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EXTREMAL SOLITONS AND EXPONENTIAL C^∞ CONVERGENCE OF THE MODIFIED CALABI FLOW ON CERTAIN $\mathbb{C}P^1$ BUNDLES

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For a certain class of completions of $\mathbb{C}^*$-bundles, we show that the existence of Calabi extremal metrics is equivalent to geodesic stability of the Kähler class, and we prove the exponential C^∞ convergence of the modified Calabi flow whenever the extremal metric exists, assuming that the manifold has hypersurface ends. In particular, we solve the problem of convergence of the modified Calabi flow on the almost homogeneous manifolds with two hypersurface ends which we dealt with in a 1995 *Transactions* paper. As a byproduct, we found a family of Kähler metrics, called extremal soliton metrics, interpolating the extremal metrics and the generalized quasi-Einstein metrics. We also proved the existence of these metrics on compact almost homogeneous manifolds of two ends. For the completions of the $\mathbb{C}^*$-bundles we consider in this paper, we define what we call the generalized Mabuchi functional; the existence of extremal soliton metrics on these manifolds is again equivalent to the geodesic stability of the Kähler class with respect to this functional.

1. Introduction

It was shown in [Guan 1993; 1995a] that in every Kähler class of a compact almost homogeneous manifold with two ends there is a unique Calabi extremal metric, and also that there is a unique extremal metric in a given Kähler class on certain completions of a $\mathbb{C}^*$-bundle if a certain function Φ defined on the bundle there is positive. In [Guan 2003] we showed that the existence of this unique extremal metric is equivalent to the geodesic stability of the Kähler class. It is natural to ask:

> If the Kähler class is geodesically stable, does the modified Calabi flow converge pointwise to the extremal metric?

The answer is yes, and we show it in this paper for manifolds with *hypersurface*

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ends. (We shall treat in a later article the case of ends of higher codimension, which is technically more involved.) Specifically, we prove:

Theorem A. *Let a compact, almost homogeneous manifold with two hypersurface ends be given. For any given initial metric fixed by a maximal compact Lie subgroup of the automorphism group and any positive integer k, the modified Calabi flow converges exponentially in the C^k norm to a unique extremal metric.*

We also explain how the positivity of the function Φ of [Guan 1995a] is equivalent to geodesic stability, and how the convergence of the modified Calabi flow is natural in this case.

Similar questions for Riemann surfaces have been solved by Chruściel [1991] and reproved in [Chen 2001; Struwe 2002]. See also [Chang 2000; 2001] and references therein. (The second of these papers dealt with a conformal version of the surface Calabi flow, which in general is not the same as the usual Calabi flow.) Our manifolds are the first examples in higher dimensions that are not related to conformal geometry.

In Section 2 we define the class of completions of $\mathbb{C}^*$-bundles on which we shall prove the existence of the extremal metrics. A Kähler metric is *extremal* if

$$R - HR = \phi,$$

where R is the scalar curvature, HR is the averaged scalar curvature and ϕ is the potential function of a holomorphic vector field.

To interpolate extremal metrics and the quasi-Einstein metrics of [Guan 1995b], which are a kind of Kähler-soliton metrics as a generalization of Ricci-soliton metrics, we define extremal solitons. A Kähler metric is an *extremal soliton* if

$$R - HR = \phi_1 + \Delta\phi_2,$$

where Δ is the Laplacian and ϕ_1, ϕ_2 are potential functions of holomorphic vector fields. Recall too that a Kähler metric is a *quasi-Einstein metric* or *Kähler-soliton* if

$$R - HR = \Delta\phi,$$

where ϕ is the potential function of a holomorphic vector field. When the Kähler class is the Ricci class or the negative Ricci class, we have exactly the Kähler Ricci-soliton. Kähler Ricci-solitons were first studied by H. D. Cao [1996], Koiso [1987; 1990] and Tian, which was motivated by Hamilton's similar work on Ricci-solitons in the Riemannian case. (In that direction, there are some interesting results in [Koiso 1990; Guan 1995b; Tian and Zhu 2002].)

Still in Section 2 we consider the existence of extremal solitons. This generalizes the results in [Guan 1995a; 1995b]. We prove there:

Theorem B. *There is a family of extremal soliton metrics in every Kähler class on a compact almost homogeneous manifolds with two ends, which interpolates the extremal metric and the generalized quasi-Einstein metric we obtained before.*

Even for extremal metrics and quasi-Einstein metrics the results are more general than those in [Guan 1995a] and are relatively new, but the methods are the same and we just use a more general setting. In Section 2 the manifold need not be a $\mathbb{C}P^1$ bundle.

In Section 3 we explain the equivalence between the existence of Calabi extremal metrics and geodesic stability. Detailed calculations can be found in [Guan 2003, p. 279–280]. We also calculated the modified Mabuchi functional for the manifolds under consideration.

Moreover, we define a new functional, as a generalization of the modified Mabuchi functional. We call it the generalized Mabuchi functional and obtain:

Theorem C. *There is an extremal soliton metric in a given Kähler class on the manifold with respect to two given holomorphic vector fields if and only if the generalized Mabuchi functional is geodesically stable.*

In Section 4 we deal with the short time existence of the modified Calabi flow for the compact Kähler manifold. We apply a linearization method there.

It turns out that there are two kind of curvature flow equations for extremal soliton metrics. The first one, which we used around 1993, is the *modified Ricci flow*

$$\frac{\partial}{\partial t}\log\det g = -R + HR + \phi_1 + \Delta\phi_2,$$

where g is the Kähler metric. This is a quasi-second-order fourth-order heat equation. It has fourth-order derivatives of the potential function of the Kähler metric, just as those in the equation of the metrics with constant scalar curvatures.[1]

One might regard the major terms as a heat equation for $\log(\det g_t/\det g_0)$. That was the motivation for our considering this equation back in 1992; see [Guan 1995b].[2] However, I could only solve the equation for metrics assuming a certain condition; the condition can be checked to hold for many concrete cases, but I could not *prove* it for *all* of our cases (see Section 11). Then I started to look at the modified Calabi flow (see below). Although the modified Ricci flow has a fourth-order equation, it behaves more like a second-order heat equation. One might apply the maximal principle. But the Calabi functional might not be decreasing under this flow; for instance, an extremal metric with a nonzero ϕ in the Ricci class might

[1]This flow was recently used by Simanca [2005] to the extremal metrics. However, there are some serious mistakes in his paper. For example, Proposition 3.10 there is not correct. See the end of our last section for counterexamples on the simplest manifold $\mathbb{C}P^1$.

[2]I also gave a talk on this matter in 1996, invited by Professor Paul Yang; see again [Guan 1995b].

achieve the minimal of the Calabi functional and yet not be a stationary metric of the Ricci flow even up to an action of a one-parameter group of holomorphic automorphisms (again, see the end of Section 11).

The second equation is the *modified Calabi flow*, also called the Calabi–Robinson–Trautment equation:

$$\frac{\partial}{\partial t}F = R - HR - \phi_1 - \Delta\phi_2,$$

where F is the Kähler potential. This is a fourth-order heat flow equation.

From Sections 5 to 10 we show the convergence of the modified Calabi flow to the extremal metrics for the special case in the second section that the ends are hypersurfaces. Because of our setting for the problem we have to deal with weighted Sobolev inequalities, which seems a little more complicated than those in [Chruściel 1991]. We believe that this is the first step toward the similar problem for the toric manifolds we dealt with in [Guan 1999]. From our argument one sees that the convergence is very natural, compared to the more complicated situation for the modified Ricci curvature flow in [Koiso 1987; 1990; Guan 1995b; ≥ 2007b]. We apply our formula of modified Mabuchi functional where the geodesic stability is hidden.

We also use a family of *higher-order Calabi functionals* which are essential for our higher order estimates.

In the last section we compare the modified Calabi flow with the modified Ricci flow and explain why we think that the modified Calabi flow is more natural.

2. Existence of the extremal solitons on certain completions of line bundles

Our results can be regarded as a continuation of [Koiso and Sakane 1986; 1988; Koiso 1990; Guan 1993; 1995a; 1995b; 1999; 2003]. For the reader unfamiliar with those papers we state, without detailed proof, several lemmas and Theorem 2.10 below, which mostly can be found in [Guan 1995a] (Lemmas 2.2 and 2.3 are from [Guan 1999]).

Let $p: L \to M$ be a holomorphic line bundle over a compact complex Kähler manifold M and h a hermitian metric of L. Let L^0 be the open complement of the 0-section of L and let $s \in C^\infty(L^0)_\mathbb{R}$ be defined by $s(l) = \log|l|_h$ ($l \in L^0$), where $|\ |_h$ is the norm defined by h. Now consider a function $\tau = \tau(s) \in C^\infty(L^0)_\mathbb{R}$ depending only on s and monotonically increasing with respect to s.

Let $\tilde{J}$ be the complex structure of L and J that of M. Consider a Riemannian metric on L^0 of the form

$$\tilde{g} = d\tau^2 + (d\tau \circ \tilde{J})^2 + g, \tag{1}$$

where $g(l) = p^* g_{\tau(s(l))}(m)$, with $m = p(l) \in M$ and g_τ a one-parameter family of Riemannian metrics on M. Define a positive function u on L^0 depending only on τ by $u(\tau)^2 = \tilde{g}(H, H)$, where H is the real vector field on L^0 corresponding to the $\mathbb{R}^*$ action on L^0.

Lemma 2.1 [Koiso and Sakane 1986; 1988; Guan 1995a, p. 2257]. *Suppose that the range of τ contains* 0. *Then $\tilde{g}$ is Kähler if and only if g_0 is Kähler and $g_\tau = g_0 - UB$, where B is the curvature of L with respect to h and $U = \int_0^\tau u(\tau)\, d\tau$.*

The following assumptions are made throughout this paper:

(1) $\hat{L}$ is a compactification of L^0 and $\tilde{g}$ is the restriction of a Kähler metric of $\hat{L}$ to L^0.

(2) The range of τ contains 0.

(3) The eigenvalues of B with respect to g_τ are constant on M.

(4) The traces of the Ricci curvature r of g on each eigenspace of B are constant. These constants are called the *trace eigenvalues*.

Condition (4) here is much more general than what's in [Guan 1993; 1995a], where we just require that the eigenvalues of r be constants.

Our results cover some results which appeared in recent years: for example, when g has a constant scalar curvature and B has only one eigenvalue.

Let $(z^1, \dots, z^n)$ be a system of holomorphic local coordinates on M, where $n = \dim_{\mathbb{C}} M$. Using a trivialization of L^0, we take a system of holomorphic local coordinates $(z^0, \dots, z^n)$ on L^0 such that $\partial/\partial z^0 = H - \sqrt{-1}\, \tilde{J} H$.

Here we notice that z^0 corresponds to w_1 in [Guan 1999, p. 552], and s can be regarded as $\mathrm{Re}(z^0)$ near the point under consideration. So s corresponds to x_1 in [Guan 1999]. As in [Guan 1995a], we let $\varphi = u^2$ as a function of U; we also let F be the Kähler potential as in [Guan 1999, p. 552]. Then, by comparing [Guan 1995a, Lemma 2] (or Lemma 2.5 below) with [Guan 1999, p. 552], we have

$$\frac{\partial^2 F}{\partial s^2} = \tilde{g}_{0\bar{0}} = 2\varphi.$$

(The map F we used in [Guan 1999] is one-quarter of the usual potential function in the Kähler geometry. The difference might cause a constant factor in the calculations, e.g., for Lemma 2.2 and the Calabi flow equation, but does not affect our conclusions.)

The preceding equation yields:

Lemma 2.2. $$2\varphi = \partial^2 F/\partial s^2.$$

From $H = 2^{-1}\dfrac{\partial}{\partial s}$ we have $\dfrac{1}{4}\Big(\dfrac{d\tau}{ds}\Big)^2 = \varphi$ and $\dfrac{d\tau}{ds} = 2u$. Hence

$$U = \int_0^\tau u\,d\tau = \int_{s(0)}^s 2u^2 ds = \int_{s(0)}^s \frac{\partial^2 F}{\partial s^2} ds,$$

so $\partial F/\partial s = y_1$ up to a constant as in [Guan 1999, p. 552]. This leads to:

Lemma 2.3. *U is the Legendre transformation of s.*

Here we use the Legendre transformation in [Guan 1999] instead of the moment map in [Guan 1995a] since we need the new insight in later sections.

Remark 2.4. We shall see in [Guan $\geq$ 2007c; 2006; $\geq$ 2007a] that the function U here, the Legendre transformation in [Guan 1999] and the miraculous function U from [Guan and Chen 2000; Guan 2003; 2006] are special cases of the parallel coordinates along the curves in the Mabuchi moduli space of Kähler metrics on compact almost homogeneous manifolds with actions of reductive groups.

Let $\hat{X}_i,\ \hat{X}_{\bar{i}}$ $(0 \leq i \leq n)$ be the partial derivatives $\partial/\partial z^i,\ \partial/\partial \bar{z}^i$ on L^0 and $X_i,\ X_{\bar{i}}$ $(1 \leq i \leq n)$ the partial derivatives $\partial/\partial z^i,\ \partial/\partial \bar{z}^i$ on M.

Lemma 2.5 [Koiso and Sakane 1986; 1988; Guan 1995a, Lemma 2]. *We have*

$$\tilde{g}_{0\bar{0}} = 2u^2,\quad \tilde{g}_{0\bar{i}} = 2u\hat{X}_{\bar{i}}\tau,\quad \tilde{g}_{i\bar{j}} = g_{i\bar{j}} + 2\hat{X}_i\tau\cdot\hat{X}_{\bar{j}}\tau,$$

where $1 \leq i, j \leq n$. For the point $P \in L^0$ considered, we can choose a local coordinate system around $m = p(P) \in M$ such that $(\partial/\partial z^i)\tau = 0$ at m, making $\hat{X}_i\tau = \hat{X}_{\bar{j}}\tau = 0$ at P. Then if f is a function on L^0 depending only on τ, we have

$$\hat{X}_0\hat{X}_{\bar{0}} f = u\frac{d}{d\tau}\Big(u\frac{df}{d\tau}\Big),\qquad \hat{X}_i\hat{X}_{\bar{0}} f = 0,\qquad \hat{X}_i\hat{X}_{\bar{j}} f = -\tfrac{1}{2}uB_{i\bar{j}}\frac{df}{d\tau},$$

if f is a function on L^0 depending only on t. The Ricci curvature at this point is

$$\tilde{r}_{0\bar{0}} = -u\frac{d}{d\tau}\Big(u\frac{d}{d\tau}\log(u^2Q)\Big),\qquad \tilde{r}_{0\bar{i}} = 0, \tilde{r}_{i\bar{j}} = p^*r_{0\,i\bar{j}} + \tfrac{1}{2}u\frac{d}{d\tau}\log(u^2Q)\cdot B_{i\bar{j}},$$

where $Q = \det(g_0^{-1}\cdot g_\tau)$. In particular, we have the scalar curvature

$$\tilde{R} = \frac{\Delta}{Q} - \frac{1}{2Q}\frac{d}{dU}\Big(\frac{d}{dU}Q\varphi\Big), \tag{2}$$

where $\varphi = u^2$ as a function of U and $\Delta(U) = Q\sum_{i,j} r_{0\,i\bar{j}}g^{i\bar{j}}_{\tau(U)}$. We also have $\varphi'(\min U) = 2$ and $\varphi'(\max U) = -2$.

Lemma 2.6 [Futaki et al. 1990; Mabuchi 1987; Guan 1995a, Lemma 3]. *We can regard U as a moment map corresponding to $(\tilde{g},\ \tilde{J}H)$ and g_τ as the symplectic reduction of $\tilde{g}$ at $U(\tau)$. Furthermore, $\tilde{g}$ is extremal if and only if $\tilde{R} = a + bU$ for some $a, b \in \mathbb{R}$.*

Set $M_0 = U^{-1}(\min U)$ and $M_\infty = U^{-1}(\max U)$. These are complex submanifolds, since they are components of the fixed point set of $H - \sqrt{-1}\,\tilde{J}H$, which is semisimple. Let D_0 and D_∞ be the codimensions of M_0 and M_∞ in $\hat{L}$.

Lemma 2.7 [Guan 1995a, Lemma 4]. *Suppose there is another Kähler metric $\tilde{g}^\vee$ on $\hat{L}$ in the same Kähler class, which is of form* (1) *on L^0. Let*

$$\tau^\vee,\ g^\vee,\ U^\vee,\ Q^\vee,\ \Delta^\vee,\ \varphi^\vee,\ u^\vee$$

be the corresponding metric and functions of s. There is a unique corresponding $\tau^\vee$ such that $g_0^\vee = g_0$. In this case, $\min U^\vee = \min U$ (*or* $\max U^\vee = \max U$) *and* $Q^\vee = Q$, $\Delta^\vee = \Delta$ *hold. So we may write $D = \max U$ and $-d = \min U$. Then*

$$Q(U) = \Bigl(1 + \frac{U}{d}\Bigr)^{D_0-1} Q_{-d} \quad \Bigl(or\ Q(U) = \Bigl(1 - \frac{U}{D}\Bigr)^{D_\infty-1} Q_D\Bigr),$$

where Q_{-d} (*or Q_D*) *is a polynomial of U such that $Q_{-d}(-d) \neq 0$* (*or $Q_D(D) \neq 0$*) *and*

$$\Delta(U) = D_0(D_0-1)\frac{1}{d}\Bigl(1 + \frac{U}{d}\Bigr)^{D_0-2} Q_{-d} \mod \Bigl(1 + \frac{U}{d}\Bigr)^{D_0-1}$$
$$\Bigl(or\ \Delta(U) = D_\infty(D_\infty-1)\frac{1}{D}\Bigl(1 - \frac{U}{D}\Bigr)^{D_\infty-2} Q_D \mod \Bigl(1 - \frac{U}{D}\Bigr)^{D_\infty-1}\Bigr).$$

Proof. Set $\tilde{g} - \tilde{g}^\vee = i\hat{\partial}\bar{\hat{\partial}}\phi$. Then

$$\tilde{g}^\vee_{i\bar{j}} = \tilde{g}_{i\bar{j}} + \tfrac{1}{2}u\frac{d\phi}{d\tau}B_{i\bar{j}} = (g_0)_{i\bar{j}} - \Bigl(U - \tfrac{1}{2}u\frac{d\phi}{d\tau}\Bigr)B_{i\bar{j}}$$

for $1 \leq i, j \leq n$. So at $\min U$ (or $\max U$) we have $\tilde{g}_{i\bar{j}} = \tilde{g}^\vee_{i\bar{j}}$, meaning there is τ_0 such that $g^\vee_{\tau^\vee(\tau_0)} = g_0$. By choosing $\tau^\vee$ such that $\tau^\vee(\tau_0) = 0$, one sees that $\min U^\vee = \min U$ and $max U^\vee = \max U$, as desired.

The last statement follows from the fact that the scalar curvature $\tilde{R}$ is finite on both M_0 and M_∞. □

To proceed, we will need normalization. By rescaling we have the following.

Lemma 2.8 [Guan 1995a, Lemma 5]. *For any given $a_1 \in \mathbb{R}$, $\tilde{g}$ is an extremal soliton if and only if $\tilde{g}^\vee = a_1^2\tilde{g}$ is an extremal solution. We can choose $U^\vee = a_1^2 U + a_2$ for any $a_2 \in \mathbb{R}$, allowing us to assume that* $\max U - \min U = 2$ *and* $\min U = -1$, *then* $\max U = 1$.

For example, if $\hat{L} = \mathbb{C}P^{n+1}$, then M_0 is a point, $M_\infty = M = \mathbb{C}P^n$. In this case $\hat{L}$ is the one point completion (compactification) of the hyperplane line bundle L over M with M_∞ as the zero section. The anticanonical line bundle is $(n+1)L$. Therefore $r_{0,ii} = n+1$ and $Q = (1+U)^n$. The Kähler metric at $U = 0$ is the curvature of L, and therefore $\Delta = n(n+1)(1+U)^{n-1}$.

From Lemma 2.6, it can be seen that, if $\tilde{g}$ is an extremal soliton metric, then

$$\tilde{R} = a + bU + c\tilde{\Delta}, U \tag{3}$$

for some $a, b, c \in \mathbb{R}$ with a $\phi_2 = cU + d$.

By Lemma 2.5 we have

$$\begin{aligned}
\tilde{\Delta} f &= \tilde{g}^{\bar{\alpha}\beta}\hat{X}_{\bar{\alpha}}\hat{X}_{\beta} f \\
&= \tilde{g}^{\bar{0}0}\hat{X}_{\bar{0}}\hat{X}_0 f + \tilde{g}^{\bar{a}0}\hat{X}_{\bar{a}}\hat{X}_0 f + \tilde{g}^{\bar{0}a}\hat{X}_{\bar{0}}\hat{X}_a f + \tilde{g}^{\bar{a}b}\hat{X}_{\bar{a}}\hat{X}_b f \\
&= \frac{1}{2u^2}(\hat{X}_{\bar{0}}\hat{X}_0 f) + 0 + 0 + \tilde{g}^{\bar{a}b}(\hat{X}_{\bar{a}}\hat{X}_b f) \\
&= \frac{1}{2u^2} u \frac{d}{d\tau}\Big(u\frac{d}{d\tau} f\Big) + g_\tau^{\bar{a}b}\Big(-2^{-1}u\frac{df}{d\tau}B_{b\bar{a}}\Big) \\
&= \frac{1}{2}\frac{d}{dU}\Big(\varphi(U)\frac{d}{dU} f\Big) - \tfrac{1}{2}\varphi(U)\Big(\frac{d}{dU} f\Big) g_t^{\bar{a}b}B_{b\bar{a}} \\
&= \frac{1}{2}\frac{d}{dU}\Big(\varphi\frac{d}{dU} f\Big) + \tfrac{1}{2}\varphi\Big(\frac{d}{dU} f\Big)\frac{1}{Q}\frac{d}{dU}Q = \frac{1}{2Q}\frac{d}{dU}\Big(\varphi Q\frac{d}{dU} f\Big),
\end{aligned}$$

from which we get

$$\begin{aligned}
\tilde{\Delta}\phi_2 &= \frac{1}{2Q}\frac{d}{dU}\Big(\varphi Q\frac{d}{dU}(cU+d)\Big) = \frac{c}{2Q}\frac{d}{dU}(\varphi Q) \\
&= \tilde{R} - \frac{\int_{-1}^1 \tilde{R} Q\, dU}{\int_{-1}^1 Q\, dU} = \frac{\Delta}{Q} - \frac{1}{2Q}\frac{d}{dU}\Big(\frac{d}{dU}Q\varphi\Big) - (a+bU).
\end{aligned}$$

Let $m = \int_{-1}^1 \tilde{R} Q\, dU / \int_{-1}^1 Q\, dU$, $\alpha = \int_{-1}^1 Q\, dU$ and $\beta = \int_{-1}^1 UQ\, dU$. Then

$$\begin{aligned}
\int_{-1}^1 \tilde{R} Q\, dU &= \int_{-1}^1 \Big(\Delta - 2^{-1}\frac{d}{dU}\Big(\frac{d}{dU}Q\varphi\Big)\Big)\, dU \\
&= \delta - \frac{1}{2}\frac{d}{dU}(Q\varphi)\Big|_{-1}^1 = \delta - \frac{1}{2}\Big(Q\frac{d}{dU}\varphi + \varphi\frac{d}{dU}Q\Big)\Big|_{-1}^1 \\
&= \delta - \tfrac{1}{2}\big(Q(1)\cdot(-2) - Q(-1)\cdot 2\big) = \delta + Q(1) + Q(-1),
\end{aligned}$$

where $\delta = \int_{-1}^1 \Delta\, dU$. Therefore, $m = \big(\delta + Q(-1) + Q(1)\big)/\alpha$.

Hence

$$c\varphi Q = -\frac{d}{dU}(Q\varphi) - 2\int_{-1}^U (a+bx)Q(x)\, dx + 2\int_{-1}^U \Delta(x)\, dx + c_1. \tag{4}$$

If $U = -1$, we have $0 = -2Q(-1) - 0 - 0 + c_1$, that is, $c_1 = 2Q(-1)$.

If $U = 1$, we have $0 = 2Q(1) - 2a\alpha - 2b\beta + 2\delta + 2Q(-1)$. Therefore, $a = m - b\beta/\alpha$.

Moreover,

$$Q\varphi = e^{-cU}\left(\int_{-1}^{U}\left(\int_{-1}^{y}\big(-2(a+bx)Q(x)+2\Delta(x)\big)\,dx+2Q(-1)\right)e^{cy}\,dy+f\right), \tag{5}$$

with a constant f. We denote the right-hand side by $\Phi(U)$.

If $U=-1$, we have $0=e^{a}(0+f)$, so $f=0$. If $U=1$, we have

$$0=\int_{-1}^{1}\left(\int_{-1}^{y}\big(-2(a+bx)Q(x)+2\Delta(x)\big)\,dx+2Q(-1)\right)e^{cy}\,dy. \tag{6}$$

Set

$$p(U)=\int_{-1}^{U}2\big(\Delta(x)-(a+bx)Q(x)\big)\,dx+2Q(-1). \tag{7}$$

Then (4) yields

$$p(U)=\int_{U}^{1}2\big((a+bx)Q(x)-\Delta(x)\big)\,dx-2Q(1).$$

By the last statement of Lemma 2.7, we know that $p(U)$ is nonnegative near -1 and nonpositive near 1. Since the right-hand side of (6) goes to $-\infty$ (or $+\infty$) when c goes to $+\infty$ (or $-\infty$), there is at least one solution c. We pick the *smallest* such c. We therefore have:

Lemma 2.9. *For any b, there is a solution c for* (6).

Theorem 2.10 [Koiso and Sakane 1986, Guan 1995a, Lemma 6]. *There is an extremal soliton metric in the Kähler class of $\tilde{g}$ for a given b provided that $\varphi^0(U)=\Phi(U)/(Qe^{cU})$ is positive on* $(-1,\ 1)$.

If we let $b=0$ we have the (generalized) quasi-Einstein metric, as in [Guan 1995b].

To obtain an extremal metric we just let $c=0$ and solve (6) to find a and b as we did in [Guan 1995a, p. 2259] (see Lemma 2.6 there). Let $c=0$, $\delta_1=\int_{-1}^{1}x\Delta\,dx$, $\gamma=\int_{-1}^{1}x^2Q\,dx$, then (6) becomes

$$\begin{aligned}
0 &= \int_{-1}^{1}\int_{-1}^{y}((a+bx)Q(x)-\Delta(x))\,dx\,dy-Q(-1)\\
&= \int_{-1}^{1}\int_{x}^{1}((a+bx)Q(x)-\Delta(x))\,dy\,dx-Q(-1)\\
&= \int_{-1}^{1}(1-x)((a+bx)Q-\Delta)\,dx-Q(-1)\\
&= a\alpha+b\beta-a\beta-b\gamma-\delta+\delta_1-Q(-1) \;=\; m\alpha+\delta_1-Q(-1)-a\beta-b\gamma-\delta\\
&= m\alpha+\delta_1-Q(-1)-m\beta+\frac{b}{\alpha}(\beta^2-\alpha\gamma)-\delta.
\end{aligned}$$

The coefficient of b cannot be zero since

$$\alpha t^2+2\beta t+\gamma=\int_{-1}^{1}(t+U)^2 Q\,dU>0$$

for any t. Therefore, there is a unique solution of b.

Lemma 2.11 Guan 1993: 1995a; 1995b]. *Let the Ricci curvature r have nonnegative trace eigenvalues. For b fixed, the function Φ as above is always positive in* $(-1, 1)$, *and the solution c in Lemma 2.9 is unique.*

Proof. Assume r has no negative trace eigenvalues (we will consider a more relaxed condition in Corollary 2.13). Since the derivative of $Q\varphi e^{cU}$ is $p(U)e^{cU}$, we have that

$$\frac{d}{dU}\Big(e^{-cU}\frac{d}{dU}(Q\varphi e^{cU})\Big)=2\Delta(U)-2(a+bU)Q(U). \tag{8}$$

Diagonalizing B, we see that Q is a product of polynomials of degree 1. Let

$$-a_1^{-1}<\cdots<-a_p^{-1}<b_1^{-1}<\cdots<b_q^{-1},$$

denote the distinct roots of Q for which some corresponding Ricci curvature $r_{i\bar{i}}$ is nonzero, where a_i, b_j are positive. Set

$$S(U)=U\prod_{i=1}^{p}(1+a_iU)\prod_{j=1}^{q}(1-b_jU),\qquad P(U)=U\,Q(U)/S(U),$$

$$\Psi(U)=\bigg(\frac{d}{dU}\Big(e^{-cU}\frac{d}{dU}\big((Q\varphi)(U)e^{cU}\big)\Big)\bigg)\Big/P(U).$$

Then Ψ is a polynomial of degree $p+q$ and $\Psi(a)=-k_aS'(a)$ for some root a of $S(U)/U$, where $k_a\in\mathbb{R}^+$ since r is nonnegative. We can see that $S'(a)$ is nonzero and that its sign is opposite the sign of S' at the roots before a and after a (if they exist). Because $S'(0)>0$, we have $S'(-a_p^{-1})<0$ and $S'(b_1^{-1})<0$, that is, $\Psi(-a_p^{-1})>0$ and $\Psi(b_1^{-1})>0$. Now there are $p-1$ (or $q-1$) zero points of Ψ in $(-a_1^{-1},-a_p^{-1})$ (or in (b_1^{-1},b_q^{-1})) if $p,\ q$ are not zero (one may also check the case of $q=0$ or $p=0$). If φ has nonpositive points in $(-1, 1)$, then in $(-1, 1)$, $Q\varphi$ has at least two maximal and one minimal points, since $\varphi(-1)=\varphi(1)=0$, $\varphi(-1+\epsilon)>0$ and $\varphi(1-\epsilon)>0$ for ϵ small enough. So there are at least 4 zero points of Ψ in $(-a_p^{-1}, b_1^{-1})$. The polynomial Ψ has at least $(p-1)+(q-1)+4=p+q+2$ zero points, i.e., $\Psi(U)=0$, $Q\varphi=c_1+c_2e^{-cU}$. But $\varphi(-1)=\varphi(1)=0$, hence $Q\varphi=0$, which is a contradiction. This proves that Φ is positive on $(-1, 1)$.

To show the uniqueness of c we only need to prove that the function $p(U)$ in (7) has only one zero in $(-1, 1)$. If $p(U)$ has two zeros in $(-1, 1)$, it has at least three, since it is nonnegative near -1 and nonpositive near 1. So Ψ has at least four zeros in $(-a_p^{-1}, b_1^{-1})$, a contradiction. □

Corollary 2.12 [Guan 1995a; 1995b]. *For every Kähler class of a compact almost homogeneous manifold with two ends, there exists an extremal soliton metric for any given b. In particular, there is always an extremal metric and a (generalized) quasi-Einstein metric.*

Proof. By [Huckleberry and Snow 1982, Theorem 3.2], every compact Kähler almost homogeneous space is a completion of a $\mathbb{C}^*$-bundle over a product of a torus A and a C-space N with two homogeneous Kähler spaces as two ends. Hence a maximal compact subgroup of the identity component of the automorphism group of this manifold is $G = A \times S \times S^1$, where A is also the Albanese torus and S is a maximal compact subgroup of the identity component of the automorphism group of N. If g is any Kähler metric, $g_G = \int_{h \in G} h^* g\, dm$ is a Kähler metric of form (1), where m is Haar measure on G; clearly g_G is invariant under G. Also the Ricci curvature of $A \times N$ is nonnegative. Now the condition in our assumption follows from the property of invariant cohomology $(1, 1)$ classes for such manifolds; see [Dorfmeister and Guan 1991, p. 326, proof of the Proposition]. □

If $b = 0$ we can show more. Say that the trace eigenvalues are *nonnegative on one side* if they are nonnegative for all $-a_i^{-1}$ or for all b_j^{-1} in the proof of the Lemma 2.11.

Corollary 2.13 [Guan 1995a; 1995b]. *If the trace eigenvalues only change sign once and are nonnegative on one side, there is a (generalized) quasi-Einstein metric. In particular, the completion of the Hodge line bundle over a Hodge manifold with a constant scalar curvature admits a (generalized) quasi-Einstein metric.*

Proof. In the proof of Lemma 2.11, if $b = 0$, the polynomial p' is one degree lower. By ignoring the root at which the trace eigenvalue is negative and changing the sign, the proof still goes through. By the argument in the proof of the [Koiso and Sakane 1986, p. 177, Theorem 5.4], we have our corollary. □

3. Geodesic stability

For any Kähler manifold X with a given Kähler form ω_0, any Kähler form in the same Kähler class can be written as

$$\omega = \omega_0 + i\partial\bar{\partial} f.$$

Therefore, a curve of Kähler forms corresponds to a family of functions f_t. The tangent corresponds to $\dot{f}$, which is also a function on X. The Mabuchi metric is

$$g_K(f_1, f_2) = \int_X f_1 f_2 \omega^n. \tag{9}$$

The Mabuchi metric to be considered as an infinite version of the Riemann metric, in [Guan 2003, p. 279–280] we found that on the manifolds considered in the previous section, the existence of the extremal metric is equivalent to a certain stability, which we called *geodesic stability*, of the Kähler class.

In [Guan 1999], we found that the geodesics of the Mabuchi metric come from linear paths of the Legendre transformation of the pair (F, s). The Legendre transformation of (F, s) is (G, U), where $G(s) = sF_s - F$, which can be considered as a function of U.

For a family of Kähler metrics in a given Kähler class, we consider G as a function of U and a time t. Let $\dot{G}(t, U)$ and $\ddot{G}(t, U)$ be the first and second partial derivatives with respect to t. Then:

Lemma 3.1 [Guan 1999, p. 552]. *The geodesic equation is* $\ddot{G}(t, U) = 0$.

Under the Legendre transformation, we always have $(\partial F/\partial t)(t, s) = -\dot{G}(t, U)$. Since along the geodesics we have

$$2\varphi = \frac{\partial^2 F}{\partial s^2} = \left(\frac{\partial^2 G}{\partial U^2}\right)^{-1}$$

(see Lemma 2.2), we conclude that $(\varphi)^{-1} = 2\partial^2 G/\partial U^2$ is linear.

In [Guan and Chen 2000, p. 819] we saw that the modified Calabi flow is the gradient flow of the modified Mabuchi functional

$$M(\omega_0, \omega_1) = -\int_a^b \int_X \dot{f}(R - HR - \phi_E)\omega_t^n dt, \tag{10}$$

where ϕ_E is the function corresponding to the extremal vector field E in [Futaki and Mabuchi 1995]. In our case, $\phi_E = a + bU - HR$ for the values a, b in (3) with $c = 0$.

For a Ricci-soliton metric, Tian and Zhu obtain a modified Futaki invariant

$$F_E(Y) = \int_X Y(h - \phi_E)e^{\phi_E}\omega^n,$$

where h satisfies $\mathrm{Ricci}(\omega) - \omega = \partial\bar{\partial}h$. This actually comes from a generalized Mabuchi functional

$$M_E(\omega_0, \omega_1) = \int_0^1 \int_X (\nabla \dot{f}, \nabla(h - \phi_E))e^{\phi_E}\omega^n dt.$$

Here we shall use a generalized Mabuchi functional with two given potential functions ϕ_1, ϕ_2 of holomorphic vector fields E_1, E_2:

$$M_{E_1,E_2}(\omega_0, \omega_1) = \int_0^1 \int_X (\nabla \dot{f}, \nabla(\gamma - \phi_2))e^{\phi_2}\omega^n dt,$$

where γ satisfies $R - HR - \phi_1 = \Delta\gamma$.

For the case of an extremal metric, we calculated in [Guan 2003, p. 280] the derivative of the modified Mabuchi functional along a geodesic. This leads to:

Lemma 3.2. *The generalized Mabuchi functional is independent of the path which we choose and is convex along the geodesics. If we formally set*

$$\tfrac{1}{2} M_{E_1,E_2}(\varphi) = \int_{-1}^{1} (\mathcal{G} - 1 - \ln \mathcal{G}) e^{\phi_2} Q \, dU, \tag{11}$$

where $\mathcal{G} = \varphi^0/\varphi$ (see Theorem 2.10 for the definition of φ^0), *then*

$$M_{E_1,E_2}(\omega_0, \omega_1) = M_{E_1,E_2}(\varphi_1) - M_{E_1,E_2}(\varphi_0).$$

Proof. First we assume the existence of an extremal soliton and deal with the extremal metric case. The derivative of the modified Mabuchi functional along the geodesic is

$$\int_{-1}^{1} h'' Q(\varphi^0 - \varphi) \, dU,$$

where $\varphi = 1/((\varphi^0)^{-1} + th'')$. Integrating over t, we obtain our formula.

In general, if there is an extremal soliton, we deduce from the equality

$$R - HR - \phi_1 = \frac{\Delta}{Q} - \frac{1}{2Q}(\varphi Q)'' - a - bU$$

that

$$\begin{aligned}
&\int_X (\nabla \dot{f}, \nabla(\gamma - \phi_2)) e^{\phi_2} \omega^n \\
&\quad = \int (\dot{f})' \varphi Q \biggl(\frac{1}{\varphi Q} \biggl(\int_{-1}^{U} (2\Delta - 2(a+bU)Q) \, dU - (\varphi Q)' - c\varphi Q \biggr) \biggr) e^{cU} dU \\
&\quad = \int (\dot{G})'(-p(U)e^{cU} + (\varphi Q e^{cU})') \, dU = \int (\dot{G})'((\varphi Q e^{cU})' - (\varphi^0 Q e^{cU})') \, dU \\
&\quad = -\int (\dot{G})''(\varphi - \varphi^0) Q e^{cU} dU = -\frac{1}{2} \int \frac{\partial}{\partial t}(\varphi^{-1})(\varphi - \varphi^0) Q e^{cU} dU \\
&\quad = \frac{1}{2} \int \dot{\varphi} \varphi^{-2} (\varphi - \varphi^0) Q e^{cU} dU.
\end{aligned}$$

Then we integrate with respect to t, and get our desired formula.

The formula implies that the functional is independent of the path chosen. It is also clear that the second derivative of the functional is

$$\int \Bigl(\frac{\partial}{\partial t}(\varphi^{-1}) \Bigr)^2 \varphi^2 Q e^{cU} dU > 0. \qquad \square$$

Corollary 3.3. *For any given E_1, E_2 there is at most one extremal soliton metric of the given form.*

For the case of extremal metric, we notice that the formula of [Guan 2003, p. 280] is true even if φ^0 in Theorem 2.10 is not positive everywhere on $(-1, 1)$. In general we have a similar formula

$$\int_{-1}^{1} h''(\varphi^0 - \varphi) Q e^{cU} \, dU$$

for the slope of the generalized Mabuchi functional; this follows from our calculation above and the fact that $h = \dot{G}$ and $\dot{G}'' = h''$. If we fix a metric with a given function φ_0, along the geodesic connecting φ_0 and φ, we have $\varphi = ((\varphi_0)^{-1} + t h'')^{-1}$. The maximal geodesic ray cannot be infinite if $h'' < 0$ at some point, and the limit of the slope is $+\infty$.

If $h'' \geq 0$, we have an infinite geodesic ray with $\int_{-1}^{1} h'' \varphi^0 Q e^{cU} \, dU$ as the limit of the slope, which is finite. We call it the generalized Futaki invariant $F(h)$ along the given maximal geodesic ray. Regarding it as a functional of h'' we see that φ^0 is positive on $(-1, 1)$ if and only if $F(h) \geq C \int_{-1}^{1} |h''| \varphi_0 Q \, dU$ for some positive number C and all h with nonnegative h''. We name this last condition *geodesic stability*. Thus the existence of the extremal metrics is equivalent to geodesic stability. Similar results are also true for extremal solitons.

Here we would like to explain the stability a little bit more. If the solution φ^0 exists, the norm $\int_{-1}^{1} |h''| \varphi^0 Q \, dU$ is equivalent to the norm

$$\int_{-1}^{1} |h''| (1+U)^{D_0} (1-U)^{D_\infty} \, dU.$$

Therefore:

Theorem 3.4 [Guan 2003, p. 280]. *For the manifolds in Section 2, a Kähler class has an extremal soliton metric with two given holomorphic vector fields if and only if the given Kähler class enjoys geodesic stability.*

Remark 3.5. With the formula in [Guan 2003, p. 279] we can also obtain a formula for the modified Mabuchi functional without referring to φ^0 by integrating as in the proof of our Lemma 3.2. From a direct method for finding the minimal of the functional we can get another method for finding φ^0.

Remark 3.6. Combining Theorem 3.4 with the results in [Guan 2006; $\geq$ 2007a; 2005b; 2005a], we can prove that a Kähler class on a compact almost homogeneous manifold of cohomogeneity 1 admits a unique extremal soliton metric with two given holomorphic vector fields if and only if the given Kähler class is geodesically stable.

Remark 3.7. Just as in [Guan 2003], the generalized Futaki invariant does not depend on the initial metrics but only depends on the direction. In the present case the moduli space is flat and the directions at each initial metric can be regarded as parallel vectors. In general, we do not expect the moduli space to be flat. Therefore, we have the *fourth geodesic stability principle*: Instead of parallel vector fields we should use maximal geodesic rays with the *same* infinite points.

4. General result on short time existence

The modified Calabi flow (see [Guan and Chen 2000, p. 820]) is

$$\dot{f} = -\Delta \log \det(g) - HR - \phi_1 - \Delta\phi_2, \tag{12}$$

where ϕ_1, ϕ_2 are the functions corresponding to two holomorphic vector fields E_1, E_2. After changing the Kähler metric by a function v, the functions ϕ_1, ϕ_2 are changed by $E_1(v),\ E_2(v)$ [Futaki and Mabuchi 1995]. Since we always consider the case in which the metrics are invariant under a maximal compact group of the holomorphic automorphism group and JE_1, JE_2 are Killing vector fields, the potentials of E_1 and E_2 are always real. We have $E_k(v) = \frac{1}{2}(E_k(v) + \bar{E}_k(v))$, $k = 1, 2$. The linearization is

$$\dot{v} = -\Delta^2 v - v_{i\bar{j}}(R^{\bar{j}i} - \phi_2^{i\bar{j}}) - \tfrac{1}{2}(E_1^i v_i + E_1^{\bar{i}} v_{\bar{i}}) - \Delta E_2(v),$$

where the indices i corresponds to z_i in the local holomorphic coordinates $(z_1, \dots, z_i, \dots, z_n)$. Multiplying by v and observing that the functions E_1^i, E_2^i, $R^{i\bar{j}}$, $\phi_2^{i\bar{j}}$ are smooth, as is the volume, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int v^2 dV \\
&\le -\int (\Delta v)^2 dV + C_1 \int v^2 dV + C_2 \int |v v_{i\bar{j}}|\, dV + C_3 \int |v_i|\, dV + C_4 \int \Delta v |v_i|\, dV \\
&\le -(1-\epsilon) \int (\Delta v)^2 dV + C_5 \int v^2 dV + C_6 \Big(\int (\Delta v)^2 dV\Big)^{3/4} \Big(\int v^2 dV\Big)^{1/4} \\
&\le -(1-\epsilon) \int (\Delta v)^2 dV + C_5 \int v^2 dV \\
&\qquad\qquad + \Big(\int (\Delta v)^2 dV\Big)^{1/2} \Big(\epsilon \Big(\int (\Delta v)^2 dV\Big)^{1/2} + \frac{C_6^2}{\epsilon} \Big(\int v^2 dV\Big)^{1/2}\Big) \\
&\le -(1-3\epsilon) \int (\Delta v)^2 dV + C \int v^2 dV \le C \int v^2 dV,
\end{aligned}$$

for some constants C_1, C_2, C_3, C_4, C_5, C_6, C, and ϵ that can be chosen as small as we want. Here we have applied [Aubin 1982, p. 93, Theorem 3.69] to the last two terms and the formula

$$\int |v_{i\bar{j}}|^2 = -\int v_i v_{\bar{i}j\bar{j}} = -\int v_i v_{j\bar{i}\bar{j}} = -\int v_i v_{j\bar{j}\bar{i}} = \int v_{i\bar{i}} v_{j\bar{j}} = \int (\Delta v)^2$$

to the third term. We also have applied repeatedly Young's inequality

$$ab \leq ca^2 + \frac{b^2}{c}$$

by choosing suitable c's which are related to ϵ.

Let $A = \int v^2 dV$, then $\dot{A} \leq CA$; then $(d/dt)(e^{-Ct}A) \leq 0$. Therefore, if $A(0) = 0$ then $A = 0$. This allows us to use the argument in [Kobayashi 1987, p. 212]. To do that we only need to replace Theorem 5.9 in that reference by [Huisken and Polden 1999, Theorem 7.9] (see also Theorem 7.14 therein; this result was hidden in Struwe's argument [2002, p. 255]). The difference between the two norms used in these references is of little import for a finite time, as explained in [Huisken and Polden 1999, p. 76, lines 23 to 25]. We initially used our method here to [Guan $\geq$ 2007c] (see our last section for example), and in that case it was enough to apply Theorem 5.9 in [Kobayashi 1987, p. 212].

We can also apply a different kind of linearization,

$$\partial_t v = -\Delta^2 v + B(t, z, g),$$

with B only related to the third derivatives of the given family of metrics g such that we can get our equation back if we let $\partial\bar{\partial} v = g - g_0$. The preceding argument shows that $A = \int v^2 dV$ is bounded for a short time, and that if $A(0) < C_1$ we have $A < C_1$ for sufficiently short interval of time $[0, T]$. In the same way, by multiplying $\Delta^2 v$, we obtain that Δv is bounded in L^2, and so is $\nabla^2 v$. Similarly, by assuming a good initial value condition we can get the higher order estimates; see [Huisken and Polden 1999, p. 76]. This enables us to apply the linearization and contraction method for our original equation (12) (compare [Struwe 2002, p. 255]). We do this as follows. Taking the initial metric as the given family of metrics $g_0(t) = g_0$ for the linearized equation we can get a solution of a family of metrics $g_1(t)$ in a short time. We then use the new family g_1 as the given family of metrics in the linearized equation and get a solution of another family $g_2(t)$ of metrics. We apply Newton's method by iterating the preceding argument and obtain $g_i(t)$. We now claim that the map from $g_{i+1} - g_i$ to $g_{i+2} - g_{i+1}$ is a contraction. More precisely, given any metric $g(t)$, we have a family of metrics $h(t) = F(g)$ such that $h(t) - g(t)$ corresponds to the solution of the linearized equation. The map $F_1(g, h) = (F(g), F(h))$ is a contraction related to a seminorm that measures the difference $h - g$. For example, as above, we have $B = \int_0^\delta (v_1 - v_2)^2 dt \leq \delta C$, where C is a constant related to the semi-B-norm of the preimage of the pair $(g_0 + i\partial\bar{\partial} v_1, g_0 + i\partial\bar{\partial} v_2)$. We see that when δ is small enough, we have a contraction. This should lead to another proof.

Therefore, we have short time existence by the usual linearization argument:

Theorem 4.1. *The short time existence of the modified Calabi flow holds for any compact Kähler manifold with two given complex conjugates of Killing vector fields*, *i.e.*, *holomorphic vector fields with real potential functions.*

5. Setting up the equation of the modified Calabi flow in our cases

From now until Section 10 we shall focus on the case $\phi_2 = 0$, that is, the extremal metric case. We recall some results on the modified Calabi flow (12). Let

$$\mathrm{Cal}(\omega) = \int_M (R - HR - \phi_E)^2 \omega^n$$

be the modified Calabi functional.

Lemma 5.1 Guan and Chen 2000, p. 820; Guan 1999, p. 550. *The modified Calabi flow is the gradient flow of the modified Mabuchi functional with respect to the Mabuchi metric on the space of Kähler metrics. Its derivative is the negative of the modified Calabi functional*, *and its second derivative is* $2\int_X |R,_{\alpha\beta}|^2\omega^n$.

In our case, the evolution equation of the Kähler potential function F along the modified Calabi flow is given by

$$-\dot{G}(t, U) = \frac{\partial F}{\partial t}(t, s) = \tilde{R} - a - bU.$$

Recall that $G = sF_s - F$ is the Legendre transformation of F (see the paragraphs before and after Lemma 3.1). Considering G as a function of t and U, we use U as the free variable. We need to estimate the function G and its derivatives.

We use $'$ and $\dot{}$ for the partial derivatives with respect to U and t, respectively.

The function φ determines the metric, as we have seen before; thus we proceed by estimating φ and its derivatives as functions of t and U. Note that if we differentiate with respect to U twice, we obtain

$$-\dot{G}'' = \frac{\dot{\varphi}}{2\varphi^2} = \tilde{R}''.$$

In this way, we change the modified Calabi flow into a flow of the function φ.

To make things simpler we *assume that*

$$D_0 = D_\infty = 1 \tag{13}$$

from now through Section 10. That is, we assume the manifolds that we are considering are just $\mathbb{C}P^1$ bundles and we defer other situations to another paper. Since the modified Calabi functional

$$\mathrm{Cal} = \int_X (R - HR - \phi_E)^2 \omega^n = A\int_{-1}^{1} (\tilde{R} - a - bU)^2 Q\, dU,$$

is decreasing with a constant $A = \int \omega^n / \int Q\, dU > 0$ and is always positive, by (2) we have

$$\int \big(2\Delta - (Q\varphi)'' - 2(a+bU)Q\big)^2 \frac{dU}{Q} < C.$$

Now, since $(Q\varphi^0)'' = 2\Delta - 2(a+bU)Q$ — see (5) — and Q is a polynomial of U, we get

$$\int \big((Q(\varphi-\varphi^0))''\big)^2 dU < B \int \big((Q(\varphi-\varphi^0))''\big)^2 \frac{dU}{Q} < C,$$

with a constant $B > Q$ (since Q is a polynomial of U).

By the Sobolev embedding theorem, $Q\varphi$ is $C^{1+\frac{1}{2}}$, since $\varphi(-1) = 0$ and $\varphi'(-1) = 2$ by the last sentence of Lemma 2.5. Therefore, a subsequence of $Q\varphi$ converges to $Q\varphi_1$ in $C^{1+\frac{1}{2}}$. We must prove that φ converges to φ_1. First we want to see that φ_1 cannot have a zero in $(-1, 1)$ since the modified Mabuchi functional is bounded and φ_1 has a continuous first derivative. If $\varphi_1(U_0) = 0$, because U_0 is the minimal point we have $\varphi_1'(U_0) = 0$ and $\varphi_1(U) < C(U - U_0)$ for a positive number C. Also $\varphi_1^{-1} > (C(U - U_0))^{-1}$ is not integrable. This will be in contradiction to the boundedness of the modified Mabuchi functional.

Now applying the Sobolev embedding theorem to the modified Mabuchi functional, which is just $\int dt \int (\dot{G})^2 dV$, we see that G is continuous on t almost everywhere, since G is given and continuous near $t = 0$. It follows that if a subsequence of φ converges to some other φ_2, then $\varphi_2 = \varphi_1$. Thus, under the modified Calabi flow, φ converges to φ_1 in $C^{1+\frac{1}{2}}$.

Set $S = \tilde{R} - a - bU$. The equation for φ is

$$\dot{\varphi} = 2\varphi^2 S'', \tag{14}$$

with $\varphi(-1) = \varphi(1) = 0$, $\varphi'(-1) = -\varphi'(1) = 2$. By our short time existence result (Theorem 4.1), this equation has a solution for $t \in [0, T)$ for a positive number T. For later reference, note that the fact that φ is C^k implies the same for the solutions of the original metric, but the converse is not true. However, by the weighted Sobolev inequalities we shall see also later on that there is an integer k_0 such that if the original metric solutions are C^{2k+k_0}, then φ is C^k. Hence there is equivalence between the C^∞ properties.

The function φ has the disadvantage that $\varphi(-1) = \varphi(1) = 0$; that is, even if φ is smooth and bounded, it might be negative at some point. Hence, we let $\varphi = \varphi^0 e^\theta$, which gives

$$\dot{\varphi} = \varphi\dot{\theta}.$$

Now

$$\dot{\theta} = 2\varphi S'',$$

and $\theta(-1) = \theta(1) = 0$.

6. Estimate of the C^0 norm of θ and the derivative of the modified Calabi functional on $[0, T)$

From [Guan 2003, p. 281] we know that the derivative of the modified Calabi functional is

$$-2\int (\varphi S'')^2 Q\, dU.$$

We'd like to see that this is bounded for $t \in [0, T)$. To see this we $A = \varphi S''$. Then

$$\begin{aligned}
\frac{d}{dt}\Big(\int A^2 Q\, dU\Big) &= 2\int A\dot{A} Q\, dU = 2\int A(\dot{\varphi}S'' + \varphi(\dot{S}''))Q\, dU \\
&= 2\int A\Big(2\varphi^2(S'')^2 - \varphi\Big(\frac{1}{2Q}(\dot{\varphi}Q)''\Big)''\Big) Q\, dU = \\
&= 4\int \varphi^3(S'')^3 Q\, dU - 2\int \varphi^2 S''\left(\frac{(\varphi^2 S'' Q)''}{Q}\right)'' Q\, dU \\
&= 4\int A^3 Q\, dU - 2\int \big((\varphi Q A)''\big)^2 \frac{dU}{Q}.
\end{aligned} \tag{15}$$

The following is similar to [Caffarelli et al. 1984, p. 262 (A)]:

Lemma 6.1. *For $u(x) = x^k v(x)$ and $v(x) \in C_0(\mathbb{R})$, $1 < r \le +\infty$, $1 \le k$,*

$$\big||x|^{-k}u\big|_{L^r} \le C\,\big||x|^{-k+1}u'\big|_{L^r}.$$

In particular,

$$\Big(\int_{-1}^{1}((1-x^2)^{-k}u)^r dx\Big)^{1/r} \le C\Big(\int_{-1}^{1}((1-x^2)^{-k+1}u')^r dx\Big)^{1/r},$$

for $u(x) = (1-x^2)^k w(x)$ and $w(x) \in C[-1, 1]$.

Proof. Without loss the generality we may assume that $u(x) = 0$ if $x < 0$ and $x > a > 0$. Integrating by parts,

$$\int (x^{-k}u)^r dx = -\int_0^a x((x^{-k}u)^r)' dx = rk\int (x^{-k}u)^r dx - r\int x^{-rk+1}u^{r-1}u' dx.$$

Therefore,

$$\begin{aligned}
(rk-1)\int (x^{-k}u)^r dx &= r\int x^{-rk+1}u^{r-1}u' dx \\
&\le r\Big(\int (x^{-k}u)^r dx\Big)^{1/r}\Big(\int \big(x^{-k+1}u'\big)^r dx\Big)^{(r-1)/r},
\end{aligned}$$

which implies

$$\Big(\frac{rk-1}{r}\Big)^{1/(r-1)}\big||x|^{-k}u\big|_{L^r} \le \big||x|^{-k+1}u'\big|_{L^r}.$$

When $r = +\infty$ we just take the limit and get $\sup |x^{-k}u| \le \sup |x^{-k+1}u'|$, proving the first part of our lemma.

For the second part, we let $v(x) = (1-x)^k w(x)$. Then we apply the argument above, and get

$$\begin{aligned}\left(\int_{-1}^{0}((1-x^2)^{-k}u)^r dx\right)^{1/r} &\le \left(\int_{-1}^{0}((1+x)^{-k}u)^r dx\right)^{1/r}\\ &\le \left(\int_{-1}^{1}((1+x)^{-k}u)^r dx\right)^{1/r}\\ &\le C\left(\int_{-1}^{1}((1+x)^{-k+1}u')^r dx\right)^{1/r}\end{aligned}$$
$$\le C\left(C_1\left(\int_{-1}^{0}((1-x^2)^{-k+1}u')^r dx\right)^{1/r} + \left(\int_{0}^{1}\left((1-x^2)^{-k+1}u'\right)^r dx\right)^{1/r}\right).$$

We apply the same argument to [0, 1], and get the second part of our lemma. □

Now we apply the condition (13), which implies that Q is a positive polynomial of U on $[-1, 1]$, and the boundary conditions of (14). We obtain

$$C_1(1-x^2) \le \varphi Q \le C_2(1-x^2),$$

for positive constants C_1, C_2 depending on t. in the next paragraph we show that on $[0, T)$ these two inequalities hold with constants independent on t. This is the same as saying that θ is uniformly bounded (we need only apply the argument below to a subsequence of $t \to T$ if needed, since if the above estimate does not hold, it does not hold for a convergent subsequence of φ with a subsequence of t).

Since $\varphi = \varphi^0 e^\theta$ with $\theta(-1) = \theta(1) = 0$, the fact that $(\varphi Q)''$ and hence φ'' has a bounded L^2 norm (this is basically from the boundedness of the modified Calabi functional, see the discussion at the bottom of page 107) implies that $(\varphi^0(e^\theta - 1))''$ has a bounded L^2 norm. Therefore $(\varphi^0)^{-1}(\varphi^0(e^\theta-1))'$ and $(\varphi^0)^{-1}(e^\theta-1)$ have bounded L^2 norms. Hence, $(e^\theta-1)' = e^\theta\theta'$ also has bounded L^2 norm, and e^θ is uniformly continuous. Therefore, e^θ can be extended to $t = T$. Because $\varphi'(-1) = -\varphi'(1) = 2$ at $t = T$, we see that

$$e^{\theta(-1)} = e^{\theta(1)} = 1.$$

By uniform continuity, this means there is $\epsilon_0 > 0$ such that when the distance $d(U, \{-1, 1\})$ is less than ϵ_0 we have $e^\theta > \epsilon_0$. By uniform convergence, this is also true for t close enough to T. Therefore, e^θ is bounded from 0 near the boundary points $-1, 1$. We also conclude that e^θ has no zero $a \in [-1+\epsilon_0, 1-\epsilon_0]$; otherwise, since $(e^\theta)'(a) = 0$, there is a number $\epsilon_1 > 0$ such that for any small $M > 0$, there is a $\delta > 0$ such that $e^\theta \le |U - a|$ whenever $\epsilon_1 > |U - a| > M$ and $T - t < \delta$, that is, the modified Mabuchi functional $\int (e^{-\theta}-1+\theta)\,dU$ will turn to $+\infty$ since $\lim_{\theta\to-\infty}(e^{-\theta}-1+\theta)/e^{-\theta}=1$ and $e^{-\theta}-1+\theta\geq 0$ always. Therefore, e^{θ} is also bounded away from zero. Thus θ is bounded and continuous, and $\theta(-1)=\theta(1)=0$ at $t=T$. This proves that the constants C_1, C_2 are independent of t.

We have

$$\int(\varphi^{-1}A)^2dU \leq C_1\int(\varphi^{-1}(\varphi QA)')^2dU \leq C_2\int((\varphi QA)'')^2dU. \tag{16}$$

Now

$$\varphi^{-1}(\varphi QA)' = QA' + \varphi^{-1}A(Q\varphi)',$$

and

$$(\varphi QA)'' = 2(\varphi Q)'A' + A(\varphi Q)'' + \varphi QA'',$$

giving

$$\begin{gathered}\int(A')^2dU \leq C_1\int((\varphi QA)'')^2dU,\\ \int A^2dU \leq C_2\int((\varphi QA)'')^2dU, \\ \int(\varphi QA'')^2dU \leq C_3\int((\varphi QA)'')^2dU.\end{gathered} \tag{17}$$

The last inequality is true because

$$\int(A(\varphi Q)'')^2dU \leq \sup\{A^2\}\int((\varphi Q)'')^2dU \leq C_1\biggl(\int|A'|\,dU\biggr)^2 \leq C_2\int(A')^2dU.$$

Similarly, we have (see [Lin 1986])

$$\begin{aligned}\int A^3dU &\leq \sup\{|A|\}\int A^2dU \leq \int|A'|\,dU\int A^2dU\\ &\leq \biggl(\int(A')^2dU\biggr)^{1/2}\int A^2dU \leq C\biggl(\int((\varphi QA)'')^2dU\biggr)^{1/2}\int A^2Q\,dU.\end{aligned}$$

Combining this with (15) and Young's inequality $ab\leq \epsilon a^2+\epsilon^{-1}b^2$, we get

$$\frac{d}{dt}\biggl(\int A^2Q\,dU\biggr) \leq C\biggl(\int A^2Q\,dU\biggr)^2 - (1-\epsilon)\int((\varphi QA)'')^2\,\frac{dU}{Q}. \tag{18}$$

If we let $L=\int A^2Q\,dU$, we get $\dot{L}\leq CL^2$ and

$$\frac{d}{dt}(\ln L) \leq CL. \tag{19}$$

But $-2L$ is the derivative of the modified Calabi functional, so $\int_0^T L\,dt$ is bounded. Integrating (19) we conclude that L is bounded on $[0,T)$.

This implies that φ_1 is $C^{3+\frac{1}{2}}$ for points other than -1 and 1. In particular, S' is continuous in $(-1, 1)$. So we have a good inner estimate.

7. Estimate of C^2 norm of φ on $[0, T)$

For the boundary estimate we shall again apply a Hardy inequality, since φ has zeros at $U = -1$ and $U = 1$. This is another place where the condition (13) applies, otherwise Q also has zeros at the boundary. In our case we should have the Hardy inequality as follows:

$$\int_{-1}^{1} (S')^2 dU \le C \int_{-1}^{1} \varphi^2 (S'')^2 Q\, dU; \tag{20}$$

see, for example, [Caffarelli et al. 1984, p. 262 (A)]. Since

$$\int_{-1}^{1} S'(1-U^2)\, dU = S(1-U^2)|_{-1}^{1} + 2\int_{-1}^{1} SU\, dU = 0,$$

we observe that $S'(a) = 0$ for some $a \in (-1, 1)$. Then

$$\begin{aligned}\int_{-1}^{a} (S')^2 dU = \int_{-1}^{a} (U+1)'(S')^2 dU &= (U+1)(S')^2\Big|_{-1}^{a} - 2\int_{-1}^{a} (U+1) S' S'' dU \\ &\le 2\left(\int_{-1}^{a} (S')^2 dU\right)^{1/2} \left(\int_{-1}^{a} ((U+1)S'')^2 dU\right)^{1/2},\end{aligned}$$

i.e., $\int_{-1}^{a} (S')^2 dU \le 4\int_{-1}^{a} ((1+U)S'')^2 dU$. Similarly,

$$\int_{a}^{1} (S')^2 dU \le 4\int_{a}^{1} ((U-1)S'')^2 dU.$$

Combining these two inequalities we obtain our Hardy inequality. That is, S is in the Hölder space $C^{1/2}$ since $\int S\, dU = 0$ and S is continuous. Therefore, φ_1 is C^2.

Even without using the fact that $S'(a) = 0$ for a point $a \in (-1, 1)$ we can still have our conclusion. In fact by [Friedman 1963, p. 19, (8.2)], we have:

Lemma 7.1. *If $u \in L^2[-1, 1]$ and*

$$\int_{-1}^{1} (x^2-1)^2 (u^{(k)})^2 dx$$

is bounded, then $u^{(k-1)}$ is L^2 bounded. In particular, $u^{(k-2)}$ is continuous.

Proof. We apply [Friedman 1963, p. 19, (8.2)] to the closed interval $[-0.5, 0.5]$ and see that both $\left|\int u^{(k-1)} dx\right|$ and $\left|\int u^{(k)} dx\right|$ are bounded. That is, the average of $u^{(k-1)}$ and the difference of $u^{(k-1)}$ at two points is bounded, which implies that

$u^{(k-1)}(0)$ is bounded. Applying the argument above we get

$$\int_{-1}^{0}\big(u^{(k-1)}\big)^2dx \le \big(u^{(k-1)}(0)\big)^2+2\bigg(\int_{-1}^{0}(u^{(k-1)})^2dx\bigg)^{1/2}\bigg(\int_{-1}^{0}((x+1)u^{(k)})^2dx\bigg)^{1/2}.$$

Now with Young's inequality we have the first part of our lemma. The second part follows from the Sobolev embedding theorem. □

8. Estimates of H^4 and C^3 norm of φ on $[0, T)$

To obtain H^4 norm estimates of φ we let O be an operator such that $O(u) = \varphi u''$. Then $A = O(S)$. We let O^* be the dual of O. If $u \in C_0^\infty(\mathbb{R})$, we have that

$$\int uO^*(v)Q\,dU = \int O(u)vQ\,dU = \int \varphi u''vQ\,dU = \int u\frac{(\varphi Qv)''}{Q}Q\,dU.$$

Therefore, $O^*(v) = (\varphi Qv)''/Q$.

Let $B = Q^{-1}(\varphi QA)'' = O^*(A)$ and $A_1 = O(B)$, $B_i = O^*(A_i)$, $A_i = O(B_{i-1})$. We shall estimate $K = \int B^2Q\,dU$. Let $L_i = \int A_i^2Q\,dU$.

We have

$$\dot K = 8\int A_1A^2Q\,dU - 2\int A_1^2Q\,dU \le C\bigg(\int A_1^2dU\bigg)^{1/2}\bigg(\int A^4dU\bigg)^{1/2} - 2L_1.$$

As before we have that

$$\begin{aligned}\int A^4dU &\le \sup\{A^2\}\int A^2dU \le \bigg(\int |A'|\,dU\bigg)^2\int A^2dU \\ &\le C\int (A')^2Q\,dU\int A^2dU \le C_1\int B^2Q\,dU.\end{aligned}\tag{21}$$

Therefore,

$$\dot K \le CL_1^{1/2}K^{1/2} - 2L_1.\tag{22}$$

Again, by Young's inequality we have $\dot K \le C_1K$ and hence $\partial(\ln K)/\partial t \le C_2$, where C_2 only depends on the number L.

Therefore, K is bounded by a function of the bound of L. Thus, $(\varphi QA)'$ has a uniform $C^{1/2}$ norm.

By (17), $\int(A')^2dU$ is bounded. Therefore A is continuous. Also $\varphi QA'$ is continuous.

By (16), $\int(S'')^2dU$ is bounded and the H^4 norm of φ is bounded. Therefore S' is continuous, since S' has a zero point. Thus $\varphi^{(3)}$ is continuous.

Since φ'' is bounded, so is $(\varphi^0(e^\theta - 1))''$. By Lemma 6.1 $(\varphi^0)^{-1}(\varphi^0(e^\theta - 1))'$ and, hence, $(\varphi^0)^{-1}(e^\theta - 1)$ are bounded. So $(e^\theta - 1)'$ is bounded, i.e., $(e^\theta)'$ and θ' are bounded.

By the boundedness of $\varphi^{(3)}$, there are two numbers c, d such that

$$e^\theta - 1 - (c + dU)(1 - U^2) = (1 - U^2)^2 f_1$$

with f_1 continuous. The numbers c, d are determined by $\theta'(-1)$ and $\theta'(1)$, which are bounded by the C^0 norm of θ'. Now,

$$\big(\varphi^0(e^\theta - 1 - (c + dU)(1 - U^2))\big)^{(3)}$$

is bounded. So are

$$(\varphi^0)^{-1}\big(\varphi^0(e^\theta - 1 - (c + dU)(1 - U^2))\big)'',$$
$$(\varphi^0)^{-2}\big(\varphi^0(e^\theta - 1 - (c + dU)(1 - U^2))\big)',$$
$$(\varphi^0)^{-2}(e^\theta - 1 - (c + dU)(1 - U^2)),$$

as are $(\varphi^0)^{-1}(e^\theta - 1 - (c + dU)(1 - U^2))'$ and $(e^\theta - 1 - (c + dU)(1 - U^2))''$. Thus $(e^\theta)''$ and θ'' are bounded.

In the same way we have:

Lemma 8.1. *If $\varphi^{(k)}$ has L^r bound, then so does $\theta^{(k-1)}$.*

Proof. We only need to prove the case in which r is a finite number. If $\varphi^{(k)}$ is L^r bounded, $\varphi^{(k-1)}$ is C^0 bounded. So is $\theta^{(k-2)}$. By applying a similar argument as above with Lemma 6.1 and L^r norms we have our Lemma. □

Therefore, the H^3 norm of θ is bounded.

9. Estimates of higher order derivatives on [0, T)

To get higher-order estimates, we try to estimate L_i and $K_i = \int B_i^2 Q\, dU$. We can regard these functionals as higher order Calabi functionals.

Lemma 9.1. *If L_i, K_j are bounded for $i \le l$, $j < l$ (resp. $i, j \le l$), then B'_{l-1} (resp. A'_l) is bounded in L^2 norm and B_{l-1} is bounded (resp. $(\varphi Q A_l)'$ is bounded). Moreover, $\varphi^{(2l+2)}$ (resp. $\varphi^{(2l+3)}$) and $\varphi^{2l+1}\varphi^{(4l+3)}$ (resp. $\varphi^{2l+2}\varphi^{(4l+5)}$) are bounded and continuous.*

Proof. This is an extension of Lemma 7.1. When L is bounded we already see that $\varphi^{(2)}$ is bounded. Also

$$(\varphi S')' = \varphi' S' + \varphi S''$$

is L^2 bounded. So $\varphi S'$ and, hence, $\varphi\varphi^{(3)}$ are bounded. When K is bounded, $(\varphi Q A)'$ is bounded, and so are $\varphi A'$ and $\varphi^2 S'''$. Therefore, $\varphi^2\varphi^{(5)}$ is bounded. The lemma is true for $l = 0$.

If L_1, L, K are bounded, then B' is bounded in the L^2 norm. B is bounded. $\varphi^{-1}(\varphi Q A)'$ is bounded. $\varphi^{-1} A = S''$ is bounded. Hence, $\varphi^{(4)}$ is. $A' = (\varphi S'')'$ is bounded, so $\varphi S'''$ is bounded. But

$$(\varphi Q A)'' = (\varphi^2 Q)'' S'' + 2(\varphi^2 Q)' S''' + \varphi^2 Q S^{(4)},$$

which means that $\varphi^2 S^{(4)}$ is bounded. Hence $\varphi^2\varphi^{(6)}$ is bounded. But the derivative

$$(\varphi B')' = \varphi' B' + \varphi B''$$

is also L^2 bounded. Therefore, $\varphi B'$ is bounded. Moreover,

$$\varphi B' = \varphi(\varphi Q A)''' = \varphi\big((\varphi^2 Q)'' S'' + 2(\varphi^2 Q)' S''' + \varphi^2 Q S^{(4)}\big)',$$

which means that $\varphi^3 Q S^{(5)}$ is bounded. Hence $\varphi^3\varphi^{(7)}$ is bounded.

If K_1, L, K, L_1 are bounded, so is $(\varphi Q A_1)'$. Hence, $Q A_1$ and $\varphi^2 B'''$, as well as B', are all bounded. Now $B' = (Q^{-1}(\varphi^2 Q S'')'')'$, $(\varphi^2 Q S'')'''$ is bounded. Since S'' is bounded, there are two numbers c, d such that $\varphi^2 S'' - (c + dU)\varphi^2 = \varphi^3 f_1$ for a continuous function f_1. Then $\varphi^{-1}(S'' - (c+dU))$ is bounded as before, so is $(S'' - (c+dU))' = S''' - d$. Therefore, S''' and $\varphi^{(5)}$ are bounded. Hence $\varphi S^{(4)}$ and $\varphi^2 S^{(5)}$ are both bounded. Therefore, because $A_1 = \varphi B''$ is bounded, so is $\varphi^3 S^{(6)}$, and because $(\varphi Q A_1)'$ is bounded, so is $\varphi^4 S^{(7)}$. So $\varphi^4\varphi^{(9)}$ is bounded.

The same argument works for all l. We can also apply the proof of the next lemma. □

Furthermore if we let $O_1(u) = \varphi u'$ and $O_2(u) = u'$, then we call O_2 the pure derivative and O_1 the coupled derivative, and we have:

Lemma 9.2. *If L_i, K_i are bounded for i, $j \le l$ (resp. $i \le l$, $j < l$), then*

$$O_{i_1} Q_{i_2} \cdots O_{i_k}\varphi$$

is bounded for at most $2l + 3$ pure derivatives O_2 and at most $2l + 2$ coupled derivatives O_1 (resp. at most $2l+2$ pure derivatives and $2l+1$ coupled derivatives) in $(O_{i_1}, \dots, O_{i_k})$. Moreover, it is L^2 bounded for at most $2l + 4$ pure derivatives and $2l + 2$ coupled derivatives (resp. $2l + 3$ pure derivatives and $2l + 1$ coupled derivatives).

Proof. If L is bounded, then $\varphi^{(3)}$ is L^2 bounded and φ'' is bounded.

$$O_2^k O_1 O_2^{2-k}\varphi = \sum_{m \le k} (O_2^m\varphi)(O_2^{3-m}\varphi).$$

When $0 < m \le 2$ the first factor is bounded and the second factor is also bounded. When $m = 0$ we have $\varphi\varphi^{(3)}$, which is also bounded. We also have that

$$O_2^k O_1 O_2^{3-k}\varphi = \sum_{m \le k} (O_2^m\varphi)(O_2^{4-m}\varphi).$$

When $0 < m \le 3$ the first factor is bounded and the second factor is L^2 bounded. When $m = 0$ the whole term is L^2 bounded since L is bounded.

If L, K are bounded we want to see that

$$O_{i_1} \cdots O_{i_k} O_1 O_2 O_{j_1} \cdots O_{j_{3-k}}\varphi,$$

with three pure derivatives and two coupled derivatives is bounded if and only if

$$O_{i_1}\cdots O_{i_k}O_2O_1O_{j_1}\cdots O_{j_{3-k}}\varphi,$$

is bounded. Actually, the difference between them is

$$\begin{aligned}O_{i_1}&\cdots O_{i_k}(\varphi' O_2O_{j_1}\cdots O_{j_{3-k}}\varphi)\\ &=\sum_{m\le k}\sum_{(k_1,\dots,k_m)\subset(i_1,\dots,i_k)}(O_{k_1}\cdots O_{k_m}\varphi')(O_{l_1}\cdots O_{l_{k-m}}O_2O_{j_1}\cdots O_{j_{3-k}}\varphi),\end{aligned}$$

where $(k_1,\dots,k_m)$ is an ordered subset of the ordered set $(i_1,\dots,i_k)$, having $(l_1,\dots,l_{k-m})$ as its ordered complement. When $0\le m<3$ the first factor is bounded, and likewise is also bounded, the second factor being a sum of products of two bounded factors. When $m=k=3$ the second factor is φ' and is bounded, and so is the first factor, being a sum of products of two bounded factors. An easier alternative proof is that we use 2 in the place of 3 in the above argument first, then do the case for 3. The proof is similar for the L^2 bounded case.

The same argument can be carried out for all l by induction. □

Lemma 9.3. *If L_i, K_j are bounded for $i, j\le l$ (resp. $i\le l$, $j<l$), then L_{l+1} (resp. K_l) is bounded.*

Proof. If L_i, K_j are bounded for $i, j\le l$, then

$$\begin{aligned}\dot{L_{l+1}}=-2K_{l+1}+4\int A_{l+1}^2AQ\,dU\\ +8\sum_{k=0}^{l}\int B_{l+1}\big((O^*O)^kO^*(AO((O^*O)^{l-k}S))\big)Q\,dU.\end{aligned}$$

Let $O=O_1O_2$ with $O_1(u)=\varphi u'$, $O_2(u)=u'$. Then $O^*=O_3O_4$ with $O_3(u)=Q^{-1}u'$ and $O_4(u)=(\varphi Qu)'$. We see that $O_i(uv)=O_i(u)v+uO_i(v)$ for $i=1, 2, 3$ and

$$O_4(uv)=(\varphi Quv)'=(\varphi Qu)'v+\varphi Quv'=O_4(u)v+QuO_1(v).$$

Let $O_5=QO_1$; then

$$\begin{aligned}(O^*O)^k&O^*(AO((O^*O)^{l-k}S))\\ &=\sum(O_{i_1}O_{i_2}\cdots O_{i_m}A)(O_{j_1}O_{j_2}\cdots O_{j_{4k-m+2}}O(O^*O)^{l-k}S),\end{aligned}$$

with half, i.e., $2k+1$, of the O_i and O_j being O_2 and O_3. If we forget the derivatives coupled with φ, then there are at most $2k+1+3$ pure derivative of φ from the first factor. Also there are at most $2k+1+1$ coupled derivatives from the first factor. Therefore, if $k<l$, then the first factor is bounded. The second factor has bounded L^2 norm since K_l is bounded.

So we only need to check the term in which $k = l$ and the first factor has $2l+4$ pure derivatives. Then the second factor has only 3 pure derivatives and at most $2l+2$ coupled derivatives. Therefore, in this case, the second factor is bounded. However, again the first factor is L^2 bounded. Therefore,

$$\dot{L_{l+1}} \leq C_1 L_{l+1} + C_2.$$

By integration, we see that $\ln(C_1 L_{l+1} + C_2)$ is bounded. Hence L_{l+1} is bounded.

If L_i, K_j are bounded for $i \leq l$, $j < l$, we have

$$\dot{K}_l = -2L_{l+1} + 8\sum_{k=0}^{l} \int A_{l+1}\big((OO^*)^k(A(OO^*)^{l-k}OS)\big)Q\,dU$$

and

$$(OO^*)^k\big(A(OO^*)^{l-k}OS\big) = \sum (O_{i_1}\cdots O_{i_m}A)\big(O_{j_1}\cdots O_{j_{4k-m}}(OO^*)^{l-k}OS\big),$$

with half, i.e., $2k$, of the O_i and O_j being O_2 and O_3. If we forget the derivatives coupled with φ, there are at most $2k+3$ pure derivatives of φ from the first factor. Also, there are at most $2k+1$ coupled derivatives from the first factor. Thus, if $k < l$, the first factor is bounded. The second factor is L^2 bounded since L_l is bounded.

Therefore, we only need to check the term in which $k = l$ and the first factor has $2l+3$ pure derivatives. Then the second factor has only 3 pure derivatives and at most $2l+1$ coupled derivatives. In the case there are $2l+1$ coupled derivatives we can treat the first one of them as pure derivative if $l > 0$, then we treat this term as having $2l$ coupled derivatives with 4 pure derivatives. Therefore the second factor is bounded. But the first factor is again L^2 bounded, so $\dot{K}_l \leq C$, showing that K_l is bounded. □

Therefore the C^k norm of the solutions are uniformly bounded on $[0, T)$. The solution can be extended to $[0, T]$.

10. Long time existence and convergence

From the general short time existence we now obtain long time existence.

The derivative of the modified Calabi functional is $-2L$ and has a subsequence of t such that L turns to zero. Therefore, the modified Calabi flow converges to φ_1 in C^1 and in C^2 with a subsequence. The modified Calabi functional is decreasing and tends to zero, giving us that $\varphi_1 = \varphi^0$.

Moreover, by (20) and the inequality

$$\dot{\mathrm{Cal}} = -2L \leq -C_1 \int (S')^2 dU \leq -C_2 \int S^2 dU = -C_2\,\mathrm{Cal}$$

we have $\mathrm{Cal} \leq \mathrm{Cal}(t_0)e^{-kt}$ with $k = C_2$, and Cal converges to zero at an exponential rate.

By integrating (19) we have

$$L(t) \leq L(t_0) \exp C \int_{t_0}^{t} L\, dt. \tag{23}$$

For any ϵ we can pick δ such that $\delta \exp \delta < \epsilon$, and then pick t_0 such that $L_{t_0} < \delta$ and $2 \int_{t_0}^{+\infty} L\, dt < (2/C)\delta$ as the modified Calabi functional at t_0. Then $L(t) < \epsilon$ for any $t > t_0$. Therefore, $\lim_{t\to+\infty} L = 0$. Now, $\sum \int_t^{t+1} L\, dt$ converges to zero at an exponential rate. With (23), we see that L also converges exponentially to zero.

In this case, (21) becomes $\int A^4 dU < \epsilon K$ when t is big enough. Therefore, (22) becomes

$$\dot{K} \leq \epsilon L_1^{1/2} K^{1/2} - L_1$$

when t is big enough. Hence $\dot{K} \leq \epsilon K$. But from (18) we have

$$2(1-\epsilon_1) \int_{t_0}^{+\infty} K\, dt + L(t_0) \leq C\epsilon \int_{t_0}^{+\infty} L\, dt \leq Ce^{-kt_0}.$$

In particular, $\int_{t_0}^{+\infty} K\, dt \leq Ce^{-kt_0}$ when t_0 is big enough. Therefore,

$$K(t) \leq K(t_0) + \epsilon \int_{t_0}^{t} K\, dt, \leq K(t_0) + Ce^{-kt_0}$$

for $t > t_0$. Picking a t_0 such that $K(t_0)$ small enough, we see that $K \leq \epsilon$. Thus $\lim_{t\to+\infty} K = 0$. Moreover, since $\int_t^{t+1} K\, dt \leq Ce^{-kt}$, we have $K \leq Ce^{-kt}$ for some positive C and k.

In particular, $\lim_{t\to+\infty} A = 0$ and A converges at an exponential rate.

Furthermore, the argument in the proof of Lemma 9.3 shows the following:

Lemma 10.1. *If $L_i, K_i \leq Ce^{-kt}$ for $i \leq l$ (resp. $L_{i+1}, K_i \leq Ce^{-kt}$ for $i < l$), then*

$$\dot{L}_{l+1} \leq -2K_{l+1} + Ce^{-kt}(K_{l+1}^{1/2} + L_{l+1}) \quad \big(\textit{resp. } \dot{K}_l \leq -2L_{l+1} + Ce^{-kt} L_{l+1}^{1/2}\big).$$

Therefore, if $L_i, K_i \leq Cr^{-kt}$ for $i \leq l$, then $\dot{L}_{l+1} \leq \epsilon(L_{l+1} + Ce^{-kt})$. So

$$L_{l+1} \leq (L_{l+1}(t_0) + Ce^{-kt_0})e^{\epsilon}. \tag{24}$$

But we also have $\dot{K}_l \leq -2(1-\epsilon_1)L_{l+1} + Ce^{-kt}$, meaning that

$$2(1-\epsilon_1) \int_t^{+\infty} L_{l+1} dt + K_l \leq \frac{C}{k} e^{-kt}.$$

Thus $\int_t^{t+1} L_{l+1} dt \leq Ce^{-kt}$, and $L_{l+1} \leq Ce^{-kt}$ for some $C, k > 0$.

Similarly if $L_{i+1}, K_i \le Ce^{-kt}$ for $i < l$, then $\dot{K}_l \le Ce^{-kt}$ and $K_l \le K_l(t_0) + (C/k)e^{-kt_0}$. But we also have $\dot{L}_l \le -2(1-\epsilon_1)K_l + Ce^{-kt}$, which implies

$$2(1-\epsilon_1)\int_t^{+\infty} K_l dt + L_l \le \frac{C}{k}e^{-kt}.$$

Hence $\int_t^{t+1} K_l dt \le (C/k)e^{-kt}$ and $K_l \le C_1 e^{-kt}$ for some $C_1, k > 0$.

From these estimates we see that φ converges exponentially to φ^0 in any C^m norm. Therefore:

Theorem 10.2. *In the cases of* $D_0 = D_\infty = 1$, *i.e.*, *when the surface has hypersurface ends*, *the modified Calabi flow converges exponentially in* C^m *norm for any* m *to the extremal metric on our manifolds whenever the Kähler class is stable.*

11. Evolution of metrics in a Kähler class along the modified Ricci flow

Now we consider the problem of finding (generalized) quasi-Einstein metrics [Guan 1995b] in a given Kähler class by the following modified Ricci flow equation:

$$\frac{\partial}{\partial t} g = -\operatorname{Ric}(g) + \operatorname{HRic}(g) + L_{V_{\mathbb{R}}} g. \tag{25}$$

By contraction we get

$$\frac{\partial}{\partial t} \log\det g = \Delta \log\det g + HR + \operatorname{tr}_g L_{V_{\mathbb{R}}} g. \tag{26}$$

We can easily see that these two equations are equivalent. Once we have a solution g_t for (26), we get

$$-\operatorname{Ric}(g) + \operatorname{HRic}(g) + L_{V_{\mathbb{R}}} g = \partial\bar{\partial} f,$$

for some f with $\int_M f dV_g = 0$ and $\Delta f = (\partial/\partial t)\log\det g$. Let $g_{t,i\bar{j}} = g_{0,i\bar{j}} + \partial_i\bar{\partial}_j u$. Then $f = (d/dt)u + C(t)$, where $C(t)$ is a function that only depends on t. Thus, $(d/dt)g_{i\bar{j}} = \partial_i\bar{\partial}_j f$, which means that g_t is a solution of (25).

Now we consider the short time existence of (26). With

$$h = (\exp(-Vt))_* g,$$

that equation is equivalent to

$$\frac{\partial}{\partial t} \log\det h = \Delta \log\det h + HR. \tag{27}$$

Set $h_t = h_0 + \partial\bar{\partial} F_t$. Then linearized equation of (27) is then

$$\frac{\partial}{\partial t}\Delta v = \Delta^2 v + R^{i\bar{j}} v_{i\bar{j}}, \tag{28}$$

where $v_{i\bar{j}} = \partial_i\bar{\partial}_j v$.

We want to prove that (28) has unique solution. Multiplying it by $2\Delta v$ and integrating, we get

$$\frac{d}{dt}\int(\Delta v)^2 = -2\int|\nabla\Delta v|^2 + 2\int \Delta v R^{i\bar{j}} v_{i\bar{j}} + 2\int \Delta F(\Delta v)^2$$
$$\leq C_1\Big(\int(\Delta v)^2\Big)^{1/2}\Big(\int\sum_{i,j}|v_{i\bar{j}}|^2\Big)^{1/2} + C_2\int(\Delta v)^2 = C\int(\Delta v)^2,$$

where C_1, C_2 are constants not depending on t and $C = C_1 + C_2$. Letting $v(t) = \int(\Delta\phi)^2$, we get $dv/dt - Cv \leq 0$, that is, $(d/dt)e^{-Ct}v \leq 0$. Thus $e^{-Ct}v$ is decreasing, so $v = 0$ if $v(0) = 0$. Thus, we have the short time existence for the evolution.

In the general case of the extremal soliton metrics, we consider the equation

$$\frac{\partial}{\partial t}\log(\det(g)) = -R + HR + \phi_1 + \Delta\phi_2.$$

The linear equation is

$$\frac{\partial}{\partial t}\Delta v = \Delta^2 v + (R^{i\bar{j}} - \phi_2^{i\bar{j}})v_{i\bar{j}} + \Delta(E_2(v)) + \tfrac{1}{2}(E_1^i v_i + E_1^{\bar{i}} v_{\bar{i}}).$$

The proof for short time existence still holds, since for the extra terms we have

$$\int \Delta v \Delta E_2(v)\,dV = -\int\big(\bar{\partial}\Delta v, \bar{\partial}E_2(v)\big)\,dV$$
$$\leq C\Big(\int|\nabla\Delta v|^2 dV\Big)^{1/2}\Big(\int|v_{j\bar{j}}|^2 dV\Big)^{1/2}$$
$$= C\Big(\int|\nabla\Delta v|^2 dV\Big)^{1/2}\Big(\int(\Delta v)^2 dV\Big)^{1/2},$$

$$\int \Delta v E_1(v)\,dV \leq C\Big(\int(\Delta v)^2 dV\Big)^{1/2}\Big(\int|v_i|^2 dV\Big)^{1/2}$$
$$\leq C\Big(\int(\Delta v)^2 dV\Big)^{3/4}\Big(\int v^2 dV\Big)^{1/4} \leq \lambda_1^{-1/2}C\int(\Delta v)^2 dV,$$

where $\lambda_1 > 0$ is the first eigenvalue of the Laplacian Δ. We also used Young's inequality in the calculation.

In this paper we only consider the special situation of the completions of $\mathbb{C}^*$-bundles studied in previous sections. To avoid confusion we use ∂_t to denote the partial derivative of t for a function of s and t, i.e., with respect to the coordinates of the manifold; and we use d/dt to denote the partial derivative of t for a function of U and t, i.e., with respect to the moving coordinates. Equation (25) can be written as

$$\frac{1}{\varphi}\partial_t\varphi(s,t)+\frac{1}{Q}\partial_t Q(s,t)=-\frac{\Delta}{Q}+\frac{1}{2Q}(\varphi Q)''+a+bU+\frac{c}{2Q}(\varphi Q)' \tag{29}$$

And since $\partial_t H(U^0(s,t))=0$, we get $\partial_t\Big(\varphi(s,t)\dfrac{dU^0}{dU}(s,t)\Big)=0$, which implies

$$\frac{d}{dU}(\partial_t U)=\varphi^{-1}\partial_t\varphi(s,t).$$

Combining these two equations we get

$$\frac{d}{dU}(\partial_t U(s,t))+\frac{Q'}{Q}\partial_t U(s,t)=\frac{\Delta}{Q}+\frac{1}{2Q}(\varphi Q)''+a+bU+\frac{c}{2Q}(\varphi Q)'.$$

We view this as a first order equation in $\partial_t U$, and get the solution

$$\partial_t U=-\frac{1}{Q}\int_{-1}^{U}\Delta(x)\,dx+\frac{1}{2Q}(\varphi Q)'+\frac{1}{Q}\int_{-1}^{U}(a+bU)Q(x)\,dx+\frac{c}{2}\varphi-\frac{d}{Q}. \tag{30}$$

Letting $U=-1$, we obtain $d=Q(-1)$. Therefore,

$$\partial_t U=-\frac{p}{2Q}+\frac{(\varphi Q)'}{2Q}+\frac{c}{2}\varphi$$

with p as in (7).

Now, we let $\varphi=(1+\theta)\varphi^0$ with φ^0 being the extremal soliton solution. Then we combine (29) and (30) to get

$$2\varphi^0\frac{d}{dt}\theta=\varphi\varphi^0\theta''+(1+\theta)\theta\Big(\varphi^0\Big(\frac{p}{Q}\Big)'-(\varphi^0)'\Big(\frac{p}{Q}\Big)\Big)+\frac{\varphi^0 p}{Q}\theta'-(\varphi^0\theta')^2. \tag{31}$$

This is basically a nonlinear heat equation, meaning that a short time solution exists. Now we want to prove that the long time solution of (31) exists and converges uniformly to 0 at an exponential rate. Set $p_1=p/Q$. The desired result can be proved by the maximum principle as in [Koiso 1990], under the following condition on C:

The function $\varphi^0(p_1)'-(\varphi^0)'p_1$ *is negative on* $[-1,1]$.

In general, this is very difficult to check since p might not be a product of linear factors, as in [Koiso 1990].

This is additional evidence that the modified Calabi flow is more natural than the modified Ricci flow.

Moreover, it has long been known that the generalized Mabuchi functional is decreasing under the modified Ricci flow but the Calabi functional is not. For example, let our manifold be $\mathbb{C}P^1$. Then $Q=1$, $\Delta=0$, $\varphi^0=1-U^2$, $p(U)=-2U$,

$2\dot{\varphi} = (1-U^2)^2(1+\theta)\theta'' - 2U(1-U^2)\theta' - (1-U^2)^2(\theta')^2 - 2(1+\theta)\theta(1+U^2)$, and $R = -2^{-1}\varphi''$, $\mathrm{Cal} = 2^{-2}\int_{-1}^{1}(\varphi'')^2 dU$,

$$\dot{\mathrm{Cal}} = \frac{1}{2}\int_{-1}^{1}\varphi''\dot{\varphi}''dU = \frac{1}{2}\int_{-1}^{1}\varphi^{(4)}\dot{\varphi}dU + \varphi''\dot{\varphi}'\Big|_{-1}^{1}.$$

We also let $\theta = A(1-U^2)$; then $\varphi^{(4)} = 24A$, $\theta' = -2AU$, $\theta'' = -2A$. Therefore,

$$\begin{aligned}2\dot{\varphi} &= -2(1-U^2)^2\big(A(1+A(1-U^2)) + 2A^2U^2 + A^2(1+U^2)\big)\\ &\qquad\qquad - 2(1-U^2)\big(-2AU^2 + A(1+U^2)\big)i\\ &= -2A(1-U^2)^2\big(1 + A(1-U^2+2U^2+1+U^2)+1\big)\\ &= -4A(1-U^2)^2(1+A(1+U^2)).\end{aligned}$$

We have $\dot{\varphi}'\big|_{-1}^{1} = 0$. Since $1+\theta > 0$ we have

$$1 > -A(1-U^2),$$

and this holds if and only if $-A < 1$, or equivalently $A > -1$. We have

$$\begin{aligned}\lim_{A\to-1}\dot{\mathrm{Cal}} &= \lim_{A\to-1}(-48A^2)\int_{-1}^{1}(1+A(1+U^2))(1-U^2)^2dU\\ &= 48\int_{-1}^{1}U^2(1-U^2)^2dU > 0.\end{aligned}$$

Therefore, there is a negative A near -1 such that the Calabi functional Cal (same for the modified Calabi functional since they are only different by adding a constant) is not decreasing under the modified Ricci flow.

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