**LOCAL CUT POINTS AND METRIC MEASURE SPACES WITH
RICCI CURVATURE BOUNDED BELOW**

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A local cut point is a point that disconnects its sufficiently small neighborhood. We show that there exists an upper bound for the degree of a local cut point in a metric measure space satisfying the generalized Bishop–Gromov inequality. As a corollary, we obtain an upper bound for the number of ends of such a space. We also obtain some obstruction conditions for the existence of a local cut point in a metric measure space satisfying the Bishop–Gromov inequality or the Poincaré inequality. For example, the measured Gromov–Hausdorff limits of Riemannian manifolds with a lower Ricci curvature bound satisfy these two inequalities.

1. Introduction

A point x in a metric space is called a *local cut point* if $U \setminus \{x\}$ is disconnected for some connected neighborhood U of x. For $n \geq 2$, there exists no local cut point in any n-dimensional Alexandrov space with curvature bounded from below. Every Gromov–Hausdorff limit of a sequence of Riemannian manifolds with a uniform lower bound on sectional curvature is such an Alexandrov space. It is therefore natural to ask whether there exists a local cut point in the limit of manifolds with a uniform lower bound on Ricci curvature unless the limit is one-dimensional. We conjecture that the limit has no local cut point.

In this paper, we consider metric measure spaces with (suitably defined) Ricci curvature bounded below. Let $\deg(x)$ denote the *degree* of a point x or the supremum of the number of connected components of $U \setminus \{x\}$ for all connected neighborhoods U of x. We give an upper bound for the degree of a local cut point. As a consequence, we obtain an upper bound for the number of ends (see Section 4.2 for the definition of an end). We also obtain some obstruction conditions for the existence of a local cut point.

Cheeger and Colding [1996; 1997; 2000a; 2000b] and Menguy [2000a; 2000b; 2001] studied the limits of manifolds with Ricci curvature bounded from below. They constructed examples of limit spaces showing that the local structure is more complicated than that of Alexandrov spaces. Recently, Lott and Villani [≥ 2007; 2007], Sturm [2006a; 2006b], and Ohta [2007] independently generalized to metric measure spaces the notion of Ricci curvature bounded below, introducing the condition that a space has N-Ricci curvature $\geq K$, the curvature-dimension condition $CD(K, N)$, and the measure contraction property $MCP(K, N)$. For each N-dimensional Riemannian manifold M, each of these conditions is equivalent to the Ricci curvature of M being bounded below by K. Moreover, the conditions are preserved under measured Gromov–Hausdorff limits.

We note that the measured Gromov–Hausdorff limits of manifolds with Ricci curvature bounded from below satisfy the generalized Bishop–Gromov inequality and a Poincaré inequality of type $(1, 1)$, as explained below.

Let (X, d, μ) be a complete, locally compact length space equipped with a Borel measure. We denote by $B_r(x)$ the open ball of radius r and centered at $x \in X$. Let $k \in \mathbb{R}$, $n \in \mathbb{N}$, $C \geq 1$. Let $0 < r \leq R$ and $x \in X$. If $C = 1$, then (1-1) is the usual Bishop–Gromov inequality with respect to lower bound $(n-1)k$ of Ricci curvature and upper bound n of dimension:

$$(1\text{-}1) \qquad \frac{\mu(B_R(x))}{\mu(B_r(x))} \leq C \, \frac{V_{k,n}(R)}{V_{k,n}(r)},$$

where $V_{k,n}(r)$ is the volume of a ball of radius r in the n-dimensional, complete, simply connected space of constant curvature k.

We consider a stronger inequality, which is the directionally restricted version of (1-1). See [Gromov 1999, Section 5. I_+] and [Cheeger and Colding 1997, Appendix 2, (A.2.2.)]. We call the inequality the *generalized Bishop–Gromov inequality with constant C* or, for short, $BG(k, n)$ with C. This inequality naturally extends to the case $n \in \mathbb{R}$ with $n \geq 1$. See Definition 3.1 for the precise definition and [Kuwae and Shioya 2001] for slightly different definitions. This inequality is preserved under the measured Gromov–Hausdorff limits; see [Gromov 1999, Section 5. I_+] and [Cheeger and Colding 1997, Theorem 1.6, Theorem 1.10]. The measure contraction property $MCP((n-1)k, n)$ implies $BG(k, n)$ with $C = 1$. For example, these metric measure spaces satisfy $MCP((n-1)k, n)$:

- n-dimensional Riemannian manifolds whose Ricci curvature is bounded below by $(n-1)k$ and that are equipped with a Riemannian measure;

- n-dimensional Alexandrov spaces whose curvature is bounded below by k and that are equipped with the n-dimensional Hausdorff measure;

- "nonbranching" metric measure spaces satisfying the curvature-dimension condition CD$((n-1)k, n)$ (see Definition 3.3 for "nonbranching").

The non-Euclidean, finite-dimensional, normed linear spaces equipped with the Lebesgue measure satisfy MCP$(0, n)$. These spaces, however, cannot arise as a Gromov–Hausdorff limit of any Riemannian manifolds with a uniform lower bound on Ricci curvature (see Proposition 3.2).

Applying the method of the proof of Theorem 5.1 in [Cheeger and Colding 2000a], we have

Theorem 1.1. *Let (X, d, μ) be a metric measure space satisfying the generalized Bishop–Gromov inequality* BG(k, n) *with constant C for some $k \in \mathbb{R}$, $n \geq 1$, and $C \geq 1$. Assume that there exists a local cut point x in X. Then* $\deg(x) \leq C^2 + 1$.

The Cheeger–Gromoll splitting theorem [1971/72] states that if a Riemannian manifold M of nonnegative Ricci curvature contains a straight line, then M is isometric to $\mathbb{R} \times N$ for some manifold N. Cheeger and Colding [1996] extended this to limit spaces of nonnegative Ricci curvature in a generalized sense; see our Theorem 2.1. For the limits of manifolds with Ricci curvature bounded from below ($C = 1$), Theorem 1.1 ($\deg(x) = 2$) is also proved by using the Cheeger–Colding splitting theorem; see Proposition 4.3 here.

By the splitting theorem, a space of nonnegative Ricci curvature has at most two ends. The splitting theorem for metric measure spaces satisfying BG$(0, n)$ with C does not necessarily hold. As a corollary of Theorem 1.1, we have

Corollary 1.2. *Let (X, d, μ) be a metric measure space satisfying* BG$(0, n)$ *with C for some $n \geq 1$ and $C \geq 1$. Then the number of ends of X is at most $C^2 + 1$.*

We investigate the geometric structure of the neighborhood of a local cut point (Theorem 4.14). As a result, the "three-pronged" space (see Figure 4) does not satisfy BG(k, n) with C. We also study the structure of the accumulation of local cut points and find that the convergent sequence of certain local cut points "stands in a line" (Corollary 4.20).

We now assume that a measure μ satisfies $0 < \mu(B_r(x)) < +\infty$ for all $x \in X$ and all $0 < r < +\infty$. Let $1 \leq p < \infty$. We say that a metric measure space (X, d, μ) satisfies a *Poincaré inequality of type* $(1, p)$ if, for all $R > 0$, there exists a constant $C_P = C_P(p, R) > 0$ depending only on p and R such that

$$(1\text{-}2) \qquad \fint_{B_r(x)} |u - u_{B_r(x)}| \, d\mu \leq C_P \, r \left(\fint_{B_r(x)} g^p \, d\mu \right)^{1/p}$$

holds for all $x \in X$, the range $0 < r \leq R$, all measurable functions u, and all "upper gradients" g of u (see Section 4), where $u_B := \fint_B u \, d\mu := \mu(B)^{-1} \int_B u \, d\mu$.

It follows from Hölder's inequality that a metric measure space satisfying a Poincaré inequality of type $(1, p)$ also satisfies one of type $(1, q)$ for all $q \geq p$. For

instance, there are many examples of metric measure spaces satisfying a Poincaré inequality of type $(1, 1)$:

- n-dimensional Riemannian manifolds with Ricci curvature bounded below by $(n-1)k$ equipped with Riemannian measure [Buser 1982]. The constant C_P in (1-2) depends on k, n, and R. If $k = 0$, then C_P depends only on n. See [Saloff-Coste 2002, Theorem 5.6.5].

- The measured Gromov–Hausdorff limits of manifolds with Ricci curvature bounded from below [Cheeger and Colding 2000b, Theorem 2.15].

- n-dimensional Alexandrov spaces equipped with the n-dimensional Hausdorff measure. See [Kuwae et al. 2001].

- Nonbranching metric measure spaces satisfying $\mathrm{CD}(K, N)$. See [von Renesse 2007, Section 3.2], and combine with [Sturm 2006b, Lemma 4.1].

- Metric measure spaces with a doubling measure and the "democratic condition" DM. See [Lott and Villani 2007, Theorem 2.5].

The space $\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + x_2^2 + \cdots + x_{n-1}^2 \le x_n^2\}$ equipped with the Euclidean distance and the n-dimensional Lebesgue measure satisfies a Poincaré inequality of type $(1, p)$ for all $p > n$ [Hajłasz and Koskela 2000, Example 4.2].

We can obtain an obstruction condition for the existence of a local cut point:

Theorem 1.3. *Let (X, d, μ) be a metric measure space satisfying a Poincaré inequality of type $(1, p)$ for some $1 \le p < \infty$. Assume that*

$$(1\text{-}3) \qquad\qquad \liminf_{r \to 0} \frac{\mu(B_r(x))}{r^p} = 0$$

for a point $x \in X$. Then x is not a local cut point.

Although the measured Gromov–Hausdorff limits of manifolds with Ricci curvature bounded from below satisfy a Poincaré inequality of type $(1, 1)$, they do not necessarily satisfy the assumption (1-3) for $p = 1$. See [Cheeger and Colding 1997, Proposition 1.22]. Theorem 1.3 is one of the geometric consequences of the Poincaré inequality. See [Hajłasz and Koskela 2000, Proposition 4.5] and [Heinonen and Koskela 1998, Theorems 3.13 and 6.15] for related results.

This paper is organized as follows: In Section 2, we recall the definitions of the Hausdorff dimension, length spaces, and the (pointed, measured) Gromov–Hausdorff convergence. In Section 3, we precisely define $\mathrm{BG}(k, n)$ with C and give its basic properties. We prove Theorem 1.1 and Corollary 1.2 in Section 4. We also investigate the geometric structure of the neighborhood of local cut points and also study the structure of the accumulation of local cut points. We prove Theorem 1.3 in Section 5.

2. Preliminaries

We first recall the definition of the Hausdorff dimension. We then define length spaces and the (pointed, measured) Gromov–Hausdorff convergence. Let (X, d) be a metric space.

2.1. *Hausdorff dimension.* We refer to [Ambrosio and Tilli 2004, Chapter 2] for details. Let $A \subset X$. For $0 \le s < \infty$ and $0 < \delta \le \infty$, we define

$$\mathcal{H}^s_\delta(A) = \omega_s \, \inf\left\{\sum_i \left(\frac{\operatorname{diam}(U_i)}{2}\right)^s \,\bigg|\, A \subset \bigcup_i U_i, \, \operatorname{diam}(U_i) \le \delta\right\},$$

where $\omega_s := \pi^{s/2} / \Gamma(s/2+1)$ and Γ is the gamma function $\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx$. If s is a positive integer, then ω_s equals the volume of a unit ball in $\mathbb{R}^s$. Note that $\mathcal{H}^s_\delta(A)$ is nonincreasing in δ. We define the *s-dimensional Hausdorff measure* of A by

$$\mathcal{H}^s(A) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(A) = \sup_{\delta > 0} \mathcal{H}^s_\delta(A).$$

Then $\mathcal{H}^s$ is a Borel regular measure on X. The *Hausdorff dimension* of A is

$$\dim_{\mathrm{H}}(A) = \inf\{0 \le s < \infty \mid \mathcal{H}^s(A) = 0\} = \sup\{0 \le s < \infty \mid \mathcal{H}^s(A) = +\infty\}.$$

2.2. *Length spaces.* We define the length of a continuous path $\gamma : [0, l] \to X$ by

$$L(\gamma) = \sup_{0=t_0 < t_1 < \cdots < t_k = l} \sum_{i=1}^k d(\gamma(t_{i-1}), \gamma(t_i)),$$

where the supremum is taken over all partitions of $[0, l]$. By the triangle inequality, we have $L(\gamma) \ge d(\gamma(0), \gamma(l))$. We say that a path $\gamma : [0, l] \to X$ is a *geodesic* if it is locally minimizing and is proportional to arclength, that is, for each $s \in [0, l]$ there exists $\epsilon = \epsilon(s) > 0$ such that

$$(2\text{-}4) \qquad\qquad d(\gamma(t), \gamma(t')) = |t - t'|$$

holds for all $t, t' \in (s - \epsilon, s + \epsilon) \cap [0, l]$. Furthermore, we say that a path $\gamma : (-\infty, \infty) \to X$ is a *straight line* if (2-4) holds for all $t, t' \in (-\infty, \infty)$. In this paper, we assume that every path is proportional to arclength.

We say that X is a *length space* if $d(x, y) = \inf_\gamma L(\gamma)$ for all $x, y \in X$, where the infimum is taken over all paths joining x and y. A metric space X is a length space if and only if, for all $x, y \in X$ and all $\epsilon > 0$, there exists a point $z \in X$ such that $\max\{d(x, z), d(z, y)\} \le d(x, y)/2 + \epsilon$. If X is a complete, locally compact length space, then all two points in X are joined by a minimal geodesic.

For more, see [Burago et al. 2001; Gromov 1999] and their references.

2.3. *Gromov–Hausdorff convergence.* Let us recall that the *Hausdorff distance* between two closed bounded subsets A and B in a metric space X is defined by

$$d_{\mathrm{H}}(A, B) = \inf\{\epsilon > 0 \mid A \subset U_\epsilon(B), \, B \subset U_\epsilon(A)\},$$

where $U_\epsilon(A)$ is the ϵ-neighborhood of A. Let $\mathscr{C}$ denote the set of isometry classes of all compact metric spaces. For $X, Y \in \mathscr{C}$, the *Gromov–Hausdorff distance* $d_{\mathrm{GH}}(X, Y)$ between X and Y is defined as the infimum of $d_{\mathrm{H}}(f(X), g(Y))$ over all metric spaces Z and all isometric embeddings $f : X \to Z$ and $g : Y \to Z$. Then d_{GH} defines a metric on $\mathscr{C}$. Let (X, d_X) and (Y, d_Y) be metric spaces. For $\epsilon > 0$, we say that a map $\varphi : X \to Y$ is an ϵ-*approximation* if

(i) $|d_X(x, y) - d_Y(\varphi(x), \varphi(y))| < \epsilon$ holds for all $x, y \in X$, and

(ii) the ϵ-neighborhood of $\varphi(X)$ coincides with Y.

Let $X_i, X \in \mathscr{C}$ for $i = 1, 2, \ldots$. The sequence $\{X_i\}$ Gromov–Hausdorff converges to X, that is, $d_{\mathrm{GH}}(X_i, X) \to 0$ as $i \to \infty$, if and only if there exist ϵ_i-approximations from X_i to X (or from X to X_i) for some $\epsilon_i \to 0$.

Let (X_i, x_i) for $i = 1, 2, \ldots$ and (X, x) be pointed metric spaces. We say that $\{(X_i, x_i)\}$ *pointed Gromov–Hausdorff converges* to (X, x) if, for each $R > 0$, there exist $R_i \searrow R$, $\epsilon_i \searrow 0$, and ϵ_i-approximations $\varphi_i : B_{R_i}(x_i) \to B_R(x)$ with $\varphi_i(x_i) = x$. A pointed Gromov–Hausdorff limit of $\{(X, r_i^{-1}d, x)\}$ as $r_i \to 0$ is called a *tangent cone* at x. The Gromov–Hausdorff limit of a sequence of length spaces is also a length space. See [Burago et al. 2001; Gromov 1999] for details.

Let (M_i, p_i) for $i = 1, 2, \ldots$ be n-dimensional, complete pointed Riemannian manifolds with Ricci curvature $\mathrm{Ric}_{M_i} \geq -(n-1)$. Gromov's compactness theorem states that $\{(M_i, p_i)\}$ pointed Gromov–Hausdorff subconverges to some pointed metric space (X, x). We then have $\dim_{\mathrm{H}}(X) \leq n$ (see Proposition 3.8). The sequence $\{M_i\}$ is said to *collapse* to X if $\dim_{\mathrm{H}}(X) < n$. In fact, if $\{M_i\}$ collapses, then $\dim_{\mathrm{H}}(X) \leq n - 1$ [Cheeger and Colding 1997, Theorem 3.1]. Fukaya conjectured [1987, Conjecture 3.13] that the Hausdorff dimension of the limit X is an integer. By Gromov's compactness theorem, there exists a tangent cone at every x in the limit X. A tangent cone at each point is not necessarily unique even in the noncollapsed case [Cheeger and Colding 1997, Example 8.41]. It does not always have a metric cone structure in the collapsed case [Cheeger and Colding 1997, Example 8.95]. The splitting theorem for the limit holds, even in the collapsed case:

Theorem 2.1 [Cheeger and Colding 1996]. *Let (X, x) be a pointed Gromov–Hausdorff limit of complete Riemannian manifolds M_i with $\mathrm{Ric}_{M_i} \geq -\delta_i$, where $\delta_i \to 0$. If X contains a straight line, then X is isometric to $\mathbb{R} \times Y$ for some length space Y.*

2.4. *Measured Gromov–Hausdorff convergence.* Fukaya [1987] introduced the concept of the measured Gromov–Hausdorff convergence. Let $\mathscr{CM}$ denote the set of all compact metric spaces X equipped with a Borel measure μ such that $\mu(X) \le 1$. A directed system $\{(X_\alpha, \mu_\alpha)\}_{\alpha \in \mathscr{A}} \subset \mathscr{CM}$ is said to *measured Gromov–Hausdorff converge* to $(X, \mu) \in \mathscr{CM}$ if, for each $\alpha \in \mathscr{A}$, there exist $\epsilon_\alpha > 0$ and a Borel measurable ϵ_α-approximation $\varphi_\alpha : X_\alpha \to X$ such that

(i) $\lim_{\alpha \in \mathscr{A}} \epsilon_\alpha = 0$;

(ii) a directed system of push-forward measures $\{(\varphi_\alpha)_* \mu_\alpha\}_\alpha$ converges vaguely to μ, that is,

$$(2\text{-}5) \qquad \lim_{\alpha \in \mathscr{A}} \int_{X_\alpha} f \circ \varphi_\alpha \, d\mu_\alpha = \int_X f \, d\mu$$

holds for all continuous functions $f : X \to \mathbb{R}$.

The measured Gromov–Hausdorff convergence induces a topology on $\mathscr{CM}$. Fukaya [1987] shows it is Hausdorff [Proposition 2.7] and the projection from $\mathscr{CM}$ to $\mathscr{C}$ is proper [Proposition 2.10]. Gromov [1999] gives other topologies on $\mathscr{CM}$.

A directed system $\{(X_\alpha, \mu_\alpha, x_\alpha)\}_{\alpha \in \mathscr{A}}$ of (possibly noncompact) pointed metric measure spaces is said to *pointed measured Gromov–Hausdorff converge* to a pointed metric measure space (X, μ, x) if, for any $R > 0$, there exist $R_\alpha \searrow R$, $\epsilon_\alpha \searrow 0$, and Borel measurable ϵ_α-approximations $\varphi_\alpha : B_{R_\alpha}(x_\alpha) \to B_R(x)$ with $\varphi_\alpha(x_\alpha) = x$ such that the directed system $\{(\varphi_\alpha)_* \mu_\alpha\}_\alpha$ converges to μ vaguely; that is, (2-5) holds for all continuous functions $f : X \to \mathbb{R}$ with compact support.

Let (M_i, p_i) for $i = 1, 2, \ldots$ be n-dimensional, complete, pointed Riemannian manifolds with $\mathrm{Ric}_{M_i} \ge -(n-1)$ and $v \le \mathrm{vol}_{M_i}(B_1(p_i)) \le V$ for some $0 < v \le V$. Assume that $\{(M_i, p_i)\}$ pointed Gromov–Hausdorff converges to a pointed metric space (X, x). Then there exists a measure μ on X such that $\{(M_i, \mathrm{vol}_{M_i}, p_i)\}$ pointed measured Gromov–Hausdorff subconverges to (X, μ, x). In the noncollapsed case, μ is unique and it coincides with the n-dimensional Hausdorff measure, up to constant multiple [Cheeger and Colding 1997, Theorem 5.9]; they show in the collapsed case that μ is not necessarily unique [Example 1.24, Section 8].

3. The generalized Bishop–Gromov inequality

In this section, we define the generalized Bishop–Gromov inequality $\mathrm{BG}(k, n)$ with constant C and give basic properties of a metric measure space satisfying $\mathrm{BG}(k, n)$ with C.

We denote by $c(c_1, \ldots, c_l)$ a positive constant depending only on $c_1, \ldots, c_l$. Let (X, d, μ) be a complete, locally compact length space equipped with a Borel measure.

3.1. *Definition of the generalized Bishop–Gromov inequality.* Let $k, n \in \mathbb{R}$ with $n \geq 1$. For $0 \leq r_1 < r_2$, we define

$$V_{k,n}(r_1, r_2) = \alpha_{n-1} \int_{r_1}^{r_2} \mathsf{s}_k(t)^{n-1} \, dt,$$

where $\alpha_{n-1} := 2\pi^{n/2}/\Gamma(n/2) = n\,\omega_n$ and $\mathsf{s}_k(t)$ for $t \geq 0$ is defined by

$$\mathsf{s}_k(t) = \begin{cases} \frac{1}{\sqrt{k}} \sin(\sqrt{k}\,t) & \text{if } k > 0, \\ t & \text{if } k = 0, \\ \frac{1}{\sqrt{-k}} \sinh(\sqrt{-k}\,t) & \text{if } k < 0. \end{cases}$$

If n is a positive integer, then $V_{k,n}(r_1, r_2)$ is equal to the volume of an annulus of radius between r_1 and r_2 in the n-dimensional, complete, simply connected space of constant curvature k. We denote by $B_r(x)$ and $\bar{B}_r(x)$ the balls of radius r centered at x that are open and closed, respectively. For a point $x \in X$, let U be a set in $A_{r_1, r_2}(x)$, where $A_{r_1, r_2}(x) = B_{r_2}(x) \setminus \bar{B}_{r_1}(x)$ and $A_{0, r}(x) = B_r(x)$. For $0 \leq s_1 < s_2$ with $s_1 \leq r_1$ and $s_2 \leq r_2$, set

$$S_{s_1, s_2}(x, U) = \left\{ y \in A_{s_1, s_2}(x) \mid d(x, y) + d(y, z) = d(x, z) \text{ for some } z \in U \right\},$$

that is, $S_{s_1, s_2}(x, U)$ is the intersection of $A_{s_1, s_2}(x)$ and all minimal geodesics between x and each point in U.

Definition 3.1. Let $C \geq 1$. We say that a metric measure space (X, d, μ) satisfies the *generalized Bishop–Gromov inequality with constant C* or, for short, BG(k, n) with C, if for all points $x \in X$ and all $0 \leq r_1 < r_2$ and $0 \leq s_1 < s_2$ with $s_1 \leq r_1$ and $s_2 \leq r_2$, the inequality

$$(3\text{-}6) \qquad \frac{\mu(U)}{\mu(S_{s_1, s_2}(x, U))} \leq C \frac{V_{k,n}(r_1, r_2)}{V_{k,n}(s_1, s_2)}$$

holds for all Borel sets U in $A_{r_1, r_2}(x)$.

BG(k, n) with $C = 1$ induces the usual Bishop–Gromov inequality. Indeed, if we choose $r_1 = s_1 = 0$, $s_2 = r \leq R = r_2$, and $U = B_R(x)$, then (3-6) implies

$$\frac{\mu(B_R(x))}{\mu(B_r(x))} \leq \frac{V_{k,n}(0, R)}{V_{k,n}(0, r)}.$$

3.2. *Examples.* We recall metric measure spaces satisfying BG(k, n) with $C = 1$, as shown in the introduction. Every n-dimensional, normed linear space equipped with the n-dimensional Lebesgue measure satisfies BG$(0, n)$ with $C = 1$.

Proposition 3.2. *Let X be an n-dimensional, normed linear space. Assume that X is a Gromov–Hausdorff limit of Riemannian manifolds M_i with Ricci curvature bounded from below. Then X is isometric to $\mathbb{R}^n$.*

Proof. By expanding the metrics of converging manifolds, we may assume that $\mathrm{Ric}_{M_i} \geq -\delta_i$, where $\delta_i \to 0$. Since X contains n orthogonal lines, the splitting theorem (Theorem 2.1) completes the proof. $\square$

Definition 3.3 [Sturm 2006a, Definition 2.8]. A metric space (X, d) is said to be *nonbranching* if for all four points $y, x_0, x_1, x_2 \in X$ such that y is a midpoint between x_0 and x_1 and, between x_0 and x_2, we have $x_1 = x_2$. A point y is called a *midpoint* between x_0 and x_1 if $2^{-1}d(x_0, x_1) = d(x_0, y)$.

A metric space X is nonbranching if and only if for all two minimal geodesics $\gamma, \gamma' : [0, l] \to X$ with $\gamma(0) = \gamma'(0)$, we have $\inf\{t > 0 \mid \gamma(t) \neq \gamma'(t)\} = 0$ or ∞, where $\inf \varnothing := \infty$. Each Alexandrov space with curvature bounded from below is nonbranching.

Remark 3.4. For nonbranching metric measure spaces, the curvature-dimension condition $CD(K, N)$ implies the measure contraction property $MCP(K, N)$; see [Sturm 2006b, Lemma 4.1, Theorem 5.4]. Moreover, $CD(K, N)$ is a weaker condition than having N-Ricci curvature $\geq K$.

All n-dimensional Alexandrov spaces with curvature bounded below by k that are given the n-dimensional Hausdorff measure $\mathcal{H}^n$ satisfy $MCP((n-1)k, n)$; see [Ohta 2007, Proposition 2.8; Sturm 2006b, Theorem 5.7; Kuwae and Shioya 2001, Lemma 6.1]. For Riemannian manifolds, a lower bound on the sectional curvature implies the Ricci curvature is also bounded from below. However, it is an open problem whether those Alexandrov spaces satisfy $CD((n-1)k, n)$ or have n-Ricci curvature $\geq (n-1)k$. In particular, we do not know whether all Alexandrov spaces can arise as Gromov–Hausdorff limits of Riemannian manifolds with Ricci curvature bounded from below.

Remark 3.5. Let (X, d, μ) be a measured Gromov–Hausdorff limit of those n-dimensional compact Riemannian manifolds M_i with $\mathrm{Ric}_{M_i} \geq (n-1)k$ and equipped with the normalized Riemannian measure. Set $m = \dim_H(X)$, which is not necessarily an integer. Cheeger and Colding [1997, Conjecture 1.34] conjectured there exists a number l with $m \leq l \leq n$ such that (3-6) holds for $\mu = \mathcal{H}^m$, $n = l$, and $C = 1$.

3.3. *Basic properties.* We begin by recalling the definition of a doubling measure. We say that a measure μ is *doubling* if for all $R > 0$ there exists a constant $C_D = C_D(R) \geq 1$ such that $\mu(B_{2r}(x)) \leq C_D \, \mu(B_r(x))$ holds for all $x \in X$ and all $0 < r \leq R$. If μ is doubling, then X is proper (that is, all closed bounded subsets are compact). We note that if (X, d, μ) satisfies $BG(k, n)$ with C, then μ is doubling.

Although the following lemmas and proposition are somewhat standard, we prove them for the completeness of this paper.

Lemma 3.6. *Let (X, d, μ) satisfy* BG(k, n) *with $C = 1$ for some $k, n \in \mathbb{R}$ with $n \geq 1$. Then we have $\mu(S_r(x)) = 0$ for all $x \in X$ and all $r > 0$, where $S_r(x)$ is the sphere of radius r and centered at x. In particular, if X does not consist of a single point, then $\mu(\{x\}) = 0$ for all $x \in X$.*

Proof. For a sufficiently small $\epsilon > 0$, we have

$$\mu(S_r(x)) \leq \mu(B_{r+\epsilon}(x)) - \mu(B_{r-\epsilon}(x)) = \mu(B_{r-\epsilon}(x))\left(\frac{\mu(B_{r+\epsilon}(x))}{\mu(B_{r-\epsilon}(x))} - 1\right)$$

$$\leq \mu(B_{r-\epsilon}(x))\left(\frac{V_{k,n}(0, r+\epsilon)}{V_{k,n}(0, r-\epsilon)} - 1\right),$$

where we have used (3-6) for the last step. Letting $\epsilon \to 0$ completes the proof. $\square$

Let $A \subset X$. We say that A is *convex* if, for all pairs $x, y \in A$, all minimal geodesics between x and y are contained in A.

Lemma 3.7. *Let (X, d, μ) satisfy* BG(k, n) *with C for some $k \in \mathbb{R}$, $n \geq 1$, and $C \geq 1$. Then:*

(1) *If $k' \leq k$ and $n' \geq n$, then (X, d, μ) satisfies* BG(k', n') *with C.*

(2) *If $a, b > 0$, then $(X, ad, b\mu)$ satisfies* BG$(k/a^2, n)$ *with C.*

(3) *If A is a convex subset in X, then (A, d, μ) also satisfies* BG(k, n) *with C.*

Proof. Fix $0 \leq r_1 < r_2$ and $0 \leq s_1 < s_2$ with $s_1 \leq r_1$ and $s_2 \leq r_2$. (1) follows because $V_{k,n}(r_1, r_2)/V_{k,n}(s_1, s_2)$ is monotone nonincreasing in k and monotone nondecreasing in n. We respectively denote by $A_{r_1, r_2}^{(d)}(x)$ and $S_{s_1, s_2}^{(d)}(x, U)$ the sets $A_{r_1, r_2}(x)$ and $S_{s_1, s_2}(x, U)$ with respect to a metric d. For all sets U in $A_{r_1, r_2}^{(ad)}(x)$, we have

$$\frac{(b\mu)(U)}{(b\mu)\left(S_{s_1, s_2}^{(ad)}(x, U)\right)} = \frac{\mu(U)}{\mu\left(S_{s_1/a, s_2/a}^{(d)}(x, U)\right)}$$

$$\leq C \frac{V_{k,n}(r_1/a, r_2/a)}{V_{k,n}(s_1/a, s_2/a)} = C \frac{V_{k/a^2, n}(r_1, r_2)}{V_{k/a^2, n}(s_1, s_2)},$$

which implies (2). Let $x \in A$ and $U \subset A_{r_1, r_2}(x)$. Since A is convex, $S_{s_1, s_2}(x, U)$ is contained in A. We hence have (3). $\square$

Let $\epsilon > 0$. A set $S \subset X$ is called an *ϵ-net* if we have $d(x, S) \leq \epsilon$ for all $x \in X$. We say S is *ϵ-separated* if $d(x, y) \geq \epsilon$ for all pairs of distinct $x, y \in S$. Clearly, each maximal ϵ-separated set is an ϵ-net.

Proposition 3.8. *Let (X, d, μ) be as in Lemma 3.7. Then we have $\dim_{\mathrm{H}}(X) \leq n$.*

Proof. We will show that $\mathcal{H}^{n'}(X) = 0$ holds for all $n' > n$. We may assume that $\mathrm{diam}(X) < \infty$. Let $\{x_i\}_{i=1}^N$ be a maximal δ-separated set in X for $\delta > 0$. Then $\{x_i\}_{i=1}^N$ is a δ-net. Note that there exists a constant $c(k, n) > 0$ depending only on

k and n such that $V_{k,n}(0, \delta) \geq c(k, n)\delta^n$ holds for all (sufficiently small if $k > 0$) $\delta > 0$. Let x_{i_0} be a point at which $\mu(B_{\delta/2}(x_{i_0}))$ attains $\min_{1 \leq i \leq N} \mu(B_{\delta/2}(x_i))$. Note that $\{B_{\delta/2}(x_i)\}_{i=1}^N$ is disjoint. Since (X, d, μ) satisfies BG(k, n) with C, we have

$$\mu(X) \geq \sum_{i=1}^N \mu(B_{\delta/2}(x_i))$$

$$\geq N\mu(B_{\delta/2}(x_{i_0})) \geq NC^{-1} \frac{V_{k,n}(0, \delta/2)}{V_{k,n}(0, \operatorname{diam}(X))} \mu\big(B_{\operatorname{diam}(X)}(x_{i_0})\big).$$

Hence $N \leq c(k, n)\delta^{-n}$, which implies

$$\min\Big\{N \in \mathbb{N} \;\Big|\; X = \bigcup_{i=1}^N B_\delta(x_i)\Big\}$$

$$\leq \max\big\{N \in \mathbb{N} \;\big|\; \{x_i\}_{i=1}^N \text{ is a maximal } \delta\text{-separated set}\big\}$$

$$\leq c(k, n)\delta^{-n}.$$

Therefore

$$\mathcal{H}_\delta^{n'}(X) = \omega_{n'} \inf\Big\{\sum_{i=1}^\infty \Big(\frac{\operatorname{diam}(U_i)}{2}\Big)^{n'} \;\Big|\; X = \bigcup_{i=1}^\infty U_i, \ \operatorname{diam}(U_i) \leq \delta\Big\}$$

$$\leq \omega_{n'} \inf\Big\{\sum_{i=1}^\infty \Big(\frac{\operatorname{diam}(B_{\delta/2}(x_i))}{2}\Big)^{n'} \;\Big|\; X = \bigcup_{i=1}^\infty B_{\delta/2}(x_i)\Big\}$$

$$\leq \omega_{n'} \Big(\frac{\delta}{2}\Big)^{n'} \min\Big\{N \in \mathbb{N} \;\Big|\; X = \bigcup_{i=1}^N B_{\delta/2}(x_i)\Big\}$$

$$\leq c(k, n, n') \,\delta^{n'-n}.$$

Letting $\delta \to 0$ completes the proof. $\qquad\square$

4. Local cut points

In this section, we first prove Theorem 1.1. In Section 4.2, we recall the definition of an end and then prove Corollary 1.2. In addition, we investigate the geometric structure of the neighborhood of a local cut point and also study the structure of the accumulation of local cut points by using the idea of the proof of Theorem 1.1. We now assume that (X, d) is a complete, locally compact length space.

4.1. *Local cut points.*

Definition 4.1 (local cut point [Gromov 1999, 3.32], r-cut point). We say that a point $x \in X$ is a *local cut point* if $U \setminus \{x\}$ is disconnected for some connected neighborhood U of x. The *degree* of x, denoted by $\deg(x)$, is defined as the

supremum of the number of connected components of $U \setminus \{x\}$ for all connected neighborhoods U of x. Let $r > 0$. We say that a point $x \in X$ is an *r-cut point* if these three conditions hold:

(i) $\overline{B}_r(x) \setminus \{x\}$ is disconnected;

(ii) the number of connected components of $\overline{B}_r(x) \setminus \{x\}$ is equal to $\deg(x)$;

(iii) $O \cap S_r(x)$ is nonempty for all connected components O of $\overline{B}_r(x) \setminus \{x\}$.

If x is a local cut point, then $U \setminus \{x\}$ is disconnected for *every* sufficiently small neighborhood U of x. The end points in a graph (one-dimensional space) are *not* local cut points. An interior point in a graph is not always a local cut point; see Example 4.2 (3). We have $\deg(x) \geq 2$ for each local cut point x. Every local cut point with finite degree is an r-cut point for some $r > 0$. Let $0 < r_1 \leq r_2$. If a point x is an r_2-cut point, then x is an r_1-cut point.

Example 4.2. (1) Consider the set $\bigcup_{i=0}^{\infty}\{r \exp(2^{-i}\pi\sqrt{-1}) \in \mathbb{C} \mid 0 \leq r \leq 2^{-i}\}$ with the induced distance [Shioya 1999, Example 9.1]. The origin is a local cut point and its degree is infinite.

(2) Consider the set $\bigcup_{i=1}^{\infty}\{(x, y) \in \mathbb{R}^2 \mid x^2 + (y - 1/i)^2 = 1/i^2\}$ with the induced distance. The origin is a local cut point and its degree is infinite.

(3) Consider the set $\{(x, 0) \mid 0 \leq x \leq 1\} \cup \{(0, y) \mid 0 \leq y \leq 1\} \cup \left(\bigcup_{i=0}^{\infty}\{(x, -x+2^{-i}) \mid 0 \leq x \leq 2^{-i})\}\right) \subset \mathbb{R}^2$ with the induced distance. The origin is *not* a local cut point.

(4) Consider the interval $[0, 1] \subset \mathbb{R}$ with the Euclidean distance. The point $1/3 \in [0, 1]$ is *not* an r-cut point for any $r > 1/3$ because (iii) is not satisfied.

(5) Consider the set $\{(x, y) \mid x^2 + y^2 = 1\} \cup \{(x, 0) \mid 1 \leq x\} \subset \mathbb{R}^2$ with the induced distance. The point $(-1, 0)$ is *not* an r-cut point for any $r \geq \pi$ because (ii) is not satisfied.

Proof of Theorem 1.1. For any positive integer d with $d \leq \deg(x)$, take a sufficiently small $r > 0$. Choose any d connected components $O_1, O_2, \ldots, O_d$ of $\overline{B}_r(x) \setminus \{x\}$ such that the degree of x in $\overline{B}_r(x) \cap (O_1 \cup \cdots \cup O_d)$ is equal to d and $O_i \cap S_r(x)$ is nonempty for all $1 \leq i \leq d$. Let l be a positive number with $0 < l \leq r/2$. For each $1 \leq i \leq d$, we choose a point $x_i \in O_i$ such that $d(x, x_i) = l$. See Figure 1.

For a sufficiently small $0 < \epsilon \ll l$, set $U = B_\epsilon(x) \cap O_1$. We claim that every minimal geodesic between any point in U and x_i ($2 \leq i \leq d$) passes through the local cut point x. Suppose that there exists a minimal geodesic $\gamma : [0, l] \to X$ from some point y in U to x_i such that γ does not pass through x. By the choice of r and l, the point $\gamma(t)$ is not contained in $B_r(x)$ for some $t \in [0, l]$. Therefore

$$d(y, x_i) = d(\gamma(0), \gamma(l)) = l \geq t = d(y, \gamma(t)) \geq d(x, \gamma(t)) - d(x, y) > r - \epsilon.$$
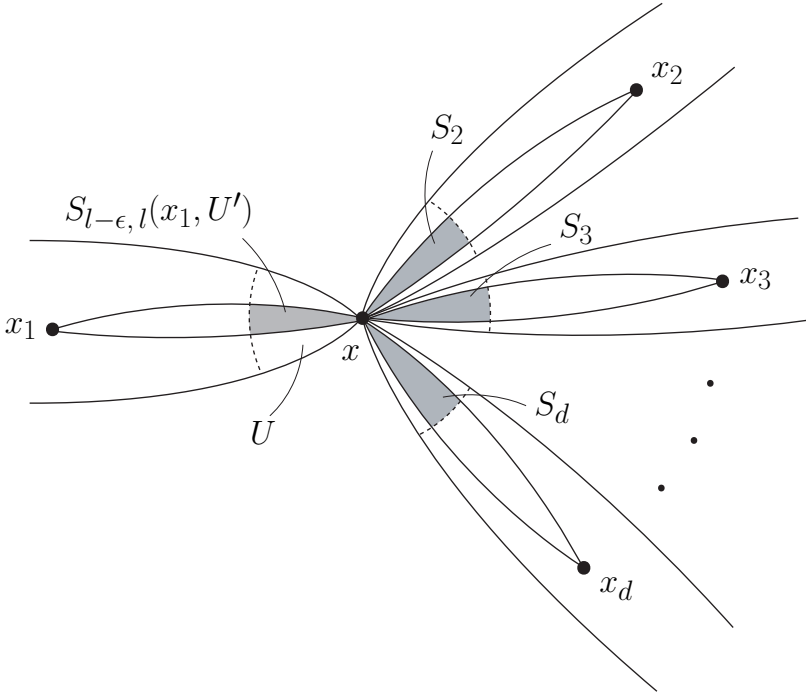
Figure 1. Proof of Theorem 1.1.

On the other hand, we have a contradiction from

$$d(y, x_i) \leq d(y, x) + d(x, x_i) < \epsilon + l \leq \epsilon + r/2.$$

We first use the generalized Bishop–Gromov inequality (3-6) with the base point x_i ($2 \leq i \leq d$). We see that $A_{l, l+\epsilon}(x_i)$ contains U for $2 \leq i \leq d$. Set $S_i = S_{l-\epsilon, l}(x_i, U)$ for $2 \leq i \leq d$. We recall that S_i is the intersection of $A_{l-\epsilon, l}(x_i)$ and all minimal geodesics from x_i to each point in U, hence to x. Then $S_i \cap S_j$ is empty for all $i \neq j$. By applying (3-6) to $s_1 = l - \epsilon$, $s_2 = r_1 = l$, $r_2 = l + \epsilon$, the point x_i, and the set U, it follows for $2 \leq i \leq d$ that

$$(4\text{-}7) \qquad C^{-1} \frac{V_{k,n}(l - \epsilon, l)}{V_{k,n}(l, l + \epsilon)} \leq \frac{\mu(S_{l-\epsilon, l}(x_i, U))}{\mu(U)} = \frac{\mu(S_i)}{\mu(U)}.$$

We now denote $U' = \bigcup_{i=2}^{d} S_i$. Since $\bigcap_{i=2}^{d} S_i = \varnothing$, we have $\sum_{i=2}^{d} \mu(S_i) = \mu(U')$. Summing up (4-7) for all $2 \leq i \leq d$, we obtain

$$(4\text{-}8) \qquad (d - 1) C^{-1} \frac{V_{k,n}(l - \epsilon, l)}{V_{k,n}(l, l + \epsilon)} \leq \frac{\sum_{i=2}^{d} \mu(S_i)}{\mu(U)} = \frac{\mu(U')}{\mu(U)}.$$

Next, we use (3-6) with the base point x_1. We see that $A_{l,l+\epsilon}(x_1)$ contains U'. Applying (3-6) to $s_1 = l - \epsilon$, $s_2 = r_1 = l$, $r_2 = l + \epsilon$, the point x_1, and the set U', we obtain

$$(4\text{-}9) \qquad \frac{\mu(U')}{\mu\big(S_{l-\epsilon,l}(x_1, U')\big)} \leq C\, \frac{V_{k,n}(l, l+\epsilon)}{V_{k,n}(l-\epsilon, l)}.$$

Note that, much as before, every minimal geodesic between x_0 and any point in U' passes through x; hence U contains $S_{l-\epsilon,l}(x_1, U')$. Combining (4-8) and (4-9), we have

$$d \leq C^2 \left(\frac{V_{k,n}(l, l+\epsilon)}{V_{k,n}(l-\epsilon, l)} \right)^2 + 1.$$

Letting $\epsilon \to 0$, we have $d \leq C^2 + 1$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

For the limits of Riemannian manifolds with Ricci curvature bounded from below, the Cheeger–Colding splitting Theorem (Theorem 2.1) implies that $\deg(x) = 2$ for any local cut point x; see Proposition 4.3 below. Since the splitting theorem for metric measure spaces satisfying $\mathrm{BG}(0, n)$ with C do not hold in general (see Proposition 3.2), the proof of Theorem 1.1 gives another proof that $\deg(x) = 2$.

Proposition 4.3. *Let (X, d) be a Gromov–Hausdorff limit of n-dimensional, complete Riemannian manifolds (M_i, d_i) with $\mathrm{Ric}_{M_i} \geq -(n-1)$, where d_i is the Riemannian distance on M_i. If there exists a local cut point x in X, then every tangent cone at x is isometric to $\mathbb{R}$; in particular, $\deg(x) = 2$.*

Proof. Take points p_i in M_i such that $\{(M_i, d_i, p_i)\}$ pointed Gromov–Hausdorff converges to (X, d, x). Let us recall that a tangent cone at x is the pointed limit space of $(X, r_i^{-1}d, x)$ as $r_i \to 0$. By passing to a subsequence, the tangent cone at x is itself the pointed limit of rescaled manifolds $(M_j, r_j^{-1}d_j, p_j)$. Then the Ricci curvature of $(M_j, r_j^{-1}d_j)$ is bounded below by $-(n-1)r_j^2$.

Because every local cut point is an interior point of some geodesic, the tangent cone at x contains a straight line (expanded from the geodesic). Applying the splitting theorem (Theorem 2.1), we see that the tangent cone at x is isometric to $\mathbb{R} \times Y$ for some length space Y. Since x is a local cut point, it follows that Y consists of a single point. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 4.4. We recall that if (X, d, μ) satisfies $\mathrm{BG}(k, n)$ with C, then μ is doubling. Although the length space in Example 4.2 (1) has the natural doubling measure, the degree of the origin is infinite.

In the case of graphs, we obtain a better estimate of the degree.

Proposition 4.5. *Let $(X, d, \mathcal{H}^1)$ be a graph, where d is the usual distance and $\mathcal{H}^1$ is the one-dimensional Hausdorff measure. Assume that $(X, d, \mathcal{H}^1)$ satisfies*

$BG(k, n)$ *with C for some $k \in \mathbb{R}$, $n \geq 1$, and $C \geq 1$. Then we have $\deg(x) \leq C + 1$ for all $x \in X$.*

Proof. Let x be a local cut point (an interior point) in X. Take a point $x_1 \in X$ with $d(x, x_1) \ll 1$, and let $l = d(x, x_1)$. Let $\gamma : [0, l] \to X$ be the minimal geodesic from x_1 to x. Set $U = B_\epsilon(x) \setminus \gamma([0, l])$ for $0 < \epsilon \ll 1$. We then have $S_{l-\epsilon, l}(x_1, U) = \gamma((l-\epsilon, l))$ because $l = d(x, x_1)$ is sufficiently small. By using (3-6) with $s_1 = l - \epsilon$, $s_2 = r_1 = l$, $r_2 = l + \epsilon$, the point x_1, and the set U, it follows that

$$(4\text{-}10) \qquad \frac{(\deg(x) - 1)\epsilon}{\epsilon} = \frac{\mathcal{H}^1(U)}{\mathcal{H}^1(S_{l-\epsilon, l}(x_1, U))} \leq C \frac{V_{k, n}(l, l + \epsilon)}{V_{k, n}(l - \epsilon, l)}.$$

Taking $\epsilon \to 0$, we obtain $\deg(x) \leq C + 1$. $\qquad \square$

Remark 4.6. In general, the converse of Proposition 4.5 does not necessarily hold.

4.2. *The number of ends.* We recall the definition of an end.

Definition 4.7. A path $\gamma : [0, \infty) \to X$ is called a *ray* if (2-4) holds for all $t, t' \in [0, \infty)$. Let $\gamma_1, \gamma_2 : [0, \infty) \to X$ be rays from the base point x. Two rays γ_1 and γ_2 are said to be *cofinal* if $\gamma_1(t)$ and $\gamma_2(t)$ lie in the same connected component of $X \setminus B_r(x)$ for all $t, r > 0$ with $t \geq r$. An equivalence class of cofinal rays is called an *end* of X.

Proof of Corollary 1.2. Suppose that the number of ends of X is greater than $C^2 + 1$. Given any sequence $\epsilon_i \to 0$ and any point $x \in X$, we put $(X_i, d_i, x_i) = (X, \epsilon_i d, x)$. Let μ_i be the push-forward of the measure $\mu(B_{1/\epsilon_i}(x))^{-1}\mu$ by the identity map from X to X_i. From Lemma 3.7 (2) the metric measure space (X_i, d_i, μ_i) satisfies $BG(0, n)$ with C. Then there is a subsequence $\epsilon_j \to 0$ such that $\{(X_j, d_j, \mu_j, x_j)\}$ pointed measured Gromov–Hausdorff converges to some pointed metric measure space $(X_\infty, d_\infty, \mu_\infty, x_\infty)$. The limit space also satisfies $BG(0, n)$ with C. Since the number of ends of X is greater than two, the point x_∞ is a local cut point. Then the degree of x_∞ is equal to the number of ends of X. This contradicts Theorem 1.1. $\qquad \square$

4.3. *Branch points.* We next give an obstruction condition for the existence of a local cut point in a metric measure space X satisfying $BG(k, n)$ with $1 \leq C < \sqrt{2}$. Let us recall that the measured Gromov–Hausdorff limits of manifolds M_i with $\mathrm{Ric}_{M_i} \geq (n - 1)k$ satisfy $BG(k, n)$ with $C = 1$. It is conjectured that the limit space has no local cut point unless the limit is one-dimensional.

Assume that X has a local cut point x. Let x be an r-cut point. Let $\gamma : [0, l] \to X$ be a minimal geodesic with $\gamma(l) = x$. Assume that l is sufficiently small (for example, $l \leq r/3$). If $\sigma : [0, L] \to X$ is a minimal geodesic such that $\sigma(0) = \gamma(0)$ and $\sigma(L) \in B_l(\gamma(l))$ for $L \geq l$, then $\sigma(l) = \gamma(l)$, that is, σ passes through x. We then define two kinds of branch points of the geodesic γ.
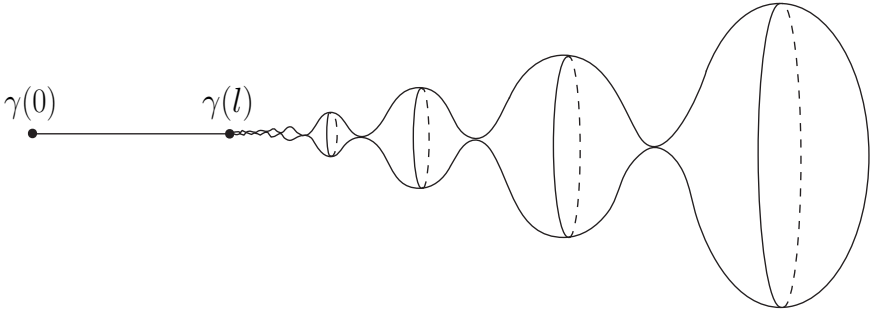
Figure 2. Weak branch point.

Definition 4.8 (branch point). Let γ be as above. We say that $\gamma(l)$ is a *branch point* of γ if for all $\epsilon > 0$ there exist two distinct points $x_1, x_2 \in B_\epsilon(\gamma(l))$ such that $d(\gamma(0), x_1) = d(\gamma(0), x_2) > l$.

Note that $\gamma(l)$ is a branch point of γ if and only if no neighborhood of $\gamma(l)$ is a segment.

Definition 4.9 (weak branch point [Cheeger and Colding 2000a, Section 5]). Let γ be as in Definition 4.8. We say that $\gamma(l)$ is a *weak branch point* of γ if, for all $\epsilon > 0$ and all two points $x_1, x_2 \in B_\epsilon(\gamma(l))$ with $d(\gamma(0), x_1), d(\gamma(0), x_2) > l$, there exist minimal geodesics $\sigma_i : [0, l_i] \to X$ for $i = 1, 2$ from $\gamma(l)$ to x_i such that $\sigma_1(s) = \sigma_2(s)$ for some $s > 0$. See Figure 2.

Example 4.10. (1) Let $X = [a, b]$. Every interior point $x \in (a, b)$ is a weak branch point of all paths $\gamma : [0, l] \to X$ with $\gamma(l) = x$ and $l \ll 1$. The point x is not a branch point of any γ.

(2) Let X be the length space in Figure 2. The intersection x of two spheres is a branch point of all paths $\gamma : [0, l] \to X$ with $\gamma(l) = x$ and $l \ll 1$. However, the point x is *not* a weak branch point. Although a branch point is not necessarily a weak branch point, we use the term "weak" as in [Cheeger and Colding 2000a, Section 5].

Assume now that there exists a local cut point in a metric measure space satisfying $\mathrm{BG}(k, n)$ with $1 \leq C < \sqrt{2}$. We give an obstruction condition for the existence of a local cut point by observing geodesics that pass through a local cut point.

Theorem 4.11 [Cheeger and Colding 2000a, Theorem 5.1]. *Let (X, d, μ) be a metric measure space satisfying the generalized Bishop–Gromov inequality $\mathrm{BG}(k, n)$ with constant C for some $k \in \mathbb{R}$, $n \geq 1$, and $1 \leq C < \sqrt{2}$. If there exists a local cut point x in X, then it is a weak branch point of all geodesics $\gamma : [0, l] \to X$ with $\gamma(l) = x$ and $l \ll 1$.*

The proof is essentially the same as in [Cheeger and Colding 2000a]. We give the proof for the completeness of the paper.

Proof. If x has a neighborhood that is a segment, then x is a weak branch point of all paths which go to the point. Assume now that no neighborhood of x is a segment. Let $\gamma : [0, l] \to X$ be a geodesic that branches at $\gamma(l) = x$, where l is sufficiently small. Suppose that x is not a weak branch point: There exist two points $x_1, x_2 \in B_l(\gamma(l))$ with $d(\gamma(0), x_1), d(\gamma(0), x_2) > l$ such that all geodesics $\sigma_i : [0, l_i] \to X$ for $i = 1, 2$ from $\gamma(l)$ to x_i have $\sigma_1(s) \neq \sigma_2(s)$ for all $0 < s \leq \min\{l_1, l_2\}$. We may assume that $d(\gamma(l), x_1) = d(\gamma(l), x_2)$ and denote both by l'.

Consider all minimal geodesics from x to x_i for $i = 1, 2$. By assumption, the set of all minimal geodesics from x to x_1 and the set of all minimal geodesics from x to x_2 have no intersection except x. We will use the generalized Bishop–Gromov inequality (3-6) with the base point x_i. Let $0 < \epsilon \ll l'$. We denote by U the connected component of $B_\epsilon(x) \setminus \{x\}$ that contains γ. We see that $A_{l', l'+\epsilon}(x_i)$ contains U. Set $S_i = S_{l'-\epsilon, l'}(x_i, U)$ for $i = 1, 2$. Let us recall that S_i is the intersection of $A_{l'-\epsilon, l'}(x_i)$ and all minimal geodesics from x_i to each point in U. Then $S_1 \cap S_2$ is empty by assumption. For $i = 1, 2$, the generalized Bishop–Gromov inequality (3-6) induces

$$(4\text{-}11) \qquad C^{-1} \frac{V_{k,n}(l'-\epsilon, l')}{V_{k,n}(l', l'+\epsilon)} \leq \frac{\mu\big(S_{l'-\epsilon, l'}(x_i, U)\big)}{\mu(U)} = \frac{\mu(S_i)}{\mu(U)}.$$

Let U' denote the union of connected components of $B_\epsilon(x) \setminus \{x\}$ that contain S_1 and S_2. We see that $A_{l, l+\epsilon}(\gamma(0))$ contains U'. Since $S_i \subset U'$ holds for $i = 1, 2$, we have $\mu(S_1) + \mu(S_2) \leq \mu(U')$. By summing up (4-11) for $i = 1, 2$, it follows that

$$(4\text{-}12) \qquad 2C^{-1} \frac{V_{k,n}(l'-\epsilon, l')}{V_{k,n}(l', l'+\epsilon)} \leq \frac{\mu(S_1) + \mu(S_2)}{\mu(U)} \leq \frac{\mu(U')}{\mu(U)}.$$

Applying (3-6) with the base point $\gamma(0)$, we obtain

$$(4\text{-}13) \qquad \frac{\mu(U')}{\mu\big(S_{l-\epsilon, l}(\gamma(0), U')\big)} \leq C \frac{V_{k,n}(l, l+\epsilon)}{V_{k,n}(l-\epsilon, l)}.$$

Since U contains $S_{l-\epsilon, l}(\gamma(0), U')$, it, together with (4-12) and (4-13), follows that

$$2C^{-1} \frac{V_{k,n}(l'-\epsilon, l')}{V_{k,n}(l', l'+\epsilon)} \leq C \frac{V_{k,n}(l, l+\epsilon)}{V_{k,n}(l-\epsilon, l)}.$$

Taking $\epsilon \to 0$, we obtain $2 \leq C^2$, which is a contradiction. $\square$

Remark 4.12. Cheeger and Colding [2000a, Theorem 5.1] studied the limit space that contains a one-dimensional piece and is not one-dimensional. It follows from Theorem 4.11 that the length space in Figure 2 can't satisfy BG(k, n) with $1 \leq C < \sqrt{2}$ for any k, n, and any measure. This does not follow from their [Theorem 5.1].
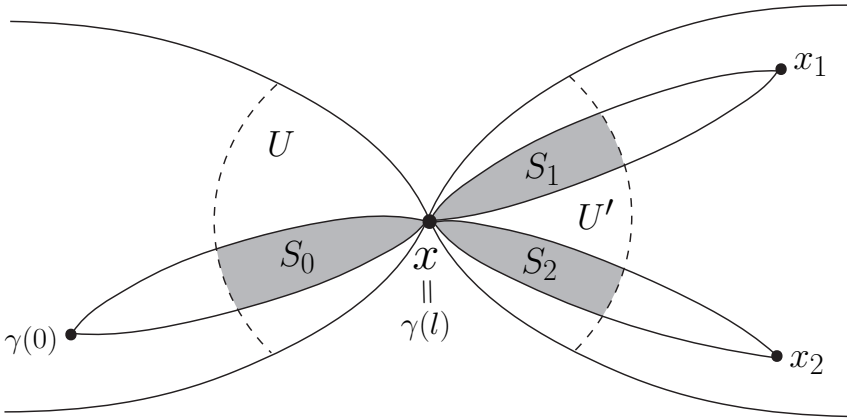
Figure 3. Proof of Theorem 4.11.

As a corollary of Theorem 4.11, we obtain a particular case of Theorem 1.1.

Theorem 4.13. *Let (X, d, μ) satisfy $\mathrm{BG}(k, n)$ with C for some $k \in \mathbb{R}$, $n \geq 1$, and $1 \leq C < \sqrt{2}$. Assume that there exists a local cut point x in X. Then $\deg(x) = 2$.*

Proof. Suppose that $\deg(x) \geq 3$. For all geodesics $\gamma : [0, l] \to X$ with $\gamma(l) = x$ and $l \ll 1$, the local cut point $x = \gamma(l)$ is not a weak branch point of γ. This is because the geodesic γ is extended to two connected components of $B_r(x) \setminus \{x\}$ which do not contain γ, where r is sufficiently small. This contradicts Theorem 4.11. $\qquad\square$

4.4. *Local geometric structure.* Recall the definition of an r-cut point (Definition 4.1). We investigate the geometric structure of the neighborhood of an r-cut point in a metric measure space X satisfying $\mathrm{BG}(k, n)$ with $1 \leq C < \sqrt{2}$.

Assume X has an r-cut point x. Then $\mathrm{diam}(O \cap S_r(x)) \leq \mathrm{diam}(\overline{B}_r(x)) \leq 2r$ for all connected components O of $\overline{B}_r(x) \setminus \{x\}$; see Figure 5. By improving the method of the proof of Theorem 4.11, we obtain a more precise estimate of the diameter:

Theorem 4.14. *For all $k \in \mathbb{R}$, $n \geq 1$, $1 \leq C < \sqrt{2}$, and all $R > 0$ there exists a constant $\delta = \delta(k, n, C, R) > 0$ depending only on k, n, C, and R such that: Let (X, d, μ) be a metric measure space satisfying the generalized Bishop–Gromov inequality $\mathrm{BG}(k, n)$ with constant C. If X has an r-cut point x with $0 < r \leq R$, then $\mathrm{diam}\big(O \cap S_r(x)\big) \leq (2 - \delta)r$ holds for all connected components O of $\overline{B}_r(x) \setminus \{x\}$.*

Remark 4.15. The constant $\delta(k, n, C, R)$ in Theorem 4.14 is independent of the metric measure space (X, d, μ). Moreover, we can calculate the precise value; see Remark 4.19. It follows from Theorem 4.14 that for fixed k, n, and C the metric space in Figure 4 cannot satisfy $\mathrm{BG}(k, n)$ with C for any measure, provided
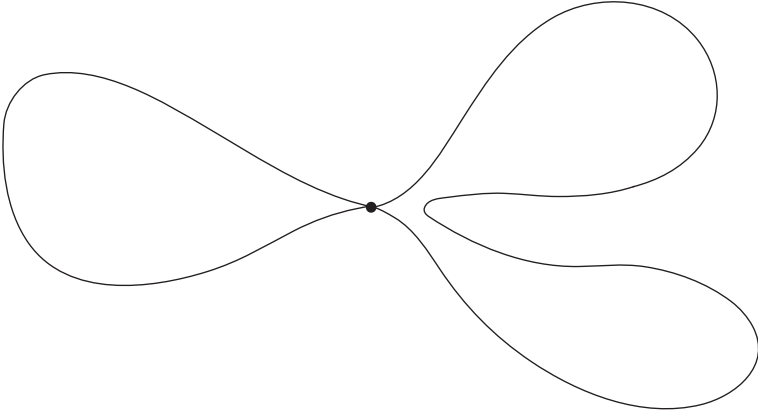
Figure 4. "Three-pronged" space.

a narrow part is sufficiently small. Roughly speaking, combined with Theorem 4.13, "three-pronged" spaces cannot satisfy BG(k, n) with $1 \leq C < \sqrt{2}$.

Proof of Theorem 4.14. For $k \in \mathbb{R}$, $n \geq 1$, $1 \leq C < \sqrt{2}$, and $R > 0$, let (X, d, μ) be a metric measure space satisfying BG(k, n) with C and $x \in X$ be an r-cut point with $0 < r \leq R$. Fix $0 < \delta \ll r$. Suppose that diam $(O \cap S_r(x)) > (2-\delta)r$ holds for some connected component O of $\bar{B}_r(x) \setminus \{x\}$. We take two points $x_1, x_2 \in O \cap S_r(x)$ with $d(x_1, x_2) > (2 - \delta)r$ and then choose minimal geodesics $\sigma_i : [0, r] \to X$ from x to x_i for $i = 1, 2$. Since $\deg(x) = 2$, we denote by O' another connected component of $\bar{B}_r(x) \setminus \{x\}$. For $0 < \epsilon < \delta r$, denote by U the connected component of $B_\epsilon(x) \setminus \{x\}$ that does not contain σ_1 and σ_2. Then $A_{r/3, r/3+\epsilon}(\sigma_i(r/3))$ contains U for $i = 1, 2$.

Claim 4.16. *Every minimal geodesic from $\sigma_i(r/3)$ to a point in U passes through x for $i = 1, 2$.*

Proof. Suppose that there exists a minimal geodesic $\gamma : [0, l] \to X$ from $\sigma_i(r/3)$ to some point y in U such that γ does not pass through x. Then, $\gamma(t) \notin B_r(x)$ for some $t \in [0, l]$ since x is an r-cut point. Therefore,

$$d(y, \sigma_i(r/3)) \geq d(y, \gamma(t)) \geq d(\gamma(t), x) - d(y, x) > r - \epsilon.$$

On the other hand, we have a contradiction from

$$d(y, \sigma_i(r/3)) \leq d(y, x) + d(x, \sigma_i(r/3)) < \epsilon + r/3. \qquad \square$$

Setting $S_i = S_{r/3-\delta r-\epsilon, r/3-\delta r}(\sigma_i(r/3), U)$ for $i = 1, 2$, we have

Claim 4.17. $S_1 \cap S_2 = \varnothing$.

Proof. By assumption, we have

$$(2-\delta)r < d(x_1, x_2) \le d(x_1, \sigma_1(r/3)) + d(\sigma_1(r/3), \sigma_2(r/3)) + d(\sigma_2(r/3), x_2)$$
$$= \frac{2r}{3} + d(\sigma_1(r/3), \sigma_2(r/3)) + \frac{2r}{3}.$$

It follows that $(2/3-\delta)r < d(\sigma_1(r/3), \sigma_2(r/3))$. Suppose that there exists a point $y \in S_1 \cap S_2$. We then have

$$d(\sigma_1(r/3), \sigma_2(r/3)) \le d(\sigma_1(r/3), y) + d(y, \sigma_2(r/3)) \le 2\left(\frac{r}{3} - \delta r\right) = \left(\frac{2}{3} - 2\delta\right)r.$$

This is a contradiction.   □

The generalized Bishop–Gromov inequality (3-6) with the base point $\sigma_i(r/3)$ gives, for $i = 1, 2$,

$$(4\text{-}14) \qquad C^{-1} \frac{V_{k,n}\big((1/3-\delta)r - \epsilon, (1/3-\delta)r\big)}{V_{k,n}(r/3, r/3+\epsilon)} \le \frac{\mu(S_i)}{\mu(U)}.$$

We set $U' = S_1 \cup S_2$. Then $S_1 \cap S_2 = \varnothing$ (Claim 4.17) gives $\mu(S_1) + \mu(S_2) \le \mu(U')$. Therefore, by summing (4-14) for $i = 1, 2$, we get

$$(4\text{-}15) \qquad 2C^{-1} \frac{V_{k,n}\big((1/3-\delta)r - \epsilon, (1/3-\delta)r\big)}{V_{k,n}(r/3, r/3+\epsilon)} \le \frac{\mu(S_1) + \mu(S_2)}{\mu(U)} \le \frac{\mu(U')}{\mu(U)}.$$

Next, we take an arbitrary point x_0 in $O' \cap S_{r/3}(x)$. Such a point exists since x is an r-cut point.

Claim 4.18. *Each minimal geodesic from x_0 to each point in U' passes through x.*

Proof. Suppose that there exists a minimal geodesic $\gamma : [0, l] \to X$ from x_0 to some point y in U' such that γ does not pass through x. Then $\gamma(t) \notin B_r(x)$ for some $t \in [0, l]$. Therefore,

$$d(x_0, y) \ge d(x_0, \gamma(t_0)) \ge d(\gamma(t_0), x) - d(x, x_0) \ge r - \frac{r}{3} = \frac{2r}{3}.$$

On the other hand, we have a contradiction from

$$d(x_0, y) \le d(x_0, x) + d(x, y) \le \frac{r}{3} + \delta r + \epsilon.   \square$$

Set $S_0 = S_{r/3-\epsilon, r/3}(x_0, U')$. The generalized Bishop–Gromov inequality (3-6) with the base point x_0 implies

$$(4\text{-}16) \qquad \frac{\mu(U')}{\mu(S_0)} \le C \frac{V_{k,n}\big((1/3+\delta)r, (1/3+\delta)r + \epsilon\big)}{V_{k,n}(r/3 - \epsilon, r/3)}.$$
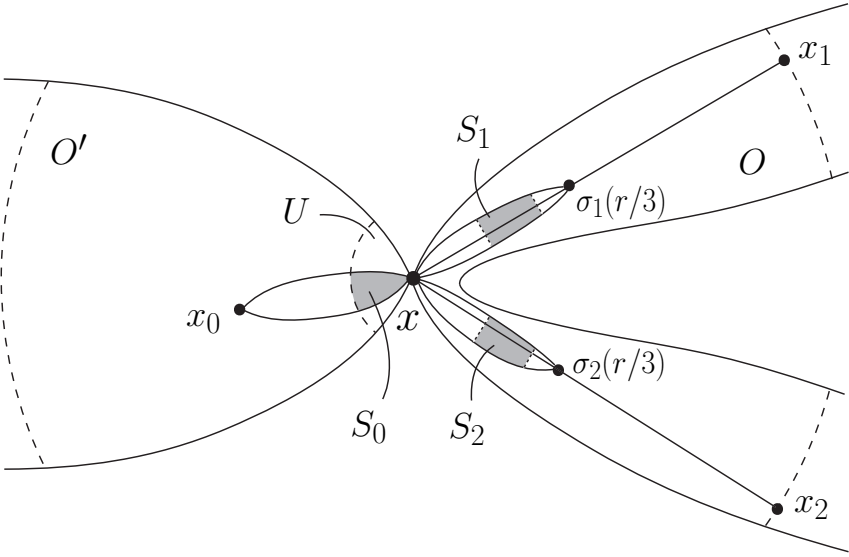
Figure 5. Proof of Theorem 4.14.

Since U contains S_0, inequalities (4-15) and (4-16) give

$$(4\text{-}17) \qquad 2 \leq C^2 \, \frac{V_{k,n}(r/3, r/3+\epsilon)}{V_{k,n}(r/3-\epsilon, r/3)} \, \frac{V_{k,n}\big((r/3+\delta)r, (r/3+\delta)r+\epsilon\big)}{V_{k,n}\big((r/3-\delta)r-\epsilon, (r/3-\delta)r\big)}.$$

After letting $\epsilon \to 0$, a sufficiently small $\delta > 0$ implies a contradiction. This completes the proof of Theorem 4.14. □

Remark 4.19. In the case $k = 0$, we have

$$\delta(0, n, C, R) = \frac{1}{4} \, \frac{(2/C^2)^{1/(n-1)} - 1}{(2/C^2)^{1/(n-1)} + 1}.$$

Indeed, the right-hand side of (4-17) is equal to

$$C^2 \, \frac{(r/3+\epsilon)^n - (r/3)^n}{(r/3)^n - (r/3-\epsilon)^n} \, \frac{[(1/3+\delta)r+\epsilon]^n - [(1/3+\delta)r]^n}{[(1/3-\delta)r]^n - [(1/3-\delta)r-\epsilon]^n},$$

which converges to $C^2[(1/3+\delta)/(1/3-\delta)]^{n-1}$ as $\epsilon \to 0$. Therefore, it suffices to determine a positive number δ such that $C^2[(1/3+\delta)/(1/3-\delta)]^{n-1} < 2$.

4.5. *Convergence of local cut points.* Using Theorem 4.14, we observe the structure of the accumulation of local cut points in a metric measure space satisfying BG(k, n) with $1 \leq C < \sqrt{2}$.

Let (X, d, μ) satisfy BG(k, n) with C for some $k \in \mathbb{R}$, $n \geq 1$, and $1 \leq C < \sqrt{2}$. Assume that there exist three r-cut points $x_1, x_2, x_3 \in X$ $(r > 0)$ such that

$d(x_i, x_j) \ll r$ for all $1 \leq i, j \leq 3$. We recall that $\deg(x_i) = 2$ (Theorem 4.13). Denote by O_1, O_1' the connected components of $\bar{B}_r(x_1) \setminus \{x_1\}$. Assume then that O_1' contains x_2 and x_3 and without loss of generality that $d(x_1, x_2) < d(x_1, x_3)$. For $i = 2, 3$, denote by O_i the connected component of $\bar{B}_r(x_i) \setminus \{x_i\}$ that contains x_1 and by O_i' the connected component of $\bar{B}_r(x_i) \setminus \{x_i\}$ that does not contain x_1. By definition, $O_2 \cap O_3$ is nonempty. We say that $\{x_1, x_2, x_3\}$ *stands in a line* if $O_2' \cap O_3'$ is nonempty.

As a corollary of Theorem 4.14, we have

Corollary 4.20. *Let (X, d, μ) be a metric measure space satisfying* $\mathrm{BG}(k, n)$ *with C for some $k \in \mathbb{R}$, $n \geq 1$, and $1 \leq C < \sqrt{2}$. Let $\delta = \delta(k, n, C, R)$ be the constant in Theorem 4.14 for $R > 0$. Assume that there exist r-cut points $x_1, x_2, x_3 \in X$ with $0 < r \leq R$. If $d(x_i, x_j) < \delta r/6$ holds for all $1 \leq i, j \leq 3$, then $\{x_1, x_2, x_3\}$ stands in a line.*

Proof. Suppose that $O_2' \cap O_3'$ is empty; see Figure 6. Since x_2 is an $r/3$-cut point, $O_2 \cap S_{r/3}(x_2) = (O_1 \cap S_{r/3}(x_2)) \cup (O_3' \cap S_{r/3}(x_2))$ has at least two connected components. Denote by O one of connected components of $O_1 \cap S_{r/3}(x_2)$ and by O' one of connected components of $O_3' \cap S_{r/3}(x_2)$.

Since x_2 is an r-cut point, all minimal geodesics from each point in O to each point in O' pass through x_1 and x_3. Therefore, we have

$$(4\text{-}18) \qquad d(O, O') = d(x_1, O) + d(x_1, x_3) + d(x_3, O').$$

Since $d(x_i, x_j) < \delta r/6$ holds by the assumption, we have

$$(4\text{-}19) \qquad d(x_1, O) \geq \frac{r}{3} - d(x_2, x_1) > \frac{r}{3} - \frac{\delta r}{6} = \left(1 - \frac{\delta}{2}\right)\frac{r}{3}.$$

Similarly, we have

$$(4\text{-}20) \qquad d(x_3, O') > \left(1 - \frac{\delta}{2}\right)\frac{r}{3}.$$

Hence, relations (4-18), (4-19) and (4-20) imply

$$\mathrm{diam}(O_2 \cap S_{r/3}(x_2)) \geq d(O, O') = d(x_1, O) + d(x_1, x_3) + d(x_3, O')$$
$$> \left(1 - \frac{\delta}{2}\right)\frac{r}{3} + \left(1 - \frac{\delta}{2}\right)\frac{r}{3}$$
$$= (2 - \delta)\frac{r}{3}.$$

This contradicts Theorem 4.14. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Assume now that there exists a sequence $\{x_i\}_{i=1}^\infty$ of r-cut points in X such that $d(x_i, x_j) \ll r$ for all i and j. Denote by O_i and O_i' the connected components of $\bar{B}_r(x_i) \setminus \{x_i\}$. Assume that O_i contains x_j for all $j < i$. We say that $\{x_i\}_{i=1}^\infty$ *stands in a line* if $O_i' \cap O_{i+1}'$ is nonempty for all i. Corollary 4.20 implies:
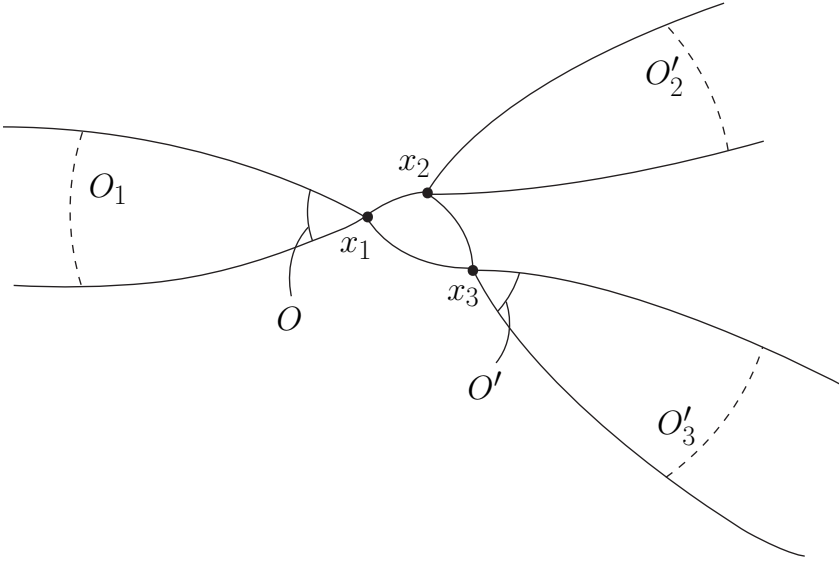
Figure 6. Proof of Corollary 4.20.

Corollary 4.21. *Let* (X, d, μ) *be as in Corollary 4.20. For each* $r > 0$*, the set of all* r*-cut points in* X *is closed.*

Proof. Assume that there exists a sequence $\{x_i\}_{i=1}^{\infty}$ of r-cut points in X that converges to a point x in X. By Corollary 4.20, $\{x_i\}_{i=N}^{\infty}$ stands in a line for a sufficiently large N. Hence the limit x is a local cut point. Since each x_i is an r-cut point, so is x. □

Remark 4.22. Let $\{x_i\}_{i=1}^{\infty}$ be a sequence of r_i-cut points, where $r_i \to 0$. It is then not necessarily true that a limit point of $\{x_i\}$ is a local cut point. Consider the sequence $\{1/i\}_{i=1}^{\infty} \subset [0, 1]$.

In fact, Corollary 4.21 holds without the assumption $\mathrm{BG}(k, n)$ with $1 \le C < \sqrt{2}$.

Proposition 4.23. *Let* (X, d) *be a complete, locally compact length space. Assume that for* $r > 0$ *there exists a sequence* $\{x_i\}_{i=1}^{\infty}$ *of* r*-cut points in* X *that converges to a point* $x \in X$*. Then, for some sufficiently large* N*, we have* $\deg(x_i) = 2$ *for all* $i \ge N$*. Moreover,* $\{x_i\}_{i=N}^{\infty}$ *stands in a line. (Hence, the limit point* x *is also an* r*-cut point. We do not always have* $\deg(x) = 2$*.)*

Proof. Suppose that for each N there exists $i \ge N$ such that $\deg(x_i) \ge 3$. We may assume that $\deg(x_i) \ge 3$ for all i and $d(x, x_i) > d(x, x_j)$ for all $i < j$. For each i, there exists a connected component O_i of $\bar{B}_r(x_i) \setminus \{x_i\}$ that does not contain x and x_{i-1}. Since $O_i \cap S_r(x_i)$ is nonempty, we have a contradiction to the local compactness at x. Thus it follows that $\deg(x_i) = 2$ for all sufficiently large i.

Next, we show that $\{x_i\}_{i=N}^{\infty}$ stands in a line for some sufficiently large N. Suppose that $\{x_i\}_{i=N}^{\infty}$ does not stand in a line for any large N. By taking a subsequence, we may assume that $O_i' \cap O_{i+1}' = \varnothing$ for $i = 1, 3, 5, \ldots$. Since $O_i' \cap S_r(x_i)$ is nonempty, we have a contradiction to the local compactness at x. $\qquad\square$

Proposition 4.23 does not necessarily hold if X is not locally compact.

Example 4.24. Consider the set $\bigcup_{i=1}^{\infty}\{(2/i, y) \mid 0 \le y \le 2\} \cup \{(x, 0) \mid 0 \le x \le 2\} \subset \mathbb{R}^2$ with the induced distance. Although the point $(2/i, 1/i)$ is a 1-cut point for every $i \in \mathbb{N}$, the limit point $(0, 0)$ is not a local cut point.

5. The Poincaré inequality

In this section we prove Theorem 1.3. We begin by recalling the definition of a Poincaré inequality of type $(1, p)$. Let (X, d) be a metric space.

Definition 5.1 (upper gradient [Heinonen and Koskela 1998]). Let u be a function on X. A Borel function $g : X \to [0, \infty]$ is called an *upper gradient* of u if for all paths $\gamma : [0, l] \to X$ (proportional to arclength), we have

$$|u(\gamma(l)) - u(\gamma(0))| \le \int_0^l g(\gamma(t))\, dt.$$

Every function has an upper gradient $g \equiv \infty$, and hence upper gradients are never unique. For a Lipschitz function $u : X \to \mathbb{R}$, we define $|\nabla u| : X \to \mathbb{R}$ by

$$|\nabla u|(x) = \limsup_{y \to x} \frac{|u(x) - u(y)|}{d(x, y)}$$

if x is not isolated, and $|\nabla u|(x) = 0$ if x is isolated.

Proposition 5.2. *If u is a Lipschitz function, then $|\nabla u|$ is an upper gradient of u.*

Proof. Let $\gamma : [0, l] \to X$ be a path. The function $u \circ \gamma$ is Lipschitz and hence is differentiable almost everywhere by Rademacher's theorem. Since γ is proportional to arclength,

$$|(u \circ \gamma)'(t)| = \lim_{s \to t} \frac{|u(\gamma(t)) - u(\gamma(s))|}{|t - s|} \le \lim_{s \to t} \frac{|u(\gamma(t)) - u(\gamma(s))|}{d(\gamma(t), \gamma(s))} \le |\nabla u|(\gamma(t))$$

holds for all differentiable points $t \in [0, l]$. Therefore

$$|u(\gamma(l)) - u(\gamma(0))| \le \int_0^l |(u \circ \gamma)'(t)|\, dt \le \int_0^l |\nabla u|(\gamma(t))\, dt. \qquad\square$$

Assume that (X, d, μ) is a complete, locally compact length space equipped with a Borel measure such that $0 < \mu(B_r(x)) < +\infty$ holds for all $x \in X$ and all

$0 < r < +\infty$. For $B \subset X$, we denote

$$u_B = \fint_B u \, d\mu = \frac{1}{\mu(B)} \int_B u \, d\mu.$$

Definition 5.3 (Poincaré inequality). Let $1 \leq p < \infty$. We say that (X, d, μ) satisfies a *Poincaré inequality of type* $(1, p)$ if, for all $R > 0$, there exists a constant $C_P = C_P(p, R) > 0$ depending only on p and R such that

$$\fint_{B_r(x)} |u - u_{B_r(x)}| \, d\mu \leq C_P \, r \left(\fint_{B_r(x)} g^p \, d\mu \right)^{1/p}$$

holds for all $x \in X$, all $0 < r \leq R$, all measurable functions u, and all upper gradients g of u.

In our setting (X is a length space), a Poincaré inequality of type $(1, p)$ is derived from a "weak" Poincaré inequality of type $(1, p)$ if we assume that μ is doubling; see [Hajłasz and Koskela 2000] for details.

Remark 5.4. It follows from Hölder's inequality that every metric measure space satisfying a Poincaré inequality of type $(1, p)$ also satisfies one of type $(1, q)$ for all $q \geq p$.

Keith and Zhong [$\geq$ 2007] proved the following: Let $p > 1$. If μ is doubling and if (X, d, μ) satisfies a Poincaré inequality of type $(1, p)$, then there is an $\epsilon > 0$ such that (X, d, μ) satisfies a Poincaré inequality of type $(1, q)$ for all $q > p - \epsilon$.

Proof of Theorem 1.3. The proof is by contradiction; suppose that there is a local cut point x_0 satisfying (1-3). Fix a sufficiently small $r > 0$ such that $B_r(x_0) \setminus \{x_0\}$ is disconnected. We choose two connected components O_1, O_2 of $B_r(x_0) \setminus \{x_0\}$. For sufficiently large numbers $N \in \mathbb{N}$, we define functions $u_N : B_r(x_0) \to \mathbb{R}$ as follows. We set $U_i = O_i \cap \left(B_r(x_0) \setminus B_{(N\mu(O_i))^{-1}}(x_0) \right)$ for $i = 1, 2$ and define

$$u_N(x) = \begin{cases} (-1)^{i+1}/\mu(O_i) & \text{on } U_i, \\ (-1)^{i+1} N d(x_0, x) & \text{on } O_i \setminus U_i, \\ 0 & \text{on } B_r(x_0) \setminus (O_1 \cup O_2). \end{cases}$$

The function u_N is Lipschitz; hence $|\nabla u_N|$ is an upper gradient of u_N (Proposition 5.2). Since (X, μ) satisfies a Poincaré inequality of type $(1, p)$, for $R \geq r$ there exists a constant $C_P = C_P(p, R) > 0$ such that

$$(5\text{-}21) \qquad \fint_{B_r(x_0)} \left| u_N - (u_N)_{B_r(x_0)} \right| d\mu \leq C_P \, r \left(\fint_{B_r(x_0)} |\nabla u_N|^p \, d\mu \right)^{1/p}.$$

We first estimate the left-hand side of (5-21) from below. We have

$$
\fint_{B_r(x_0)} \left| u_N - (u_N)_{B_r(x_0)} \right| d\mu \geq \fint_{B_r(x_0)} |u_N| \, d\mu - \left| (u_N)_{B_r(x_0)} \right|
$$

$$
= \frac{1}{\mu(B_r(x_0))} \left(\left(\frac{\mu(U_1)}{\mu(O_1)} + \int_{O_1 \setminus U_1} N d(x_0, x) \, d\mu(x) + \frac{\mu(U_2)}{\mu(O_2)} + \int_{O_2 \setminus U_2} N d(x_0, x) \, d\mu(x) \right) \right.
$$

$$
\left. - \left| \frac{\mu(U_1)}{\mu(O_1)} + \int_{O_1 \setminus U_1} N d(x_0, x) \, d\mu(x) - \frac{\mu(U_2)}{\mu(O_2)} - \int_{O_2 \setminus U_2} N d(x_0, x) \, d\mu(x) \right| \right)
$$

$$
\geq \frac{1}{\mu(B_r(x_0))} \left(\left(\frac{\mu(U_1)}{\mu(O_1)} + \int_{O_1 \setminus U_1} N d(x_0, x) \, d\mu(x) + \frac{\mu(U_2)}{\mu(O_2)} + \int_{O_2 \setminus U_2} N d(x_0, x) \, d\mu(x) \right) \right.
$$

$$
\left. - \left(\left| \frac{\mu(U_1)}{\mu(O_1)} - \frac{\mu(U_2)}{\mu(O_2)} \right| + \int_{O_1 \setminus U_1} N d(x_0, x) \, d\mu(x) + \int_{O_2 \setminus U_2} N d(x_0, x) \, d\mu(x) \right) \right)
$$

$$
= \frac{1}{\mu(B_r(x_0))} \left(\frac{\mu(U_1)}{\mu(O_1)} + \frac{\mu(U_2)}{\mu(O_2)} - \left| \frac{\mu(U_1)}{\mu(O_1)} - \frac{\mu(U_2)}{\mu(O_2)} \right| \right).
$$

Since N is sufficiently large, $\mu(U_i)/\mu(O_i)$ is approximately equal to one ($i = 1, 2$). Therefore, we see that the left-hand side of (5-21) is bounded below by the positive number $\mu(B_r(x_0))^{-1}$, which is independent of N.

Next, we bound the right-hand side of (5-21) from above. Note that

$$
|\nabla u_N| = \begin{cases} N & \text{on } (O_1 \setminus U_1) \cup (O_2 \setminus U_2), \\ 0 & \text{otherwise,} \end{cases}
$$

and $(O_1 \setminus U_1) \cup (O_2 \setminus U_2) \subset B_{r_N}(x_0)$ for $r_N := \max\{(N\mu(O_1))^{-1}, (N\mu(O_2))^{-1}\}$. It follows that

$$
C_P r \left(\fint_{B_r(x_0)} |\nabla u_N|^p \, d\mu \right)^{1/p} = C_P r \left(\frac{N^p}{\mu(B_r(x_0))} \mu \big((O_1 \setminus U_1) \cup (O_2 \setminus U_2) \big) \right)^{1/p}
$$

$$
\leq C_P r \left(\frac{N^p}{\mu(B_r(x_0))} \mu \big(B_{r_N}(x_0) \big) \right)^{1/p},
$$

which goes to zero as $N \to \infty$, by assumption (1-3). This is a contradiction. $\qquad \square$

Let $\alpha > 0$. Consider the metric measure space

$$
\left\{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + x_2^2 + \cdots + x_{n-1}^2 \leq x_n^{2\alpha} \right\}
$$

equipped with the Euclidean distance and the n-dimensional Lebesgue measure $\mathscr{L}^n$. The origin $o = (0, 0, \ldots, 0)$ is a local cut point. We have $\mathscr{L}^n(B_r(o)) =$

$c(n)\, r^{\alpha(n-1)+1}$. It follows from Theorem 1.3 that, if the metric measure space satisfies a Poincaré inequality of type $(1, p)$, then we have

$$\alpha \le \frac{p-1}{n-1}.$$

Acknowledgments

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