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**EARTHQUAKES AND THURSTON'S BOUNDARY  
FOR THE TEICHMÜLLER SPACE  
OF THE UNIVERSAL HYPERBOLIC SOLENOID**

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# EARTHQUAKES AND THURSTON'S BOUNDARY FOR THE TEICHMÜLLER SPACE OF THE UNIVERSAL HYPERBOLIC SOLENOID

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A measured lamination on the universal hyperbolic solenoid  $\mathcal{S}$  is, by our definition, a leafwise measured lamination with appropriate continuity for the transverse variations. An earthquake on the universal hyperbolic solenoid  $\mathcal{S}$  is uniquely determined by a measured lamination on  $\mathcal{S}$ ; it is a leafwise earthquake with the leafwise earthquake measure equal to the leafwise measured lamination. Leafwise earthquakes fit together to produce a new hyperbolic metric on  $\mathcal{S}$  which is transversely continuous, and we show that any two hyperbolic metrics on  $\mathcal{S}$  are connected by an earthquake. We also establish the space of projective measured lamination  $PML(\mathcal{S})$  as a natural Thurston-type boundary to the Teichmüller space  $T(\mathcal{S})$  of the universal hyperbolic solenoid  $\mathcal{S}$ . The baseleaf-preserving mapping class group  $MCG_{BLP}(\mathcal{S})$  acts continuously on the closure  $T(\mathcal{S}) \cup PML(\mathcal{S})$  of  $T(\mathcal{S})$ . Moreover, the set of transversely locally constant measured laminations on  $\mathcal{S}$  is dense in  $ML(\mathcal{S})$ .

## 1. Introduction

The universal hyperbolic solenoid  $\mathcal{S}$  is the inverse limit of the system of all finite sheeted unbranched pointed covers of a compact surface of genus greater than one. Sullivan [1993] introduced the solenoid  $\mathcal{S}$  as the *universal compact surface*, that is, the universal object in the category of finite unbranched covers of a compact surface.  $\mathcal{S}$  has a rich deformation theory; its Teichmüller space  $T(\mathcal{S})$  is complex being in fact the first example of a Teichmüller space which is separable but not finite dimensional. Recall that Teichmüller spaces of compact Riemann surfaces with finite points possibly removed are finite-dimensional complex manifolds, and that Teichmüller spaces of geometrically infinite Riemann surfaces are nonseparable infinite-dimensional complex Banach manifolds.

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Sullivan started in the same paper the study of the complex structure and Teichmüller metric on  $T(\mathcal{S})$ . Unlike most laminations, the universal hyperbolic solenoid  $\mathcal{S}$  has a transverse measure, which is used in [Šarić 2004b] to continue the investigation of Teichmüller metric on  $T(\mathcal{S})$ . The results in that paper are used in [Epstein et al. 2006] to show that generic points in  $T(\mathcal{S})$  do not have Teichmüller-type Beltrami coefficient representatives, which sharply contrasts the situations for finite surfaces, where all points have Teichmüller type representatives, and for infinite surfaces, where a dense, open subset of the Teichmüller space does. Thus, unlike Riemann surfaces, most points in  $T(\mathcal{S})$  are not connected to the basepoint by a nice Teichmüller geodesic obtained by stretching horizontal and shrinking vertical foliations of a holomorphic quadratic differential on  $\mathcal{S}$  (see [Lakic 1997] for the statement for infinite surfaces).

According to the results of [Candel 1993], a conformal structure on  $\mathcal{S}$  contains a unique hyperbolic metric. The present paper starts the investigation of Teichmüller space  $T(\mathcal{S})$  of the solenoid  $\mathcal{S}$  using the hyperbolic structures on  $\mathcal{S}$ . The approach to study the decorated Teichmüller space of the related punctured solenoid via hyperbolic structures is made in [Penner and Šarić 2005; Bonnot et al. 2006]. That approach is specific to the punctured solenoid while the results in this paper hold for both universal hyperbolic solenoid and punctured solenoid.

Our first result concerns earthquakes between two hyperbolic solenoids. We recall the notion of an earthquake; see [Thurston 1986] for details. An earthquake on a hyperbolic surface is a piecewise isometric bijective map from the hyperbolic surface to another hyperbolic surface, which is not necessarily continuous. The support of an earthquake is a geodesic lamination, called the earthquake fault, along which the quaking, that is the discontinuity, appears. The restriction of an earthquake to each stratum that is a leaf of the support or a connected component of the complement of the support is an isometry, and each stratum is moved to the left when viewed from any other stratum. Given an earthquake on a hyperbolic surface, there exists a unique transverse measure, called an earthquake measure, to its support which determines the earthquake. An earthquake measure is identified with a positive Radon measure, called a measured lamination, on the space of geodesics of a surface whose support is the support geodesic lamination; the measure of a bunch of geodesics in the support is given by the measure of an arc which intersects them and which does not intersect other geodesics in the support. The main result concerning the earthquakes on hyperbolic surfaces is that any two hyperbolic metrics are connected by an earthquake.

We introduce a proper notion of an earthquake measure on the universal hyperbolic solenoid  $\mathcal{S}$ . An earthquake measure on  $\mathcal{S}$  is an assignment of an earthquake measure on each leaf, isometric to a hyperbolic plane, of  $\mathcal{S}$  such that the measures vary continuously for the transverse variations in an appropriate topology

(see [Definition 4.2](#) for more details). The support of leafwise earthquake measures are leafwise geodesic laminations which do not vary continuously for the transverse variations (see [Example 4.3](#)). However, the continuity of measures in the appropriate Fréchet topology guarantees that the leafwise earthquakes on  $\mathcal{S}$  piece together a new hyperbolic structure on  $\mathcal{S}$ , which is continuous for the transverse variations. Therefore, earthquake measures on  $\mathcal{S}$  produce an earthquake of  $\mathcal{S}$ . We establish the transitivity statement for earthquakes on hyperbolic structures of the universal hyperbolic solenoid  $\mathcal{S}$ , analogous to the case of hyperbolic surfaces.

**Theorem 5.1.** *A measured lamination  $\mu$  on a solenoid  $X$  with an arbitrary hyperbolic metric gives an earthquake map  $E_\mu$  of  $X$  into another solenoid  $Y$  with a hyperbolic metric such that there exists a (differentiable) quasiconformal map  $f : X \rightarrow Y$  whose extension to the boundary of leaves coincides with the extension of  $E_\mu$ . Any two points in the Teichmüller space  $T(\mathcal{S})$  of the universal hyperbolic solenoid  $\mathcal{S}$  are connected by a unique earthquake along a measured lamination.*

To prove this, we show that if two quasisymmetric maps of the unit circle  $S^1$  are close then their corresponding earthquake measures are close in the Fréchet topology. The opposite is false by an easy example. Therefore, the earthquake map from the space of bounded measured laminations in the unit disk onto the Teichmüller space is bijective but not a homeomorphism for the Fréchet topology on measured lamination. In the case of compact surfaces, the earthquake map is a homeomorphism if measured laminations are equipped with the weak-\* topology, which is equivalent to the Fréchet topology for compact surfaces [[Kerckhoff 1983](#)]. We establish similar result for the solenoid  $\mathcal{S}$ .

**Corollary 5.2.** *The earthquake map that assigns to each bounded measured lamination on the universal hyperbolic solenoid  $\mathcal{S}$  the corresponding marked hyperbolic solenoid is a homeomorphism between the space  $ML(\mathcal{S})$  of bounded measured laminations and the Teichmüller space  $T(\mathcal{S})$ .*

[Thurston \[1988\]](#) (see also [[Fathi et al. 1979](#)]) introduced a natural boundary for the Teichmüller space of a compact surface by *adding at infinity* to the space of projective measured laminations. The mapping class group acts continuously on the closure and there is a classification of its elements according to their actions on the boundary. [Bonahon \[1988\]](#) gave an alternative description of Thurston's boundary to the Teichmüller space of a compact surface using the Liouville map which embeds the Teichmüller space into the space of measures on the space of geodesics of the surface. The boundary points at infinity are asymptotic rays to the image of the Teichmüller space. In [[Šarić 2005](#)] we used the idea of the Liouville embedding to give a Thurston-type boundary to the Teichmüller space of any Riemann surface, possibly geometrically infinite. We extend this result to the Teichmüller space  $T(\mathcal{S})$  of the universal hyperbolic solenoid  $\mathcal{S}$ .

Biswas, Nag and Mitra introduced in [Biswas et al. 1999] the direct limit of projective measured laminations on the compact surfaces as boundary at infinity of the direct limit of Teichmüller spaces of compact surfaces covering a fixed compact surface of genus at least two. Since  $T(\mathcal{S})$  contains the above direct limit of Teichmüller spaces of compact surfaces as a dense subset, the authors remark that Thurston's boundary for  $T(\mathcal{S})$  should be a completion of the union of the projective measured laminations on all compact surfaces. We give a proper analytical description of the Thurston's boundary that answers their question about the completion. The main point is to properly define continuity for the transverse variations of various spaces of measures and distributions on the space  $\mathcal{G}(\mathcal{S})$  of geodesics on the universal hyperbolic solenoid  $\mathcal{S}$ . We establish this goal using the Fréchet topology on the *enveloping* space of Hölder distributions  $H(\mathcal{S})$  in Section 6.

**Theorem 6.2.** *The Liouville map  $\mathcal{L}_{\mathcal{S}} : T(\mathcal{S}) \rightarrow H(\mathcal{S})$  is a homeomorphism onto its image. The set of asymptotic rays to  $\mathcal{L}_{\mathcal{S}}(T(\mathcal{S}))$  is homeomorphic to  $PML(\mathcal{S})$ , the space of projective measured laminations on  $\mathcal{S}$ . The baseleaf-preserving mapping class group  $MCG_{BLP}(\mathcal{S})$  acts continuously on the closure  $T(\mathcal{S}) \cup PML(\mathcal{S})$  of the Teichmüller space  $T(\mathcal{S})$  of the universal hyperbolic solenoid  $\mathcal{S}$ .*

A lift of a measured lamination on a compact surface of genus at least two to the universal hyperbolic solenoid  $\mathcal{S}$  is a measured lamination on  $\mathcal{S}$ . Such a measured lamination is locally constant for transverse variations and is called a transversely locally constant (TLC) measured lamination. We show that an arbitrary measured lamination on  $\mathcal{S}$  is the limit of TLC measured laminations in the Fréchet topology.

**Theorem 6.3.** *The subset of all measured laminations on the universal hyperbolic solenoid  $\mathcal{S}$  that are locally transversely constant, is dense in the space of all measured laminations  $ML(\mathcal{S})$  on  $\mathcal{S}$  for the Fréchet topology.*

In Section 7, we introduce the space of compactly supported measured laminations  $PML_0(\mathcal{S}_p)$  on the punctured solenoid  $\mathcal{S}_p$  and extend the transitivity of earthquakes and Thurston's boundary for the punctured solenoid by replacing  $ML(\mathcal{S})$  with  $ML_0(\mathcal{S}_p)$ .

## 2. Preliminaries

We recall several definitions: the universal hyperbolic solenoid, earthquakes of the unit disk, the Fréchet topology on the measures on the space of geodesics of the unit disk.

**The universal hyperbolic solenoid.** Let  $(S, x)$  be a fixed compact surface of genus at least two with the basepoint  $x \in S$ . Consider all finite sheeted unbranched covers  $(S_i, x_i)$  by compact surfaces with basepoints such that the covering maps  $\pi_i : S_i \rightarrow S$  satisfy  $\pi_i(x_i) = x$ . There is a natural partial ordering  $\leq$  on the set



A *complex structure* on the universal hyperbolic solenoid  $\mathcal{S}$  is a collection of local charts such that the transition maps are holomorphic in the disk direction and they vary continuously for topology of uniform convergence in the transverse Cantor direction of the local charts [Sullivan 1993]. Complex structures on  $\mathcal{S}$  are in one to one correspondence with conformal structures on  $\mathcal{S}$  that are continuous for the variations in the transverse direction, via the continuous dependence on the parameters of the solution of Beltrami equation. Any conformal structure on  $\mathcal{S}$  contains a unique hyperbolic metric which is continuous for the variations in the transverse direction [Candel 1993]. A hyperbolic metric (or a complex structure) on  $\mathcal{S}$ , which is transversely locally constant for some choice of charts on  $\mathcal{S}$ , is called a TLC hyperbolic metric (or a TLC complex structure) on  $\mathcal{S}$  [Sullivan 1993]. Any TLC hyperbolic metric (or TLC complex structure) on  $\mathcal{S}$  is obtained by lifting a hyperbolic metric (or a complex structure) from a finite cover of  $S$  to  $\mathcal{S}$  [Nag and Sullivan 1995]. Note that we fix a TLC complex structure on  $\mathcal{S}$  coming from Riemann surface  $D/G$  by identifying  $\mathcal{S}$  with  $(D \times \hat{G})/G$ . A (differentiable) quasiconformal map  $f : \mathcal{S} \rightarrow X$  from the fixed TLC complex solenoid  $\mathcal{S}$  to an arbitrary complex solenoid  $X$  is a homeomorphism which is  $C^\infty$ -differentiable in the disk direction in local charts and varies continuously in the transverse direction for  $C^\infty$ -topology on  $C^\infty$ -maps [Sullivan 1993; Šarić 2004b]. We note that the use of differentiable quasiconformal maps as opposed to only quasiconformal maps is necessary in order for the compositions of quasiconformal maps to be continuous in the transverse direction. We define the Teichmüller space  $T(\mathcal{S})$  of the universal hyperbolic solenoid  $\mathcal{S}$  as follows.

**Definition 2.3.** The *Teichmüller space*  $T(\mathcal{S})$  of the universal hyperbolic solenoid  $\mathcal{S}$  consists of all quasiconformal maps  $f : \mathcal{S} \rightarrow X$  up to an equivalence. Two quasiconformal maps  $f : \mathcal{S} \rightarrow X$  and  $g : \mathcal{S} \rightarrow X_1$  are equivalent if there exists a conformal map  $c : X \rightarrow X_1$  such that  $g^{-1} \circ c \circ f : \mathcal{S} \rightarrow \mathcal{S}$  is homotopic to the identity. The equivalence class of the identity  $\text{id} : \mathcal{S} \rightarrow \mathcal{S}$  is called the *basepoint* of  $T(\mathcal{S})$ .

**Earthquakes in the unit disk.** We define earthquakes of the unit disk  $D$  and recall their main properties. A *geodesic lamination* in the unit disk  $D$  is a closed subset of  $D$  which is foliated by geodesics for the hyperbolic metric on  $D$ , or equivalently, it is a closed subset of the space  $\mathcal{G}(D)$  of geodesics in  $D$  such that no two geodesics in the subset intersect in  $D$ . Some familiar examples of geodesic laminations in  $D$  are: a set of finitely many nonintersecting geodesics in  $D$ ; a countable, discrete set of nonintersecting geodesics; a foliation of  $D$  by geodesics.

**Definition 2.4.** An *earthquake measure* on the unit disk is a positive Radon measure on the space of geodesic  $\mathcal{G}(D)$  of the unit disk  $D$ , whose support is a geodesic lamination.

Note that  $\mathcal{G}(\mathbf{D})$  is homeomorphic to  $(S^1 \times S^1 - \text{diag})/\mathbb{Z}_2$  by mapping a geodesic in  $\mathbf{D}$  to the unordered pair of its ideal endpoints in  $S^1$ . Earthquakes were introduced in [Thurston 1986] as follows (this is the reference for all results on this page unless otherwise specified).

**Definition 2.5.** An *earthquake*  $E : \mathbf{D} \rightarrow \mathbf{D}$  of the unit disk  $\mathbf{D}$  is a bijective map which maps a fixed geodesic lamination  $\lambda$ , called the *support* of  $E$ , in  $\mathbf{D}$  onto another geodesic lamination  $\lambda'$ . A geodesic from  $\lambda$  or a connected component of  $\mathbf{D} - \lambda$  is called a *stratum* of  $E$ . The restriction of the earthquake  $E$  to each stratum is a hyperbolic isometry with the additional property that for any two strata  $A, B$  of  $E$ , the *comparison isometry*  $E|_B \circ (E|_A)^{-1}$  is a hyperbolic translation whose axis separates  $A$  from  $B$ , and which translates  $B$  to the left as seen from  $A$ .

Each earthquake  $E : \mathbf{D} \rightarrow \mathbf{D}$  continuously extends to a homeomorphism of  $\partial\mathbf{D} \equiv S^1$ , which we denote by  $E|_{S^1} : S^1 \rightarrow S^1$ . An important theorem due to Thurston is that each orientation-preserving homeomorphism of  $S^1$  is obtained as the extension of an earthquake to  $S^1$ . Given an earthquake  $E$  of  $\mathbf{D}$ , there exists a unique corresponding earthquake measure  $\mu$  on  $\mathcal{G}(\mathbf{D})$  supported on  $\lambda$  determined by the following condition. Consider a subset of  $\lambda$  consisting of geodesics which intersect a closed arc  $I$ . Choose finitely many strata of  $E$  intersecting  $I$ . The  $\mu$  measure of the subset is approximated by the sum of translation lengths between comparison isometries of adjacent strata of the above chosen finitely many strata of  $E$  intersecting the arc  $I$ , when the distance between adjacent strata goes to zero. An alternative description is to consider the measure  $\mu$  to be a family of measures on arcs  $I$  in  $\mathbf{D}$  which are invariant under homotopies of arcs preserving the leaves of  $\lambda$ . If an earthquake measure  $\mu$  corresponds to an earthquake as above, we denote the corresponding earthquake by  $E_\mu$ . Two homeomorphisms have the same corresponding earthquake measures if and only if they differ by a postcomposition with a hyperbolic isometry of  $\mathbf{D}$ .

We say that an earthquake measure  $\mu$  is *bounded* if the *norm*  $\|\mu\|$  satisfies

$$\|\mu\| := \sup_I \{\mu(I)\} < \infty,$$

where the supremum is over all geodesic arcs in  $\mathbf{D}$  of length 1. If  $\mu$  is a bounded earthquake measure, then there exists earthquake  $E_\mu$  corresponding to  $\mu$ . A homeomorphism  $h : S^1 \rightarrow S^1$  is *quasisymmetric* if and only if the corresponding earthquake measure  $\mu$  is bounded, where  $E_\mu|_{S^1} = h$  [Šarić 2006; 2007].

**The Fréchet topology.** We recall the definition of the Fréchet topology on the space of Hölder distributions  $H(\mathbf{D})$  of the unit disk  $\mathbf{D}$  from [Šarić 2004a]. The space of bounded positive measures on  $\mathcal{G}(\mathbf{D})$ , and the space of bounded earthquake measures on  $\mathcal{G}(\mathbf{D})$ , are subsets of  $H(\mathbf{D})$ .





as  $n \rightarrow \infty$  because  $h_n \rightarrow h$  in the quasimetric topology and by the invariance of Liouville measure under hyperbolic isometries. Thus  $\bar{h} \circ \bar{f}^{-1}$  preserves Liouville measure and fixes  $1, i, -1$ . Therefore,  $\bar{h} = \bar{f}$ .

Let  $\mu'_n := A_n^*(\mu_n)$  and  $\sigma_n := A_n^*(\mu)$ . Then there exists a sequence

$$(\varphi'_n, (1, i) \times (-1, -i)) \in \text{test}(v)$$

such that

$$(1) \quad |\mu'_n(\varphi'_n) - \sigma'_n(\varphi'_n)| \geq m > 0.$$

Since  $\mu'_n$  is the pushforward by a hyperbolic isometry of  $\mu_n$ ,  $\|\mu'_n\| = \|\mu_n\|$ . Moreover, since  $\mu_n$  are earthquake measures for  $h_n$  and  $h_n$  converges in the quasimetric topology, it follows that  $\mu_n$  are uniformly bounded measures, and that the same holds for  $\mu'_n$ . The sequence  $\sigma_n$  is also uniformly bounded because it is the pushforward of a single measure by hyperbolic isometries. Both sequences  $\mu'_n$  and  $\sigma_n$  converge to bounded earthquake measures  $\mu'$  and  $\sigma$  such that  $\bar{h} = E_{\mu'}|_{S^1}$  and  $\bar{f} = E_{\sigma}|_{S^1}$  by [Proposition 3.1](#). By (1), we conclude that  $\mu' \neq \sigma$ . In fact, the weak-\* convergence on a fixed box  $(1, i) \times (-1, -i)$  is equivalent to the uniform convergence with respect to all  $(\varphi, (1, i) \times (-1, -i)) \in \text{test}(v)$  [[Šarić 2004a](#)]. But this is a contradiction with  $\bar{h} = \bar{f}$  by the uniqueness of earthquake measures [[Thurston 1986](#)].  $\square$

We remark that the inverse of [Proposition 3.2](#) is not true. This is easily seen by an example. Take a fixed geodesic with a positive weight as one lamination. Take a convergent sequence in Fréchet topology, that consists of geodesics sharing exactly one endpoint with the above geodesic and taking the same positive weight. It is obvious that the extension of the earthquakes to  $S^1$  corresponding to the sequence does not converge to the extension of the earthquake to  $S^1$  corresponding to the limit in the quasimetric topology. Note that they do converge pointwise. This is in contrast with the statement in [Proposition 3.1](#) which gives the equivalence. However, if we restrict ourselves to the earthquakes on compact surfaces then the equivalence holds. One of the main results in [Section 4, 5](#) is that the equivalence holds for the universal hyperbolic solenoid as well.

#### 4. Measured laminations on the universal hyperbolic solenoid

Recall that a leaf of the universal hyperbolic solenoid  $\mathcal{S}$  intersects any local chart countably many times. Each intersection is a 2-disk, which is called a *local leaf*. Given two local leaves of two global leaves, there exists an identification isometry of the global leaves given as follows. Since the hyperbolic metric in the local charts is continuous for the trivial (vertical) identification, we can choose two points sitting one above the other and two unit tangent vectors based at the points whose

directions get vertically identified. The two vectors are not necessarily vertically identified because the hyperbolic metrics are not constant in the transverse direction. The isometric identification is uniquely determined by requiring to map one point onto the other such that the unit vector is mapped onto the unit vector. The identification depends on the chart and the choice of two points while the choice of tangent vectors does not affect it.

Fix one local leaf  $l$  and consider a sequence of local leaves  $l_n$  approaching  $l$ . Suppose we choose two different isometric identifications  $f_n : \tilde{l} \rightarrow \tilde{l}_n$  and  $g_n : \tilde{l} \rightarrow \tilde{l}_n$  of the global leaves  $\tilde{l}, \tilde{l}_n$  containing local leaves  $l, l_n$ . Note that the identifications differ by the choice of points in the local leaves. Then  $g_n^{-1} \circ f_n$  is an isometry of  $\tilde{l}$  which converges to the identity as  $n \rightarrow \infty$  because of the continuity in the transverse direction of the hyperbolic metrics. This implies that any two identifications of two global leaves differ by an isometry which is close to the identity when corresponding local leaves are close. Therefore, it makes sense to compare objects preserved by isometries on two nearby leaves as well as maps from leaves.

**Definition 4.1.** A (transversely continuous) *geodesic lamination* on the universal hyperbolic solenoid  $\mathcal{S}$  is an assignment of a geodesic lamination to each leaf which is continuous, for Hausdorff distance between closed subset of  $\mathcal{G}(\mathbf{D})$  defined using the angle metric  $d$  on  $S^1 \times S^1$ , with respect to the transverse variations given by each local chart as above.

Namely, we consider the isometric identifications for any local chart as above. The geodesic laminations on global leaves can be mapped to the unit disk  $\mathbf{D}$  by the identifications. Thus we obtain a map from the local transverse set, obtained by considering each local leaf in the chart as a point, to the space of geodesic laminations on the unit disk  $\mathbf{D}$ . We require that this map be continuous for the Hausdorff topology on the space of geodesic laminations.

This definition certainly seems to be in the spirit of transverse continuity of the hyperbolic metrics on  $\mathcal{S}$ . However, we introduce below measured laminations on  $\mathcal{S}$  in terms of the continuity of measures. It turns out that the support of measured laminations on  $\mathcal{S}$  is not geodesic lamination as above, even though the restriction to each leaf is a geodesic lamination.

**Definition 4.2.** A (transversely continuous) *measured lamination*  $\mu$  on  $\mathcal{S}$  is an assignment of a bounded measured lamination to each leaf of  $\mathcal{S}$  such that it is continuous for the transverse variations with respect to Fréchet topology on the space of measured laminations on the unit disk.

Given a local chart  $D \times T$ , where  $D$  is a 2-disk and  $T$  a transverse Cantor set, the measured lamination  $\mu$  on  $\mathcal{S}$  gives a map  $\mu : T \rightarrow ML_{\text{bdd}}(\mathbf{D})$  using the identifications of leaves induced by the local chart. We denote by  $ML_{\text{bdd}}(\mathbf{D})$  the space of all bounded measured laminations on the unit disk  $\mathbf{D}$ . In the definition

above, we require that  $\|\mu(t) - \mu(t')\|_\nu \rightarrow 0$  as  $t' \rightarrow t$  for each  $t \in T$  and for each  $0 < \nu \leq 1$ .

The definition of measured laminations on  $\mathcal{S}$  does not specify the support. To give an example of a measured lamination on  $\mathcal{S}$ , fix a measured lamination  $\sigma$  on a compact surface  $S$  of genus at least two. Since each leaf of  $\mathcal{S}$  is a universal cover of the compact surface  $S$ , we can lift  $\sigma$  to a measured lamination  $\tilde{\sigma}$  on each leaf of  $\mathcal{S}$ . The lifts  $\tilde{\sigma}$  are locally constant for the transverse variations in the local charts of a TLC hyperbolic metric coming from the hyperbolic metric on  $S$ . Thus  $\tilde{\sigma}$  defines a geodesic lamination on  $\mathcal{S}$ .

As we mention above, there are measured laminations on  $\mathcal{S}$  whose supports are not a geodesic laminations on  $\mathcal{S}$ . We give an example of such a measured lamination.

**Example 4.3.** We identify  $\mathcal{S}$  with  $(\mathbf{D} \times \hat{G})/G$ , for a Fuchsian group  $G$  uniformizing a compact hyperbolic surface  $S = \mathbf{D}/G$ . We define a measured lamination  $\tilde{\mu}$  on  $\mathbf{D} \times \hat{G}$  which is invariant under  $G$ . Let  $G_i$  be a decreasing sequence of finite index normal subgroups of  $G = G_1$  such that  $\bigcap_{i=1}^\infty G_i = \{\text{id}\}$ . We fix two simple closed curves  $\gamma_1, \gamma_2$  in  $S$  which intersect in one point and we fix two lifts  $\tilde{\gamma}_1, \tilde{\gamma}_2$  of  $\gamma_1, \gamma_2$  in the universal cover  $\mathbf{D}$  such that  $|\tilde{\gamma}_1 \cap \tilde{\gamma}_2| = 1$ . Denote by  $C_1, C_2$  primitive hyperbolic translations in  $G$  whose axes are  $\tilde{\gamma}_1, \tilde{\gamma}_2$  respectively. Let  $C_1^{r_i}$  and  $C_2^{t_i}$  be primitive elements in  $G_i$ . We further require that the group  $\tilde{G}_{i+1}$  generated by  $C_1^{r_i}, C_2^{t_i}$  and  $G_{i+1}$  is of index at least 3 in  $G_i$ .

Consider the cosets  $a_0^i \tilde{G}_{i+1}, a_1^i \tilde{G}_{i+1}, \dots, a_{k_i}^i \tilde{G}_{i+1}, a_0^i = \text{id}$ , of  $\tilde{G}_{i+1}$  in  $G$ . Since  $[G_i : \tilde{G}_{i+1}] \geq 3$ , there are at least two cosets different from  $\tilde{G}_{i+1}$  which lie in  $G_i$ . Denote them by  $a_1^i \tilde{G}_{i+1}, a_2^i \tilde{G}_{i+1} \subset G_i$ . Then  $(a_1^i G_{i+1}) \cdot (a_2^i G_{i+1})^{-1}$  are not of the form  $C_j^k G_{i+1}$ , for  $j = 1$  or  $j = 2$  and for some  $k \in \mathbb{Z}$ . To see this, first note that if  $a_1^i (a_2^i)^{-1}$  is a power of  $C_1$  or  $C_2$  then it has to be a power of primitive elements  $C_1^{r_i}$  or  $C_2^{t_i}$  in  $G_i$ . Otherwise  $(a_1^i G_{i+1}) \cdot (a_2^i G_{i+1})^{-1} \notin G_i/G_{i+1}$ , which is a contradiction. On the other hand, by our choice of  $\tilde{G}_{i+1}$  and cosets  $a_1^i \tilde{G}_{i+1}, a_2^i \tilde{G}_{i+1}$ , we get that  $a_1^i (a_2^i)^{-1}$  is not a power of primitive elements  $C_1^{r_i}$  or  $C_2^{t_i}$ .

Let  $\delta_{\tilde{\gamma}_i, t}$  denote a unit mass measure on the space of geodesics of  $\mathbf{D} \times \hat{G}$  supported on the geodesic  $(\tilde{\gamma}_i, t) \subset \mathbf{D} \times \{t\}$ , for  $t \in \hat{G}$ . We define

$$\tilde{\mu}' := \sum_i \left( \sum_{t \in a_1^i \hat{G}_{i+1}} m_i \delta_{\tilde{\gamma}_1, t} + \sum_{t \in a_2^i \hat{G}_{i+1}} m_i \delta_{\tilde{\gamma}_2, t} \right)$$

where  $m_i > 0, m_i \rightarrow 0$  as  $i \rightarrow \infty$  and  $\hat{G}_{i+1} < \hat{G}$  is the profinite completion of  $G_{i+1}$ . The measured lamination  $\tilde{\mu}'$  varies continuously in the transverse direction for the Fréchet topology on measured laminations of the unit disk. The continuity is immediate at any  $t \in \hat{G}$  because  $\tilde{\mu}'$  is locally constant.

We define

$$\tilde{\mu} := \sum_{A \in G} A^*(\tilde{\mu}').$$

Then  $\tilde{\mu}$  is invariant under the action of  $G$ . We first show that the support of  $\tilde{\mu}$  on each leaf  $\mathbf{D} \times \{t\}$ ,  $t \in \hat{G}$ , is a geodesic lamination. Let  $\omega$  be a fundamental polygon for the action of  $G$  in  $\mathbf{D}$ . Then  $\omega \times \hat{G}$  is a fundamental set for the action of  $G$  on  $\mathbf{D} \times \hat{G}$ . It is enough to show that the support of  $\tilde{\mu}$  has no self-intersections on  $\omega \times \hat{G}$ . Note that the support of  $\tilde{\mu}'$  has no self-intersections because the cosets of  $\hat{G}$  which contain copies of  $\tilde{\gamma}_1$  and  $\tilde{\gamma}_2$  in the support are chosen to be disjoint. The only possibility for the support of  $\tilde{\mu}$  to have a self-intersection is the case that a coset  $a_1^i \hat{G}_{i+1}$  is mapped onto the coset  $a_2^j \hat{G}_{i+1}$  by  $C_1^k$  for some  $k \in \mathbb{Z}$ , and similar for  $C_2$ . This is impossible by our choice of cosets.

We claim that  $\tilde{\mu}$  is continuous for the transverse variations in the Fréchet topology on the space of measured laminations of the unit disk  $\mathbf{D}$ . First, we can assume that different lifts of  $\gamma_j$ ,  $j = 1, 2$ , in  $\mathbf{D}$  do not belong to a single box of geodesics  $Q$  with  $L(Q) = \log 2$  by appropriate choice of the group  $G$ . If we show continuity in this case, the result follows because the convergence of measured laminations in the Fréchet topology is independent of the hyperbolic metric. We already conclude that  $\tilde{\mu}'$  is continuous for the transverse variations. By taking the pushforward of  $\tilde{\mu}'$  by  $G$ , we add some extra support of  $\tilde{\mu}$  intersecting fundamental set  $\omega \times \hat{G}$ . The extra support is obtained by adding  $(\tilde{\gamma}_1, t)$  for  $t \in C_1^k a_1^i \hat{G}_{i+1}$ ,  $k \in \mathbb{Z}$  and  $i = 1, 2, \dots$ , and by adding  $(\tilde{\gamma}_2, t)$  for  $t \in C_2^k a_2^j \hat{G}_{i+1}$ ,  $k \in \mathbb{Z}$  and  $i = 1, 2, \dots$ . It is obvious that the restriction of  $\tilde{\mu}$  to the part which intersects  $\omega \times \hat{G}$  is continuous for the transverse variations at any  $t \in \hat{G} - \{\text{id}\}$ , similar to  $\tilde{\mu}'$ . The continuity at  $t = \text{id}$  follows because  $m_i \rightarrow 0$ . Our assumption that the orbit of  $\tilde{\gamma}_i$  does not contain two geodesic which lie in a box  $Q$  with  $L(Q) = \log 2$  implies the continuity of  $\tilde{\mu}$ .

The measured lamination  $\tilde{\mu}$  on  $\mathbf{D} \times \hat{G}$  descends to a measured lamination  $\mu$  on the universal hyperbolic solenoid  $\mathcal{S}$ . The continuity of  $\mu$  for the transverse variations follows by the continuity of  $\tilde{\mu}$ . It is clear that the support of  $\mu$  is not a geodesic lamination on  $\mathcal{S}$  as in [Definition 4.1](#) because it is not a closed set. Moreover, the closure in  $\mathcal{S}$  of the support of  $\mu$  is not a geodesic lamination because, on the baseleaf, it consists of the full preimage of the two intersecting geodesics  $\gamma_1$  and  $\gamma_2$  on the closed surface  $S$ .

## 5. Earthquake theorem

We show that any two points in the Teichmüller space  $T(\mathcal{S})$  of the universal hyperbolic solenoid  $\mathcal{S}$  are connected by an earthquake. First, we need to recall certain facts from [[Šarić 2005](#)] about arbitrary points in  $T(\mathcal{S})$ .

A TLC solenoid  $\mathcal{S}$  is homeomorphic to  $(\mathbf{D} \times \hat{G})/G$ , where  $\mathbf{D}$  is the unit disk. The space  $\mathbf{D} \times \hat{G}$  is considered as a universal cover of  $\mathcal{S}$ . Denote the covering map by  $\pi : \mathbf{D} \times \hat{G} \rightarrow \mathcal{S}$ . The action of  $G$  is given by

$$A(z, t) := (A(z), tA^{-1}),$$

where  $A$  can be considered as an element of  $\hat{G}$ .

A point in  $T(\mathcal{S})$  is given by a (differentiable) quasiconformal map  $f : \mathcal{S} \rightarrow X$ , where  $X$  is the universal hyperbolic solenoid with an arbitrary hyperbolic metric, not necessarily TLC. We introduced the universal (hyperbolic) cover to  $X$  and the covering group as follows [Šarić 2005]. The action by  $G$  does not introduce identifications to the set  $\{0\} \times \hat{G}$  where  $0 \in \mathbf{D}$ . Consider a local chart  $D \times T$  for  $X$  which contains  $f(\pi(\{0\} \times \hat{G}))$  as a vertical set, where  $T \equiv \hat{G}$  and  $D$  is a 2-disk with center at 0. To fix the notation, we assume that  $f(\pi(\{0\} \times \hat{G}))$  corresponds to  $\{0\} \times T \subset D \times T$  in the local chart. We fix unit tangent vectors at the points  $f(\pi(\{0\} \times \hat{G}))$  corresponding to the unit tangent vectors under the chart map at the points  $\{0\} \times T$  along the positive axis in the chart  $D \times T$ .

The universal cover for  $X$  is, by the definition,  $\mathbf{D} \times T$  and the covering map  $\pi_X : \mathbf{D} \times T \rightarrow X$  is given by isometrically mapping each  $\mathbf{D} \times \{t\}$  onto the leaf containing  $f(\pi(0, t))$  such that the origins are mapped onto the origins and the unit tangent vectors along positive axes are mapped onto the unit tangent vectors along the positive axes when considered in the chart  $D \times T$ . The map  $f : \mathcal{S} \rightarrow X$  lifts to a quasiconformal map

$$\tilde{f} : \mathbf{D} \times \hat{G} \rightarrow \mathbf{D} \times T$$

of the universal covers.

The action of  $G$  is conjugated by  $\tilde{f}$  to an action of a group  $G_X$  on  $\mathbf{D} \times T$ . An element  $A$  of  $G$  acts on  $\mathbf{D} \times \hat{G}$  by  $A(z, t) = (A(z), tA^{-1})$ , namely it acts on the unit disk component by hyperbolic isometry and it shifts the leaf  $\mathbf{D} \times \{t\}$  onto the leaf  $\mathbf{D} \times \{tA^{-1}\}$ . In particular, the action on the disk coordinate is independent of the leaf as the second coordinate. We define a covering transformation  $A_f$  for  $X$  by the formula

$$(A_f \circ \tilde{f})(z, t) = (\tilde{f} \circ A)(z, t).$$

An equivalent definition for  $A_f$  is

$$A_f(z, t) := (\pi_X)^{-1}(\pi_X(z, t), tA^{-1}),$$

where  $\pi_X(z, t) \in \mathcal{S}$  and  $\pi_X(\cdot, tA^{-1})$  stands for the inverse of the covering map restricted to  $\mathbf{D} \times \{tA^{-1}\}$ . Recall that, when restricted to each leaf, the covering map is an isometry for the hyperbolic metric on leaves. Thus  $A_f$  is an isometry on each leaf of  $\mathbf{D} \times T$ ; yet it varies with leaves. The group of covering maps for the



between leaves. The maps  $A_*$ , for all  $A_f \in G_f$ , form a group  $G_*$  isomorphic to  $G_f$  under the conjugation by  $h$ .

We claim that  $A_*(t)$  is continuous in  $t$  for the standard topology on the space of hyperbolic isometries of  $\mathbf{D}$ . To see this, it is enough to show that the images of three fixed point on  $S^1$  vary continuously in  $t$ . By the above equation, we get

$$A_*(t) = h_{tA^{-1}} \circ A_f(t) \circ h_t^{-1}.$$

Since  $\tilde{\mu}(t)$  varies continuously in  $t$  and since  $h_t, h_{tA^{-1}}$  are properly normalized earthquakes, we conclude that  $h_t, h_{tA^{-1}}$  are continuous in  $t$  for the topology of pointwise convergence by [Šarić 2007, Proposition 3.3]. By our assumption,  $A_f(t)$  is continuous in  $t$ . Thus  $A_*(t)$  is continuous in  $t$ .

We need to show that the quotient  $(\mathbf{D} \times T)/G_*$  is quasiconformally equivalent to  $X$ , i.e., there exists a homeomorphism  $g : X \rightarrow (\mathbf{D} \times T)/G_*$  which is a differentiable quasiconformal map on each leaf and which varies continuously in the transverse direction in the  $C^\infty$ -topology. Let  $g_t = ex(h_t : \mathbf{D} \rightarrow \mathbf{D})$  be the barycentric extension of  $h_t : S^1 \rightarrow S^1$  (see [Douady and Earle 1986] for this definition and for the rest of this paragraph). Since the family  $h_t, t \in T$ , is continuous in the pointwise convergence topology, the family  $g_t$  of barycentric extensions is continuous for the  $C^\infty$ -topology of  $C^\infty$ -maps over compact subsets of  $\mathbf{D}$ .

We claim that  $f_t$  is also continuous in the parameter  $t$  for the quasiconformal topology. (That the earthquake measures converge in the Fréchet topology does not imply that the extensions of earthquakes to  $S^1$  converge in the quasisymmetric topology, by the example in Section 3. At this point, we rely heavily on the compactness of the solenoid  $X$ .) Recall from [Šarić 2004b] that  $G_X$  has a compact fundamental set for the action on  $\mathbf{D} \times T$ , given by the image under  $\tilde{f}$  of the fundamental set  $\omega \times \hat{G}$  for the action of  $G$  on  $\mathbf{D} \times \hat{G}$ . Thus the Beltrami coefficients of the family  $g_t$  are continuous in  $t$  for the supremum norm over the fundamental set of  $X$ . By the invariance of the quasisymmetric family  $h_t$  and by the conformal naturality of barycentric extension [Douady and Earle 1986], we obtain

$$g_{tA^{-1}} \circ A_f(t) = A_*(t) \circ g_t,$$

for  $t \in T$ . The invariance under  $G_f$  and the continuity of Beltrami coefficients of  $g_t$  on the fundamental domain of  $G_f$  imply that the Beltrami coefficients of  $g_t$  are continuous in  $t$  for the essential supremum norm on the unit disk. Thus we have obtained a differentiable quasiconformal homeomorphism

$$\tilde{g} : \mathbf{D} \times T \rightarrow \mathbf{D} \times T, \quad \tilde{g}(\cdot, t) := g_t(\cdot),$$

which conjugates  $G_f$  onto  $G_*$ . Consequently, it projects onto a quasiconformal homeomorphism  $g : X \rightarrow (\mathbf{D} \times T)/G_*$ . Thus the earthquake  $E_\mu$  defines a new hyperbolic solenoid  $Y := (\mathbf{D} \times T)/G_*$  which is the image of  $X$ . By its definition,

the boundary values of each  $g_t$  agree with  $E_{\tilde{\mu}(t)}|_{S^1}$ . This finishes the proof of the first part of the Theorem.

It remains to show that any two points  $[f : \mathcal{S} \rightarrow X], [g : \mathcal{S} \rightarrow Y]$  are connected by an earthquake along a measured lamination on  $X$ . In other words, we need to find a measured lamination  $\mu$  on  $X$  such that  $E_\mu$  maps  $X$  onto  $Y$  and that the extensions of  $E_\mu$  to the boundaries of leaves are equal to the extensions of  $g \circ f^{-1}$ . We lift the maps  $f$  and  $g$  to the maps

$$\tilde{f} : \mathbf{D} \times T \rightarrow \mathbf{D} \times T, \quad \text{and} \quad \tilde{g} : \mathbf{D} \times T \rightarrow \mathbf{D} \times T$$

of the universal covers of  $X$  and  $Y$ . Let  $h_t := \tilde{g} \circ \tilde{f}^{-1}|_{\mathbf{D} \times \{t\}}$ . Note that the family of quasimetric maps  $h_t, t \in T$ , is continuous in  $t$  for the quasimetric topology. By Thurston's earthquake theorem for the unit disk [1986], there exists a measured lamination  $\tilde{\mu}(t)$  such that  $E_{\tilde{\mu}(t)}|_{S^1} = h_t$ . Moreover,  $\tilde{\mu}(t)$  is a bounded earthquake measure on  $\mathbf{D}$ . Since  $h_t$  varies continuously, we get that  $\tilde{\mu}(t)$  varies continuously in  $t$  for the Fréchet topology by Proposition 3.2. The family  $h_t$  satisfies invariance properties with respect to  $G_X$ . Therefore, by the uniqueness of earthquake measures [Thurston 1986], the family of corresponding earthquake measures  $\tilde{\mu}(t)$  also satisfies invariance properties. Thus it descends to the desired earthquake measure  $\mu$  on  $X$ .  $\square$

We recall that Proposition 3.2 states that if quasimetric maps are close in the quasimetric topology then the corresponding earthquake measures are close in the Fréchet topology. The inverse is false in general. However, we show above that the compactness of the universal hyperbolic solenoid  $\mathcal{S}$  forces the continuity of quasimetric maps on nearby leaves obtained by earthquaking along transversely continuous measured laminations. The proof extends along the same lines to show that if two measured laminations on the solenoid  $\mathcal{S}$  are close in the Fréchet topology then the extension of earthquake maps to the boundary leaves are close in the quasimetric topology. We obtain:

**Corollary 5.2.** *The earthquake map which assigns to each bounded measured lamination on the universal hyperbolic solenoid  $\mathcal{S}$  the corresponding marked hyperbolic solenoid is a homeomorphism between the space  $ML(\mathcal{S})$  of bounded measured laminations and the Teichmüller space  $T(\mathcal{S})$ .*

## 6. Thurston's boundary for $T(\mathcal{S})$

We recall the definition of the Liouville map  $\mathcal{L} : T(\mathbf{D}) \rightarrow H(\mathbf{D})$  from the universal Teichmüller space  $T(\mathbf{D})$  to the space of Hölder distributions  $H(\mathbf{D})$  of the unit disk  $\mathbf{D}$ . Liouville map first appears in [Bonahon 1988] in the case of the Teichmüller space of a compact Riemann surface and it is used in [Šarić 2005] to introduce Thurston-type boundary to the universal Teichmüller space  $T(\mathbf{D})$ .







groups [Nag and Sullivan 1995], that is the closure of the union

$$\bigcup_{[G:K] < \infty} \tilde{T}(\mathbf{D}/K)$$

of the lifts  $\tilde{T}(\mathbf{D}/K)$  to  $T(\mathbf{D})$  of all Teichmüller spaces  $T(\mathbf{D}/K)$  of finite degree unbranched covers  $\mathbf{D}/K$  of  $\mathbf{D}/G$ . By the characterization of the image of  $\mathcal{L}$  from [Bonahon 1988], the image  $\mathcal{L}(\bigcup_{[G:K] < \infty} \tilde{T}(\mathbf{D}/K))$  of the union consists of all bounded Hölder distributions  $\alpha$  which are positive measures invariant under finite index subgroups of  $G$  and which satisfy

$$e^{-\alpha([a,b] \times [c,d])} + e^{-\alpha([b,c] \times [d,a])} = 1$$

for all  $a, b, c, d \in S^1$  given in counterclockwise order.

Since  $\mathcal{L}$  is a homeomorphism onto its image and  $\mathcal{L}(T(\mathbf{D}))$  is closed in  $H(\mathbf{D})$  [Šarić 2005], it follows that  $\mathcal{L}(T_{\text{restr}}(\mathbf{D}))$  equals to the closure in the Fréchet topology of  $\mathcal{L}(\bigcup_{[G:K] < \infty} \tilde{T}(\mathbf{D}/K))$ . By [Bonahon 1988] or [Šarić 2005], the asymptotic rays to  $\mathcal{L}(\bigcup_{[G:K] < \infty} \tilde{T}(\mathbf{D}/K))$  contain all the projective measured laminations that are invariant under all finite index subgroups  $K$  of  $G$ . The Liouville map  $\mathcal{L}$  composed with the projection  $pr : H(\mathcal{S}) \rightarrow S_\nu$  to the unit sphere  $S_\nu$  in  $H(\mathcal{S})$  for a fixed  $\nu$ -norm is a homeomorphism [Šarić 2005]. This implies that the points in the closure of  $pr \circ \mathcal{L}(T_{\text{restr}}(\mathbf{D}))$  which are not in  $pr \circ \mathcal{L}(T_{\text{restr}}(\mathbf{D}))$  are projectivized asymptotic rays to  $\mathcal{L}(T_{\text{restr}}(\mathbf{D}))$ . Any such point in the closure of  $pr \circ \mathcal{L}(T_{\text{restr}}(\mathbf{D}))$  is approximated by a projective measure in  $pr \circ \mathcal{L}(T_{\text{restr}}(\mathbf{D}))$  that is invariant under finite index subgroups of  $G$ . Thus the closure of all invariant, projective measured laminations under finite index subgroups of  $G$  contains all asymptotic rays to  $\mathcal{L}(T_{\text{restr}})$ . The measured laminations invariant under finite index subgroups of  $G$  lift to locally transversely constant measured laminations on  $\mathcal{S}$ . Thus the limits of locally transversely constant measured laminations on  $\mathcal{S}$  give all transversely continuous measured laminations on  $\mathcal{S}$ .  $\square$

**Remark 6.4.** The set of asymptotic rays to  $\mathcal{L}(T_{\text{restr}}(\mathbf{D}))$  in  $H(\mathbf{D})$  is equal to the restriction to the baseleaf of the set of asymptotic rays to  $\mathcal{L}(T(\mathcal{S}))$  in  $H(\mathcal{S})$ . This is a consequence of the proof of Theorem 6.3, since each asymptotic ray for  $\mathcal{L}(T_{\text{restr}}(\mathbf{D}))$  is the limit in the Fréchet topology of the asymptotic rays invariant under finite index subgroups of  $G$ .

## 7. The punctured solenoid

We sketch an extension of our results to the punctured solenoid  $\mathcal{S}_p$  defined in [Penner and Šarić 2005]. We first recall the definition of  $\mathcal{S}_p$ .

Let  $H$  be the subgroup of  $PSL_2(\mathbb{Z})$  such that  $\mathbf{D}/H$  is a once punctured torus. Denote by  $\hat{H}$  the profinite completion of  $H$ . We define the punctured solenoid by

$$\mathcal{S}_p := (\mathbf{D} \times \hat{H})/H,$$

where the action of  $H$  is given by  $A(z, t) := (A(z), tA^{-1})$ , for  $A \in H$ .

A leafwise measured lamination on  $\mathcal{S}_p$  is an assignment of a bounded measured lamination to each leaf of  $\mathcal{S}_p$  which varies continuously in the transverse direction. The support of a leafwise measured lamination on  $\mathcal{S}_p$  is a leafwise geodesic lamination which is not necessarily continuous for the transverse variations. We say that the support of a leafwise measured lamination on  $\mathcal{S}_p$  is compact if, when restricted to each leaf of  $\mathcal{S}_p$ , the support geodesic lamination is a precompact subset of  $\mathcal{S}_p$ . The earthquake theorem ([Theorem 5.1](#)) holds for  $\mathcal{S}_p$  when we use the space of measured laminations with compact support  $ML_0(\mathcal{S}_p)$ .

If a simple geodesic on a punctured surface enters a definite neighborhood of a puncture, then it has an endpoint at the puncture. We recall a standard proof of this fact in the upper halfplane model of the hyperbolic plane. Let  $A : z \mapsto z + 1$  be the parabolic element corresponding to the puncture. If a lift of the geodesic entering a neighborhood of a puncture on the surface is a Euclidean halfcircle with radius greater than  $1/2$ , then the translate of the lift of the geodesic by  $A$  intersects the lift of the geodesic. Thus the geodesic is not simple. Contradiction. Therefore, there exists a definite neighborhood of a puncture where no simple geodesic enters unless it ends at the puncture.

Assume that a measured lamination  $\mu$  on  $\mathcal{S}_p$  does not have a compact support. Then there exists a leaf  $l$  of  $\mathcal{S}_p$  such that the restriction  $\mu|_l$  of  $\mu$  contains a geodesic  $g \in l$  with an endpoint at the puncture. By the continuity for the transverse variations, the support of  $\mu|_l$  contains the translates of  $g$  in  $l$  by parabolic elements  $A^{kn} \in H$ , for a fixed  $k > 0$  and for all  $n \in \mathbb{N}$ , where  $A$  fixes the endpoint of  $g$ . Then  $g$  has to be isolated in  $l$  because otherwise the translates under  $A^{kn}$  of the geodesics in the support of  $\mu|_l$  converging to  $g$  would intersect the geodesics converging to  $g$  similar to the punctured surface case. Thus  $\mu|_l$  would not be a measured lamination. The remaining possibility is that  $g$  is isolated with atomic measure. Then the translates  $A^{kn}(g)$  also have atomic measure approximately equal to the atomic measure of  $g$  for  $n$  large and they share the same endpoint. Then the measured lamination  $\mu|_l$  is not bounded, which is a contradiction with the definition of a measured lamination on  $\mathcal{S}_p$ . Therefore, a measured lamination  $\mu$  on  $\mathcal{S}_p$  always has a compact support.

As in the proof of [Theorem 5.1](#), it follows that the extension to the boundary of each quasiconformal map  $f : \mathcal{S}_p \rightarrow X_p$  can be achieved by leafwise earthquakes whose measures vary continuously in the transverse direction for the Fréchet topology on earthquake measures. On the other hand, a measured lamination  $\mu$  on  $\mathcal{S}_p$



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