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MOONJA JEONG AND VLADIMIR MITYUSHEV

# THE BERGMAN KERNEL FOR CIRCULAR MULTIPLY CONNECTED DOMAINS

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**The Bergman kernel is written in terms of a Poincaré series for every circular multiply connected domain in the plane. The proposed exact formula is based on the Green's function constructed by Mityushev and S. V. Rogosin for multiply connected domains. First we verify the relation between the Green's function and the Bergman kernel for unbounded domains. Then we apply it to construct the Bergman kernel.**

## 1. Introduction

The Bergman kernel plays an important role in studies of the boundary behavior of biholomorphic mappings, and of the potential theory in one complex variable. It is related to the Green's function and to other domain functions such as the Neumann function, harmonic measures, canonical mapping functions, and extremal problems [Bergman 1950; Bell 1992; Aharonov et al. 2005]. Therefore, explicit expressions of the Bergman kernels yield profound progress in the theory of functions.

The Bergman kernel represents the Bergman projection, which reproduces all holomorphic square integrable functions. In the unit disc  $U$  in  $\mathbb{C}$  with area measure  $dx \wedge dy = i/2 dz \wedge d\bar{z}$ , the Bergman kernel is given by

$$(1-1) \quad K_U(z, \zeta) = \frac{1}{\pi} \frac{1}{(1 - z\bar{\zeta})^2}, \quad z, \zeta \in U.$$

The Bergman kernel  $K_\Omega(z, \zeta)$  associated with a simply connected planar domain  $\Omega$  is expressed by the Riemann mapping  $f : \Omega \rightarrow U$

$$(1-2) \quad K_\Omega(z, \zeta) = \frac{1}{\pi} \frac{f'(z)\overline{f'(\zeta)}}{(1 - f(z)\overline{f(\zeta)})^2}, \quad z, \zeta \in \Omega.$$

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Conversely, the derivative of  $f(z)$  is determined through the Bergman kernel by the formula

$$(1-3) \quad f'(z) = K_{\Omega}(z, \zeta) \sqrt{\frac{\pi}{K_{\Omega}(\zeta, \zeta)}},$$

where the point  $\zeta$  is transformed onto zero by the mapping  $f$  with  $f'(\zeta) > 0$ .

The Bergman kernel for a circular annulus  $A_{\rho} = \{z \in \mathbb{C} : \rho < |z| < 1\}$  was calculated in [Bergman 1950, p. 10]

$$(1-4) \quad K_{A_{\rho}}(z, \zeta) = \frac{1}{\pi z \bar{\zeta}} \left( \mathcal{P}(\ln z \bar{\zeta}) + \frac{\eta_1}{\pi i} - \frac{1}{2 \ln \rho} \right),$$

where  $\mathcal{P}$  is the Weierstrass function with the periods  $\omega_1 = \pi i$ ,  $\omega_2 = \ln \rho$ , and  $\eta_1$  is the half-increment of the Weierstrass  $\zeta$ -function related to the period  $\omega_1$ . Ball and Clancey [1996] gave a formula for the Bergman kernel for multiply connected domains in terms of theta functions defined on the Jacobian variety of the Schottky double of domain. However, this formula involves the principal functionals of the Schottky double of domain for which only theorems of existence are known.

Note that in the several dimensional complex space, the Bergman kernel was calculated explicitly for the following domains: the unit ball, the polydisc, the Thullen domain [D'Angelo 1994], convex domains [Boas et al. 1999], the Lie ball [Hua 1963], the minimal ball [Oeljeklaus et al. 1997], and so on. For example, the Bergman kernel associated with the Thullen domain  $\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^{2p} + |z_2|^2 < 1\}$ , with any positive integer  $p$  is

$$\sum_{k=1}^2 c_k \frac{(1 - z_2 \bar{\zeta}_2)^{-2+(k/p)}}{((1 - z_2 \bar{\zeta}_2)^{1/p} - z_1 \bar{\zeta}_1)^{k+1}}$$

where  $c_1 = (p - 1)/(\pi^2 p)$  and  $c_2 = 2/(\pi^2 p)$ .

Let  $M$  be a 1-dimensional complex manifold and  $H^2(M)$  the Hilbert space of holomorphic 1-forms  $h$  on  $M$  such that

$$(1-5) \quad \frac{i}{2} \int_M h \wedge \bar{h} < \infty.$$

Let  $h_1, h_2, \dots$  be an orthonormal basis for  $H^2(M)$ . The Bergman kernel  $\mathcal{H}_M$  associated with  $M$  can be defined as follows:

$$(1-6) \quad \mathcal{H}_M = \frac{i}{2} \sum_{k=1}^{\infty} h_k \wedge \bar{h}_k.$$

In this paper, we consider domains with  $C^\infty$  smooth boundary. The relation between the Bergman kernel and the Green's function is given by the following proposition:

**Proposition 1.1** [Schiffer 1946; Bell 1992, p. 131]. *Suppose  $\Omega$  is a bounded domain with  $C^\infty$  smooth boundary. The Green's function  $G_\Omega(z, w)$  and the Bergman kernel  $K_\Omega(z, w)$  associated with  $\Omega$  are related via*

$$(1-7) \quad K_\Omega(z, \zeta) = -\frac{2}{\pi} \frac{\partial^2 G_\Omega(z, \zeta)}{\partial z \partial \bar{\zeta}}.$$

In this paper we write the Bergman kernel explicitly for every circular multiply connected domain. Our work is based on the Green function constructed in [Mityushev and Rogosin 2000] for multiply connected domains. In Section 2 we verify the relation (1-7) for unbounded domains. In Section 3 we apply it to the construction of the Bergman kernel.

## 2. Green's function and the Bergman kernel

On the first page of his book, B. Epstein [Epstein 1965] writes “Let  $D$  denote any bounded\* domain...”, and in the footnote: “The restriction of boundedness is made for the sake of simplicity; all arguments to be presented in the present chapter and in the following chapters are valid, with at most minor modifications, for an unbounded domain whose boundary contains at least one component consisting of more than a single point.” Since the present paper directly concerns such unbounded domains, we include in this section the formal derivation of the statements that we will need, as they seem not to be explicitly available elsewhere.

First, the definition of a biholomorphic mapping for unbounded domains is given. Let  $\bar{\mathbb{C}}$  denote the extended complex plane  $\mathbb{C} \cup \{\infty\}$ . Following [Shabat 1985] we define a subset  $U$  of  $\bar{\mathbb{C}}$  to be open if

- (1)  $U \cap \mathbb{C}$  is open in  $\mathbb{C}$ , and
- (2) if  $\infty \in U$ , then for some  $R > 0$ , the set  $\{z \in \mathbb{C} : |z| > R\}$  is contained in  $U$ .

Hence, the mapping of  $\bar{\mathbb{C}}$  to  $\bar{\mathbb{C}}$  given by  $z \mapsto 1/z$  is continuous. With the topology in  $\bar{\mathbb{C}}$  and with the associated idea of continuity, we may think of meromorphic functions as holomorphic functions with values in  $\bar{\mathbb{C}}$ . According to [Greene and Krantz 2002, p. 143], we define a continuous function  $F : U \rightarrow \bar{\mathbb{C}}$  on an open subset  $U$  of  $\bar{\mathbb{C}}$  to be holomorphic if

- (1)  $F|_{F^{-1}(\mathbb{C})}$  is holomorphic and
- (2)  $F^{-1}(\{\infty\})$  is a discrete set in  $U$ , and  $1/F$  is holomorphic on some open neighborhood of  $F^{-1}(\{\infty\})$ .

Let  $\Omega$  be a bounded domain with  $C^\infty$  smooth boundary. For a fixed point  $\zeta \in \Omega$ , the Green's function  $G_\Omega(z, \zeta)$  associated with  $\Omega$  is defined by

$$G_\Omega(z, \zeta) = -\ln |z - \zeta| + g_\zeta(z),$$

where  $g_\zeta(z)$  is the harmonic function of  $z$  on  $\Omega$  which solves the Dirichlet problem

$$g_\zeta(z)|_{\partial\Omega} = \ln |z - \zeta|.$$

Now we consider unbounded circular multiply connected domains. Introduce mutually disjoint discs

$$(2-1) \quad D_k = \{z \in \mathbb{C} : |z - a_k| < r_k\}, \quad k = 1, 2, \dots, n,$$

and the complement of these discs to the extended complex plane

$$(2-2) \quad D = \overline{\mathbb{C}} - \bigcup_{k=1}^n (D_k \cup \partial D_k).$$

The domain  $D$  is called a circular multiply connected domain.

The transformation formula for the Bergman kernels holds under biholomorphic mappings between two bounded domains. It is proved by a change of variable produced by a biholomorphic mapping between two bounded domains; see [Greene and Krantz 2002, p. 434]. The same arguments can be applied to a biholomorphic mapping between an unbounded multiply connected domain and a bounded multiply connected domain. For convenience, we present this result in the following lemma and sketch the proof below.

**Lemma 2.1.** *Suppose that  $f : D \rightarrow \Omega$  is a biholomorphic mapping where  $D$  is defined by (2-1)–(2-2) and  $\Omega$  is a bounded domain. The Bergman kernels  $K_D$  and  $K_\Omega$  associated with  $D$  and  $\Omega$  respectively, are related via*

$$(2-3) \quad K_D(z, \zeta) = f'(z) K_\Omega(f(z), f(\zeta)) \overline{f'(\zeta)} \quad z, \zeta \in D.$$

*Proof.* Introduce the function  $\tilde{K}(z, \zeta) = f'(z) K_\Omega(f(z), f(\zeta)) \overline{f'(\zeta)}$  for  $z, \zeta \in D$ . Then

$$(2-4) \quad \int_D |\tilde{K}(z, \zeta)|^2 dA_z = \int_D |f'(z)|^2 |K_\Omega(f(z), f(\zeta))|^2 |f'(\zeta)|^2 dA_z.$$

By a change of variable, this becomes

$$(2-5) \quad \begin{aligned} \int_\Omega |f'(f^{-1}(w))|^2 |K_\Omega(w, f(\zeta))|^2 |f'(\zeta)|^2 |(f^{-1})'(w)|^2 dA_w \\ = |f'(\zeta)|^2 \int_\Omega |K_\Omega(w, f(\zeta))|^2 dA_w < \infty. \end{aligned}$$

Now for any holomorphic function  $\varphi \in L^2(D)$ ,

$$(2-6) \quad \int_D \tilde{K}(z, \zeta) \varphi(\zeta) dA_\zeta = \int_D f'(z) K_\Omega(f(z), f(\zeta)) \overline{f'(\zeta)} \varphi(\zeta) dA_\zeta$$

$$\begin{aligned}
&= \int_{\Omega} f'(z) K_{\Omega}(f(z), \eta) \overline{f'(f^{-1}(\eta))} \varphi(f^{-1}(\eta)) |(f^{-1})'(\eta)|^2 dA_{\eta} \\
&= f'(z) \int_{\Omega} K_{\Omega}(f(z), \eta) \{(\varphi \circ f^{-1})(\eta)(f^{-1})'(\eta)\} dA_{\eta} \\
&= f'(z) \{(\varphi \circ f^{-1})(f(z))(f^{-1})'(f(z))\},
\end{aligned}$$

since  $(\varphi \circ f^{-1})(\eta)(f^{-1})'(\eta)$  is holomorphic in  $\Omega$  and belongs to  $L^2(\Omega)$ . Hence,

$$\int_D \tilde{K}(z, \zeta) \varphi(\zeta) dA_{\zeta} = \varphi(z).$$

By the uniqueness of the Bergman kernel, we establish that (2-3) is valid.  $\square$

More generally, Lemma 2.1 can be applied to a biholomorphic mapping between two complex manifolds with the same dimensions. Suppose that  $f : M_1 \rightarrow M_2$  is a biholomorphic mapping where  $M_1$  and  $M_2$  are 1-dimensional complex manifolds. The Bergman kernels  $\mathcal{H}_{M_1}$  and  $\mathcal{H}_{M_2}$  associated with  $M_1$  and  $M_2$  respectively, are related via

$$(2-7) \quad \mathcal{H}_{M_1} = (f \times f)^* \mathcal{H}_{M_2},$$

where  $f \times f$  is a biholomorphic mapping from  $M_1 \times M_1$  to  $M_2 \times M_2$ .

**Example 2.2.** We apply (2-3) to get the Bergman kernel associated with the upper half plane  $\mathbb{C}^+$ . Let  $f : \mathbb{C}^+ \rightarrow U$  be the bilinear transformation

$$f(z) = \frac{A(z - z_0)}{z - \bar{z}_0},$$

where  $\text{Im } z_0 > 0$  and  $|A| = 1$ . Then  $f'(z) = A(z_0 - \bar{z}_0)/(z - \bar{z}_0)^2$  and

$$(2-8) \quad K_U(f(z), f(\zeta)) = \frac{1}{\pi(1 - f(z)\overline{f(\zeta)})^2} = \frac{(z - \bar{z}_0)^2(\bar{\zeta} - z_0)^2}{\pi(\bar{\zeta} - z)^2(z_0 - \bar{z}_0)^2}.$$

Therefore

$$(2-9) \quad K_{\mathbb{C}^+}(z, \zeta) = f'(z) K_U(f(z), f(\zeta)) \overline{f'(\zeta)} = -\frac{1}{\pi(z - \bar{\zeta})^2}.$$

The Bergman kernel function (2-9) does vanish at  $z = \infty$ , while the Bergman kernel form  $-dz d\bar{\zeta}/(\pi(z - \bar{\zeta})^2)$  is nonvanishing on the closure of  $\mathbb{C}^+ \times \mathbb{C}^+$ .

The Green's function is invariant under biholomorphic mappings between two bounded domains with  $C^\infty$  smooth boundaries; see [Conway 1978, p. 276; Bell 1992, p. 128]. We extend this result to the case of biholomorphic mappings between  $D$  and bounded domains with  $C^\infty$  smooth boundaries.

**Lemma 2.3.** *Let the domain  $D$  be defined by (2-1)–(2-2) and  $\Omega$  be a bounded domain in the complex plane with  $C^\infty$  smooth boundary. If  $f$  is a biholomorphic mapping of  $D$  onto  $\Omega$ , then the Green's functions  $G_D$  and  $G_\Omega$  associated with  $D$  and  $\Omega$  respectively, satisfy the identity*

$$(2-10) \quad G_D(z, \zeta) = G_\Omega(f(z), f(\zeta)), \quad z, \zeta \in D.$$

*Proof.* It is enough to show that  $G_\Omega(f(z), f(\zeta))$  is the Green's function for  $D$  with singularity at  $z = \zeta \in D$ . Note that  $f \in C^\infty(\overline{D})$  since  $D$  has  $C^\infty$  smooth boundary and  $\overline{D}$  is compact. We can write  $f(z) - f(\zeta) = (z - \zeta)H_\zeta(z)$  where  $H_\zeta(z)$  is a nonvanishing holomorphic function in  $A^\infty(D) = C^\infty(\overline{D}) \cup \mathcal{H}(D)$ . Here  $\mathcal{H}(D)$  represents the set of holomorphic functions on  $D$ . Hence,

$$\ln |f(z) - f(\zeta)| = \ln |z - \zeta| + \ln |H_\zeta(z)|$$

where  $\ln |H_\zeta(z)|$  is a harmonic function in  $C^\infty(\overline{D})$ . Hence  $G_\Omega(f(z), f(\zeta)) + \ln |z - \zeta|$  is a harmonic function of  $z$  on  $D$  with

$$G_\Omega(f(z), f(\zeta))|_{z \in \partial D} = 0.$$

Therefore, by the uniqueness of Green's functions,  $G_\Omega(f(z), f(\zeta))$  is the Green's function associated with  $D$ ; see [Conway 1978, p. 276].  $\square$

Using Lemma 2.3 we obtain the following result.

**Theorem 2.4.** *Let the domain  $D$  be defined by (2-1)–(2-2). The Bergman kernel  $K_D$  and the Green's function  $G_D(z, \zeta)$  associated with  $D$  are related via*

$$(2-11) \quad K_D(z, \zeta) = -\frac{2}{\pi} \frac{\partial^2 G_D(z, \zeta)}{\partial z \partial \bar{\zeta}}.$$

*Proof.* Let  $f : D \rightarrow \Omega$  be a biholomorphic mapping of  $D$  onto a bounded domain  $\Omega$  with  $C^\infty$  smooth boundary. By Proposition 1.1,

$$(2-12) \quad K_\Omega(w, \eta) = -\frac{2}{\pi} \frac{\partial^2 G_\Omega(w, \eta)}{\partial w \partial \bar{\eta}}$$

for  $w, \eta \in \Omega$  where  $K_\Omega(w, \eta)$  is the Bergman kernel and  $G_\Omega(w, \eta)$  is the Green's function associated with  $\Omega$ . Application of Lemma 2.1 and (2-12) yields

$$K_D(z, \zeta) = f'(z)K_\Omega(w, \eta)\overline{f'(\zeta)} = f'(z) \left( -\frac{2}{\pi} \frac{\partial^2 G_\Omega(w, \eta)}{\partial w \partial \bar{\eta}} \right) \overline{f'(\zeta)},$$

where  $w = f(z)$ ,  $\eta = f(\zeta)$ . By a change of variables, this becomes

$$-\frac{2}{\pi} f'(z) \frac{\partial^2 G_D(z, \zeta)}{\partial z \partial \bar{\zeta}} (f^{-1})'(w) \overline{(f^{-1})'(\eta)} \overline{f'(\zeta)} = -\frac{2}{\pi} \frac{\partial^2 G_D(z, \zeta)}{\partial z \partial \bar{\zeta}}. \quad \square$$

### 3. Exact formula for the Bergman kernel

Let  $D$  be a circular multiply connected domain described by (2-1)–(2-2). Hereafter we deal only with the domain  $D$ , hence for brevity we omit the corresponding indexes in the Green and Bergman functions.

**Schottky group.** Consider inversions with respect to the circles  $|z - a_k| = r_k$  and their compositions

$$(3-1) \quad z_{(k)}^* := \frac{r_k^2}{z - a_k} + a_k, \quad z_{(k_s, k_{s-1} \dots k_1)}^* := \left( z_{(k_{s-1} \dots k_1)}^* \right)_{(k_s)}^*.$$

In the sequence  $k_1, k_2, \dots, k_s$  no two neighboring numbers are equal. When  $s$  is even, these are Möbius transformations in  $z$ . If  $s$  is odd, we have transformations in  $\bar{z}$ . Hence, the mappings can be written in the form

$$(3-2) \quad \begin{aligned} \gamma_j(z) &= (e_j z + b_j) / (c_j z + d_j), \quad s \in 2\mathbb{Z}, \\ \gamma_j(\bar{z}) &= (e_j \bar{z} + b_j) / (c_j \bar{z} + d_j), \quad s \in 2\mathbb{Z} + 1, \end{aligned}$$

where  $e_j d_j - b_j c_j = 1$ . Here  $\gamma_0(z) := z$  (identity mapping),  $\gamma_1(\bar{z}) := z_{(1)}^*, \dots, \gamma_n(\bar{z}) := z_{(n)}^*$  ( $n$  simple inversions),  $\gamma_{n+1}(z) := z_{(12)}^*, \gamma_{n+2}(z) := z_{(13)}^*, \dots, \gamma_{n^2}(z) := z_{(n, n-1)}^*$  ( $n^2 - n$  pairs of inversions),  $\gamma_{n^2+1}(\bar{z}) := z_{(121)}^*, \dots$  and so on. The number  $s$  is called the level of the mapping  $\gamma_j$ . The indexes  $j$  of  $\gamma_j$  are ordered in such a way that the level is not decreasing. The functions (3-1) or (3-2) generate a Schottky group  $\mathcal{G}$ . Thus, each element of  $\mathcal{G}$  is presented in the form of the composition of inversions (3-1) or in the form of linearly ordered functions (3-2). Let  $\mathcal{G}_m$  be the subset of  $\mathcal{G} - \{\gamma_0\}$  consisting of mappings whose last inversion is different from  $z_{(m)}^*$ , i.e.,  $\mathcal{G}_m = \{z_{(k_s, k_{s-1} \dots k_1)}^* : k_s \neq m\}$ .

**Harmonic measures.** For a fixed point  $z_0 \in D - \{\infty\}$ , we introduce the functions

$$(3-3) \quad \psi_m(z) = \ln \left( \prod_{\gamma_j \in \mathcal{G}_m} \psi_m^{(j)}(z) \right),$$

where we have, for even level of  $\gamma_j \in \mathcal{G}$

$$(3-4) \quad \psi_m^{(j)}(z) = \frac{\gamma_j(z) - a_m}{\gamma_j(z_0) - a_m},$$

and for odd level of  $\gamma_j \in \mathcal{G}$

$$(3-5) \quad \psi_m^{(j)}(z) = \frac{\overline{\gamma_j(\bar{z}_0) - a_m}}{\gamma_j(\bar{z}) - a_m}.$$



The infinite product (3-3) converges uniformly in every compact subset of  $\bar{D}$ ; see [Mityushev and Rogosin 2000, p. 153]. The multipliers in the infinite product (3-3) are ordered according to increasing level.

Consider a fixed  $s$  from the set  $\{1, 2, \dots, n\}$ . The harmonic measure  $\alpha_s(\zeta)$  associated to  $D$  satisfies the boundary conditions

$$(3-6) \quad \alpha_s(z) = \delta_{sk}, \quad |z - a_k| = r_k, \quad k = 1, 2, \dots, n,$$

where  $\delta_{sk}$  is the Kronecker symbol.  $\alpha_s(z)$  is explicitly written in [Mityushev and Rogosin 2000, p. 153, Theorem 4.10] as follows:

$$(3-7) \quad \alpha_s(z) = \sum_{m=1}^n A_{sm} \operatorname{Re}[\psi_m(z) + \ln(z - a_m)] + A_s,$$

where  $\psi_m(z)$  has the form (3-3), and the real constants  $A_{sm}$  and  $A_s$  are described in the theorem just cited.

Following Chapter 4 of [Mityushev and Rogosin 2000] we briefly describe how to deduce (3-7). The harmonic measure  $\alpha_s(z)$  as a harmonic function in  $D$  can be represented in the form

$$(3-8) \quad \alpha_s(z) = \operatorname{Re} \varphi(z) + \sum_{m=1}^n A_{sm} \ln |z - a_m| + A_s,$$

where  $\varphi(z)$  is analytic in  $D$  and continuously differentiable in its closure ( $\varphi \in A^1(D)$ ) and

$$(3-9) \quad \varphi(z_0) = 0.$$

$A_{sm}$  are real constants having zero sum. The real constant  $A_s$  is introduced for convenience. It can be included into  $\operatorname{Re} \varphi(z)$ . Substitution of (3-8) into (3-6) yields the boundary value problem

$$(3-10) \quad \operatorname{Re} \varphi(z) + \sum_{m=1}^n A_{sm} \ln |z - a_m| + A_s = \delta_{sk}, \quad |z - a_k| = r_k, \quad k = 1, 2, \dots, n.$$

It can be reduced, first, to the  $\mathbb{R}$ -linear conjugation problem

$$(3-11) \quad \varphi(z) = \varphi_k(z) - \overline{\varphi_k(z)} + f(z), \quad |z - a_k| = r_k, \quad k = 1, 2, \dots, n,$$

where  $\varphi_k \in A^1(D_k)$ ,  $f(z) = \delta_{sk} - A_s - A_{sk} \ln r_k - \sum_{m \neq k} A_{sm} \ln(z - a_m)$ ,  $z \in D_k$ .

The problem (3-10) is equivalent to the problem (3-11), i.e., the problem (3-10) is solvable if and only if (3-11) is solvable. If  $\varphi(z)$  is a solution of (3-10), it is also a solution of (3-11) in  $D$ , and a solution of (3-11) in  $D_k$  can be found from

the following simple problem for the disc  $D_k$  with respect to function  $2 \operatorname{Im} \varphi_k(z)$  harmonic in  $D_k$

$$(3-12) \quad 2 \operatorname{Im} \varphi_k(z) = \operatorname{Im}(\varphi(z) - f(z)), \quad |z - a_k| = r_k.$$

The problem (3-12) has a unique solution up to an arbitrary additive real constant.

We introduce the following function:

$$\Phi(z) := \begin{cases} \varphi_k(z) + \sum_{m \neq k} \left( \overline{\varphi_m(z_{(m)}^*)} - \overline{\varphi_m((z_0)_{(m)}^*)} \right) - \overline{\varphi_k((z_0)_{(k)}^*)} + f(z), & |z - a_k| \leq r_k, \\ \varphi(z) + \sum_{m=1}^n \left( \overline{\varphi_m(z_{(m)}^*)} - \overline{\varphi_m((z_0)_{(m)}^*)} \right), & z \in D. \end{cases}$$

The jump across the circle  $|t - a_k| = r_k$  can be calculated as

$$\Delta_k := \Phi^+(t) - \Phi^-(t),$$

where  $\Phi^+(t) := \lim_{z \rightarrow t, z \in D} \Phi(z)$  and  $\Phi^-(t) := \lim_{z \rightarrow t, z \in D_k} \Phi(z)$ . Using (3-11) we get  $\Delta_k = 0$ . It follows from the analytic continuation principle that  $\Phi(z)$  is analytic in the extended complex plane. Then Liouville's theorem implies that  $\Phi(z)$  is a constant. Using (3-9) we calculate  $\Phi(z_0) = 0$ , hence  $\Phi(z) \equiv 0$ . The definition of  $\Phi(z) \equiv 0$  in  $|z - a_k| \leq r_k$  yields the following system of functional equations:

$$(3-13) \quad \varphi_k(z) = - \sum_{m \neq k} \left( \overline{\varphi(z_{(m)}^*)} - \overline{\varphi((z_0)_{(m)}^*)} \right) + g(z), \quad |z - a_k| \leq r_k, \quad k = 1, 2, \dots, n,$$

where  $g(z) = -f(z) + \overline{\varphi((z_0)_{(k)}^*)}$ . Equations (3-13)  $\mathbb{R}$ -linearly relate the values of  $\varphi_k$  at  $z \in \bar{D}_k$  with the values of  $\varphi_m$  ( $m \neq k$ ) at the symmetric points  $z_{(m)}^* \in D_m$ . If  $\varphi_k(z)$  is known,  $\varphi(z)$  has the form

$$(3-14) \quad \varphi(z) = - \sum_{m \neq k} \left( \overline{\varphi(z_{(m)}^*)} - \overline{\varphi((z_0)_{(m)}^*)} \right), \quad z \in D.$$

**Lemma 3.1** [Mityushev and Rogosin 2000, p. 147]. *The system of functional equations (3-13) for any  $g \in A^1(\bigcup_{m=1}^n \bar{D}_m)$  has a unique solution. This solution can be found by the method of successive approximations. The approximations converge uniformly in  $\bigcup_{m=1}^n \bar{D}_m$ .*

The straightforward application of successive approximations to (3-13), combined with the use of (3-14), yield (3-7) with (3-3)–(3-5).

**Green's function.** Consider the function  $\omega(z, \zeta)$  defined by the formulas (4.4.27) or (4.4.28) from [Mityushev and Rogosin 2000, p. 156]

$$(3-15) \quad \omega(z, \zeta) = \ln \prod_{j=1}^{\infty} \omega_j(z, \zeta),$$

where

$$(3-16) \quad \omega_j(z, \zeta) = \begin{cases} \frac{\zeta - \gamma_j(z)}{\zeta - \gamma_j(z_0)}, & \text{if level of } \gamma_j \text{ is even,} \\ \frac{\overline{\zeta - \gamma_j(\bar{z}_0)}}{\zeta - \gamma_j(\bar{z})}, & \text{if level of } \gamma_j \text{ is odd.} \end{cases}$$

The infinite product (3-15) converges uniformly on  $z$  in  $\bar{D}$  for each fixed  $\zeta \in D$  (uniformly on each compact subset of  $\bar{D} - \{\zeta\}$  if  $\zeta \in \partial D$ ). Let  $\mathcal{G} \subset \mathcal{S}$  be the subgroup consisting of  $\gamma_j$  of the even level,  $\mathcal{F} \subset \mathcal{S}$  be the set of  $\gamma_j$  of the odd level.

**Lemma 3.2** [Mityushev and Rogosin 2000, p. 154]. *The complex Green's function  $M(z, \zeta)$  associated with  $D$  is represented in the form*

$$(3-17) \quad M(z, \zeta) = \sum_{k=1}^n \alpha_k(\zeta) [\psi_k(z) + \ln(z - a_k)] - \omega(z, \zeta) - \ln(z - \zeta) + A(\zeta).$$

The proof of this lemma is based on the method of functional equations described in the previous subsection.

Let  $G(z, \zeta)$  be the real Green's function. Then

$$(3-18) \quad G(z, \zeta) = \frac{1}{2} [M(z, \zeta) + \overline{M(z, \zeta)}].$$

We want to calculate  $\frac{\partial^2 G}{\partial z \partial \bar{\zeta}}$ . The function  $M(z, \zeta)$  is analytic on  $z$ , i.e.,

$$(3-19) \quad \frac{\partial \overline{M}}{\partial z} = 0.$$

It follows from (3-18) and (3-19) that

$$(3-20) \quad \frac{\partial^2 G}{\partial z \partial \bar{\zeta}} = \frac{1}{2} \frac{\partial^2 M}{\partial z \partial \bar{\zeta}}.$$

By using (3-17), we obtain

$$(3-21) \quad \frac{\partial^2 M}{\partial z \partial \bar{\zeta}} = \sum_{k=1}^n \frac{\partial \alpha_k(\zeta)}{\partial \bar{\zeta}} \left( \psi'_k(z) + \frac{1}{z - a_k} \right) - \frac{\partial^2 \omega(z, \zeta)}{\partial z \partial \bar{\zeta}}.$$

Here we use the evident equalities

$$(3-22) \quad \frac{\partial \ln(z - \zeta)}{\partial \bar{\zeta}} = 0, \quad \frac{\partial A(\zeta)}{\partial z} = 0.$$

Differentiating (3-7) we obtain

$$(3-23) \quad \frac{\partial \alpha_k}{\partial \bar{\zeta}} = \frac{1}{2} \sum_{m=1}^n A_m \left( \overline{\psi'_m(\zeta)} + \frac{1}{\zeta - a_m} \right),$$

where the prime means the derivative on  $z$  or on  $\zeta$  (but not on complex conjugated variables). Then (3-21) becomes

$$(3-24) \quad \frac{\partial^2 M}{\partial z \partial \bar{\zeta}} = \sum_{k=1}^n \sum_{m=1}^n A_m \left( \overline{\psi'_m(\zeta)} + \frac{1}{\zeta - a_m} \right) \left( \psi'_k(z) + \frac{1}{z - a_k} \right) - \frac{\partial^2 \omega(z, \zeta)}{\partial z \partial \bar{\zeta}}.$$

We calculate  $\frac{\partial^2 \omega(z, \zeta)}{\partial z \partial \bar{\zeta}}$  using the formulas (3-15) and (3-16)

$$(3-25) \quad \frac{\partial^2 \omega(z, \zeta)}{\partial z \partial \bar{\zeta}} = - \sum_{\gamma_j \in \mathcal{F}} \frac{\bar{\gamma}'_j(z)}{(\bar{\zeta} - \bar{\gamma}_j(z))^2}.$$

We now proceed to calculate  $\psi'_m(z)$ , where  $\psi_m(z)$  has the form (3-3). Differentiate the logarithms of (3-4)–(3-5)

$$(3-26) \quad \Psi_m^{(j)}(z) := (\ln \psi_m^{(j)})'(z) = \begin{cases} \frac{\gamma'_j(z)}{\gamma_j(z) - a_m} & \text{for } \gamma_j \in \mathcal{G}, \\ -\frac{\bar{\gamma}'_j(z)}{\bar{\gamma}_j(z) - a_m} & \text{for } \gamma_j \in \mathcal{F}. \end{cases}$$

Then (3-20) and (3-24) yield

$$(3-27) \quad \frac{\partial^2 G}{\partial z \partial \bar{\zeta}} = \frac{1}{2} \sum_{k=1}^n \sum_{m=1}^n A_m \overline{\sum_{\gamma_j \in \mathcal{S}_m} \Psi_m^{(j)}(\zeta)} \sum_{\gamma_j \in \mathcal{S}_m} \Psi_m^{(j)}(z) + \frac{1}{2} \sum_{\gamma_j \in \mathcal{F}} \frac{\bar{\gamma}'_j(z)}{(\bar{\zeta} - \bar{\gamma}_j(z))^2},$$

where  $\Psi_m^{(j)}(z)$  has the form (3-26).

**Bergman kernel.** Using (2-11) and (3-27) we obtain the following result.

**Theorem 3.3.** *The Bergman kernel  $K(z, w)$  associated with  $D$  is given by*

$$(3-28) \quad K(z, \zeta) = -\frac{1}{\pi} \sum_{k=1}^n \sum_{m=1}^n A_m \overline{\sum_{\gamma_j \in \mathcal{S}_m} \Psi_m^{(j)}(\zeta)} \sum_{\gamma_j \in \mathcal{S}_m} \Psi_m^{(j)}(z) - \frac{1}{\pi} \sum_{\gamma_j \in \mathcal{F}} \frac{\bar{\gamma}'_j(z)}{(\bar{\zeta} - \bar{\gamma}_j(z))^2},$$

where  $\Psi_m^{(j)}(z)$  has the form (3-26).

**Example 3.4.** Consider the simply connected domain  $D = \{z \in \bar{\mathbb{C}} : |z| > 1\}$ . Then we have the two-element group of inversions  $\gamma_0(z) = z$ ,  $\gamma_1(\bar{z}) = 1/\bar{z}$ . The constant  $A_1$  is equal to zero (see (4.4.3) of [Mityushev and Rogosin 2000, p. 149]) and (3-28) becomes

$$(3-29) \quad K(z, \zeta) = -\frac{1}{\pi} \frac{\bar{\gamma}'_1(z)}{(\bar{\zeta} - \bar{\gamma}_1(z))^2} = \frac{1}{\pi} \frac{1}{(1 - z\bar{\zeta})^2}.$$

We are not aware that any short demonstration exists which shows that [Theorem 3.3](#) yields the well-known formula (1-4) for doubly connected domains under appropriate conformal mappings.

Bell [1995] showed that the Bergman kernel is rational if and only if the associated domain is simply connected and there is a rational biholomorphic mapping of the given domain onto the unit disc. He proved [1999] that when the given domain is a finitely connected planar domain such that no boundary component is a point, the corresponding Bergman kernel is algebraic if and only if there is a proper holomorphic mapping of the given domain onto the unit disc which is algebraic.

Thus, for a circular domain  $D$  defined by (2-1)–(2-2), if  $n = 1$ ,  $D$  is biholomorphic to the unit disc with a rational biholomorphic mapping  $f(z) = 1/(z - a_1)$ , and hence the Bergman kernel is rational; see [Example 3.4](#) above. We know that every unbounded  $n$ -connected domain  $D$  is biholomorphic to a bounded  $n$ -connected domain. Hence, if  $n > 1$ , the Bergman kernel for  $D$  is not rational, but is algebraic if there is an algebraic proper holomorphic mapping from  $D$  onto the unit disc.

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MOONJA JEONG  
DEPARTMENT OF MATHEMATICS  
THE UNIVERSITY OF SUWON  
SUWON P.O.BOX 77  
KYUNGKIDO 440-600  
KOREA  
[mjeong@suwon.ac.kr](mailto:mjeong@suwon.ac.kr)

VLADIMIR MITYUSHEV  
DEPARTMENT OF MATHEMATICS  
PEDAGOGICAL ACADEMY  
UL. PODCHORAZYCH 2  
KRAKÓW 30-084  
POLAND  
[vmityu@yahoo.com](mailto:vmityu@yahoo.com)