ROKHLIN ACTIONS AND SELF-ABSORBING C*-ALGEBRAS

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Let $\mathcal{A}$ be a unital separable C^*-algebra and $\mathcal{D}$ a K_1-injective strongly self-absorbing C^*-algebra. We show that if $\mathcal{A}$ is $\mathcal{D}$-absorbing, then the crossed product of $\mathcal{A}$ by a compact second countable group or by $\mathbb{Z}$ or by $\mathbb{R}$ is $\mathcal{D}$-absorbing as well, assuming the action satisfies a Rokhlin property. In the case of a compact Rokhlin action we prove a similar statement about approximate divisibility.

1. Introduction

Following the terminology in [Toms and Winter 2007], we call a separable, unital C^*-algebra $\mathcal{D}$ *strongly self-absorbing* if it is infinite-dimensional and the map $\mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$ given by $d \mapsto d \otimes 1$ is approximately unitarily equivalent to an isomorphism (we note that strongly self-absorbing algebras are always nuclear, so there is no ambiguity when we talk of tensor products). Currently, the only known examples of such algebras are the Jiang–Su algebra $\mathscr{Z}$ [Jiang and Su 1999], the Cuntz algebras $\mathcal{O}_2$ and $\mathcal{O}_\infty$, UHF algebras of infinite type (i.e., those where all the primes which occur in the relevant supernatural number do so with infinite multiplicity) and tensor products of $\mathcal{O}_\infty$ by such UHF algebras. Those algebras exhaust the possible Elliott invariants for strongly self-absorbing algebras, and thus one might hope that this list is complete.

While only few algebras are strongly self-absorbing, many C^*-algebras $\mathcal{A}$ are $\mathcal{D}$-absorbing for a strongly self-absorbing algebra $\mathcal{D}$ (i.e. $\mathcal{A} \cong \mathcal{A} \otimes \mathcal{D}$), and such algebras seem to enjoy nice regularity properties — see [Rørdam 1991; 1992] for absorption of UHF algebras, and [Gong, Jiang and Su 2000; Rørdam 2004; Toms and Winter 2005] for absorption of the Jiang–Su algebra. Absorption of $\mathcal{O}_\infty$ and $\mathcal{O}_2$ plays a central role in the classification theorems of Kirchberg and Phillips [Kirchberg and Phillips 2000; Kirchberg ≥ 2007; Phillips 2000], and is the focus of further study (see for instance [Kirchberg and Rørdam 2002]). It thus seems interesting to study the permanence properties of $\mathcal{D}$-absorption. In [Toms and Winter 2007], it was shown that if $\mathcal{D}$ is strongly self-absorbing and K_1-injective (i.e., the

canonical map $\mathcal{U}(\mathcal{D})/\mathcal{U}_0(\mathcal{D}) \to K_1(\mathcal{D})$ is injective, a condition which is automatically fulfilled for the known examples mentioned above), then the property of being (separable and) strongly self-absorbing is closed under passing to hereditary subalgebras, quotients, inductive limits and extensions (for $\mathcal{D} = \mathcal{O}_2$ and $\mathcal{D} = \mathcal{O}_\infty$ these results had already been shown by Kirchberg; see [Kirchberg 2006]). Related results have been obtained in [Hirshberg, Rørdam and Winter 2007], [Hirshberg and Winter 2007], and [Dadarlat and Winter 2007b].

This note concerns the question of permanence under formation of crossed products. One cannot expect permanence to hold in general. Indeed, $\mathcal{O}_2$ is $\mathcal{O}_2$-absorbing; however, there are actions of $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ on $\mathcal{O}_2$ such that the crossed product has nontrivial K-theory (see [Izumi 2004a]); in particular, the crossed product algebra cannot be $\mathcal{O}_2$-absorbing. If α is such an order 2 automorphism, then $\mathcal{O}_2 \times_\alpha \mathbb{Z}_2$ is a quotient of $\mathcal{O}_2 \times_\alpha \mathbb{Z}$. Since $\mathcal{O}_2$-absorption passes to quotients, we see that $\mathcal{O}_2$-absorption is not permanent under crossed products by $\mathbb{Z}$ either.

In the present paper we show that this phenomenon does not occur if the group action satisfies certain extra conditions. More precisely, we show the following:

Theorem 1.1. *Let $\mathcal{A}$ and $\mathcal{D}$ be separable unital C^*-algebras; let $\mathcal{D}$ be strongly self-absorbing. Suppose $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ is a strongly continuous action of a group G, where G is second countable compact Hausdorff, $\mathbb{Z}$ or $\mathbb{R}$, which satisfies the respective Rokhlin property (to be made precise in the subsequent sections).*

(1) *If G is compact and $\mathcal{A}$ is $\mathcal{D}$-absorbing, then $\mathcal{A} \times_\alpha G$ is $\mathcal{D}$-absorbing. If $\mathcal{A}$ is approximately divisible, then so is $\mathcal{A} \times_\alpha G$.*

(2) *If $G = \mathbb{Z}$ or $G = \mathbb{R}$, $\mathcal{A}$ is $\mathcal{D}$-absorbing and $\mathcal{D}$ is K_1-injective, then $\mathcal{A} \times_\alpha G$ is $\mathcal{D}$-absorbing.*

There are many examples of C^*-algebras with Rokhlin actions; see [Izumi 2004a; 2004b] for the finite group case and [Kishimoto 2000; 2001] for the case where $G = \mathbb{Z}$ or $\mathbb{R}$. Rokhlin flows and a number of striking applications to the theory of purely infinite C^*-algebras have been studied in [Kishimoto 1996] and [Bratteli, Kishimoto and Robinson 2007] and the references therein. Izumi [2001] provides a survey (mainly in the case where $G = \mathbb{Z}$ or $\mathbb{R}$) and describes a number of applications to the theory of von Neumann algebras and to Elliott's classification program.

2. Central sequence embeddings and crossed products

Notation 2.1. Let $\mathcal{A}$ be a C^*-algebra. We denote

$$\mathcal{A}_\infty = \ell^\infty(\mathbb{N}, \mathcal{A})/\mathscr{C}_0(\mathbb{N}, \mathcal{A}).$$

$\mathcal{A}$ may be embedded into $\ell^\infty(\mathbb{N}, \mathcal{A})$ and into $\mathcal{A}_\infty$ in a canonical way (as constant sequences); we shall write $\iota_\mathcal{A}$ for both these embeddings — although we will

sometimes find it convenient not to state them explicitly.

We write $\mathscr{A}_\infty \cap \mathscr{A}'$ for the central sequence algebra of $\mathscr{A}$, i.e., the relative commutant of $\mathscr{A}$ in $\mathscr{A}_\infty$.

If $\alpha : G \to \mathrm{Aut}(\mathscr{A})$ is a strongly continuous action of a locally compact group G on a C^*-algebra $\mathscr{A}$, then we have naturally induced actions of G on $\ell^\infty(\mathbb{N}, \mathscr{A})$, $\mathscr{A}_\infty$ and $\mathscr{A}_\infty \cap \mathscr{A}'$ — those actions will all be denoted by $\bar{\alpha}$. They are in general not strongly continuous. We denote by $\ell^{\infty,(\alpha)}(\mathbb{N}, \mathscr{A})$ the set of elements of $\ell^\infty(\mathbb{N}, \mathscr{A})$ on which the action $\bar{\alpha}$ of G is continuous — this is clearly a C^*-algebra, which contains $\mathscr{A}$ as the constant sequences, as well as $\mathscr{C}_0(\mathbb{N}, \mathscr{A})$. We denote

$$\mathscr{A}_\infty^{(\alpha)} = \ell^{\infty,(\alpha)}(\mathbb{N}, \mathscr{A})/\mathscr{C}_0(\mathbb{N}, \mathscr{A})$$

(we stress that $\mathscr{A}_\infty^{(\alpha)}$ is not defined as the set of all elements in $\mathscr{A}_\infty$ on which G acts continuously — it is a-priori smaller). We denote by $\mathcal{M}(\mathscr{A})$ the multiplier algebra of $\mathscr{A}$. To avoid notational nuisances, we follow in this paper the convention that 0 is not considered to be a unital C^*-algebra (i.e., we require that $1 \neq 0$).

We have the following characterization of $\mathscr{D}$-absorption (based on ideas of Elliott and of Kirchberg), which appears as Theorem 7.2.2 in [Rørdam 2002]. The statement there refers to the relative commutant of $\mathscr{A}$ in an ultrapower of $\mathcal{M}(\mathscr{A})$; however, it is easy to see that the characterization still holds as stated below.

Theorem 2.2. *Let $\mathscr{D}$ be a strongly self-absorbing C^*-algebra and $\mathscr{A}$ any separable C^*-algebra. $\mathscr{A}$ is $\mathscr{D}$-absorbing if and only if $\mathscr{D}$ admits a unital $*$-homomorphism to $\mathcal{M}(\mathscr{A})_\infty \cap \mathscr{A}'$.*

Since any strongly self-absorbing C^*-algebra has to be simple, it follows that unless $\mathscr{A} = 0$, the $*$-homomorphism above must be an embedding (of course, 0 is $\mathscr{D}$-absorbing for any $\mathscr{D}$).

The following gives us a sufficient condition for $\mathscr{D}$-absorption of the crossed product of $\mathscr{A}$ by a group.

Lemma 2.3. *Let $\mathscr{A}, \mathscr{D}$ be unital separable C^*-algebras. Let G be a locally compact Hausdorff group with a strongly continuous action $\alpha : G \to \mathrm{Aut}(\mathscr{A})$. Suppose there is a unital homomorphism $\mathscr{D} \to \mathscr{A}_\infty \cap \mathscr{A}'$ whose image is fixed under the induced action of G. Then $\mathscr{D}$ admits a unital homomorphism into $(\mathcal{M}(\mathscr{A} \times_\alpha G))_\infty \cap (\mathscr{A} \times_\alpha G)'$. In particular, if $\mathscr{D}$ is strongly self-absorbing, then $\mathscr{A} \times_\alpha G$ is $\mathscr{D}$-absorbing.*

Proof. By the universal property of $\mathscr{A} \times_\alpha G$, there are a unital $*$-homomorphism

$$\pi : \mathscr{A} \to \mathcal{M}(\mathscr{A} \times_\alpha G)$$

and a strictly continuous unitary representation

$$u : G \to \mathcal{M}(\mathscr{A} \times_\alpha G)$$

such that

$$u_g \pi(a) u_{g^{-1}} = \pi \alpha_g(a) \quad \text{for all } a \in \mathcal{A}, \ g \in G.$$

π and u induce maps

$$\tilde{\pi} : \ell^\infty(\mathbb{N}, \mathcal{A}) \to \ell^\infty(\mathbb{N}, \mathcal{M}(\mathcal{A} \times_\alpha G)), \ \tilde{u} : G \to \ell^\infty(\mathbb{N}, \mathcal{M}(\mathcal{A} \times_\alpha G))$$

which in turn induce maps

$$\bar{\pi} : \mathcal{A}_\infty \to (\mathcal{M}(\mathcal{A} \times_\alpha G))_\infty, \ \bar{u} : G \to (\mathcal{M}(\mathcal{A} \times_\alpha G))_\infty.$$

Let $\varphi : \mathcal{D} \to \mathcal{A}_\infty \cap \mathcal{A}' \subseteq \mathcal{A}_\infty$ be the given homomorphism, and consider $\bar{\pi} \circ \varphi :$ $\mathcal{D} \to (\mathcal{M}(\mathcal{A} \times_\alpha G))_\infty$. We claim that the image of the unital $*$-homomorphism $\bar{\pi} \circ \varphi$ commutes with the images of $\bar{\pi} \circ \iota_\mathcal{A}$ and $\bar{u}$. That it commutes with the image of $\bar{\pi} \circ \iota_\mathcal{A}$ is immediate. As for $\bar{u}$, fix $g \in G$, $d \in \mathcal{D}$. Lift $\varphi(d)$ to $(d_1, d_2, \dots) \in$ $\ell^\infty(\mathbb{N}, \mathcal{A})$. We have $u_g \pi(d_n) u_{g^{-1}} = \pi(\alpha_g(d_n))$ for all n, so, $\tilde{u}_g \tilde{\pi}(d_1, d_2, \dots) \tilde{u}_{g^{-1}} = \tilde{\pi}(\alpha_g(d_1), \alpha_g(d_2), \dots)$. We know that

$$(\alpha_g(d_1), \alpha_g(d_2), \dots) - (d_1, d_2, \dots) \in \mathscr{C}_0(\mathbb{N}, \mathcal{A}),$$

so

$$\tilde{u}_g \tilde{\pi}(d_1, d_2, \dots) \tilde{u}_{g^{-1}} - \tilde{\pi}(d_1, d_2, \dots) \in \mathscr{C}_0(\mathbb{N}, \mathcal{M}(\mathcal{A} \times_\alpha G))$$

and thus $\bar{u}_g \bar{\pi}(\varphi(d)) \bar{u}_{g^{-1}} - \bar{\pi}(\varphi(d)) = 0$, as required.

It follows that if $f \in \mathscr{C}_c(G, \mathcal{A})$ (i.e. if f is a compactly supported continuous $\mathcal{A}$-valued function), then $\bar{\pi}(\varphi(d))$ commutes with $\int_G f(g) \bar{u}_g dg$ for any $d \in \mathcal{D}$. However, the elements of the form $\int_G f(g) \bar{u}_g dg$ for $f \in \mathscr{C}_c(G, \mathcal{A})$ generate the copy of $\mathcal{A} \times_\alpha G$ in $(\mathcal{M}(\mathcal{A} \times_\alpha G))_\infty$, and therefore, $\bar{\pi} \circ \varphi(\mathcal{D})$ commutes with the image of $\mathcal{A} \times_\alpha G$ in $(\mathcal{M}(\mathcal{A} \times_\alpha G))_\infty$, as required. The second statement follows from Theorem 2.2. $\qquad \square$

In fact, to deduce $\mathcal{D}$-absorption, it is enough to show that the conditions of Lemma 2.3 hold only approximately, as made precise in the following lemma.

Lemma 2.4. *Let $\mathcal{A}, \mathcal{B}$ be unital separable C^*-algebras. Let G be a second-countable locally compact Hausdorff group with a strongly continuous action $\alpha :$ $G \to \mathrm{Aut}(\mathcal{A})$. Suppose that for any finite sets $B_0 \subseteq \mathcal{B}, A_0 \subseteq \mathcal{A}$, any compact subset $K_0 \subseteq G$ and any $\varepsilon > 0$ there is a completely positive contraction $\varphi : \mathcal{B} \to \mathcal{A}_\infty^{(\alpha)}$ such that for all $b, b' \in B_0, a \in A_0, g \in K_0$ we have*

(i) $\|\bar{\alpha}_g(\varphi(b)) - \varphi(b)\| < \varepsilon$,

(ii) $\|\varphi(1) - 1\| < \varepsilon$,

(iii) $\|\varphi(b)\varphi(b') - \varphi(bb')\| < \varepsilon$,

(iv) $\|[\varphi(b), a]\| < \varepsilon$.

Then there is a unital homomorphism $\mathcal{B} \to \mathcal{A}_\infty \cap \mathcal{A}'$ whose image is fixed under the induced action of G.

Proof. Pick increasing sequences of finite sets $A_1 \subseteq A_2 \subseteq \cdots \subseteq \mathcal{A}$, $B_1 \subseteq B_2 \subseteq \cdots \subseteq \mathcal{B}$ such that $\mathcal{A}_0 = \bigcup_{n=1}^\infty A_n$, $\mathcal{B}_0 = \bigcup_{n=1}^\infty B_n$ form dense unital self-adjoint subrings of $\mathcal{A}$ and $\mathcal{B}$, respectively, and pick an increasing sequence $K_1 \subseteq K_2 \subseteq \cdots \subseteq G$ of compact subsets such that $\bigcup_{n=1}^\infty K_n = G$.

For each n, pick a map $\varphi_n : \mathcal{B} \to \mathcal{A}_\infty^{(\alpha)}$ which satisfies the conditions listed in the statement, with respect to A_n, B_n, K_n and $\varepsilon = 1/n$.

For each φ_n, pick a linear self-adjoint lifting (possibly unbounded)

$$\tilde{\varphi}_n = (\varphi_n(1), \varphi_n(2), \dots) : \mathcal{B} \to \ell^{\infty,(\alpha)}(\mathbb{N}, \mathcal{A}).$$

We claim that we can find a sequence p_k of natural numbers such that the following hold for all $k \in \mathbb{N}$:

(1) $\|\varphi_k(p_k)(b) - \alpha_g(\varphi_k(p_k)(b))\| < 1/k$ for all $b \in B_k$, $g \in K_k$.

(2) $\|\varphi_k(p_k)(1) - 1\| < 1/k$.

(3) $\|\varphi_k(p_k)(b_1 b_2) - \varphi_k(p_k)(b_1)\varphi_k(p_k)(b_2)\| < 1/k$ for all $b_1, b_2 \in B_k$.

(4) $\|[\varphi_k(p_k)(b), a]\| < 1/k$ for all $b \in B_k$, $a \in A_k$.

(5) $\|\varphi_k(p_k)(b)\| < \|b\| + 1/k$ for all $b \in B_k$.

For condition (1), we view $\ell^{\infty,(\alpha)}(\mathbb{N}, \mathcal{A})$ as the subspace of constant functions in $\mathcal{C}(K_k, \ell^{\infty,(\alpha)}(\mathbb{N}, \mathcal{A}))$.

Consider the linear self-adjoint map $\tilde{\varphi}'_k : \mathcal{B} \to \mathcal{C}(K_k, \ell^{\infty,(\alpha)}(\mathbb{N}, \mathcal{A}))$ given by

$$\tilde{\varphi}'_k(b)(g) = \bar{\alpha}_g(\tilde{\varphi}_k(b)).$$

By condition (i) in the statement, we know that $\|\pi_{K_k}(\tilde{\varphi}_k(b) - \tilde{\varphi}'_k(b))\| < 1/k$ for all $b \in B_k$, where π_{K_k} denotes the quotient map onto $\mathcal{C}(K_k, \mathcal{A}_\infty^{(\alpha)})$. Thus, for all but finitely many $p \in \mathbb{N}$, we have $\|\varphi_k(p)(b) - \alpha_g(\varphi_k(p)(b))\| < 1/k$ for all $b \in B_k$.

As for conditions (2)–(5), we know, from conditions (ii)–(iv) in the statement (and since the φ_n are contractions), that for all but finitely many $p \in \mathbb{N}$, conditions (2)–(5) hold with p standing for p_k. Thus, we can pick some p_k such that all the conditions hold.

Denoting $\pi : \ell^\infty(\mathbb{N}, \mathcal{A}) \to \mathcal{A}_\infty$ the quotient map, we see that

$$\pi \circ (\varphi_1(p_1), \varphi_2(p_2), \dots)$$

gives us a contractive unital self-adjoint homomorphism of $\mathcal{B}_0$ into $\mathcal{A}_\infty$ — it is a *-ring homomorphism, since all the maps involved are *-linear, and condition (3) ensures multiplicativity; it is unital by condition (2), and is contractive by condition (5). This homomorphism thus extends to a homomorphism from $\mathcal{B}$. By conditions (1) and (4), the image of $\mathcal{B}_0$ commutes with $\mathcal{A}_0$, and is fixed under the action of G.

Since $\mathcal{B}_0$ is dense in $\mathcal{B}$ and $\mathcal{A}_0$ is dense in $\mathcal{A}$, this homomorphism has the required properties.   □

3. Compact Rokhlin actions

We first recall the definition of the Rokhlin property for finite groups (see [Izumi 2004a]).

Definition 3.1. Let $\mathcal{A}$ be a unital separable C^*-algebra, G a finite group, and $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ an action. α is said to have the *Rokhlin property* if there is a partition of $1_{\mathcal{A}_\infty}$ into projections $\{e_g \mid g \in G\} \subseteq \mathcal{A}_\infty \cap \mathcal{A}'$ such that $\bar{\alpha}_g(e_h) = e_{gh}$ for all $g, h \in G$.

This definition can be generalized in a straightforward way to the case of compact Hausdorff second-countable groups, as follows.

Definition 3.2. Let $\mathcal{A}$ be a unital separable C^*-algebra, G a compact Hausdorff second-countable group, and $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ a strongly continuous action. α is said to have the *Rokhlin property* if there is a unital embedding $\mathscr{C}(G) \to \mathcal{A}_\infty^{(\alpha)} \cap \mathcal{A}'$ such that for any $f \in \mathscr{C}(G) \subseteq \mathcal{A}_\infty^{(\alpha)} \cap \mathcal{A}'$ (under this embedding), we have $\bar{\alpha}_g(f)(h) = f(g^{-1}h)$ for all $g, h \in G$.

We are indebted to the referee for pointing out to us a small mistake in an earlier version of the proof of the following theorem.

Theorem 3.3. *Let $\mathcal{A}$ be a separable unital C^*-algebra, let G be a compact Hausdorff second-countable group, and let $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ be an action satisfying the Rokhlin property. If $\mathcal{B}$ is a unital separable C^*-algebra which admits a central sequence of unital homomorphisms into $\mathcal{A}$, then $\mathcal{B}$ admits a unital homomorphism into the fixed point subalgebra of $\mathcal{A}_\infty \cap \mathcal{A}'$.*

Proof. Fix a unital embedding $\mathscr{C}(G) \to \mathcal{A}_\infty^{(\alpha)} \cap \mathcal{A}'$, as in the definition of the Rokhlin property (to lighten notation, we shall view $\mathscr{C}(G)$ as embedded in $\mathcal{A}_\infty^{(\alpha)} \cap \mathcal{A}'$ in this fashion). Let $B_0 \subseteq \mathcal{B}$, $A_0 \subseteq \mathcal{A}$ be compact subsets.

We shall find a unital homomorphism φ of $\mathcal{B}$ into the fixed point subalgebra of $\mathcal{A}_\infty^{(\alpha)}$ such that $\|[\varphi(b), a]\| < \varepsilon$ for all $b \in B_0$, $a \in A_0$. The theorem will follow, then, from Lemma 2.4.

We may assume, without loss of generality, that $\alpha_g(A_0) \subseteq A_0$ for all $g \in G$ (otherwise replace A_0 by $\bigcup_{g \in G} \alpha_g(A_0)$, which is compact, as A_0 and G are compact).

Pick a unital homomorphism $\psi : \mathcal{B} \to \mathcal{A}$ such that $\|[\psi(b), a]\| < \varepsilon$ for all $b \in B_0$, $a \in A_0$. Note that $\|[\alpha_g(\psi(b)), a]\| < \varepsilon$ for all $g \in G$ as well. We view $\mathcal{A}$ as embedded in $\mathcal{A}_\infty^{(\alpha)}$ in the usual way, and think of ψ as a homomorphism of $\mathcal{B}$ into $\mathcal{A} \subseteq C^*(\mathcal{A} \cup \mathscr{C}(G)) \subseteq \mathcal{A}_\infty^{(\alpha)}$.

Since the copies of $\mathcal{A}$ and $\mathcal{C}(G)$ commute, we have a unital homomorphism

$$\theta : \mathcal{C}(G) \otimes \mathcal{A} \to C^*(\mathcal{A} \cup \mathcal{C}(G)) \subset \mathcal{A}_\infty^{(\alpha)},$$

given by $f \otimes a \mapsto f \cdot a$ for $f \in \mathcal{C}(G)$ and $a \in \mathcal{A}$.

We now define $\varphi' : \mathcal{B} \to \mathcal{C}(G, \mathcal{A}) \cong \mathcal{C}(G) \otimes \mathcal{A}$ by

$$\varphi'(b)(g) = \alpha_g(\psi(b)).$$

Using that $\bar{\alpha}_g(\theta(f \otimes a)) = \bar{\alpha}_g(f) \cdot \alpha_g(a)$, it is straightforward to check that $\bar{\alpha}_g(\theta \circ \varphi'(b)) = \theta \circ \varphi'(b)$ for all $g \in G$, and that $\|[\theta \circ \varphi'(b), a]\| < \varepsilon$ for all $b \in B_0$, $a \in A_0$. Setting $\varphi := \theta \circ \varphi'$, we obtain the required homomorphism. $\qquad\square$

Recall that a separable unital C^*-algebra $\mathcal{A}$ is said to be *approximately divisible* if there is a finite-dimensional C^*-algebra $\mathcal{B}$ with no abelian summands, which admits a unital homomorphism into $\mathcal{M}(\mathcal{A})_\infty \cap \mathcal{A}'$, or, equivalently, if there is a central sequence of unital homomorphisms of $\mathcal{B}$ into $\mathcal{M}(\mathcal{A})$. We have the following straightforward consequence of Theorem 3.3 and Lemma 2.3.

Corollary 3.4. *Let $\mathcal{A}$ be a separable unital C^*-algebra, let G be a compact Hausdorff second-countable group, and let $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ be a strongly continuous action satisfying the Rokhlin property.*

(1) *If $\mathcal{D}$ is a strongly self-absorbing C^*-algebra and $\mathcal{A}$ is $\mathcal{D}$-absorbing, then $\mathcal{A} \times_\alpha G$ is $\mathcal{D}$-absorbing.*

(2) *If $\mathcal{A}$ is approximately divisible, then $\mathcal{A} \times_\alpha G$ is approximately divisible.*

4. Rokhlin Actions of $\mathbb{Z}$

We recall the definition of the Rokhlin property of an automorphism (see for instance [Kishimoto 2000]).

Definition 4.1. Let $\mathcal{A}$ be a unital separable C^*-algebra, and $\alpha \in \mathrm{Aut}(\mathcal{A})$. α is said to have the *Rokhlin property* if for any n there is a partition of $1_{\mathcal{A}_\infty}$ into projections $e_0, \ldots, e_{n-1}, f_0, \ldots, f_n \in \mathcal{A}_\infty \cap \mathcal{A}'$ such that $\bar{\alpha}(e_j) = e_{j+1}$, for any $j < n-1$, and $\bar{\alpha}(f_j) = f_{j+1}$ for all $j < n$.

Sometimes, also the conditions $\bar{\alpha}(e_{n-1}) = e_n$ and $\bar{\alpha}(f_n) = f_0$ are required, whereas the preceding definition only implies $\bar{\alpha}(e_{n-1} + f_n) = e_0 + f_0$. We are indebted to the referee for pointing out to us that our Theorem 4.4 below also follows with this weaker version of the Rokhlin property.

We first note the following slight enhancement of the Rokhlin property. The proof is a simple Cantor-diagonalization type trick, which we leave to the reader.

Lemma 4.2. *If $\mathcal{C}$ is a separable subspace of $\mathcal{A}_\infty \cap \mathcal{A}'$, and α is an automorphism of $\mathcal{A}$ satisfying the Rokhlin property, then the projections as in the Rokhlin property can be chosen to furthermore commute with $\mathcal{C}$.*

In assertion (2) of Theorem 1.1 we assumed $\mathcal{D}$ to be K_1-injective. In fact, a little less will do:

Definition 4.3. We say that a separable unital C^*-algebra $\mathcal{D}$ has $\overline{\mathrm{Inn}_0}$ *half-flip* if there is a sequence of unitaries $u_1, u_2, \cdots \in \mathcal{U}_0(\mathcal{D} \otimes \mathcal{D})$ such that for all $d \in \mathcal{D}$, we have $u_n(d \otimes 1)u_n^* \to 1 \otimes d$.

It was shown in [Toms and Winter 2007], Proposition 1.13, that if $\mathcal{D}$ is a strongly self-absorbing C^*-algebra which is K_1-injective, then it has $\overline{\mathrm{Inn}_0}$ half-flip (a review of the proof shows that, in fact, it is enough to know that $u \otimes u^* \in \mathcal{U}_0(\mathcal{D} \otimes \mathcal{D})$ for all $u \in \mathcal{U}(\mathcal{D})$). In [Dadarlat and Winter 2007a], it was even shown that any two unital endomorphisms of $\mathcal{D}$ are asymptotically (and not only approximately) unitarily equivalent.

The purpose of this section is to prove the following.

Theorem 4.4. *Let $\mathcal{D}$ be a strongly self-absorbing C^*-algebra with an $\overline{\mathrm{Inn}_0}$ half-flip, and let $\mathcal{A}$ be a separable unital $\mathcal{D}$-absorbing algebra. Let α be an automorphism of $\mathcal{A}$. If α has the Rokhlin property then $\mathcal{D}$ admits a unital $*$-homomorphism into the fixed point subalgebra of $\mathcal{A}_\infty \cap \mathcal{A}'$. In particular, $\mathcal{A} \times_\alpha \mathbb{Z}$ is $\mathcal{D}$-absorbing.*

Before proving Theorem 4.4, we need a technical lemma.

Lemma 4.5. *Let $\mathcal{A}$, $\mathcal{B}$ be unital separable C^*-algebras. If $\mathcal{B}$ admits a unital homomorphism into $\mathcal{A}_\infty \cap \mathcal{A}'$, and $\mathcal{C}$ is a separable subalgebra of $\mathcal{A}_\infty \cap \mathcal{A}'$, then there is a unital homomorphism of $\mathcal{B}$ into $\mathcal{A}_\infty \cap \mathcal{A}' \cap \mathcal{C}'$.*

Proof. We denote by π the projection $\ell^\infty(\mathbb{N}, \mathcal{A}) \to \mathcal{A}_\infty$. Let $c_1, c_2, \ldots$ be a dense sequence in $\mathcal{C}$. We wish to find a unital homomorphism of $\mathcal{B}$ into $\mathcal{A}_\infty \cap \mathcal{A}'$ which commutes with all the elements of this sequence. Let $\varphi : \mathcal{B} \to \mathcal{A}_\infty \cap \mathcal{A}'$ be a unital homomorphism, and let $\tilde{\varphi} = (\varphi_1, \varphi_2, \ldots) : \mathcal{B} \to \ell^\infty(\mathbb{N}, \mathcal{A})$ be a linear lifting. For any subsequence $(\varphi_{k_1}, \varphi_{k_2}, \ldots)$, composition with π will give us another unital homomorphism of $\mathcal{B}$ into $\mathcal{A}_\infty \cap \mathcal{A}'$. We can thus use a Cantor-diagonalization type argument to construct such a subsequence, such that the image $\pi \circ (\varphi_{k_1}, \varphi_{k_2}, \ldots)$ will commute with $\mathcal{C}$. More specifically, note first that for any $a \in \mathcal{A}$, $b \in \mathcal{B}$, we have that $\|[\varphi_n(b), a]\| \to 0$. For each c_k we pick a lifting $\tilde{c}_k = (c_k(1), c_k(2), \ldots) \in \ell^\infty(\mathbb{N}, \mathcal{A})$. Let $b_1, b_2, \ldots$ be a dense sequence in $\mathcal{B}$. For each n, choose k_n such that $\|[\varphi_{k_n}(b_j), c_i(n)]\| < 1/n$ for all $i, j \leq n$ (and we may assume that this sequence k_n is increasing). Thus, $\pi \circ (\varphi_{k_1}, \varphi_{k_2}, \ldots)$ gives us a homomorphism with the required properties. $\qquad\square$

Proof of Theorem 4.4. It suffices, by Lemmas 2.4 and 2.3, to prove that for any finite subset $F \subseteq \mathcal{D}$ and $\varepsilon > 0$ there is a unital homomorphism

$$\varphi : \mathcal{D} \to \mathcal{A}_\infty \cap \mathcal{A}'$$

such that

$$\|\bar{\alpha}(\varphi(x)) - \varphi(x)\| < \varepsilon$$

for all $x \in F$. Fix such F, ε.

Fix a homomorphism $\iota : \mathcal{D} \to \mathcal{A}_\infty \cap \mathcal{A}'$. $C^* \left(\bigcup_{k=-\infty}^{\infty} \bar{\alpha}^k(\iota(\mathcal{D})) \right)$ is separable, so by Lemma 4.5, there is a unital homomorphism $\eta : \mathcal{D} \to \mathcal{A}_\infty \cap \mathcal{A}'$ whose image commutes with it. Let

$$\mathcal{B} := C^* \left(\eta(\mathcal{D}) \cup \bigcup_{k=-\infty}^{\infty} \bar{\alpha}^k(\iota(\mathcal{D})) \right) ;$$

note that $\mathcal{B} \cong \mathcal{B} \otimes \mathcal{D}$.

Choose a unitary $w \in \mathcal{U}_0(\mathcal{D} \otimes \mathcal{D})$ such that

$$\| w(x \otimes 1)w^* - 1 \otimes x \| < \frac{\varepsilon}{4}$$

for all $x \in F$. w can be connected to $1_{\mathcal{D} \otimes \mathcal{D}}$ via a rectifiable path. Let L be the length of such a path. Choose n such that $L\|x\|/n < \varepsilon/8$ for all $x \in F$.

Consider the homomorphisms $\iota, \bar{\alpha}^n \circ \iota : \mathcal{D} \to \mathcal{B}$. Define homomorphisms

$$\rho, \rho' : \mathcal{D} \otimes \mathcal{D} \to \mathcal{B}$$

by

$$\rho(x \otimes y) = \iota(x)\eta(y), \quad \rho'(x \otimes y) = \bar{\alpha}^n(\iota(x))\eta(y) .$$

Pick unitaries $1 = w_0, w_1, \ldots, w_n = w \in \mathcal{U}_0(\mathcal{D} \otimes \mathcal{D})$ such that $\|w_k - w_{k+1}\| \leq L/n$ for $k = 0, \ldots, n-1$. Now, let $u_k = \rho(w_k)^* \rho'(w_k)$. Note that

$$\| u_k - u_{k+1} \| \leq \frac{2L}{n}$$

for $k = 0, \ldots, n-1$, that $u_0 = 1$, and that

$$\| u_n \bar{\alpha}^n(\iota(x))u_n^* - \iota(x) \| < \frac{\varepsilon}{2}$$

for all $x \in F$.

Similarly, we choose unitaries $1 = v_0, v_1, \ldots, v_{n+1} \in \mathcal{B}$ such that

$$\| v_k - v_{k+1} \| \leq \frac{2L}{n+1}$$

for $k = 0, \ldots, n$ and

$$\| v_{n+1} \bar{\alpha}^{n+1}(\iota(x))v_{n+1}^* - \iota(x) \| < \frac{\varepsilon}{2}$$

for all $x \in F$.

We use the Rokhlin property (and Lemma 4.2) to find a partition of $1_{\mathcal{A}_\infty}$ into projections

$$e_i, f_j \in \mathcal{A}_\infty \cap \mathcal{A}' \cap \mathcal{B}' \cap \{\bar{\alpha}^{k-n}(u_k), \bar{\alpha}^{l-n-1}(v_j) \mid k = 1, \ldots, n-1, l = 1, \ldots, n\}',$$

$i = 0, \ldots, n-1$, $j = 0, \ldots, n$, such that $\bar{\alpha}(e_j) = e_{j+1}$ for all $j < n-1$ and $\bar{\alpha}(f_j) = f_{j+1}$ for all $j < n$ (we recall again that $\bar{\alpha}(e_{n-1} + f_n) = e_0 + f_0$). We then

define $\varphi : \mathcal{D} \to \mathcal{A}_\infty \cap \mathcal{A}'$ by

$$\varphi(x) = \sum_{k=0}^{n-1} e_k \bar{\alpha}^{k-n}(u_k) \bar{\alpha}^k(\iota(x)) \bar{\alpha}^{k-n}(u_k^*) e_k$$
$$+ \sum_{k=0}^{n} f_k \bar{\alpha}^{k-n-1}(v_k) \bar{\alpha}^k(\iota(x)) \bar{\alpha}^{k-n-1}(v_k^*) f_k.$$

By the choice of the projections $e_0, \ldots, e_{n-1}, f_0, \ldots, f_n$, φ is indeed a unital homomorphism, and a simple computation shows that for any $x \in F$, we have

$$\|\bar{\alpha}(\varphi(x)) - \varphi(x)\| < \frac{\varepsilon}{2} + \frac{4L\|x\|}{n} < \varepsilon,$$

as required. $\square$

5. Rokhlin flows

Recall from [Kishimoto 1996] the definition of a Rokhlin flow:

Definition 5.1. A strongly continuous action $\alpha : \mathbb{R} \to \mathrm{Aut}(\mathcal{A})$ on a unital separable C^*-algebra $\mathcal{A}$ is said to have the Rokhlin property if, for any $p \in \mathbb{R}$, there is a unitary $v \in \mathcal{A}_\infty^{(\alpha)} \cap \mathcal{A}'$ such that $\bar{\alpha}_t(v) = e^{itp} \cdot v$.

Our main theorem for this section is the following.

Theorem 5.2. *Let $\mathcal{A}$ and $\mathcal{D}$ be separable unital C^*-algebras with $\mathcal{D}$ strongly self-absorbing with an $\overline{\mathrm{Inn}}_0$ half-flip, and suppose $\mathcal{A}$ is $\mathcal{D}$-absorbing; let $\alpha : \mathbb{R} \to \mathrm{Aut}(\mathcal{A})$ be a Rokhlin flow. Then there is a unital $*$-homomorphism*

$$\varphi : \mathcal{D} \to \mathcal{A}_\infty \cap \mathcal{A}'$$

which is invariant under $\bar{\alpha}$. In particular, $\mathcal{A} \times_\alpha \mathbb{R}$ is $\mathcal{D}$-absorbing.

It was shown in [Kishimoto 1996] that if $\mathcal{A}$ is simple and purely infinite with a Rokhlin flow α, then the crossed product is simple and purely infinite again. In the case where $\mathcal{A}$ is simple, unital and nuclear, for $\mathcal{D} = \mathcal{O}_\infty$ our result coincides with Kishimoto's theorem (although the proofs are very different), since Kirchberg has shown that being purely infinite and absorbing $\mathcal{O}_\infty$ are equivalent conditions when $\mathcal{A}$ is nuclear (see [Kirchberg $\geq$ 2007; Kirchberg and Phillips 2000]).

The basic idea of the proof of Theorem 5.2 is similar to that of Theorem 4.4; we organize it differently, though. The main difference stems from the fact that the induced action of $\mathbb{R}$ on $\mathcal{A}_\infty \cap \mathcal{A}'$ is generally discontinuous—a problem which is irrelevant in the case of $\mathbb{Z}$. Since embeddings into $\mathcal{A}_\infty \cap \mathcal{A}'$ arise as approximate embeddings into $\mathcal{A}$, we work here with maps into $\mathcal{A}$ and into $C^*(\mathcal{A}, v) \subseteq \mathcal{A}_\infty^{(\alpha)}$, where v is a unitary arising from the Rokhlin property, which behave approximately like their analogues from the previous section. This allows us to circumvent the

continuity problems, but at the cost of having messier estimates. The algebra $C^*(v)$, which is in fact isomorphic to $\mathscr{C}(\mathbb{T})$, here will play an analogous role to that of $\mathscr{C}(G)$ in the compact case.

Lemma 5.3. *Let $\mathscr{D}$ be a strongly self-absorbing C*-algebra, and let $\mathscr{A}$ be a unital separable $\mathscr{D}$-absorbing C*-algebra. For any compact subset $K \subset \mathscr{A}$ and $\varepsilon > 0$ there is a unital expectation $\theta : \mathscr{A} \to \mathscr{A}$ such that $\|x - \theta(x)\| < \varepsilon$ for all $x \in K$, and such that $\mathscr{D}$ admits a unital embedding into $\mathscr{A} \cap \theta(\mathscr{A})'$.*

Proof. We have

$$\mathscr{A} \cong \mathscr{A} \otimes \mathscr{D}^{\otimes \infty} = \varinjlim (\mathscr{A} \otimes \mathscr{D}^{\otimes k}) \,.$$

Under this identification, there is some k such that $\operatorname{dist}(K, \mathscr{A} \otimes \mathscr{D}^{\otimes k} \otimes 1) < \varepsilon$, where 1 here denotes the identity of the copy of $\mathscr{D}^{\otimes \infty}$ sitting in $\mathscr{A} \otimes \mathscr{D}^{\otimes \infty}$ as $1_{\mathscr{A}} \otimes 1_{\mathscr{D}^{\otimes k}} \otimes \mathscr{D}^{\otimes \infty}$. If we fix some arbitrary state τ of $\mathscr{D}^{\otimes \infty}$, the map $\theta = \operatorname{id}_{\mathscr{A} \otimes \mathscr{D}^{\otimes k}} \otimes \tau$ will satisfy the required properties. $\square$

In the following lemma, we think of $\mathscr{C}(\mathbb{T}, \mathscr{A})$ as the space of $\mathscr{A}$-valued $2M$-periodic functions on $\mathbb{R}$ (where M is selected in the lemma).

Lemma 5.4. *Let $\mathscr{A}$ and $\mathscr{D}$ be separable unital C*-algebras; suppose that $\mathscr{D}$ is strongly self-absorbing with an $\overline{\operatorname{Inn}}_0$ half-flip and that $\mathscr{A} \cong \mathscr{A} \otimes \mathscr{D}$. Let α be a strongly continuous action of $\mathbb{R}$ on $\mathscr{A}$. For any $\varepsilon > 0$, any compact sets $D_0 \subseteq \mathscr{D}$, $A_0 \subseteq \mathscr{A}$ and any $\mu > 0$ there are an $M > \mu$ and a completely positive contraction $\beta : \mathscr{D} \to \mathscr{C}(\mathbb{T}, \mathscr{A})$, with the following properties:*

(1) $\|\alpha_t(\beta(d)(s-t)) - \beta(d)(s)\| < \varepsilon$

(2) $\|\beta(1) - 1\| < \varepsilon$

(3) $\|\beta(dd') - \beta(d)\beta(d')\| < \varepsilon$

(4) $\|[\beta(d), a]\| < \varepsilon$ *(where a is thought of as a constant function on $\mathbb{T}$)*

for all $d, d' \in D_0, a \in A_0, t \in [-\mu, \mu], s \in [-M, M]$.

Since the proof of this lemma is somewhat lengthy, we shall first show how it is used to prove Theorem 5.2.

Proof of Theorem 5.2. Fix compact sets $D_0 \subseteq \mathscr{D}$, $A_0 \subseteq \mathscr{A}$, $[-\mu, \mu] \subseteq \mathbb{R}$ ($\mu > 0$) and $\varepsilon > 0$. By Lemmas 2.3 and 2.4, it will suffice to find a completely positive contraction $\varphi : \mathscr{D} \to \mathscr{A}_\infty^{(\alpha)}$ such that, for any $d, d' \in D_0, a \in A_0, t \in [-\mu, \mu]$, the expressions $\|\bar{\alpha}_t(\varphi(d)) - \varphi(d)\|$, $\|\varphi(1) - 1\|$, $\|\varphi(d)\varphi(d') - \varphi(dd')\|$ and $\|[\varphi(d), a]\|$ are all bounded above by ε.

By the Rokhlin property of α, for any $M > 0$ there is a unitary $v \in \mathscr{A}_\infty^{(\alpha)} \cap \mathscr{A}'$ satisfying

$$\bar{\alpha}_t(v) = e^{-\pi i t/M} \cdot v \quad \text{for all } t \in \mathbb{R} \,.$$

We can define a $*$-homomorphism

$$\sigma : \mathscr{C}(\mathbb{T}) \otimes \mathscr{A} \to C^*(v, \mathscr{A}) \subseteq \mathscr{A}_\infty^{(\alpha)}$$

by setting

$$\sigma(f \otimes x) = f(v)x \quad \text{for all } x \in \mathscr{A}, \ f \in \mathscr{C}(\mathbb{T}).$$

We then have

$$\bar{\alpha}_t \circ \sigma(f \otimes x) = \sigma(f_{-t} \otimes \alpha_t(x))$$

for all $t \in \mathbb{R}$, $f \in \mathscr{C}(\mathbb{T})$ and $x \in \mathscr{A}$, where f_{-t} denotes the function f translated by $-t$. We write $\tilde{\alpha}_t$ for the action on $\mathscr{C}(\mathbb{T}) \otimes A$ given by $f \otimes x \mapsto f_{-t} \otimes \alpha_t(x)$. As before, we identify $\mathscr{C}(\mathbb{T}) \otimes A \cong \mathscr{C}(\mathbb{T}, \mathscr{A})$ with the space of $\mathscr{A}$-valued $2M$-periodic functions on $\mathbb{R}$.

Select M, β as in Lemma 5.4 and define

$$\varphi = \sigma \circ \beta.$$

We only check that $\|\bar{\alpha}_t(\varphi(d)) - \varphi(d)\| < \varepsilon$ for $d \in D_0$, $t \in [-\mu, \mu]$; the other requirements above are straightforward to verify:

$$\begin{aligned}
\|\bar{\alpha}_t(\varphi(d)) - \varphi(d)\| &\leq \|\tilde{\alpha}_t \circ \beta(d) - \beta(d)\| \\
&= \sup_{s \in [-M, M]} \|(\tilde{\alpha}_t \circ \beta(d))(s) - \beta(d)(s)\| \\
&= \sup_{s \in [-M, M]} \|\alpha_t(\beta(d)(s - t)) - \beta(d)(s)\| \underbrace{<}_{5.4(1)} \varepsilon. \qquad \square
\end{aligned}$$

It remains to prove Lemma 5.4.

Proof of Lemma 5.4. We may assume, without loss of generality, that $\|x\| \leq 1$ for all $x \in D_0 \cup A_0$, and that $1_{\mathscr{D}} \in D_0$, $1_{\mathscr{A}} \in A_0$ (in particular, we have $D_0 \subseteq D_0^2 = \{xy \mid x, y \in D_0\}$).

Since $\mathscr{D}$ is strongly self-absorbing with $\overline{\mathrm{Inn}_0}$ half-flip, there is a unitary $w \in \mathscr{U}_0(\mathscr{D} \otimes \mathscr{D})$ such that

$$(1) \qquad \|w(d \otimes 1_{\mathscr{D}})w^* - 1_{\mathscr{D}} \otimes d\| < \frac{\varepsilon}{40} \quad \text{for all } d \in D_0^2.$$

There is thus a continuous path of some finite length L in the unitary group of $\mathscr{D} \otimes \mathscr{D}$ connecting w and $1_{\mathscr{D} \otimes \mathscr{D}}$.

Fix some $M > \max\{\mu, 40\mu L/\varepsilon\}$. Pick a continuous path $(w_t)_{t \in [-M, M]} \subseteq \mathscr{U}_0(\mathscr{D} \otimes \mathscr{D})$, such that $w_{-M} = 1_{\mathscr{D} \otimes \mathscr{D}}$, $w_M = w$ and

$$\|w_t - w_s\| \leq \frac{L|t - s|}{2M} \quad \text{for all } s, t \in [-M, M].$$

We find $x_1, \ldots, x_p, y_1, \ldots, y_p \in \mathscr{D}$ of norm ≤ 1, and $\lambda_j(i) \in \mathbb{C}$, $j = 1, \ldots, p$, $i = 1, \ldots, P$ for some P, such that for any $t \in [-M, M]$ there is an $i \in \{1, \ldots, P\}$

such that

$$(2) \qquad \left\| \sum_{j=1}^{p} \lambda_j(i) x_j \otimes y_j - w_t \right\| < \frac{\varepsilon}{400}$$

and $\left\| \sum_{j=1}^{p} \lambda_j(i) x_j \otimes y_j \right\| \leq 1$ for all i. Let $\Lambda = \max\{|\lambda_j(i)| + 1 \mid j \leq p, i \leq P\}$.
Pick a unital embedding $\psi : \mathcal{D} \to \mathcal{A}$ such that

$$(3) \qquad \|[\psi(x), a]\| < \frac{\varepsilon}{30p\Lambda} < \frac{\varepsilon}{30}$$

for all $a \in \bigcup_{t=-2M}^{2M} \alpha_t(A_0)$ and all $x \in \{x_1, \dots, x_p, x_1^*, \dots, x_p^*\} \cup D_0$. Use [Lemma 5.3](#) to find an expectation map $\theta : \mathcal{A} \to \mathcal{A}$ such that

$$(4) \qquad \|\theta(x) - x\| < \frac{\varepsilon}{400p^2\Lambda^2} < \frac{\varepsilon}{400}$$

for all $x \in \left(\bigcup_{t=-3M}^{3M} \alpha_t(\psi(\{x_1, \dots, x_p, x_1^*, \dots, x_p^*\} \cup D_0) \cup A_0) \right)^3$, and a unital embedding $\eta : \mathcal{D} \to \mathcal{A} \cap \theta(\mathcal{A})'$.

Notice that for all $x, y, z \in \bigcup_{t=-M}^{M} \alpha_t(\psi(\{x_1, \dots, x_p, x_1^*, \dots, x_p^*\} \cup D_0) \cup A_0)$, we have

$$(5) \qquad \|\theta(xyz) - \theta(x)\theta(y)\theta(z)\| < \frac{4\varepsilon}{400p^2\Lambda^2} < \frac{4\varepsilon}{400}.$$

Define unital completely positive maps $\varrho, \varrho' : \mathcal{D} \otimes \mathcal{D} \to \mathcal{A}$ by

$$\varrho(d_1 \otimes d_2) := \theta(\alpha_{-M}(\psi(d_1))) \cdot \eta(d_2) \ , \ \varrho'(d_1 \otimes d_2) := \theta(\alpha_M(\psi(d_1))) \cdot \eta(d_2)$$

for $d_1, d_2 \in \mathcal{D}$. For any $t \in [-M, M]$ and $d_1, d_2 \in D_0^2$, we find $i \in \{1, \dots, P\}$ such that $\left\| \sum_{j=1}^{p} \lambda_j(i) x_j \otimes y_j - w_t \right\| < \varepsilon/400$; we see then that

$$(6) \quad \|\varrho(w_t)\varrho(d_1 \otimes d_2)\varrho(w_t^*) - \varrho(w_t(d_1 \otimes d_2)w_t^*)\|$$

$$\underset{(2)}{\leq} \left\| \varrho\left(\sum_{j=1}^{p} \lambda_j(i) x_j \otimes y_j \right) \varrho(d_1 \otimes d_2) \varrho\left(\sum_{j=1}^{p} \lambda_j(i) x_j \otimes y_j \right)^* \right.$$
$$\left. - \varrho\left(\left(\sum_{j=1}^{p} \lambda_j(i) x_j \otimes y_j \right)(d_1 \otimes d_2)\left(\sum_{j=1}^{p} \lambda_j(i) x_j \otimes y_j \right)^* \right) \right\| + \frac{4\varepsilon}{400}$$

$$\leq \sum_{j,k=1}^{p} \left| \lambda_j(i)\overline{\lambda_k(i)} \right| \left\| \theta(\alpha_{-M}(\psi(x_j)))\theta(\alpha_{-M}(\psi(d_1)))\theta(\alpha_{-M}(\psi(x_k^*))) \right.$$
$$\left. - \theta(\alpha_{-M}(\psi(x_j))\alpha_{-M}(\psi(d_1))\alpha_{-M}(\psi(x_k^*))) \right\| \left\| \eta(y_j d_2 y_k^*) \right\| + \frac{4\varepsilon}{400}$$

$$\underset{(5)}{\leq} \frac{4\varepsilon}{400} + \frac{4\varepsilon p^2\Lambda^2}{400p^2\Lambda^2} = \frac{\varepsilon}{50},$$

and the same estimate holds if we replace ϱ by ϱ', and if we interchange w_t and w_t^*. In particular, setting $d_1 = d_2 = 1$, we see that for all $t \in [-M, M]$

(7) $\|\varrho(w_t^*)\varrho(w_t) - 1\|, \ \|\varrho(w_t)\varrho(w_t^*) - 1\|,$

$$\|\varrho'(w_t^*)\varrho'(w_t) - 1\| \, , \ \|\varrho'(w_t)\varrho'(w_t^*) - 1\| < \frac{\varepsilon}{50}.$$

Similarly, we estimate for any $t \in [-2M, 2M]$, $s \in [-M, M]$, and $a \in A_0$ (for a suitable $i \in \{1, \ldots, P\}$):

(8) $$\|[\varrho(w_s), \alpha_t(a)]\| \underset{(2)}{\leq} \left\| \left[\varrho \left(\sum_{j=1}^{p} \lambda_j(i) x_j \otimes y_j \right), \alpha_t(a) \right] \right\| + \frac{2\varepsilon}{400}$$

$$\underset{(4)}{\leq} \left\| \left[\varrho \left(\sum_{j=1}^{p} \lambda_j(i) x_j \otimes y_j \right), \theta(\alpha_t(a)) \right] \right\| + \frac{4\varepsilon}{400}$$

$$\leq \ p\Lambda \max_{j \leq p} \|[\theta(\alpha_{-M}(\psi(x_j))), \theta(\alpha_t(a))]\| + \frac{4\varepsilon}{400}$$

$$\underset{(5)}{\leq} \ p\Lambda \max_{j \leq p} \|\theta([\psi(x_j), \alpha_{t+M}(a)])\| + \frac{12\varepsilon}{400}$$

$$\underset{(3)}{<} \ \frac{\varepsilon}{30} + \frac{12\varepsilon}{400} < \frac{\varepsilon}{15},$$

and the same estimate holds with w_s^* instead of w_s, and with ϱ' instead of ϱ.

Define a continuous path $(u_t)_{t \in [-M, M]} \subseteq \mathscr{A}$ by

$$u_t := \varrho(w_t)^* \varrho'(w_t).$$

The u_t are not necessarily unitary, but one checks that they satisfy

(9) $$\|u_t u_t^* - 1\|, \ \|u_t^* u_t - 1\| < \frac{2\varepsilon}{50} = \frac{\varepsilon}{25}.$$

Note also that

(10) $$\|u_t - u_s\| \leq \frac{2L}{2M} \cdot |t - s| \quad \text{for all } s, t \in [-M, M]$$

and that $u_{-M} = 1$. Let $d \in D_0^2$. Using the inequalities

$$\|\alpha_M(\psi(d)) - \varrho'(d \otimes 1_{\mathscr{D}})\| < \frac{\varepsilon}{400} \quad \text{and} \quad \|\alpha_{-M}(\psi(d)) - \varrho(d \otimes 1_{\mathscr{D}})\| < \frac{\varepsilon}{400},$$

we check that

$$(11) \quad \|u_M \alpha_M(\psi(d)) u_M^* - \alpha_{-M}(\psi(d))\|$$

$$< \quad \|\varrho(w_M)^* \varrho'(w_M) \varrho'(d \otimes 1_{\mathcal{D}}) \varrho'(w_M)^* \varrho(w_M) - \varrho(d \otimes 1_{\mathcal{D}})\| + \frac{2\varepsilon}{400}$$

$$\underbrace{<}_{(6)} \quad \|\varrho(w_M)^* \varrho'(w_M(d \otimes 1_{\mathcal{D}}) w_M^*) \varrho(w_M) - \varrho(d \otimes 1_{\mathcal{D}})\| + \frac{2\varepsilon}{400} + \frac{\varepsilon}{50}$$

$$\underbrace{<}_{(1)} \quad \|\varrho(w_M)^* \varrho(1_{\mathcal{D}} \otimes d) \varrho(w_M) - \varrho(d \otimes 1_{\mathcal{D}})\| + \frac{2\varepsilon}{400} + \frac{\varepsilon}{50} + \frac{\varepsilon}{40}$$

$$< \quad \frac{2\varepsilon}{400} + 2 \cdot \left(\frac{\varepsilon}{50} + \frac{\varepsilon}{40} \right) < \frac{\varepsilon}{10} \, .$$

Furthermore,

$$(12) \quad \|[\alpha_t(a), u_s]\| = \|[\alpha_t(a), \varrho(w_s^*) \varrho'(w_s)]\|$$

$$\leq \|[\alpha_t(a), \varrho(w_s)]\| + \|[\alpha_t(a), \varrho'(w_s)]\| \underbrace{<}_{(8)} \frac{2\varepsilon}{15}$$

for $a \in A_0$, $s \in [-M, M]$, $t \in [-2M, 2M]$, and the same estimate holds for u_s^* in place of u_s.

For each $d \in \mathcal{D}$, $t \in [-M, M]$, we define

$$h(d, t) := \alpha_{t-M}(u_t) \alpha_t(\psi(d)) \alpha_{t-M}(u_t^*) \, ,$$

then

$$h(d, M) = u_M \alpha_M(\psi(d)) u_M^*$$

and

$$h(d, -M) = \alpha_{-M}(\psi(d)) \quad \text{for all } d \in \mathcal{D} \, ,$$

whence

$$(13) \quad \|h(d, M) - h(d, -M)\| < \frac{\varepsilon}{10} \quad \text{for all } d \in D_0^2 \, .$$

For any d_1, d_2 in the unit ball of $\mathcal{D}$ and for any $t \in [-M, M]$, we have

$$(14) \quad \|h(d_1, t) h(d_2, t) - h(d_1 d_2, t)\| \leq \|u_t^* u_t - 1\| \underbrace{<}_{(9)} \frac{\varepsilon}{25} \, .$$

For any $a \in A_0$, $d \in D_0$, $t \in [-M, M]$, we have

$$
\begin{aligned}
(15) \quad \big\| [h(d, t), a] \big\| &\leq \big\| [\alpha_{t-M}(u_t), a] \big\| + \big\| [\alpha_t(\psi(d)), a] \big\| + \big\| [\alpha_{t-M}(u_t^*), a] \big\| \\
&= \big\| [u_t, \alpha_{M-t}(a)] \big\| + \big\| [\psi(d), \alpha_{-t}(a)] \big\| + \big\| [u_t^*, \alpha_{M-t}(a)] \big\| \\
&\underset{(3),\,(12)}{<} \frac{2\varepsilon}{15} + \frac{\varepsilon}{30} + \frac{2\varepsilon}{15} < \frac{\varepsilon}{3}.
\end{aligned}
$$

We can view h as a completely positive contraction from $\mathscr{D}$ to $\mathscr{C}([-M, M], \mathscr{A})$. $\mathscr{C}(\mathbb{T}, \mathscr{A})$ may be identified with the subalgebra of $\mathscr{C}([-M, M], \mathscr{A})$ consisting of functions which agree on $-M$ and M. We shall now perturb h so as to ensure that its range is in $\mathscr{C}(\mathbb{T}, \mathscr{A})$. For each $\delta > 0$, define a continuous function $g_\delta \in \mathscr{C}([-M, M])$ by

$$
g_\delta(t) := \begin{cases} 0 & \text{if } t = -M, \\ 1 & \text{if } -M + \delta \leq t \leq M, \\ \text{linear} & \text{elsewhere.} \end{cases}
$$

Now, let

$$
h_\delta(d, t) := g_\delta(t) \cdot h(d, t) + (1 - g_\delta(t)) \cdot h(d, M).
$$

For any $d \in \mathscr{D}$, $h_\delta(d, t)$ is a continuous function of t, and satisfies $h_\delta(d, -M) = h_\delta(d, M)$. h_δ, thus, can be regarded as a map $h_\delta : \mathscr{D} \to \mathscr{C}(\mathbb{T}, \mathscr{A})$. This map can readily be seen to be a completely positive contraction.

Fix $\delta > 0$ such that

$$
\| h(d, t) - h_\delta(d, t) \| < \frac{\varepsilon}{4} \quad \text{for all } d \in D_0^2,\ t \in [-M, M];
$$

this is possible by (13). We let

$$
\beta := h_\delta.
$$

The proof will be complete once we check that β satisfies the four conditions required in the statement.

Condition (1): We wish to show that

$$
\big\| \alpha_t(\beta(d, s - t)) - \beta(d, s) \big\| < \varepsilon \quad \text{for all } d \in D_0,\ s \in [-M, M],\ t \in [-\mu, \mu].
$$

It will clearly be enough to consider $t \in [0, \mu]$. (The case $t < 0$ will follow with t replaced by $-t$ and s replaced by $s - t$.) Fix $d \in D_0$.

Case I: $s \in [-M + t, M]$. We have

$$\big\| \alpha_t(h(d, s - t)) - h(d, s) \big\|$$
$$= \big\| \alpha_{s-M}(u_{s-t})\alpha_s(\psi(d))\alpha_{s-M}(u_{s-t}^*) - \alpha_{s-M}(u_s)\alpha_s(\psi(d))\alpha_{s-M}(u_s^*) \big\|$$
$$\leq 2\|u_{s-t} - u_s\| \underbrace{\leq}_{(10)} \frac{2L|t|}{M} \leq \frac{2L\mu}{M} < \frac{\varepsilon}{20},$$

from which it follows that $\|\alpha_t(\beta(d, s - t)) - \beta(d, s)\| < 2\varepsilon/4 + \varepsilon/20 < \varepsilon$.

Case II: $s \in [-M, -M + t]$. In this case, there are $0 \leq t_0, t_1 \leq t$ such that $t = t_0 + t_1$ and $s = -M + t_0$. Note that $\|\beta(d, s - t) - h(d, M - t_1)\| < \varepsilon/4$. Thus

$$\big\| \alpha_t(\beta(d, s - t)) - \beta(d, s) \big\|$$
$$\leq \big\| \alpha_t(h(d, M - t_1)) - h(d, s) \big\| + \frac{2\varepsilon}{4}$$
$$\leq \big\| \alpha_{t_0}(h(d, M)) - h(d, s) \big\| + \big\| \alpha_{t_1}(h(d, M - t_1)) - h(d, M) \big\| + \frac{2\varepsilon}{4}$$
$$< \big\| \alpha_{-M+t_0}(\psi(d)) - h(d, s) \big\| + \frac{3\varepsilon}{4} + \frac{\varepsilon}{20}$$
$$\leq 2\|u_{-M} - u_s\| + \frac{3\varepsilon}{4} + \frac{\varepsilon}{20} < \frac{\varepsilon}{20} + \frac{3\varepsilon}{4} + \frac{\varepsilon}{20} < \varepsilon.$$

Condition (2): Indeed,

$$\|\beta(1) - 1\|$$
$$< \sup_{t \in [-M, M]} \|h(1, t) - 1\| + \frac{\varepsilon}{4} = \sup_{t \in [-M, M]} \|\alpha_{t-M}(u_t u_t^*) - 1\| + \frac{\varepsilon}{4} < \frac{\varepsilon}{25} + \frac{\varepsilon}{4} < \varepsilon.$$

Condition (3): Let $d_1, d_2 \in D_0$. We have

$$\big\| \beta(d_1)\beta(d_2) - \beta(d_1 d_2) \big\|$$
$$\leq \sup_{t \in [-M, M]} \big\| \beta(d_1, t)\beta(d_2, t) - \beta(d_1 d_2, t) \big\|$$
$$< \sup_{t \in [-M, M]} \big\| h(d_1, t)h(d_2, t) - h(d_1 d_2, t) \big\| + \frac{3\varepsilon}{4} \underbrace{<}_{(14)} \frac{\varepsilon}{25} + \frac{3\varepsilon}{4} < \varepsilon.$$

Condition (4): For each $d \in D_0$, $a \in A_0$, we have

$$\big\| [\beta(d), a] \big\| < \sup_{t \in [-M, M]} \big\| [h(d, t), a] \big\| + \frac{2\varepsilon}{4} \underbrace{<}_{(15)} \frac{\varepsilon}{3} + \frac{2\varepsilon}{4} < \varepsilon. \qquad \square$$

Acknowledgments

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