and (ii), and choose $x \in N$. Then we have

$$\phi(\phi^n(x)^* \circ \phi^n(x)) = \phi^{n+1}(x)^* \circ \phi^{n+1}(x) \quad \text{for } n \geq 0,$$

because ϕ is a Jordan endomorphism of N ; hence $N \subseteq B_\phi$. It follows that $N = \phi^n(N) \subseteq \phi^n(B_\phi)$ for every $n \geq 0$, so that $N \subseteq \bigcap_n \phi^n(B_\phi) = C_\phi$. \square

The following result associates the multiplicative core with asymptotic lifts that are embeddable as Jordan subalgebras:

Theorem 4.2. *For every UP map $\phi : M \rightarrow M$ acting on a von Neumann algebra M , the tail operator system M_∞ contains the multiplicative core C_ϕ .*

Assume that the asymptotic lift embeds in M . Then M_∞ is a Jordan subalgebra of M precisely when $M_\infty = C_\phi$.

Proof. By definition, $C_\phi = \bigcap_n \phi^n(B_\phi)$, and ϕ restricts to a Jordan endomorphism of B_ϕ . Since each power ϕ^n of ϕ restricts to a Jordan endomorphism on B_ϕ and a Jordan homomorphism maps the unit ball of the self-adjoint part of its domain onto the unit ball of the self-adjoint part of its range, we have $\phi^n(\text{ball}_r B_\phi^{\text{sa}}) = \text{ball}_r(\phi^n(B_\phi)^{\text{sa}})$ for every $r > 0$ and every $n = 1, 2, \dots$; hence

$$\text{ball}_r C_\phi^{\text{sa}} = \bigcap_{n=1}^{\infty} \text{ball}_r \phi^n(B_\phi^{\text{sa}}) = \bigcap_{n=1}^{\infty} \phi^n(\text{ball}_r B_\phi^{\text{sa}}) \subseteq M_\infty.$$

The asserted inclusion $C_\phi = C_\phi^{\text{sa}} + iC_\phi^{\text{sa}} \subseteq M_\infty$ follows after taking the union over all positive r .

Assume that the asymptotic lift of $\phi : M \rightarrow M$ embeds in M . If M_∞ is closed under the Jordan multiplication of M , then by [Theorem 3.3](#)(iii), $\phi \upharpoonright_{M_\infty}$ is an order automorphism of M_∞ , and an application of Kadison's Schwarz inequality for positive linear maps implies that ϕ is a Jordan automorphism of M_∞ . By [Lemma 4.1](#), $M_\infty \subseteq C_\phi$, and we conclude that $M_\infty = C_\phi$. The converse is trivial. \square

It is significant that when M_∞ is not itself closed under the Jordan product, it contains a largest JW*-algebra that is characterized as follows.

Proposition 4.3. *Let $\phi : M \rightarrow M$ be a UP map whose asymptotic lift embeds in M . Then the weak*-closed linear span of all projections in M_∞ is a JW*-algebra.*

Proof. By [Corollary 3.4](#), there are a JW*-algebra B and an order isomorphism α of B onto M_∞ . Viewing α as a UP map of B into the JW*-algebra M , it makes sense to speak of its definite set B_α . Since B_α is a JW*-subalgebra of B and the restriction of α to B_α is a Jordan homomorphism, it follows that $\alpha(B_\alpha) \subseteq M_\infty$ is a JW*-subalgebra of M .

Being a JW*-algebra, $\alpha(B_\alpha)$ is generated by its projections. Conversely, we claim that every projection $e \in M_\infty$ belongs to $\alpha(B_\alpha)$. To see that, fix such an e and choose $f \in B$ such that $\alpha(f) = e$. Then $0 \leq f \leq \mathbf{1}$ because α is a UP order isomorphism, so, by Kadison's Schwarz inequality,

$$0 = e - e^2 = \alpha(f) - \alpha(f)^2 \geq \alpha(f - f^2) \geq 0,$$

hence $\alpha(f - f^2) = 0$ and finally $f = f^2$. Then $\alpha(f^2) = \alpha(f)^2$ shows f belongs to the definite set of α ; hence $e = \alpha(f) \in \alpha(B_\alpha)$. \square

We conclude the section by describing intrinsic conditions on a UP map which imply that its asymptotic lift is embeddable as the multiplicative core; note that the sufficient conditions of [Theorem 4.5](#) are clearly necessary as well.

For a linear functional ρ on M and a set of operators $S \subseteq M$, we write $\rho \perp S$ whenever $\rho(S) = \{0\}$. We require the following result characterizing the equality $M_\infty = C_\phi$ in terms of the action of ϕ on the predual of M :

Lemma 4.4. *Let $\phi : M \rightarrow M$ be a UP map. Then $M_\infty = C_\phi$ if and only if for every $\rho \in M_*$ satisfying $\rho \perp C_\phi$, one has*

$$(4-7) \quad \lim_{n \rightarrow \infty} \|\rho \circ \phi^n\| = 0.$$

Proof. We claim first that $\rho \in M_*$ satisfies (4-7) if and only if $\rho \perp M_\infty$. Indeed, setting $M_\infty(r) = \bigcap_n \phi^n(\text{ball}_r M)$ for $r > 0$, and noting that the compact convex sets $\phi^n(\text{ball}_r M)$ decrease to $M_\infty(r)$ as $n \uparrow \infty$, we can apply [Arveson 2007, Lemma 3.5] to conclude that (4-7) is equivalent to the assertion

$$\begin{aligned} \sup\{|\rho(x)| : x \in M_\infty(r)\} &= \lim_{n \rightarrow \infty} \sup\{|\rho(x)| : x \in \phi^n(\text{ball}_r M)\} \\ &= r \cdot \lim_{n \rightarrow \infty} \|\rho \circ \phi^n\| = 0 \end{aligned}$$

for every $r > 0$. Noting that $M_\infty = \bigcup_{r>0} M_\infty(r)$, the claim follows.

The preceding paragraph, together with a standard separation theorem, shows that the assertion

$$(4-8) \quad \rho \perp C_\phi \implies \lim_{n \rightarrow \infty} \|\rho \circ \phi^n\| = 0$$

is equivalent to the claim $\overline{M_\infty}^{\text{w}*} \subseteq C_\phi$. Since we generally have $C_\phi \subseteq M_\infty \subseteq \overline{M_\infty}^{\text{w}*}$, (4-8) is equivalent to $C_\phi = M_\infty$. \square

Theorem 4.5. *Let $\phi : M \rightarrow M$ be a UP map such that*

- (i) *The positive linear map obtained by restricting ϕ to C_ϕ is faithful.*
- (ii) *$\lim_{n \rightarrow \infty} \|\rho \circ \phi^n\| = 0$ for every $\rho \in M_*$ satisfying $\rho \perp C_\phi$.*

Then $M_\infty = C_\phi$, the restriction of ϕ to C_ϕ is a Jordan automorphism of C_ϕ , and the asymptotic lift of ϕ is $(C_\phi, \phi \upharpoonright_{C_\phi}, \iota)$.

Proof. Hypothesis (i) implies that the restriction of ϕ to C_ϕ is a Jordan automorphism. Thus, the triple $(C_\phi, \phi \upharpoonright_{C_\phi}, \iota)$ is a nondegenerate reversible lift of ϕ . To show it is the asymptotic lift, we must establish for every $\rho \in M_*$ the inequality

$$(4-9) \quad \lim_{n \rightarrow \infty} \|\rho \circ \phi^n\| \leq \|\rho \upharpoonright_{C_\phi}\|.$$

For that, fix ρ , and, for every $n = 1, 2, \dots$, choose an element $x_n \in M$ satisfying $\|x_n\| = 1$ and $|\rho(\phi^n(x_n))| = \|\rho \circ \phi^n\|$. We can find a subsequence $n_1 < n_2 < \dots$ such that $\phi^{n_k}(x_{n_k})$ converges to $y \in M$ weak* as $k \rightarrow \infty$. Since the weak*-compact sets $\phi^n(\text{ball } M)$ decrease with increasing n , y must belong to their intersection $\bigcap_n \phi^n(\text{ball } M) = M_\infty$, and of course $\|y\| \leq 1$.

Making use of hypothesis (ii) and [Lemma 4.4](#), we conclude that $M_\infty = C_\phi$, so that $y \in \text{ball } C_\phi$. Hence

$$\lim_{n \rightarrow \infty} \|\rho \circ \phi^n\| = \lim_{k \rightarrow \infty} \|\rho \circ \phi^{n_k}\| = \lim_{k \rightarrow \infty} |\rho(\phi^{n_k}(x_{n_k}))| = |\rho(y)| \leq \|\rho \upharpoonright_{C_\phi}\|,$$

and the desired inequality [\(4-9\)](#) follows. \square

Remark 4.6. The conclusion of [Theorem 4.5](#) can be significantly strengthened whenever there is a normal positive projection E of M onto C_ϕ — in particular, whenever M is a finite von Neumann algebra. In such cases, a simple argument (which we omit) shows that the automorphism $\alpha = \phi \upharpoonright_{C_\phi}$ of C_ϕ satisfies

$$\lim_{n \rightarrow \infty} \|\rho \circ \phi^n - \rho \circ \alpha^n \circ E\| = 0 \quad \text{for } \rho \in M_*.$$

5. UP maps on finite-dimensional algebras

In this section we show that the asymptotic lift of every UP map acting on a finite-dimensional algebra M embeds in M . We identify M_∞ with the multiplicative core whenever there is a faithful ϕ -invariant state, and more generally, we identify the multiplicative core when ϕ is faithful. We conclude with an elementary example exhibiting nontrivial asymptotic dynamics, for which M_∞ is not closed under the Jordan product of M and hence differs from the multiplicative core.

Theorem 5.1. *Let $\phi : M \rightarrow M$ be a UP map on a finite-dimensional von Neumann algebra. Then $(M_\infty, \phi \upharpoonright_{M_\infty}, \iota)$ is the asymptotic lift of ϕ . If, in addition, for every positive operator $x \in M$, one has*

$$(5-10) \quad \lim_{n \rightarrow \infty} \|\phi^n(x)\| = 0 \quad \text{implies } x = 0,$$

then M_∞ is the multiplicative core C_ϕ . Condition [\(5-10\)](#) is satisfied whenever there is a faithful state ρ of M satisfying $\rho \circ \phi = \rho$.

The proof requires a known elementary result:

Lemma 5.2 [[Effros and Størmer 1979](#)]. *Let M be a unital C^* -algebra and let $E : M \rightarrow M$ be an idempotent UP map that is faithful, that is, $x \in M^+$ and $E(x) = 0$ implies $x = 0$. Then $E(M)$ is a Jordan subalgebra of M .*

Proof. Choose a self-adjoint element $x \in E(M)$. Then $E(E(x^2) - x^2) = E(x^2) - E(x^2) = 0$. By Kadison's Schwarz inequality, $x^2 = E(x)^2 \leq E(x^2)$, so that $E(x^2) - x^2 \geq 0$. Since E is faithful, $E(x^2) = x^2 \in E(M)$. This shows that $E(M)^{\text{sa}}$ is closed under the Jordan product; hence $E(M)$ is a Jordan subalgebra of M . \square

Proof of [Theorem 5.1](#). There is a sequence $n_1 < n_2 < \dots$ of positive integers such that ϕ^{n_k} converges to a unique idempotent E ; this is a result of Kuperberg, see

[Arveson 2004, Theorem 4.1 and what follows]. Note that $E(M) = M_\infty$. Indeed, for every $x \in M$,

$$E(x) = \lim_k \phi^{n_k}(x) \in \overline{\bigcap_n \{\phi^n(x), \phi^{n+1}(x) \dots\}} \subseteq \bigcap_n \phi^n(\text{ball}_{\|x\|} M);$$

hence $E(x) \subseteq M_\infty$. For the opposite inclusion, choose $y \in M_\infty$. Then there is a bounded sequence $x_k \in M$ such that $y = \phi^{n_k}(x_k)$ for every k . Let k' be a subsequence of k such that $x_{k'}$ converges to $x \in M$. Then

$$\|y - \phi^{n_{k'}}(x)\| \leq \|y - \phi^{n_{k'}}(x_{k'})\| + \|\phi^{n_{k'}}(x_{k'}) - \phi^{n_{k'}}(x)\| \leq \|x_{k'} - x\|,$$

and the right side tends to zero as $k' \rightarrow \infty$.

[Arveson 2004] also shows that ϕ restricts to a surjective isometry on M_∞ and that the powers of ϕ tend to zero on $\ker E$, so that

$$\lim_{n \rightarrow \infty} \|\phi^n \circ E - \phi^n\| = 0.$$

It follows that for every bounded linear functional ρ on M , we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \left| \|\rho \upharpoonright_{M_\infty}\| - \|\rho \circ \phi^n\| \right| &= \limsup_{n \rightarrow \infty} \left| \|\rho \circ \phi^n \circ E\| - \|\rho \circ \phi^n\| \right| \\ &\leq \lim_{n \rightarrow \infty} \|\rho \circ \phi^n \circ E - \rho \circ \phi^n\| = 0. \end{aligned}$$

The latter implies that $(M_\infty, \phi \upharpoonright_{M_\infty}, \iota)$ is the asymptotic lift of ϕ .

Assuming that (5-10) is satisfied, we claim that $M_\infty = C_\phi$. By Theorem 4.2, it is enough to show that M_∞ is closed under the Jordan multiplication of M . Note that E must be faithful. Indeed, if $x \geq 0$ and $E(x) = 0$, then

$$\lim_k \|\phi^{n_k}(x)\| = \|E(x)\| = 0,$$

and since the sequence of norms $\|\phi_n(x)\|$ is decreasing, (5-10) implies that $x = 0$. By Lemma 5.2, M_∞ is a Jordan subalgebra of M .

Finally, assuming that there is a faithful ϕ -invariant state ρ , we claim that (5-10) holds. Indeed, if x is a positive operator satisfying $\|\phi^n(x)\| \rightarrow 0$ as $n \rightarrow \infty$, then

$$|\rho(x)| = |\rho \circ \phi^n(x)| \leq \|\phi^n(x)\| \rightarrow 0$$

as $n \rightarrow \infty$. Hence $\rho(x) = 0$, and $x = 0$ follows because ρ is faithful. \square

We conclude with a result which identifies the multiplicative core C_ϕ in many cases where $C_\phi \neq M_\infty$.

Corollary 5.3. *Let $\phi : M \rightarrow M$ be a faithful UP map on a finite-dimensional von Neumann algebra. Then the multiplicative core of ϕ is the linear space spanned by the projections in M_∞ .*

Proof. Let A be the linear span of all projections in M_∞ . We claim that $\phi(A) = A$. Indeed, if e is a projection in A , then, since ϕ is an order automorphism of M_∞ , there is an operator $f \in M_\infty$ such that $0 \leq f \leq \mathbf{1}$ and $\phi(f) = e$. As in the proof of [Proposition 4.3](#), this implies $\phi(f - f^2) = 0$; hence $f = f^2$ is a projection because ϕ is faithful. This implies that $A \subseteq \phi(A)$ and, since A is finite-dimensional, $\phi(A) = A$.

[Proposition 4.3](#) implies that A is the largest Jordan algebra in M_∞ . The multiplicative core C_ϕ is a Jordan algebra in M_∞ by [Theorem 4.2](#); hence $C_\phi \subseteq A$. [Lemma 4.1](#) now implies that $C_\phi = A$. \square

Example 5.4. We conclude by describing an example of a UP map on the 3-dimensional commutative C^* -algebra $M = \mathbb{C}^3$ for which M_∞ is not closed under the ambient Jordan multiplication of M . While there are simpler examples with that specific property, this one exhibits nontrivial asymptotic dynamics that are not detected by the multiplicative core. Viewing the elements of M as column vectors, the map ϕ is multiplication by the stochastic matrix

$$\phi = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The even and odd powers of ϕ are

$$\phi^{2n} = \begin{pmatrix} \frac{1}{9^n} & \frac{1}{2} - \frac{1}{2 \cdot 9^n} & \frac{1}{2} - \frac{1}{2 \cdot 9^n} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \phi^{2n+1} = \begin{pmatrix} \frac{1}{3 \cdot 9^n} & \frac{1}{2} - \frac{1}{6 \cdot 9^n} & \frac{1}{2} - \frac{1}{6 \cdot 9^n} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

and the unique *idempotent* limit point of $\{\phi, \phi^2, \phi^3, \dots\}$ is given by

$$E = \lim_{n \rightarrow \infty} \phi^{2n} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The range of E is the two-dimensional space (written as row vectors)

$$M_\infty = E(M) = \left\{ \left(\frac{a+b}{2}, a, b \right) : a, b \in \mathbb{C} \right\}.$$

We summarize the basic properties of this example without proof: *Relative to the intrinsic (Jordan) multiplication defined by $x \circ y = E(xy)$, the space M_∞ is isomorphic to the two-dimensional commutative C^* -algebra \mathbb{C}^2 . This identification implements a conjugacy of $\phi \upharpoonright_{M_\infty}$ with the order 2 automorphism $(a, b) \mapsto (b, a)$ for $a, b \in \mathbb{C}$. The multiplicative core of ϕ is the one-dimensional C^* -algebra $C_\phi = \mathbb{C} \cdot \mathbf{1}$.*

Remark 5.5 (Invariant states of ϕ). The above UP map $\phi : \mathbb{C}^3 \rightarrow \mathbb{C}^3$ has a unique invariant state, namely

$$\rho(a, b, c) = \frac{b+c}{2} \quad \text{for } (a, b, c) \in \mathbb{C}^3.$$

This state is not faithful of course, so there is no conflict with [Theorem 5.1](#).

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