Pacific Journal of Mathematics

GLOBAL CONIC SHOCK WAVE FOR THE STEADY SUPERSONIC FLOW PAST A CONE: ISOTHERMAL CASE

DACHENG CUI AND HUICHENG YIN

Volume 233 No. 2 December 2007

PACIFIC JOURNAL OF MATHEMATICS
Vol. 233, No. 2, 2007

GLOBAL CONIC SHOCK WAVE FOR THE STEADY SUPERSONIC FLOW PAST A CONE: ISOTHERMAL CASE

DACHENG CUI AND HUICHENG YIN

We establish the global existence and stability of a steady symmetric conic shock wave for the perturbed supersonic isothermal flow past an infinitely long circular cone with an arbitrary vertex angle. The flow is assumed to be described by a steady potential equation. By establishing the uniform weighted energy estimate on the linearized problem, we show that the symmetric conic shock attached at the vertex of the cone exists globally in the whole space when the speed of the supersonic incoming flow is appropriately large.

1. Introduction

We study the steady conic shock wave problem for the symmetrically perturbed supersonic gas past an infinitely long circular cone. Such a problem has been extensively studied both computationally and theoretically under some suitable conditions. See [Bertin 1994; Chen et al. 2003; Chen et al. 2002; Courant and Friedrichs 1948; Cox and Crabtree 1965; Cui and Yin 2006; Keyfitz and Warnecke 1991; Li and Yu 1985; Lien and Liu 1999; Tsien 1946; Xin and Yin 2006; Yin 2006; 2002; Zheng 2001] and the references therein. As noted by Courant and Friedrichs [1948], if there is a uniform supersonic flow $(0, 0, q_0)$ with constant density $\rho_0 > 0$ coming from minus infinity, and the flow hits the circular cone $(x_1^2+x_2^2)^{1/2} = b_0x_3$ in the direction of the x_3-axis, a conic shock $(x_1^2+x_2^2)^{1/2} = s_0x_3$ for $s_0 > b_0$ will arise and attach to the cone's apex when b_0 is less than a critical value b^* (b^* is determined by the parameters of the incoming flow).

When the supersonic incoming flow is perturbed, a natural problem arises: is the conic shock globally stable? Our goal is to establish the global existence and stability of a conic shock for the perturbed hypersonic isothermal gas past an infinitely long conic body $(x_1^2+x_2^2)^{1/2} = b_0x_3$ with any fixed constant b_0 (this means that the

MSC2000: primary 35L70, 35L65, 35L67; secondary 76N15.

Keywords: isothermal gas, potential equation, supersonic incoming flow, hypersonic flow, conic shock, global existence, tangent vector field.

This research was supported by the National Natural Science Foundation of China #10571082 and the National Basic Research Programm of China #2006CB805902.

critical value b^* can be arbitrarily large for the isothermal gas with an appropriately hypersonic flow). The so-called isothermal gas means that the pressure P and the density ρ of the gas are described by the state equation $P = A\rho$ for some constant $A > 0$. In this case, the sound speed is a constant independent of the density ρ.

We will use the potential equation to describe the motion of the supersonic isothermal gas; this model is also recommended, for example, in [Majda 1991; 1984; Majda and Thomann 1987]. Let $\Phi(x)$ be the potential of velocity $u = (u_1, u_2, u_3)$, that is, $u_i = \partial_i \Phi$. Then it follows from Bernoulli's law that

$$\tfrac{1}{2}|\nabla\Phi|^2 + h(\rho) = C_0, \tag{1-1}$$

where $h(\rho) = A \ln \rho$ is the specific enthalpy, $\nabla = (\partial_1, \partial_2, \partial_3)$, and $C_0 = \frac{1}{2}q_0^2 + h(\rho_0)$ is Bernoulli's constant, which is determined by the uniform supersonic flow coming from negative infinity with velocity $(0, 0, q_0)$ and density $\rho_0 > 0$.

By using (1-1), we can express the density function $\rho(x)$ as

$$\rho = \exp\left(\frac{C_0 - \frac{1}{2}|\nabla\Phi|^2}{A}\right) \equiv H(\nabla\Phi). \tag{1-2}$$

Substituting (1-2) into the mass conservation equation $\sum_{i=1}^{3} \partial_i(\rho u_i) = 0$ yields

$$\begin{aligned} ((\partial_1\Phi)^2 - A)\partial_{11}^2\Phi + ((\partial_2\Phi)^2 - A)\partial_{22}^2\Phi + ((\partial_3\Phi)^2 - A)\partial_{33}^2\Phi \\ + 2\partial_1\Phi\partial_2\Phi\partial_{12}^2\Phi + 2\partial_1\Phi\partial_3\Phi\partial_{13}^2\Phi + 2\partial_2\Phi\partial_3\Phi\partial_{23}^2\Phi = 0. \end{aligned} \tag{1-3}$$

Due to the geometric property of the cone surface $(x_1^2 + x_2^2)^{1/2} = b_0 x_3$ and the symmetric property of the perturbed incoming flow which we will discuss later, as in [Chen et al. 2002], we may assume $\Phi(x) = \Phi(x_3, r)$. It is convenient to introduce cylindrical coordinates (x_3, r) as

$$r = \sqrt{x_1^2 + x_2^2} \quad \text{and} \quad x_3 = x_3. \tag{1-4}$$

Set $\Phi = q_0 x_3 + \varphi$ with $\varphi(x) = \varphi(x_3, r)$. Then, in the new coordinates (1-4), Equation (1-3) can be rewritten as

$$((q_0+\partial_3\varphi)^2 - A)\partial_3^2\varphi + ((\partial_r\varphi)^2 - A)\partial_r^2\varphi + 2\partial_r\varphi(q_0+\partial_3\varphi)\partial_{r3}^2\varphi - \frac{A}{r}\partial_r\varphi = 0. \tag{1-5}$$

Denote by $\varphi^-(x_3, r)$ the flow field before the possible shock front $r = \chi(x_3)$ with $\chi(0) = 0$, and let φ^+ denote the corresponding flow field behind the front. Then the system (1-5) can be split into two equations. The equation for $\varphi^-(x_3, r)$

is

$$((q_0+\partial_3\varphi^-)^2-A)\partial_3^2\varphi^-+((\partial_r\varphi^-)^2-A)\partial_r^2\varphi^- + 2\partial_r\varphi^-(q_0+\partial_3\varphi^-)\partial_{r3}^2\varphi^- - \frac{A}{r}\partial_r\varphi^- = 0 \tag{1-6}$$

and holds for $r > \chi(x_3)$ and $x_3 > 0$, or $x_3 \le 0$. The equation for $\varphi^+(x_3, r)$ is

$$((q_0+\partial_3\varphi^+)^2-A)\partial_3^2\varphi^+ + ((\partial_r\varphi^+)^2-A)\partial_r^2\varphi^+ + 2\partial_r\varphi^+(q_0+\partial_3\varphi^+)\partial_{r3}^2\varphi^+ - \frac{A}{r}\partial_r\varphi^+ = 0 \tag{1-7}$$

and holds for $r < \chi(x_3)$.

It is easy to verify that (1-6) and (1-7) are strictly hyperbolic with respect to x_3 for $\partial_3\Phi^\pm > \sqrt{A}$, where $\sqrt{A}$ is the sound speed.

On the cone surface $r = b_0x_3$, the flow field φ^+ satisfies

$$-(q_0+\partial_3\varphi^+)b_0+\partial_r\varphi^+=0 \quad \text{for } r=b_0x_3. \tag{1-8}$$

Meanwhile, on the possible conic shock $r = \chi(x_3)$, we apply Equation (1-3) and the symmetric form of $\varphi(x_3, r)$ to the Rankine–Hugoniot condition to obtain

$$-((q_0+\partial_3\varphi)H)\chi'(x_3)+\partial_r\varphi H=0 \quad \text{for } r=\chi(x_3). \tag{1-9}$$

Also, the potential $\varphi^+(x_3, r)$ is continuous across the shock:

$$\varphi^+(x_3,\chi(x_3))=\varphi^-(x_3,\chi(x_3)). \tag{1-10}$$

In addition, we impose initial conditions on $\varphi^-(x_3, r)$ as

$$\varphi^-(0,r)=\varepsilon\varphi_0^-(r) \quad \text{and} \quad \partial_3\varphi^-(0,r)=q_0+\varepsilon\varphi_1^-(r), \tag{1-11}$$

where $\varepsilon > 0$ a small constant and $\varphi_0^-(r), \varphi_1^-(r) \in C_0^\infty(0, l)$ with some fixed positive number $l > 0$.

Our main result is this:

Theorem 1.1. *Under the above assumptions, for any given $b_0 > 0$ and appropriately large constant q_0, there exists a small constant ε_0 such that the problem (1-6)–(1-7) with (1-8)–(1-11) has a global C^∞ shock solution $(\varphi^\pm(x_3, r), \chi(x_3))$ for $\varepsilon < \varepsilon_0$. Also, the quantities $(\nabla_{x_3,r}\varphi^+(x_3, r), \chi(x_3))$ tend to the corresponding ones for the uniform supersonic incoming flow $(0, 0, q_0; \rho_0)$ past the circular cone $r = b_0x_3$ with the rate $(1+x_3)^{\mu/2-1}$, where μ is any fixed constant with $0 < \mu < 2$.*

Remark 1.2. By Remark 2.2, for any fixed $b_0 > 0$, we know that there appears a unique supersonic conic shock $r = s_0x_3$ when the uniform supersonic incoming flow $(0, 0, q_0; \rho_0)$ hits the cone $r = b_0x_3$ and q_0 is appropriately large. This implies that the critical value b^* can be large for the isothermal gas and hypersonic incoming flow (the main reason is that the sound speed is a constant independent of the density). However, for the polytropic gas described by the state equation $P(\rho) = A\rho^\gamma$ with $1 < \gamma < 3$, there exists a critical value

$$b_* = \sqrt{\frac{1}{2}\Big(\sqrt{\frac{\gamma+7}{\gamma-1}} - 1\Big)}$$

such that the attached supersonic shock $r = s_0 x_3$ with $u_3 > c(\rho)$ only appears for the potential equation and hypersonic flow when b_0 becomes less than b_*. See [Cui and Yin 2006] for details.

Remark 1.3. As in [Chen et al. 2002; Cui and Yin 2006; Godin 1997; Xin and Yin 2006; Yin 2006], we emphasize that there are no other discontinuities in the solution other than the main conic shock front. This means that the conic shock is structurally stable in the whole space for the isothermal gas. This coincides with the phenomena observed in physical experiments and numerical computations.

Remark 1.4. If there is no main shock for the Equation (1-7), then its classical solution will blow up; see for example [Godin 2005; Sideris 1985]. This means that the main shock can absorb possible compressions of the flow and prevent the formation of new shocks between the main shock and the fixed boundary.

Now we cite work directly related to this paper. Chen et al. [2002], assuming a uniform supersonic incoming flow and a sharp angle for the curved conic body, showed that a curved conic shock exists globally when the supersonic polytropic flow passes a symmetrically curved cone. On the other hand, Z. Xin and H. Yin [2006] established the global existence of a multidimensional conic shock for the uniform supersonic incoming flow past a generally curved sharp cone under a certain boundary condition on the conic surface (physically, that boundary condition meant that the body is perforated or porous). In addition, by using Glimm's scheme, W. C. Lien and T. P. Liu [1999] obtained the global existence of a weak solution and the long distance asymptotic behavior in the symmetric case under the suitable conditions on the large Mach number, the sharp vertex angle and the shock strength. Our main interest here is to establish the global existence of a shock for the perturbed supersonic isothermal gas past an infinitely long conic body with an arbitrarily large angle when the speed of the incoming flow is large; especially, we remove the smallness assumption on the sharp cone. This assumption, used in [Chen et al. 2002; Lien and Liu 1999; Xin and Yin 2006; Yin 2002], was essential to the proofs there.

Next, we comment on the proof of Theorem 1.1. To prove it, we intend to use the continuity method to establish a priori estimates for the solution and its derivatives. To achieve this as in [Chen et al. 2002], [Cui and Yin 2006; Godin 1997] and [Xin and Yin 2006; Yin 2006], we need to derive some global uniform weighted energy estimates for the linearized problem of (1-7)–(1-10). Based on such estimates we can obtain the existence and the asymptotic behavior of the solution to the perturbed nonlinear problem. The main method for obtaining the weighted energy estimates is to search for appropriate multipliers. As in [Chen et al. 2002; Xin and Yin 2006], finding such a multiplier is complicated by our self-similar background solution which strongly depends on the vertex angle of the cone, the Mach number of incoming flow, and the state equation of the gas.

Although some strategies for proving Theorem 1.1 are similar to those of [Chen et al. 2002; Xin and Yin 2006], many new difficulties appear in the isothermal case, and the possible largeness of b_0 must be overcome; in contrast, the smallness of b_0 plays an essential role in those works. Also, it seems difficult to find the "dissipative" property — needed for deriving uniform a priori estimates on the solutions — on the boundary conditions (1-8)–(1-10) as in the polytropic case of [Cui and Yin 2006], because, in our case, the higher-order asymptotic expansions of the background solutions are of exponent-type. The polynomial properties of higher order asymptotic expansions on the polytropic gas play the key role in the analysis of [Cui and Yin 2006].

The paper is organized as follows. In Section 2, we show for large q_0 that there exists an attached shock $r = s_0 x_3$ for any fixed $b_0 > 0$, and we then derive some basic estimates for the background self-similar solution, which will help subsequently find the appropriate multiplier. In Section 3, we reformulate the problem (1-7)–(1-10) and derive some useful estimates on the coefficients of the resulted nonlinear equation and its boundary conditions. In Section 4, we establish the weighted energy estimate for the linearized problem, where the appropriate multiplier is given. Also, we will explain in detail the sometimes subtle search for the multiplier. Using the energy estimate in Section 4, we prove Theorem 1.1 in Section 5. Some complicated and tedious computations are given in the Appendix.

We will use the following notations: $O(q_0^{-\nu})(\nu > 0)$ and $O(e^{-\mu q_0^2})(\mu > 0)$ denote the bounded quantities, which admit the bound $|O(q_0^{-\nu})| \le M_1 q_0^{-\nu}$ and $|O(e^{-\mu q_0^2})| \le M_2 e^{-\mu q_0^2}$, where the generic constants M_1 and M_2 depend only on b_0 and A. $O(\varepsilon)$ means there exists a generic constant M_3, such that $|O(\varepsilon)| \le M_3 \varepsilon$, where M_3 depends on b_0, q_0, and A.

2. The self-similar solution and some of its properties

Here, we will discuss for large q_0 the existence of a self-similar shock solution for the supersonic isothermal flow past the circular cone $(x_1^2 + x_2^2)^{1/2} = b_0 x_3$ with fixed constant b_0. Meanwhile, we will give some precise estimates and detailed properties of the background solution by using the expressions for q_0 and b_0. These estimates will play an important role in obtaining a priori estimates in the subsequent sections.

Suppose that a uniform supersonic isothermal flow with velocity $(0, 0, q_0)$ and density $\rho_0 > 0$ comes from minus infinity, and suppose the flow hits the circular cone $(x_1^2+x_2^2)^{1/2} = b_0 x_3$ along its axis. From Lemma 2.1, we can show that there will appear a conic shock $r = s_0 x_3$ (for $s_0 > b_0$) attached at the tip of the cone for large q_0. Also the corresponding density and velocity functions (ρ, u_1, u_2, u_3) are self-similar, that is, in the cylindrical coordinates (x_3, r), these functions between the shock front and the surface of cone have the form $\rho = \rho(s)$, $u_1 = U(s)x_1/r$, $u_2 = U(s)x_2/r$, and $u_3 = u_3(s)$, where $s = r/x_3$. It follows the steady compressible Euler equation that $(\rho(s), U(s), u_3(s))$ satisfies for $b_0 \le s \le s_0$ the nonlinear ordinary differential system

$$
\begin{aligned}
\rho'(s) &= -\frac{\rho U(su_3-U)}{s\big(A(1+s^2)-(su_3-U)^2\big)},\\
U'(s) &= -\frac{AU}{s\big(A(1+s^2)-(su_3-U)^2\big)},\\
u_3'(s) &= \frac{AU}{A(1+s^2)-(su_3-U)^2}.
\end{aligned}
\tag{2-1}
$$

From Lemma 2.1, it can be shown that the denominator satisfies $A(1+s^2) - (su_3 - U)^2 > 0$ for $b_0 \le s \le s_0$. This means that the system (2-1) makes sense.

On the shock front $r = s_0 x_3$, it follows from the Rankine–Hugoniot conditions and Lax's geometric entropy conditions on the 2-shock that

$$
\rho U - s_0 \rho u_3 = 0 \quad \text{and} \quad u_3 + s_0 U = 0 \tag{2-2}
$$

and

$$
\lambda_1(s_0) < s_0 < \lambda_2(s_0) \quad \text{and} \quad \frac{\sqrt{A}}{\sqrt{q_0^2 - A}} < s_0, \tag{2-3}
$$

where $\lambda_{1,2}(s) = (U(s)u_3(s) \mp \sqrt{A}(U^2(s) + u_3^2(s) - A)^{1/2})/(u_3^2(s) - A)$.

Additionally, the flow satisfies on $s = b_0$ the fixed boundary condition

$$
U(s) = b_0 u_3(s). \tag{2-4}
$$

For large q_0 and fixed b_0, now we show the existence of the solution to system (2-1) with (2-2)–(2-4) and give some needed estimates.

Lemma 2.1. . *If $u_3(b_0) > \sqrt{A}$, then the free boundary problem (2-1)–(2-4) has a smooth supersonic shock solution for $b_0 \le s \le s_0$. Also, one has*

(i) $U'(s) < 0$, $u_3'(s) > 0$, $\rho'(s) < 0$, *and* $u_3(s) > \sqrt{A}$;

(ii) $U(s) > 0$ *and* $A(1+s^2) - (su_3(s) - U(s))^2 > 0$.

Remark 2.2. For any fixed $b_0 > 0$, if q_0 is appropriately large, then we can verify that $u_3(b_0) > \sqrt{A}$ and $u_3(s) > \sqrt{A}$; see Remark 2.4. Thus, the problem (2-1)–(2-4) has a smooth supersonic shock solution in this case.

Proof of Lemma 2.1. Set $U_+ = \lim_{s\to s_0-0} U(s)$ and $u_{3+} = \lim_{s\to s_0-0} u_3(s)$. Also set $\rho_+ = \lim_{s\to s_0-0} \rho(s)$. Then from the Rankine–Hugoniot conditions (2-2) and Bernoulli's law (1-1), we arrive at

$$
\begin{aligned}
U_+ &= \frac{s_0q_0(\rho_+-\rho_0)}{(1+s_0^2)\rho_+},\\
u_{3+} &= q_0 - \frac{s_0^2q_0(\rho_+-\rho_0)}{(1+s_0^2)\rho_+},
\end{aligned}
\qquad h(\rho_+) - h(\rho_0) = \frac{s_0^2q_0^2(\rho_+^2-\rho_0^2)}{2(1+s_0^2)\rho_+^2}. \tag{2-5}
$$

By the entropy condition (2-3) (which leads to $\rho_0 < \rho_+$), we find $U_+ > 0$. Also, from (2-5) and a direct computation, we can derive that

$$
s_0u_{3+} - U_+ = \frac{s_0\rho_0q_0}{\rho_+} > 0.
$$

Since $(s_0u_{3+} - U_+)/(1+s_0^2)^{1/2}$ is the normal velocity on the shock front, the entropy condition also implies

$$
\frac{s_0u_{3+} - U_+}{(1+s_0^2)^{1/2}} < \sqrt{A}. \tag{2-6}
$$

The physical explanation to (2-6) is that across the shock front the normal velocity of the supersonic flow becomes subsonic.

By the continuity of $\rho(s)$, $U(s)$, and $u_3(s)$, (2-6) is also valid in $s_0-\delta_0 \le s \le s_0$ with small $\delta_0 > 0$, and then (2-1) makes sense in this interval. Also, by (2-1), we obtain $\rho'(s) < 0$, $U'(s) < 0$, and $u_3'(s) > 0$ in $s_0-\delta_0 \le s \le s_0$. Hence the function $\sqrt{A} - (s_0u_3(s) - U(s))/(1+s^2)^{1/2}$ is a decreasing function of s in $s_0-\delta_0 \le s \le s_0$. From this and (2-6), we find

$$
\begin{aligned}
&A(1+s^2) - (su_3(s) - U(s))^2\\
&\qquad = (1+s^2)\Big(\sqrt{A} - \frac{su_3(s)-U(s)}{\sqrt{1+s^2}}\Big)\Big(\sqrt{A} + \frac{su_3(s)-U(s)}{\sqrt{1+s^2}}\Big)\\
&\qquad \ge \sqrt{A}(1+b_0^2)\Big(\sqrt{A} - \frac{su_{3+}-U_+}{\sqrt{1+s^2}}\Big) > 0.
\end{aligned}
$$

This result means that the denominator in (2-1) is bounded away from zero as long as the solution of (2-1) exists. Therefore, by the continuity extension method we know that this result holds in the whole interval $[b_0, s_0]$, and the solution of (2-1) exists and satisfies

$$
U'(s) < 0, \quad u_3'(s) > 0, \quad \rho'(s) < 0.
$$

By a direct computation, we have

$$\big(u_3(s)-\sqrt{A}\big)' = \frac{AU}{A(1+s^2)-(su_3-U)^2} > 0,$$

and this yields

$$u_3(s)-\sqrt{A} > u_3(b_0)-\sqrt{A} > 0.$$

Thus (2-1)–(2-4) has a supersonic shock solution. $\square$

For the isothermal gas and large q_0, the following estimates on the background solution are fundamental to our subsequent analysis.

Lemma 2.3. *If q_0 is large and $b_0 \le s \le s_0$, then, with $m_0 = b_0^2/(4A(1+b_0^2))$,*

(i) $s_0 = b_0 + O(e^{-m_0 q_0^2})$;

(ii) $U(s) = \dfrac{b_0 q_0}{1+b_0^2}\big(1+O(e^{-m_0 q_0^2})\big)$;

(iii) $u_3(s) = \dfrac{q_0}{1+b_0^2}\big(1+O(e^{-m_0 q_0^2})\big)$.

Remark 2.4. It follows from (iii) of Lemma 2.3 that the assumption $u_3(b_0) > \sqrt{A}$ in Lemma 2.1 holds when q_0 is large.

Proof of Lemma 2.3. It follows the third equation in (2-5) that

$$\ln\rho_+ - \ln\rho_0 = \frac{s_0^2 q_0^2}{2A(1+s_0^2)}\Big(1-\frac{\rho_0^2}{\rho_+^2}\Big).$$

Setting $\Lambda = \rho_+/\rho_0$, we get

$$\frac{\Lambda^2}{\Lambda^2-1}\ln\Lambda = \frac{s_0^2 q_0^2}{2A(1+s_0^2)},$$

which implies

$$\Lambda = \exp\Big(\frac{s_0^2}{2A(1+s_0^2)} q_0^2\big(1+O(e^{-2m_0 q_0^2})\big)\Big) \quad \text{and} \quad \frac{1}{\Lambda} = O(e^{-m_0 q_0^2}),$$

where m_0 is as assumed.

This, together with (2-5), yields

$$\begin{aligned} U_+ &= \frac{s_0 q_0}{1+s_0^2}\Big(1-\frac{1}{\Lambda}\Big) = \frac{s_0 q_0}{1+s_0^2} + q_0 O(e^{-m_0 q_0^2}),\\ u_{3+} &= \frac{q_0}{1+s_0^2}\Big(1+\frac{s_0^2}{\Lambda}\Big) = \frac{q_0}{1+s_0^2} + q_0 O(e^{-m_0 q_0^2}). \end{aligned} \tag{2-7}$$

Following from the monotone property of $U(s)$ in Lemma 2.1, we have

$$U_+ \le U(s) \le U(b_0) = b_0 u_3(b_0) \le b_0 u_{3+}. \tag{2-8}$$

By using (2-7) and (2-8), we get

$$s_0 = b_0 + O(e^{-m_0 q_0^2}).$$

This proves (i). Then (ii) and (iii) follow from (i) and (2-7)–(2-8). □

In contrast to the situation in [Chen et al. 2002] and [Xin and Yin 2006; Yin 2006], the proofs of Lemmas 2.1 and 2.3 do not require the smallness of b_0 for the isothermal hypersonic gas.

Lemma 2.5. *Under the assumptions of Lemma 2.3, we have*

(i) $U'(s) = -\dfrac{q_0}{(1+b_0^2)^2}\big(1+O(e^{-m_0 q_0^2})\big);$

(ii) $u_3'(s) = \dfrac{b_0 q_0}{(1+b_0^2)^2}\big(1+O(e^{-m_0 q_0^2})\big).$

Proof. According to the system (2-1) and Lemma 2.3, we obtain (i) as

$$\begin{aligned} U'(s) &= -\frac{\frac{A b_0 q_0}{1+b_0^2}\big(1+O(e^{-m_0 q_0^2})\big)}{A b_0(1+b_0^2) + q_0^2 O(e^{-2m_0 q_0^2}) + q_0 O(e^{-m_0 q_0^2})} \\ &= -\frac{q_0}{(1+b_0^2)^2}\big(1+O(e^{-m_0 q_0^2})\big). \end{aligned}$$

By (2-1), we have $u_3'(s) = -sU'(s)$, and then (ii) follows from (i). □

We next estimate the eigenvalues $\lambda_1(s)$ and $\lambda_2(s)$ in (2-3) when q_0 is large.

Lemma 2.6. *For large q_0 and fixed b_0, we obtain*

$$\begin{aligned} \lambda_1(s) &= b_0 - \frac{\sqrt{A}(1+b_0^2)^{3/2}}{q_0} + O(q_0^{-2}) + O(e^{-m_0 q_0^2}), \\ \lambda_2(s) &= b_0 + \frac{\sqrt{A}(1+b_0^2)^{3/2}}{q_0} + O(q_0^{-2}) + O(e^{-m_0 q_0^2}). \end{aligned}$$

Hence, $\lambda_2(s) > s_0$ and $\lambda_1(s) < s$ for large q_0.

Proof. By using the expression for $\lambda_1(s)$, Lemma 2.3 and Taylor's formula, we arrive at

$$\begin{aligned}\lambda_1(s) &= \left(\frac{b_0 q_0^2}{(1+b_0^2)^2} - \sqrt{A}\sqrt{\frac{q_0^2}{1+b_0^2} - A}\right) \Big/ \left(\frac{q_0^2}{(1+b_0^2)^2} - A\right) + O(e^{-m_0 q_0^2}) \\ &= \frac{\frac{b_0}{(1+b_0^2)^2} - \frac{\sqrt{A}}{q_0}\sqrt{\frac{1}{1+b_0^2}}\sqrt{1 - \frac{A(1+b_0^2)}{q_0^2}}}{\frac{1}{(1+b_0^2)^2}\left(1 - \frac{A(1+b_0^2)^2}{q_0^2}\right)} + O(e^{-m_0 q_0^2}) \\ &= b_0 - \frac{\sqrt{A}(1+b_0^2)^{3/2}}{q_0} + O(q_0^{-2}) + O(e^{-m_0 q_0^2}).\end{aligned}$$

We estimate $\lambda_2(s)$ similarly. □

3. Reformulating problem (1-6)–(1-11)

Here, we first prove the global existence of a smooth solution to Equation (1-6) with conditions (1-11). Next, we reformulate the problem (1-7)–(1-10) by decomposing its solution as a sum of the background solution with a small perturbation. Finally, using the analysis of the background solution from Section 2, we estimate the coefficients that in the reformulated problem when q_0 is large.

We have this global existence result for (1-6) and (1-11) on the left hand side of the shock:

Lemma 3.1. *Equation (1-6) with initial data (1-11) has a C^∞ solution $\varphi^-(x_3, r)$ in the domain $\Omega_- = \{(x_3, r) : x_3 \geq 0, r \geq \chi(x_3)\}$. Also $\varphi^-(x_3, r) \in C_0^\infty(\Omega_-)$, and there exists a positive constant C_k independent of ε such that*

$$\|\varphi^-(x_3, r)\|_{C^k(\Omega_-)} \leq C_k \varepsilon \quad \textit{for any fixed } k \in \mathbb{N}.$$

Proof. The system (1-6) is quasilinear strictly hyperbolic in the x_3-direction for the supersonic flow $u_3^- > \sqrt{A}$. The initial condition (1-11) is of a small perturbation with compact support. The lemma then follows for large q_0 from the entropy condition (2-3), the finite propagation property of hyperbolic equations, and the Picard iteration (or see [Li and Yu 1985; Smoller 1983]). □

We now start reformulating the nonlinear problem (1-7)–(1-10). For convenience, we henceforth omit all the "+" superscripts in (1-7)–(1-10).

Because the denominator of the system (2-1) is positive in $[b_0, s_0]$, we can use (2-1) to extend ρ, U, u_3, and φ to $[s_0, s_0 + \eta_0]$ for small η_0 satisfying $0 < \eta_0 \leq e^{-m_0 q_0^2}(s_0 - b_0)$. We shall denote the extensions of ρ, U, u_3, and φ in the domain $\{(x_3, r) : x_3 > 0, b_0 x_3 \leq r \leq (s_0 + \eta_0) x_3\}$ by $\hat{\rho}$, $\hat{U}$, $\hat{u_3}$, and $\hat{\varphi}$, respectively.

Let φ be the solution of (1-7)–(1-10), and let $\dot\varphi$ be the perturbation of the background solution, that is, $\dot\varphi = \varphi - \hat\varphi$. Then by a direct computation similar to [Chen et al. 2002], Equation (1-6) can be reduced to

$$L\dot\varphi = f_{11}\Big(\frac{r}{x_3}, \nabla_{x_3,r}\dot\varphi\Big)\partial_3^2\dot\varphi + f_{12}\Big(\frac{r}{x_3}, \nabla_{x_3,r}\dot\varphi\Big)\partial_{r3}^2\dot\varphi + f_{22}\Big(\frac{r}{x_3}, \nabla_{x_3,r}\dot\varphi\Big)\partial_r^2\dot\varphi + \frac{1}{r} f_0\Big(\frac{r}{x_3}, \nabla_{x_3,r}\dot\varphi\Big), \tag{3-1}$$

for $b_0x_3 \le r \le \chi(x_3)$, where

$$\begin{aligned}
L\dot\varphi &= \partial_3^2\dot\varphi + 2P_1\Big(\frac{r}{x_3}\Big)\partial_{r3}^2\dot\varphi + P_2\Big(\frac{r}{x_3}\Big)\partial_r^2\dot\varphi + P_3(x_3,r)\partial_3\dot\varphi + P_4(x_3,r)\partial_r\dot\varphi,\\
P_1(s) &= \frac{\hat u_3(s)\hat U(s)}{(\hat u_3(s))^2 - A},\\
P_2(s) &= \frac{\hat U^2(s) - A}{(\hat u_3(s))^2 - A},\\
P_3(x_3,r) &= -\frac{2s^2\hat u_3(s)\hat u_3'(s) + 2s^2\hat U(s)\hat U'(s)}{r\big((\hat u_3(s))^2 - A\big)} \equiv \frac{\tilde P_3(s)}{r},\\
P_4(x_3,r) &= \frac{2s\hat U(s)\hat U'(s) - A - 2s^2\hat u_3(s)\hat U'(s)}{r\big((\hat u_3(s))^2 - A\big)} \equiv \frac{1}{r}\tilde P_4(s).
\end{aligned}$$

Also $f_{ij}(s,0,0) = 0$, and $f_0(s,0,0) = \nabla_q f_0(s,q_1,q_2)|_{q=0} = 0$.

Meanwhile, on $r = b_0x_3$, we have

$$\partial_r\dot\varphi = b_0\partial_3\dot\varphi. \tag{3-2}$$

On the free boundary $r = \chi(x_3)$, we can use the continuity condition (1-10) to rewrite (1-9) as

$$H(\nabla\varphi)\big((\partial_r\varphi)^2 + (\partial_3\varphi)^2 + q_0\partial_3\varphi\big) - \rho_0q_0\partial_3\varphi = 0 \quad \text{for } r = \chi(x_3). \tag{3-3}$$

As in [Chen et al. 2002; Godin 1997], we introduce the notation

$$\xi(x_3) = \frac{\chi(x_3) - s_0x_3}{x_3}.$$

Then we can rewrite (3-3) as

$$B_1\partial_r\dot\varphi + B_2\partial_3\dot\varphi + B_3\xi = \kappa(\xi, \nabla_{x_3,r}\dot\varphi) \quad \text{for } r = \chi(x_3), \tag{3-4}$$

where

$$
\begin{aligned}
B_1 &= -\frac{\rho_+}{A}\big(U_+^2+u_{3+}(u_{3+}-q_0)\big)U_+ + 2\rho_+U_+,\\
B_2 &= -\frac{\rho_+}{A}\big(U_+^2+u_{3+}(u_{3+}-q_0)\big)u_{3+} + 2\rho_+(u_{3+}-q_0)+(\rho_+-\rho_0)q_0,\\
B_3 &= \rho_+\big(2U_+\hat{U}'(s_0)+2(u_{3+}-q_0)\hat{u_3}'(s_0)+q_0\hat{u_3}'(s_0)\big)\\
&\qquad\qquad + \hat{\rho}'(s_0)\big(U_+^2+u_{3+}(u_{3+}-q_0)\big)-\rho_0 q_0\hat{u_3}'(s_0),
\end{aligned}
$$

and the generic function $\kappa(\xi,\nabla_{x_3,r}\dot{\varphi})$ will be used to denote the quantity dominated by $C|(\xi,\nabla_{x_3,r}\dot{\varphi})|^2$, where the generic constant C doesn't depend on ε.

By Lemma 3.2 below, we know that $B_1 \neq 0$ in (3-4) for large q_0. Thus Equation (3-4) can be reduced to

$$
B\dot{\varphi}+\mu_2\xi=\kappa(\xi,\nabla_{x_3,r}\dot{\varphi}) \quad \text{for } r=\chi(x_3), \tag{3-5}
$$

where $B\dot{\varphi}=\partial_r\dot{\varphi}+\mu_1\partial_3\dot{\varphi}$ with $\mu_1=B_2/B_1$ and $\mu_2=B_3/B_1$.

Besides, (1-10) implies $\chi'(x_3)=-\partial_3\varphi/\partial_r\varphi$ for $r=\chi(x_3)$. By an analogous computation in [Chen et al. 2002], we have

$$
\partial_3\Big(x_3\xi+\frac{1}{U_+}\dot{\varphi}(x_3,\chi(x_3))\Big)=\kappa(\xi,\nabla_{x_3,r}\dot{\varphi}). \tag{3-6}
$$

In addition, it follows from Lemma 3.1 for the solution $\varphi^-(x_3,r)\in C_0^\infty(\Omega_-)$ in (1-6) that, near the vertex of the cone $r=b_0x_3$, by the hyperbolicity of (1-7) with respect to x_3 and the finite propagation property of hyperbolic equations, the solution $\varphi^+(x_3,r)$ is actually the background solution $\hat{\varphi}(x_3,r)$. To prove Theorem 1.1, we use that the local existence and stability result for the shock solution from the appendices of [Godin 1997] or [Godin 2005] imply that we need only solve the problem (3-1) with the boundary conditions (3-2), (3-5)–(3-6), and the small initial data of $\dot{\varphi}(x_3,r)|_{x_3=1}$, $\partial_3\dot{\varphi}(x_3,r)|_{x_3=1}$, and $\xi(x_3)|_{x_3=1}$ in the domain $\{(x_3,r): x_3\ge 1,\ b_0x_3\le r\le\chi(x_3)\}$. Here the smallness means that

$$
\sum_{l\le k_0}|\nabla_{x_3,r}^l\dot{\varphi}| \quad\text{and}\quad \sum_{l\le k_0}\Big|\frac{d^l\xi}{dx_3^l}\Big|\le C\varepsilon \quad \text{for } x_3=1, \tag{3-7}
$$

where $k_0\in\mathbb{N}$ with $k_0\ge 5$.

For later use, now we give detailed estimates of the coefficients in (3-1), (3-4), and (3-5).

Lemma 3.2. *If q_0 is large and $b_0 \le s \le s_0+\eta_0$, then*

$$P_1(s)=b_0+\frac{Ab_0(1+b_0^2)^2}{q_0^2}+O(q_0^{-4}),$$

$$P_1'(s)=-1-\frac{A(1+b_0^2)(1+3b_0^2)}{q_0^2}+O(q_0^{-4}),\quad \tilde{P}_3(s)=\frac{1}{q_0^2}O(e^{-m_0q_0^2}),$$

$$P_2(s)=b_0^2-\frac{A(1-b_0^2)(1+b_0^2)^2}{q_0^2}+O(q_0^{-4}),\quad \tilde{P}_4(s)=-\frac{A(1+b_0^2)^2}{q_0^2}+O(q_0^{-4}).$$

$$P_2'(s)=-2b_0-\frac{4Ab_0^3(1+b_0^2)}{q_0^2}+O(q_0^{-4}),$$

Proof. We only compute $P_1'(s)$, as the other terms can be treated similarly. Since $(\hat{U}(s),\hat{u}_3(s))$ is the extension of $(U(s),u_3(s))$ in $[b_0,s_0+\eta_0]$ and also $\eta_0 < e^{-m_0q_0^2}(s_0-b_0)$, it's enough for our computations to use $(U(s),u_3(s))$ instead of $(\hat{U}(s),\hat{u}_3(s))$.

It follows a direct computation and Lemmas 2.3 and 2.5 that

$$\begin{aligned}
P_1'(s)&=\frac{u_3'(s)U(s)+u_3(s)U'(s)}{u_3^2(s)-A}-\frac{2u_3^2(s)U(s)u_3'(s)}{(u_3^2(s)-A)^2}\\
&=\frac{b_0^2-1}{(1+b_0^2)\big(1-\frac{A(1+b_0^2)^2}{q_0^2}\big)}-\frac{2b_0^2}{(1+b_0^2)\big(1-\frac{A(1+b_0^2)^2}{q_0^2}\big)^2}+O(e^{-m_0q_0^2})\\
&=-1-\frac{A(1+4b_0^2+3b_0^4)}{q_0^2}+O(q_0^{-4}).
\end{aligned}$$

□

With respect to B_i for $i=1,2,3$) in (3-4) and μ_j for $j=1,2$) in (3-5), we have this:

Lemma 3.3. *If q_0 is large, then*

$$B_1=\frac{2b_0\rho_+q_0}{1+b_0^2}\big(1+O(e^{-m_0q_0^2})\big),$$

$$B_2=\frac{(1-b_0^2)\rho_+q_0}{1+b_0^2}\big(1+O(e^{-m_0q_0^2})\big),$$

$$B_3=-\frac{b_0\rho_+q_0^2}{(1+b_0^2)^2}\big(1+O(e^{-m_0q_0^2})\big),$$

$$\mu_1=\frac{1-b_0^2}{2b_0}\big(1+O(e^{-m_0q_0^2})\big),$$

$$\mu_2=-\frac{q_0}{2(1+b_0^2)}\big(1+O(e^{-m_0q_0^2})\big).$$

Proof. The lemma follows from a direct computation, which we omit, using Lemmas 2.3 and 2.5. □

4. Uniform estimate on problem (3-1) with conditions (3-2) and (3-5)–(3-7)

Here, as in [Chen et al. 2002] and [Xin and Yin 2006], we will choose the appropriate multiplier to derive an energy estimate for the linearized problem, that is the linear parts of problem (3-1) with conditions (3-2) and (3-5)–(3-7). Finding such a multiplier is rather complicated for the following reasons. First, to verify the global existence, we need to establish a global estimate independent of x_3 for the solution and its derivatives on the boundary and in the interior of its domain. This yields several restrictions on the multiplier and makes the computation very delicate. Second, because our background solution is self-similar on a fixed domain and depends on the vertex angle of the cone and the speed of the incoming flow, the coefficients of the linearized problem are variable. This implies that we should search for the multiplier by solving a system of ordinary differential inequalities with very complicated coefficients. Third, the boundary condition on the surface of body is of Neumann type rather than Dirichlet type, and the coefficients of the nonlinear problem depend on the incoming flow. These two facts induce some essential differences with the treatment of [Godin 1997], (although the ideas there will give us some heuristics). In particular, Godin could assume that the shock is arbitrarily close to the fixed boundary, a fact essential to his analysis. In addition, he could use the Poincare inequality because of the Dirichlet boundary value; however, this is not the case in our problem. Finally, compared with the methods in [Chen et al. 2002; Xin and Yin 2006], we need to choose the multiplier to overcome the new difficulties induced by the possible largeness of b_0 and the special asymptotic expansion of the background solution in the isothermal case.

Theorem 4.1 (Energy estimate). *Set $D_T=\{1\le x_3\le T,\ b_0x_3\le r\le \chi(x_3)\}$ for any $T>1$. Assume that $\dot\varphi\in C^2(D_T)$ satisfies the boundary conditions (3-2) and (3-5), and $|\xi(x_3)|+|x_3\xi'(x_3)|\le C\varepsilon$ for some small $\varepsilon>0$ and $x_3\in[1,T]$. Then for fixed $\mu\in\mathbb{R}$, we can choose a multiplier $M\dot\varphi=ra(\frac{r}{x_3})\partial_3\dot\varphi+x_3b(\frac{r}{x_3})\partial_r\dot\varphi$ such that*

$$
\begin{aligned}
(4\text{-}1)\quad & C_1T^{-\mu+1}\int_{b_0T}^{\chi(T)}|\nabla_{x_3,r}\dot\varphi(T,r)|^2dr+C_2\int\int_{D_T}x_3^{-\mu}|\nabla_{x_3,r}\dot\varphi|^2\,dr\,dx_3\\
&\qquad+C_3\int_{\Gamma_T}x_3^{-\mu+1}|\partial_3\dot\varphi|^2dl+C_4\int_{B_T}x_3^{-\mu+1}|\partial_3\dot\varphi|^2dl\\
&\le\int\int_{D_T}x_3^{-\mu}L\dot\varphi M\dot\varphi\,dr\,dx_3+C_5\int_{\Gamma_T}x_3^{-\mu+1}|B\dot\varphi|^2dl\\
&\qquad+C_6\int_{b_0}^{\chi(1)}\big(|\dot\varphi(1,r)|^2+|\partial_{x_3}\dot\varphi(1,r)|^2\big)dr,
\end{aligned}
$$

where $\Gamma_T=\{1\le x_3\le T,\ r=\chi(x_3)\}$, $B_T=\{1\le x_3\le T,\ r=b_0x_3\}$, and the positive constants C_i for $i=1,\dots,6$ are independent of T and ε (but depend on b_0, q_0 and

μ). *Also,*

$$C_4 = \frac{\sqrt{A}b_0(1+b_0^2)^{7/2}}{2q_0} + O(q_0^{-2}) + O(e^{-m_0q_0^2}),$$

$$C_5 = \frac{C(b_0,\mu)}{q_0^2} + O(q_0^{-3}) + O(e^{-m_0q_0^2}),$$

with $C(b_0,\mu) > 0$ a constant depending only on b_0 and μ.

Remark 4.2. The values of constants C_4 and C_5 will play an important role in the energy estimates for the nonlinear problem (3-1), (3-2), (3-5)–(3-7); see Section 5. By the choices of C_4 and C_5, together with the shock equations (3-5) and (3-6), we can show that the term $C_5\int_{\Gamma_T} x_3^{-\mu+1}|B\dot{\varphi}|^2 dl$ can be absorbed by the left side of (4-1).

Proof. Integrating by parts, we have

$$\begin{aligned}(4\text{-}2)\quad \iint_{D_T} x_3^{-\mu} L\dot{\varphi} M\dot{\varphi}\, dr\, dx_3 &= T^{-\mu}\int_{b_0T}^{\chi(T)} K_3(T,r)dr - \int_{b_0}^{\chi(1)} K_3(1,r)dr \\ &\quad + \int_{B_T} x_3^{-\mu}(b_0K_3 - K_4)dl + \int_{\Gamma_T} x_3^{-\mu}(K_4 - \chi' K_3)dl. \\ &\quad + \iint_{D_T} x_3^{-\mu}\big(K_0(\partial_3\dot{\varphi})^2 + K_1(\partial_r\dot{\varphi})^2 + K_2\partial_3\dot{\varphi}\partial_r\dot{\varphi}\big)dr\,dx_3,\end{aligned}$$

where

$$K_0 = -\frac{r\partial_3 a}{2} + \frac{x_3\partial_r b}{2} - \partial_r(rP_1a) + rP_3a + \frac{\mu ra}{2x_3},$$

$$K_1 = -\partial_3(x_3P_1b) + \frac{r}{2}\partial_3(P_2a) - \frac{x_3}{2}\partial_r(P_2b) + x_3P_4b + \frac{\mu}{2x_3}(2x_3P_1b - rP_2a),$$

$$K_2 = -\partial_3(x_3b) - \partial_r(rP_2a) + x_3P_3b + rP_4a + \mu b,$$

$$K_3 = \frac{ra}{2}(\partial_3\dot{\varphi})^2 + x_3b\partial_3\dot{\varphi}\partial_r\dot{\varphi} + (x_3P_1b - \frac{rP_2a}{2})(\partial_r\dot{\varphi})^2,$$

$$K_4 = (rP_1a - \frac{x_3b}{2})(\partial_3\dot{\varphi})^2 + rP_2a\partial_3\dot{\varphi}\partial_r\dot{\varphi} + \frac{x_3P_2b}{2}(\partial_r\dot{\varphi})^2.$$

We intend to find the positive functions $a(r/x_3)$ and $b(r/x_3)$ such that all integrals on D_T, B_T, and $t = T$ in the right hand side of (4-2) are definitely positive and the integral on Γ_T gives a suitable control on $\dot{\varphi}$. This will be done by following the steps in [Chen et al. 2002]. However, as already mentioned, many concrete computations will be different from those in [Chen et al. 2002] because of the possible largeness of b_0 and the special asymptotic expansion of the background solution in Lemmas 2.3 and 2.6.

For the notational convenience, we set

$$\tilde{\lambda}(s) = \frac{b(s)}{sa(s)}.$$

Step 1. Estimate of $\int_{B_T} x_3^{-\mu}(b_0K_3-K_4)\,dl$.

Using $\partial_r\dot{\varphi}=b_0\partial_3\dot{\varphi}$ for $r=b_0x_3$, a direct computation yields

$$b_0K_3-K_4=x_3(\partial_3\dot{\varphi})^2\Big(\big(b(b_0)-b_0^2a(b_0)\big)\big(\tfrac{1}{2}+b_0^2+b_0^3P_1(b_0)-\tfrac{1}{2}P_2(b_0)b_0^2\big)$$
$$+b_0a(b_0)\big(b_0+b_0^3+b_0^4P_1(b_0)-b_0^3P_2(b_0)-P_1(b_0)-b_0P_2(b_0)\big)\Big).$$

It then follows from Lemma 3.2 and a direct computation that

$$\tfrac{1}{2}+b_0^2+b_0^3P_1(b_0)-\tfrac{1}{2}P_2(b_0)b_0^2$$
$$=\frac{(1+b_0^2)^2}{2}+\frac{Ab_0^2(1+b_0^2)^3}{2q_0^2}+O(q_0^{-4})+O(e^{-m_0q_0^2})>0.$$

In addition, as in [Chen et al. 2002], we have

$$b_0+b_0^3+b_0^4P_1(b_0)-b_0^3P_2(b_0)-P_1(b_0)-b_0P_2(b_0)=0.$$

Thus

$$b_0K_3-K_4=x_3(\partial_3\dot{\varphi})^2b_0a(b_0)(\tilde{\lambda}(b_0)-b_0)\big(\tfrac{1}{2}+b_0^2+b_0^3P_1(b_0)-\tfrac{1}{2}P_2(b_0)b_0^2\big)>0$$

provided that

$$\tilde{\lambda}(b_0)>b_0, \tag{4-3}$$

which gives a constraint for $a(s)$ and $b(s)$ on $s=b_0$.

In this case, for large q_0, we have

$$\int_{B_T}x_3^{-\mu}(b_0K_3-K_4)dl=a(b_0)(\tilde{\lambda}(b_0)-b_0) \tag{4-4}$$
$$\times\bigg(\frac{b_0(1+b_0^2)^2}{2}+\frac{Ab_0^3(1+b_0^2)^3}{2q_0^2}+O(q_0^{-4})+O(e^{-m_0q_0^2})\bigg)\int_{B_T}x_3^{-\mu+1}|\partial_3\dot{\varphi}|^2dl.$$

Step 2. Estimate of $T^{-\mu}\int_{b_0T}^{\chi(T)}K_3(T,r)\,dl$.

On the line $x_3=T$, we can write $K(T,r)$ as

$$K_3(T,r)=sa(s)T\Big(\tfrac{1}{2}(\partial_3\dot{\varphi})^2+\tilde{\lambda}(s)\partial_3\dot{\varphi}\partial_r\dot{\varphi}+\big(P_1\tilde{\lambda}(s)-\tfrac{1}{2}P_2\big)(\partial_r\dot{\varphi})^2\Big).$$

To ensure $K(T,r)>0$, we require

$$\triangle=\tilde{\lambda}^2(s)-2(P_1\tilde{\lambda}(s)-\tfrac{1}{2}P_2)<0.$$

This leads to

$$\lambda_1(s)<\tilde{\lambda}(s)<\lambda_2(s). \tag{4-5}$$

Also,

$$T^{-\mu}\int_{b_0T}^{\chi(T)} K_3(T,r)\,dr \geq C_1T^{-\mu+1}\int_{b_0T}^{\chi(T)} |\nabla_{x_3,r}\dot{\varphi}(T,r)|^2dr, \tag{4-6}$$

where $C_1>0$ is a constant depending on q_0, b_0, and μ.

Step 3. Positivity of the integral on D_T.

We will choose $a(s)$ and $b(s)$ such that

$$K_0(\partial_3\dot{\varphi})^2+K_1(\partial_r\dot{\varphi})^2+K_2\partial_3\dot{\varphi}\partial_r\dot{\varphi}\geq C_2((\partial_3\dot{\varphi})^2+\partial_r\dot{\varphi})^2),$$

where $C_2>0$ is a constant depending on q_0, b_0, and μ.

The above estimate holds if K_0, K_1, and K_2 satisfy

$$K_0>0 \quad \text{and} \quad K_2^2-4K_0K_1<0.$$

Set

$$\begin{aligned}
Q_0&=a(s)(-P_1's-P_1+\tilde{P}_3+\tfrac{1}{2}\mu s),\\
Q_1&=a(s)\big(-\tfrac{1}{2}P_2's^2-\tfrac{1}{2}\mu P_2s\big)+b(s)\big(P_1's-\tfrac{1}{2}P_2'+(\mu-1)P_1+\frac{\tilde{P}_4}{s}\big),\\
Q_2&=a(s)(-P_2-P_2's+\tilde{P}_4)+b(s)\big(\mu-1+\frac{\tilde{P}_3}{s}\big).
\end{aligned}$$

Then a direct computation yields

$$\begin{aligned}
K_0&=(\tfrac{1}{2}s^2-P_1s)a'(s)+\tfrac{1}{2}b'(s)+Q_0,\\
K_1&=-\tfrac{1}{2}P_2s^2a'(s)+(P_1s-\tfrac{1}{2}P_2)b'(s)+Q_1,\\
K_2&=-P_2sa'(s)+sb'(s)+Q_2,
\end{aligned}$$

and

$$\begin{aligned}
K_2^2-4K_0K_1=\big(-P_2sa'(s)+sb'(s)\big)^2+2Q_2\big(-P_2sa'(s)+sb'(s)\big)+Q_2^2-4Q_0Q_1\\
-4\big((\tfrac{1}{2}s^2-P_1s)a'(s)+\tfrac{1}{2}b'(s)2\big)\big(-\tfrac{1}{2}P_2s^2a'(s)+(P_1s-\tfrac{1}{2}P_2)b'(s)\big)\\
+4Q_0\big(\tfrac{1}{2}P_2s^2a'(s)-(P_1s-\tfrac{1}{2}P_2)b'(s)\big)-4Q_1\big((\tfrac{1}{2}s^2-P_1s)a'(s)+\tfrac{1}{2}b'(s)\big)
\end{aligned}$$

As in [Chen et al. 2002, Step 3], we introduce the notations

$$\begin{aligned}
a_1&=-P_2Q_2s+P_2Q_0s^2-Q_1(s^2-2P_1s),\\
a_2&=Q_2s-Q_0(2P_1s-P_2)-Q_1,
\end{aligned}$$

and

$$Y_1 = a'(s) + \frac{a_1 + a_2 P_1 s}{\tilde{A} s^2 D_1},$$
$$Y_2 = -P_1 s a'(s) + b'(s) - \frac{a_2(s)}{\tilde{A}},$$
$$Y_3 = -\Big(Q_0 + \frac{a_2(s)}{2\tilde{A}} - \frac{(s-P_1)(a_1 + a_2 P_1 s)}{2\tilde{A} s D_1}\Big),$$

where $\tilde{A} = -(P_2 + s^2 - 2P_1 s)$ and $D_1 = P_1^2 - P_2$. Then $K_0 > 0$ and $K_2^2 - 4K_0K_2 < 0$ are equivalent to

$$(s^2 - P_1 s)Y_1 + Y_2 - 2Y_3 > 0, \qquad \tilde{A} s^2 D_1 Y_1^2 - \tilde{A} Y_2^2 + 4D_1 Y_3^2 < 0. \tag{4-7}$$

To solve the system (4-7) as in [Chen et al. 2002], we must to study the solvability of the differential system

$$a'(s) + \frac{a_1 + a_2 P_1 s}{\tilde{A} s^2 D_1} = -\frac{s - P_1}{s D_1}\Big(b'(s) - P_1 s a'(s) - \frac{a_2}{\tilde{A}}\Big) + k(s) a(s), \tag{4-8}$$
$$b'(s) - P_1 s a'(s) - \frac{a_2}{\tilde{A}} = \frac{D_1}{\tilde{A}}\Big(\sqrt{\delta_0(s) + k^2(s) s^2 a^2(s) D_1 + 4Y_3^2} - k(s) s a(s)(s - P_1)\Big),$$

where the new functions $\delta_0(s) > 0$ and $k(s) > 0$ will be determined together with $a(s)$ and $b(s)$.

In light of Lemma 2.6, if we assume

$$s < \tilde{\lambda}(s) < \lambda_2(s) \tag{4-9}$$

then (4-3) and (4-5) hold simultaneously.

We note that the first equation in (4-8) is equivalent to

$$\Big(1 + \frac{(s-P_1)(\tilde{\lambda}(s) - P_1)}{D_1}\Big) a'(s) = -\frac{s - P_1(s)}{s D_1}\Big((\tilde{\lambda}(s) + s\tilde{\lambda}'(s)) a(s) - \frac{a_2}{\tilde{A}}\Big) + k(s) a(s) - \frac{a_1 + a_2 P_1 s}{\tilde{A} s^2 D_1}. \tag{4-10}$$

Under the restrictions of (4-9), Lemma A.1 shows that for large q_0

$$1 + \frac{(s-P_1)(\tilde{\lambda}(s) - P_1)}{D_1} > 0.$$

Thus (4-10) can be written as

$$a'(s) = \Big(\tilde{Q}_0(s) + \frac{D_1}{D_1 + (s-P_1)(\tilde{\lambda}(s) - P_1)} k(s)\Big) a(s), \tag{4-11}$$

where

$$\tilde{Q}_0(s) = \frac{-\frac{s-P_1}{s}\left(\tilde{\lambda}(s)+s\tilde{\lambda}'(s)-\frac{a_2(s)}{\tilde{A}a(s)}\right)-\frac{a_1(s)+a_2(s)P_1 s}{\tilde{A}s^2 a(s)}}{D_1+(s-P_1)(\tilde{\lambda}(s)-P_1)}.$$

If we set $a(b_0)=1$, then (4-11) has a unique positive solution $a(s)$ in $[b_0, s_0+\eta_0]$, as follows:

$$a(s) = \exp\left\{\int_{b_0}^{s}\left(\tilde{Q}_0(s)+\frac{D_1}{D_1+(s-P_1)(\tilde{\lambda}(s)-P_1)}k(s)\right)ds\right\}.$$

By the analogous treatment in [Chen et al. 2002], the second equation in (4-8) can be changed to

$$A_0(s)k^2(s)+A_1(s)k(s) = A_2(s). \tag{4-12}$$

where

$$A_0(s) = \left(\frac{s(\tilde{\lambda}(s)-P_1)D_1}{D_1+(s-P_1)(\tilde{\lambda}(s)-P_1)}+\frac{D_1}{\tilde{A}}s(s-P_1)\right)^2-\frac{D_1^3 s^2}{\tilde{A}^2},$$

$$A_1(s) = 2\left(\frac{s(\tilde{\lambda}(s)-P_1)D_1}{D_1+(s-P_1)(\tilde{\lambda}(s)-P_1)}+\frac{D_1}{\tilde{A}}s(s-P_1)\right) \times\left(\tilde{\lambda}(s)+s\tilde{\lambda}'(s)-\frac{a_2(s)}{a(s)\tilde{A}}+s(\tilde{\lambda}(s)-P_1)\tilde{Q}_0(s)\right),$$

$$A_2(s) = \frac{D_1^2}{\tilde{A}^2}\left(\frac{\delta_0(s)}{a^2(s)}+\frac{4Y_3^2(s)}{a^2(s)}\right)-\left(\tilde{\lambda}(s)+s\tilde{\lambda}'(s)-\frac{a_2(s)}{a(s)\tilde{A}}+s(\tilde{\lambda}(s)-P_1)\tilde{Q}_0(s)\right)^2.$$

The Equation (4-12) will have a positive solution $k(s)$ if $A_0(s)<0$ and $A_2(s)<0$.

For the negativity of $A_0(s)$, refer to Lemma A.5, and see Lemma A.6 for the estimate of $A_1(s)$. To ensure the negativity of $A_2(s)$, as in [Chen et al. 2002], we will choose $\tilde{\lambda}(s)$ such that

$$\frac{4D_1^2 Y_3^2(s)}{\tilde{A}^2 a^2(s)} < \left(\tilde{\lambda}(s)+s\tilde{\lambda}'(s)+s(\tilde{\lambda}(s)-P_1)\tilde{Q}_0(s)-\frac{a_2(s)}{a(s)\tilde{A}}\right)^2. \tag{4-13}$$

More concretely, we set $A_2(s) = -1$.

To assure (4-3), (4-5), (4-9), (4-13) and related properties hold, we choose $\tilde{\lambda}(s)$ as

$$\tilde{\lambda}(s) = \frac{s_0+\eta_0-s}{s_0+\eta_0-b_0}\left(1-\frac{1}{q_0^2}\right)\lambda_2(s_0)+\frac{s-b_0}{s_0+\eta_0-b_0}\left(s_0+\frac{1}{q_0^2}(\lambda_2(s_0)-s_0)\right), \tag{4-14}$$

where η_0 is a constant given in Section 3.

In this case, Equation (4-12) has the algebraic solution

$$k(s) = \frac{-A_1(s)-\sqrt{A_1^2(s)-4A_0(s)}}{2A_0(s)} > 0.$$

The corresponding function $\delta_0(s)$ can be chosen as

$$\delta_0(s)=\frac{a^2(s)\tilde{A}^2}{D_1^2}\Big(\Big(\tilde{\lambda}(s)+s\tilde{\lambda}'(s)+s(\tilde{\lambda}(s)-P_1)\tilde{Q}_0(s)-\frac{a_2(s)}{a(s)\tilde{A}}\Big)^2-\frac{4D_1^2Y_3^2(s)}{\tilde{A}^2a^2(s)}-1\Big),$$

where

$$a(s)=\exp\Big\{\int_{b_0}^{s}\Big(\tilde{Q}_0(s)+\frac{D_1}{D_1+(s-P_1)(\tilde{\lambda}(s)-P_1)}k(s)\Big)ds\Big\},$$
$$b(s)=s\tilde{\lambda}(s)a(s).$$

Lemma A.4 shows that $\delta_0(s)>0$. Finally, we arrive at

$$\int\int_{D_T}x_3^{-\mu}(K_0(\partial_3\dot{\varphi})^2+K_1(\partial_r\dot{\varphi})^2+K_2\partial_3\dot{\varphi}\partial_r\dot{\varphi})\,dx_3\,dr \geq C\int\int_{D_T}x_3^{-\mu}|\nabla_{x_3,r}\dot{\varphi}|^2\,dx_3\,dr. \tag{4-15}$$

The function $\tilde{\lambda}(s)$ in (4-14) is chosen to be different from that in [Chen et al. 2002], because we must overcome the new difficulties incurred by the possible largeness of b_0 and working with an isothermal gas. Next we specify the form of $\tilde{\lambda}(s)$. First, we let its value be close to $\lambda_2(s_0)$ on the boundary $r=b_0x_3$, so that the coefficient of $x_3(\partial_3\dot{\varphi})^2$ in $b_0K_3-K_4$ of *Step 1* is of $O(1/q_0)$. This eliminates the influence of the possible largeness of b_0. Second, we choose its value near s_0 on the shock $r=\chi(x_3)$ so that the coefficient of $\int_{\Gamma_T}x_3^{-\mu+1}|B\dot{\varphi}|^2dl$ in the right side of (4-1) is as "small" as $O(1/q_0^2)$. This will imply that the term $\int_{\Gamma_T}x_3^{-\mu+1}|B\dot{\varphi}|^2dl$ can be absorbed by the left hand side of (4-1), together with the shock equations (3-5) and (3-6); for details see Section 5. Third, we let the derivative of $\tilde{\lambda}(s)$ be large, so that (4-13) holds. Finally, the differences of $O(1/q_0)$ between $\tilde{\lambda}(s_0)-s_0$ and $\tilde{\lambda}(b_0)-b_0$ (see the second and third parts of Lemma A.2) will play a crucial role for overcoming the difficulties induced by the possible largeness of b_0; see Lemma A.8, (4-4) and the expressions for C_4 and C_5 in (4-1). This will allow the right side term $\int_{\Gamma_T}x_3^{-\mu+1}|B\dot{\varphi}|^2dl$ to be absorbed by the left side term $\int_{B_T}x_3^{-\mu+1}|\partial_3\dot{\varphi}|^2dl$ together with Equation (3-6).

Step 4. The estimate of $\int_{\Gamma_T}x_3^{-\mu}(K_4-\chi'K_3)dl$.

By the assumption on $\xi(x_3)$ in Theorem 4.1, it follows from the expressions of K_3 and K_4 that for $r=\chi(x_3)$

$$K_4-\chi'K_3=x_3b_0a(s_0)(I+II),\quad \text{with}\quad \begin{aligned} I&=\beta_0(\partial_3\dot{\varphi})^2+\beta_1\partial_3\dot{\varphi}\partial_r\dot{\varphi}+\beta_2(\partial_r\dot{\varphi})^2,\\ II&=\big(O(e^{-m_0q_0^2})+O(\varepsilon)\big)|\nabla_{x_3,r}\dot{\varphi}|^2,\end{aligned}$$

where

$$\begin{aligned}\beta_0 &= P_1(s_0) - s_0 + \tfrac{1}{2}(s_0 - \tilde{\lambda}(s_0)),\\ \beta_1 &= P_2(s_0) - s_0^2 + s_0(s_0 - \tilde{\lambda}(s_0)),\\ \beta_2 &= s_0 P_2(s_0) - s_0^2 P_1(s_0) - \big(\tfrac{1}{2}P_2(s_0) - s_0 P_1(s_0)\big)\big(s_0 - \tilde{\lambda}(s_0)\big).\end{aligned}$$

Noting $\partial_r\dot{\varphi} = B\dot{\varphi} - \mu_1\partial_3\dot{\varphi}$, we have

$$I = \big(\beta_0 - \mu_1\beta_1 + \mu_1^2\beta_2\big)(\partial_3\dot{\varphi})^2 + \big(\beta_1 - 2\mu_1\beta_2\big)B\dot{\varphi}\partial_3\dot{\varphi} + \beta_2(B\dot{\varphi})^2.$$

By Lemma A.8, we have

$$\begin{aligned}\beta_0 - \mu_1\beta_1 + \mu_1^2\beta_2 &= A(1+b_0^2)^4/(4b_0q_0^2) + O(q_0^{-3}) + O(e^{-m_0q_0^2}),\\ \beta_1 - 2\mu_1\beta_2 &= -\sqrt{A}b_0(1+b_0^2)^{5/2}/(2q_0^3) + O(q_0^{-4}) + O(e^{-m_0q_0^2}),\\ \beta_2 &= -Ab_0(1+b_0^2)^2/(q_0^2) + O(q_0^{-3}) + O(e^{-m_0q_0^2}).\end{aligned}$$

Noting that inequalities $\partial_3\dot{\varphi}B\dot{\varphi} \le \frac{1}{2}((B\dot{\varphi})^2 + (\partial_3\dot{\varphi})^2)$ and $\beta_1 - 2\mu_1\beta_2 < 0$, we arrive at

$$\begin{aligned}I \ge \Big(\frac{A(1+b_0^2)^4}{4b_0q_0^2} + O(q_0^{-3}) + O(e^{-m_0q_0^2})\Big)(\partial_3\dot{\varphi})^2&\\ - \Big(\frac{Ab_0(1+b_0^2)^2}{q_0^2} + O(q_0^{-3}) + O(e^{-m_0q_0^2})\Big)(B\dot{\varphi})^2&.\end{aligned}$$

Thus, for $r = \chi(x_3)$, we have

$$\begin{aligned}\int_{\Gamma_T} x_3^{-\mu}(K_4 - \chi' K_3)\,dl \ge \big(O(e^{-m_0q_0^2}) + O(\varepsilon)\big)\int_{\Gamma_T} x_3^{-\mu+1}|\nabla_{x_3,r}\dot{\varphi}|^2\,dl&\\ + a(s_0)\int_{\Gamma_T} x_3^{-\mu+1}\Big(\Big(\frac{A(1+b_0^2)^4}{4q_0^2} + O(q_0^{-3}) + O(e^{-m_0q_0^2})\Big)(\partial_3\dot{\varphi})^2&\\ - \Big(\frac{Ab_0^2(1+b_0^2)^2}{q_0^2} + O(q_0^{-3}) + O(e^{-m_0q_0^2})\Big)(B\dot{\varphi})^2\Big)\,dl&\end{aligned} \tag{4-16}$$

with $1 - C(b_0,\mu)/q_0 \le a(s_0) \le C(b_0,\mu)$; see Lemma A.7.

Substituting (4-4), (4-6), (4-15) and (4-16) into (4-2), together with Lemma A.2, yields

$$
\begin{aligned}
&C_1 T^{-\mu+1}\int_{b_0T}^{\chi(T)}|\nabla_{x_3,r}\dot\varphi(T,r)|^2dr + C_2\int\int_{D_T} x_3^{-\mu}|\nabla_{x_3,r}\dot\varphi|^2dr\,dx_3\\
&\qquad+\Big(\frac{A(1+b_0^2)^4}{4q_0^2}+O(q_0^{-3})+O(e^{-m_0q_0^2})\Big)\int_{\Gamma_T} x_3^{-\mu+1}|\partial_3\dot\varphi|^2dl\\
&\qquad+\Big(\frac{\sqrt{A}b_0(1+b_0^2)^{7/2}}{2q_0}+O(q_0^{-2})+O(e^{-m_0q_0^2})\Big)\int_{B_T} x_3^{-\mu+1}|\partial_3\dot\varphi|^2dl\\
&\le \int\int_{D_T} x_3^{-\mu}L\dot\varphi M\dot\varphi\,dr\,dx_3 + C_6\int_{b_0}^{\chi(1)}(|\dot\varphi(1,r)|^2+|\partial_3\dot\varphi(1,r)|^2)dr\\
&\qquad+\Big(\frac{C(b_0,\mu)}{q_0^2}+O(q_0^{-3})+O(e^{-m_0q_0^2})\Big)\int_{\Gamma_T} x_3^{-\mu+1}|B\dot\varphi|^2dl,
\end{aligned}
$$

where the constants C_1, C_2 and C_6 are independent of T and ε but depends on q_0, b_0 and μ. Therefore, Theorem 4.1 is proved. □

5. Higher-order energy estimates and the proof of Theorem 1.1

To prove Theorem 1.1, we need to obtain higher-order energy estimates, so that we can derive the decay properties of $\nabla_{x_3,r}\dot\varphi$ and $\xi(x_3)$ for large x_3.

Theorem 5.1 (Higher-order energy estimates). *Assume that $\dot\varphi \in C^{k_0}(D_T)$ and $\xi(x_3)\in C^{k_0}[1,T]$ with $k_0 \ge 5$ is a solution of (3-1) with the initial boundary conditions (3-2), (3-5)–(3-7). In addition, assume $|\xi(x_3)|+|x_3\xi'(x_3)|\le C\varepsilon$ in $[1,T]$,*

$$\sum_{0\le l\le[k_0/2]} x_3^l|\nabla_{x_3,r}^{l+1}\dot\varphi(x_3,r)|\le C\varepsilon,$$

and $\varepsilon>0$ is sufficiently small. Then for $\mu>0$,

$$
\begin{aligned}
&\int_{b_0T}^{\chi(T)}\sum_{0\le l\le k_0-1} T^{2l-\mu+1}|\nabla_{x_3,r}^{l+1}\dot\varphi(T,r)|^2dr+\int\int_{D_T}\sum_{0\le l\le k_0-1} x_3^{2l-\mu}|\nabla_{x_3,r}^{l+1}\dot\varphi|^2dr\,dx_3\\
&\qquad+\int_{\Gamma_T}\sum_{0\le l\le k_0-1} x_3^{2l-\mu+1}|\nabla_{x_3,r}^{l+1}\dot\varphi|^2dl+\int_{B_T}\sum_{0\le l\le k_0-1} x_3^{2l-\mu+1}|\nabla_{x_3,r}^{l+1}\dot\varphi|^2dl\\
&\le C\Big(\int_{b_0}^{\chi(1)}\sum_{0\le l\le k_0}|\nabla_{x_3,r}^{l}\dot\varphi(1,r)|^2dr+\dot\varphi^2(1,\chi(1))+\dot\varphi^2(1,b_0)+\xi^2(1)\Big).
\end{aligned}
$$

Here and below $C>0$ denotes a generic constant depending on q_0, b_0, and μ.

Remark 5.2. As in [Chen et al. 2002; Godin 1997; Klainerman and Sideris 1996], we will use the vector fields that are tangent to the surface of the cone and nearly tangential to the shock front. The higher-order energy estimate then follows by the standard commutation argument.

Proof. Since the vector field $S = x_3\partial_3 + r\partial_r$ is tangent to the boundary $r = b_0 x_3$, then we have $\partial_r S^m\dot{\varphi} = b_0\partial_3 S^m\dot{\varphi}$ on $r = b_0 x_3$ in view of the boundary condition (3-2). So we can apply Theorem 4.1 to $S^m\dot{\varphi}(0 \le m \le k_0 - 1)$ and derive

$$
\begin{aligned}
(5\text{-}1)\quad & T^{-\mu+1}\int_{b_0 T}^{\chi(T)} \sum_{0\le m\le k_0-1} |\nabla_{x_3,r} S^m\dot{\varphi}(T,r)|^2 dr + \int_{B_T} x_3^{-\mu+1} \sum_{0\le m\le k_0-1} |\nabla_{x_3,r} S^m\dot{\varphi}|^2 dl \\
&+ \iint_{D_T} x_3^{-\mu} \sum_{0\le m\le k_0-1} |\nabla_{x_3,r} S^m\dot{\varphi}|^2 dr\,dx_3 + \int_{\Gamma_T} x_3^{-\mu+1} \sum_{0\le m\le k_0-1} |\nabla_{x_3,r} S^m\dot{\varphi}|^2 dl \\
\le{}& C(q_0, b_0, \mu)\biggl(\iint_{D_T} x_3^{-\mu} \sum_{0\le m\le k_0-1} L S^m\dot{\varphi} M S^m\dot{\varphi}\, dr\,dx_3 \\
&+ \int_{\Gamma_T} x_3^{-\mu+1} \sum_{0\le m\le k_0-1} |B S^m\dot{\varphi}|^2 dl + \int_{b_0}^{\chi(1)} \sum_{0\le m\le k_0-1} |\nabla^m_{x_3,r}\dot{\varphi}(1,r)|^2 dr\biggr).
\end{aligned}
$$

Following the proof procedure in Theorem 5.1 of [Chen et al. 2002], we can obtain

$$
\begin{aligned}
(5\text{-}2)\quad & \int_{b_0 T}^{\chi(T)} \sum_{0\le l\le k_0-1} T^{2l-\mu+1} |\nabla^{l+1}_{x_3,r}\dot{\varphi}(T,r)|^2 dr + \iint_{D_T} \sum_{0\le l\le k_0-1} x_3^{2l-\mu} |\nabla^{l+1}_{x_3,r}\dot{\varphi}|^2 dr\,dx_3 \\
&+ \int_{\Gamma_T} \sum_{0\le l\le k_0-1} x_3^{2l-\mu+1} |\nabla^{l+1}_{x_3,r}\dot{\varphi}|^2 dl + \int_{B_T} \sum_{0\le l\le k_0-1} x_3^{2l-\mu+1} |\nabla^{l+1}_{x_3,r}\dot{\varphi}|^2 dl \\
\le{}& C(q_0, b_0, \mu)\biggl(\int_{\Gamma_T} \sum_{0\le l\le k_0-2} x_3^{2l-\mu+1} |\nabla^{l+1}_{x_3,r}\dot{\varphi}|^2 dl + \int_{\Gamma_T} x_3^{-\mu+1} |\xi|^2 dl \\
&+ \int_{b_0}^{\chi(1)} \sum_{0\le l\le k_0} |\nabla^l_{x_3,r}\dot{\varphi}(1,r)|^2 dr\biggr).
\end{aligned}
$$

In particular for $k_0 = 1$, the boundary condition (3-5) and Lemma 3.3 give

$$
(5\text{-}3)\qquad |B\dot{\varphi}(x_3,r)|^2 \le \frac{q_0^2}{4(1+b_0^2)^2}\bigl(1 + O(e^{-m_0 q_0^2})\bigr)|\xi(x_3)|^2 + C\varepsilon^2 |\nabla_{x_3,r}\dot{\varphi}|^2.
$$

Thus it follows from (4-1) and (5-3) that

$$
\begin{aligned}
(5\text{-}4)\quad & C_1 T^{-\mu+1}\int_{b_0T}^{\chi(T)} |\nabla_{x_3,r}\dot\varphi(T,r)|^2 + C_2\int\int_{D_T} x_3^{-\mu}|\nabla_{x_3,r}\dot\varphi|^2 dr dx_3\\
&\quad + \Big(\frac{A(1+b_0^2)^4}{4q_0^2}+O(q_0^{-3})+O(e^{-m_0q_0^2})\Big)\int_{\Gamma_T} x_3^{-\mu+1}|\partial_3\dot\varphi|^2 dl\\
&\quad + \Big(\frac{\sqrt{A}b_0(1+b_0^2)^{7/2}}{2q_0}+O(q_0^{-2})+O(e^{-m_0q_0^2})\Big)\int_{B_T} x_3^{-\mu+1}|\partial_3\dot\varphi|^2 dl\\
\le & \Big(C(b_0,\mu)+O(q_0^{-1})+O(e^{-m_0q_0^2})\Big)\int_{\Gamma_T} x_3^{-\mu+1}|\xi(x_3)|^2 dl\\
&\quad + C_6\int_{b_0}^{\chi(1)} (|\dot\varphi(1,r)|^2+|\partial_3\dot\varphi(1,r)|^2) dr.
\end{aligned}
$$

It then follows from (5-2) and the induction argument that the crucial step for proving (5-1) is to estimate the first term on the right side of (5-4). We note that it has a constant factor $C(b_0,\mu)$ (not a small constant for large b_0). We will manage to show that this term can be absorbed into the left side of (5-4) by using Equation (3-6).

By the assumptions on $\xi(x_3)$, we have

$$
(5\text{-}5)\quad \int_{\Gamma_T} x_3^{-\mu+1}|\xi(x_3)|^2 dl = \Big(\sqrt{1+b_0^2}+O(\varepsilon)+O(e^{-m_0q_0^2})\Big)\int_1^T x_3^{-\mu+1}|\xi(x_3)|^2 dx_3.
$$

The term $\int_1^T x_3^{-\mu+1}|\xi(x_3)|^2dx_3$ can be treated as follows:

$$
\begin{aligned}
\int_1^T x_3^{-\mu+1}|\xi(x_3)|^2 dx_3 &= \int_1^T x_3^{-\mu-1}|x_3\xi(x_3)|^2 dx_3\\
&\le 2\int_1^T x_3^{-\mu-1}\Big|x_3\xi(x_3)+\frac{1}{U_+}\dot\varphi(x_3,\chi(x_3))\Big|^2 dx_3\\
&\quad + 2\int_1^T x_3^{-\mu-1}\Big|\frac{1}{U_+}\dot\varphi(x_3,\chi(x_3))\Big|^2 dx_3 \equiv \mathrm{I}+\mathrm{II}.
\end{aligned}
$$

Here and below, we use the inequality $(x+y)^2\le 2x^2+2y^2$ repeatedly.

For I, as in [Chen et al. 2002; Godin 1997] for $\mu>0$, we use the Hardy type inequality to obtain

$$
\int_1^T z^{-\mu-1}u^2(z)dz \le \frac{8}{\mu^2}\int_1^T z^{-\mu+1}|u'(z)|^2 dz + \frac{2}{\mu}u^2(1).
$$

With the boundary condition (3-6) and the assumptions in Theorem 5.1, we have

$$
\begin{aligned}
\text{(5-6)}\quad |\mathrm{I}| &\le \frac{16}{\mu^2}\int_1^T x_3^{-\mu+1}\Big|\partial_3\Big(x_3\xi(x_3)+\frac{1}{U_+}\dot\varphi(x_3,\chi(x_3))\Big)\Big|^2 dx_3 \\
&\qquad\qquad + \frac{2}{\mu}\big(\xi^2(1)+\dot\varphi^2(1,\chi(1))\big) \\
&\le C(b_0,\mu)\Big(\varepsilon^2\int_1^T x_3^{-\mu+1}(|\xi(x_3)|^2+|\nabla_{x_3,r}\dot\varphi(x_3,\chi(x_3))|^2)dx_3 \\
&\qquad\qquad + \xi^2(1)+\dot\varphi^2(1,\chi(1))\Big).
\end{aligned}
$$

Now we split II into $\mathrm{II}_1+\mathrm{II}_2$ so that we can estimate II by using the line integral on $r=b_0x_3$ and the double integral in the interior of D_T, where

$$
\begin{aligned}
\mathrm{II}_1 &= \frac{4}{U_+^2}\int_1^T x_3^{-\mu-1}|\dot\varphi(x_3,\chi(x_3))-\dot\varphi(x_3,b_0x_3)|^2 dx_3, \\
\mathrm{II}_2 &= \frac{4}{U_+^2}\int_1^T x_3^{-\mu-1}|\dot\varphi(x_3,b_0x_3)|^2 dx_3.
\end{aligned}
$$

II_1 can be treated as follows:

$$
\begin{aligned}
\text{(5-7)}\quad |\mathrm{II}_1| &= \frac{4(1+b_0^2)^2}{b_0^2q_0^2}(1+O(e^{-m_0q_0^2}))\int_1^T x_3^{-\mu-1}\Big(\int_{b_0x_3}^{\chi(x_3)}\partial_r\dot\varphi(x_3,r)dr\Big)^2 dx_3 \\
&\le \frac{4(1+b_0^2)^2}{b_0^2q_0^2}(1+O(e^{-m_0q_0^2}))\int_1^T x_3^{-\mu}\int_{b_0x_3}^{\chi(x_3)}|\partial_r\dot\varphi(x_3,r)|^2dr\Big|\frac{\chi(x_3)-b_0x_3}{x_3}\Big|dx_3 \\
&\le \big(O(\varepsilon)+O(e^{-m_0q_0^2})\big)\iint_{D_T} x_3^{-\mu}|\partial_r\dot\varphi(x_3,r)|^2 dr\,dx_3.
\end{aligned}
$$

Using again the Hardy type inequality and the boundary condition (3-2), we have

$$
\begin{aligned}
\text{(5-8)}\quad |\mathrm{II}_2| &\le \frac{C(b_0,\mu)}{\mu q_0^2}\dot\varphi^2(1,b_0)+\frac{32(1+b_0^2)^2}{\mu^2b_0^2q_0^2}(1+O(e^{-m_0q_0^2})) \\
&\qquad\times\int_1^T x_3^{-\mu+1}\big|b_0\partial_r\dot\varphi(x_3,b_0x_3)+\partial_3\dot\varphi(x_3,b_0x_3)\big|^2dx_3 \\
&= \frac{C(b_0,\mu)}{\mu q_0^2}\dot\varphi^2(1,b_0)+\frac{32(1+b_0^2)^4}{\mu^2b_0^2q_0^2}(1+O(e^{-m_0q_0^2}))\int_1^T x_3^{-\mu+1}|\partial_3\dot\varphi(x_3,b_0x_3)|^2dx_3.
\end{aligned}
$$

Substituting (5-6), (5-7), and (5-8) into (5-5) and (5-1), we have

$$
\begin{aligned}
\text{(5-9)}\quad & C_1T^{-\mu+1}\int_{b_0T}^{\chi(T)}|\nabla_{x_3,r}\dot\varphi(T,r)|^2+C_2\int\int_{D_T}x_3^{-\mu}|\nabla_{x_3,r}\dot\varphi|^2dr\,dx_3\\
&\quad+\Big(\frac{A(1+b_0^2)^4}{4q_0^2}+O(q_0^{-3})+O(e^{-m_0q_0^2})\Big)\int_{\Gamma_T}x_3^{-\mu+1}|\partial_3\dot\varphi|^2dl\\
&\quad+\Big(\frac{\sqrt{A}b_0(1+b_0^2)^{7/2}}{2q_0}+O(q_0^{-2})+O(e^{-m_0q_0^2})\Big)\int_{B_T}x_3^{-\mu+1}|\partial_3\dot\varphi|^2dl\\
\le{}&\Big(\frac{C(b_0,\mu)}{q_0^2}+O(q_0^{-3})+O(e^{-m_0q_0^2})+O(\varepsilon)\Big)\int_{B_T}x_3^{-\mu+1}|\partial_3\dot\varphi|^2dl\\
&\quad+C_6\int_{b_0}^{\chi(1)}(|\dot\varphi(1,r)|^2+|\partial_3\dot\varphi(1,r)|^2)dr.
\end{aligned}
$$

In this inequality, the main coefficients of $\int_{B_T}x_3^{-\mu+1}|\partial_3\dot\varphi|^2dl$ on the left and right side are respectively $\sqrt{A}b_0(1+b_0^2)^{7/2}/(2q_0)$ and $C(b_0,\mu)/q_0^2$. Then for large q_0, with fixed b_0 and μ, one has

$$
\text{(5-10)}\qquad \frac{\sqrt{A}b_0(1+b_0^2)^{7/2}}{2q_0}>\frac{C(b_0,\mu)}{q_0^2}.
$$

Thus Theorem 5.1 follows from (5-9) and (5-10). □

Proof of Theorem 1.1.

The local existence of the solution to (1-6)–(1-11) can be obtained as mentioned in Section 3, provided the initial data is smooth and satisfies the compatibility conditions. If we can show that the maximum norm of $\dot\varphi$, ξ, and their derivatives decays with a suitable rate in x_3, then the solution can be extended continuously to the whole domain.

It follows from the Sobolev imbedding theorem (see [Godin 1997, Lemma 14]) and the assumptions of Theorem 5.1 that for $b_0x_3\le r\le\chi(x_3)$ and $1\le x_3\le T$, we have

$$
\text{(5-11)}\qquad \sum_{0\le l\le k_0-2}|x_3^l\nabla_{x_3,r}^{l+1}\dot\varphi|^2\le Cx_3^{-1}\int_{b_0x_3}^{\chi(x_3)}\sum_{0\le l\le k_0-1}|x_3^l\nabla_{x_3,r}^{l+1}\dot\varphi(x_3,r)|^2dr.
$$

In addition, Theorem 5.1 implies for $\mu>0$ that

$$
\text{(5-12)}\qquad \int_{b_0x_3}^{\chi(x_3)}\sum_{0\le l\le k_0-1}|x_3^l\nabla_{x_3,r}^{l+1}\dot\varphi(x_3,r)|^2dr\le C\varepsilon^2x_3^{\mu-1}.
$$

Hence, combining (5-11) with (5-12) yields $\sum_{0\le l\le k_0-2}|x_3^l\nabla_{x_3,r}^{l+1}\dot\varphi|^2\le C\varepsilon^2x_3^{\mu-2}$ for $b_0x_3\le r\le\chi(x_3)$ and $1\le x_3\le T$. By using $k_0-2\ge[k_0/2]+1$, we derive

$\sum_{l\le[k_0/2]+1}|x_3^l\nabla_{x_3,r}^{l+1}\dot{\varphi}|\le C\varepsilon x_3^{\mu/2-1}$. This, together with the equations (3-5) and (3-6), yields $|\xi(x_3)|+|x_3\xi'(x_3)|\le C\varepsilon x_3^{\mu/2-1}$. Noting that the constant C is independent of T and choosing the constant μ with $0<\mu<2$, we obtain a global C^2 solution to the problem (1-6) and (1-7) with conditions (1-8)–(1-11). Since the initial boundary values are C^∞ in (1-8)–(1-11), the regularity of solution can be improved to C^∞, and the proof of Theorem 1.1 is completed. □

Appendix

Here we prove several results used earlier. For simplicity, we denote by C various positive constants which are independent of q_0 but may depend on b_0 and μ.

Lemma A.1. *Assume that $\tilde{\lambda}(s)$ is given in (4-14), then for large q_0, the coefficient of $a'(s)$ in (4-10) is positive. Specifically,*

$$1+\frac{(s-P_1)(\tilde{\lambda}(s)-P_1)}{D_1}=1+O(q_0^{-1})>0,\quad \textit{where } D_1=P_1^2-P_2.$$

Proof. It follows from Lemma 3.2, the expression for $\tilde{\lambda}(s)$, and a direct computation that for large q_0

$$\begin{aligned} D_1(s)&=\frac{A(1+b_0^2)^3}{q_0^2}+O(q_0^{-4})+O(e^{-m_0q_0^2})>0,\\ s-P_1(s)&=-\frac{Ab_0(1+b_0^2)^2}{q_0^2}+O(q_0^{-4})+O(e^{-m_0q_0^2})<0,\end{aligned}$$

and $\tilde{\lambda}(s)-P_1=O(q_0^{-1})$. The claim follows. □

Lemma A.2. *If q_0 is large and $\tilde{\lambda}(s)$ is given in (4-14), then for $b_0\le s\le s_0+\eta_0$ we have*

$$\begin{aligned} &s<\tilde{\lambda}(s)<\lambda_2(s),\\ &\tilde{\lambda}(s_0)-s_0=\frac{\sqrt{A}(1+b_0^2)^{3/2}}{q_0^3}+O(q_0^{-4})+O(e^{-m_0q_0^2}),\\ &\tilde{\lambda}(b_0)-b_0=\frac{\sqrt{A}(1+b_0^2)^{3/2}}{q_0}+O(q_0^{-2})+O(e^{-m_0q_0^2}),\\ &\tilde{\lambda}'(s)=-\frac{1}{s_0+\eta_0-b_0}\left(\frac{\sqrt{A}(1+b_0^2)^{3/2}}{q_0}+O(q_0^{-2})+O(e^{-m_0q_0^2})\right)\le-\frac{C}{q_0}e^{m_0q_0^2}.\end{aligned}$$

Proof. The claims follow directly from Lemmas 2.3–2.6. □

Lemma A.3. *If q_0 is large, then*

$$\tilde{A} = \frac{A(1+b_0^2)^3}{q_0^2} + O(q_0^{-4}) + O(e^{-m_0 q_0^2}),$$

$$Q_0 = a(s)\Big(\frac{\mu}{2}b_0 + \frac{2Ab_0^3(1+b_0^2)}{q_0^2} + O(q_0^{-4}) + O(e^{-m_0 q_0^2})\Big),$$

$$\begin{aligned} Q_1 = a(s)\Big(&\frac{2-\mu}{2}b_0^3 + \frac{Ab_0(1+b_0^2)\big(\mu-(\mu-4)b_0^4\big)}{2q_0^2} + O(q_0^{-4}) + O(e^{-m_0 q_0^2})\Big) \\ &+ b(s)\Big((\mu-1)b_0 - \frac{A(1+b_0^2)^2(1-(\mu-2)b_0^2)}{b_0 q_0^2} + O(q_0^{-4}) + O(e^{-m_0 q_0^2})\Big), \end{aligned}$$

$$\begin{aligned} Q_2 = a(s)\Big(&b_0^2 + \frac{Ab_0^2(-1+2b_0^2+3b_0^4)}{q_0^2} + O(q_0^{-4}) + O(e^{-m_0 q_0^2})\Big) \\ &+ b(s)\Big(\mu - 1 + \frac{1}{q_0^2}O(e^{-m_0 q_0^2})\Big), \end{aligned}$$

Also, modulo additive terms of $O(q_0^{-4}) + O(e^{-m_0 q_0^2})$,

$$a_1(s) = \frac{Ab_0(1+b_0^2)^2\big((\mu-2)b(s) + b_0^2\big(a(s)+(2\mu-3)b(s)\big) - (\mu-2)b_0^4 a(s)\big)}{q_0^2},$$

$$a_2(s) = \frac{A(1+b_0^2)^2\big(b(s) - b_0^2\big(a(s)\mu + (\mu-2)b(s)\big) - b_0^4 a(s)\big)}{b_0 q_0^2}.$$

Proof. Using the expressions for $\tilde{A}$, Q_i for $i=0,1,2$, and $a_j(s)$ for $j=1,2$, a tedious but direct computation verifies the claims; here we also omit the details. ☐

Lemma A.4. *If we choose $\tilde{\lambda}(s)$ as in (4-14), then for fixed $\mu \in \mathbb{R}$*

$$\frac{4D_1^2 Y_3^2(s)}{\tilde{A}^2 a^2(s)} + 1 < \Big(\tilde{\lambda}(s) + s\tilde{\lambda}'(s) + s(\tilde{\lambda}(s) - P_1)\tilde{Q}_0(s) - \frac{a_2(s)}{a(s)\tilde{A}}\Big)^2. \tag{A-1}$$

Proof. By Lemmas 3.2 and A.2–A.3, we have the estimates

$$P_1 - s \sim \frac{C}{q_0^2}, \qquad \tilde{A} \sim \frac{C}{q_0^2}, \qquad \Big|\frac{a_i(s)}{a(s)}\Big| \sim \frac{C}{q_0^2} \text{ for } i = 1, 2,$$
$$D_1 + (s-P_1)(\tilde{\lambda}(s) - P_1) \sim \frac{C}{q_0^2}, \qquad \tilde{\lambda}'(s) \le -\frac{C}{q_0}e^{m_0 q_0^2}.$$

Since

$$\tilde{Q}_0(s) = \frac{(P_1-s)\tilde{\lambda}'(s) + \frac{P_1-s}{s}\tilde{\lambda}(s) - \frac{(P_1-s)a_2(s)}{a(s)\tilde{A}s} - \frac{a_1(s)+a_2(s)P_1 s}{a(s)\tilde{A}s^2}}{D_1 + (s-P_1)(\tilde{\lambda}(s) - P_1)},$$

we know that the coefficient of $(\tilde{\lambda}'(s))^2$ in the right-hand side of (A.2) is

$$\Big(\frac{sD_1}{D_1+(s-P_1)(\tilde{\lambda}(s)-P_1)}\Big)^2 \sim b_0^2.$$

This means for large q_0 the right side of (A-1) is not less than $(C/q_0^2)e^{2m_0q_0^2}$.

In addition,

$$\Big|\frac{Y_3}{a(s)}\Big| = \Big|\frac{b(s)}{a(s)}\Big(\frac{P_1's-\frac{1}{2}P_2'+\frac{\tilde{P}_4}{s}-\frac{P_1\tilde{P}_3}{s}}{2D_1}\Big) + \Big(\frac{P_1P_2's-P_2P_1's+P_2\tilde{P}_3-P_1\tilde{P}_4-\frac{1}{2}P_2's^2}{2D_1}\Big)\Big| = \Big|\frac{b(s)}{2b_0a(s)}+O(e^{-m_0q_0^2})\Big| \le C.$$

So the left side of (A-1) is less than C. Thus (A-1) holds for large q_0. □

Lemma A.5. *For large q_0, $A_0(s) \sim -C/q_0^2$.*

Proof. As in [Chen et al. 2002], we factorize $A_0(s) = A_0^1(s)A_0^2(s)$, where

$$A_0^1(s) = \frac{s(\tilde{\lambda}(s)-P_1)D_1}{D_1+(s-P_1)(\tilde{\lambda}(s)-P_1)} + \frac{D_1}{\tilde{A}}s(s-P_1) + \frac{D_1}{\tilde{A}}s\sqrt{D_1},$$
$$A_0^2(s) = \frac{s(\tilde{\lambda}(s)-P_1)D_1}{D_1+(s-P_1)(\tilde{\lambda}(s)-P_1)} + \frac{D_1}{\tilde{A}}s(s-P_1) - \frac{D_1}{\tilde{A}}s\sqrt{D_1}.$$

Since $\tilde{A} = (\lambda_2(s)-s)(s-\lambda_1(s))$ and $\lambda_1(s) = P_1-\sqrt{D_1}$, we find

$$A_0^1(s) = \frac{sD_1\big((\tilde{\lambda}(s)-P_1)(P_1-s+\sqrt{D_1})+(s-P_1)(\tilde{\lambda}(s)-P_1)+D_1\big)}{(\lambda_2(s)-s)(D_1+(s-P_1)(\tilde{\lambda}(s)-P_1))}$$
$$= \frac{sD_1^{3/2}(\tilde{\lambda}(s)-\lambda_1(s))}{(\lambda_2(s)-s)\big(D_1+(s-P_1)(\tilde{\lambda}(s)-P_1)\big)}.$$

By Lemma 3.2 and Lemma A.1-Lemma A.3, we arrive at

$$D_1^{3/2} \sim \frac{C}{q_0^3}, \qquad \lambda_2(s)-s \sim \frac{C}{q_0}, \qquad \tilde{\lambda}(s)-\lambda_1(s) \sim \frac{C}{q_0}.$$

This yields $A_0^1(s) \sim C/q_0$.

In addition,

$$A_0^2(s) = \frac{sD_1^{3/2}(\tilde{\lambda}(s)-\lambda_2(s))}{(\lambda_2(s)-s)(D_1+(s-P_1)(\tilde{\lambda}(s)-P_1))},$$

then by a similar computation, one has $A_0^2(s) \sim -C/q_0$. □

Lemma A.6. *If q_0 is large, we have $|A_1(s)| \le (C/q_0^2)\exp(m_0q_0^2)$.*

Proof. Set $A_1(s) = 2sD_1A_1^1(s)A_1^2(s)$ with

$$A_1^1(s) = \frac{\tilde{\lambda}(s) - P_1}{D_1 + (s - P_1)(\tilde{\lambda}(s) - P_1)} + \frac{s - P_1}{\tilde{A}},$$
$$A_1^2(s) = \tilde{\lambda}(s) + s\tilde{\lambda}'(s) - \frac{a_2(s)}{a(s)\tilde{A}} + s(\tilde{\lambda}(s) - P_1)\tilde{Q}_0(s).$$

It follows from Lemmas 3.2, 2.6 and A.1–A.3 that $\tilde{\lambda}(s) - P_1 = \tilde{\lambda}(s) - s + s - P_1 = O({q_0}^{-1})$ and $|A_1^1(s)| \le Cq_0$.

We now treat the term $A_1^2(s)$. By Lemmas 3.2 and A.1–A.3, we directly observe that $sD_1\tilde{\lambda}'(s)/(D_1 + (s - P_1)(\tilde{\lambda}(s) - P_1))$ is the main term of $A_1^2(s)$. Moreover, it follows from the fourth relation of Lemma A.2 that $\tilde{\lambda}'(s) \le -(C/q_0)e^{m_0q_0^2}$. This means that for large q_0 $|A_1^2(s)| \le (C/q_0)e^{m_0q_0^2}$.

The claim now follows from Lemma A.1 and the expression for $A_1(s)$. □

Lemma A.7. *For large q_0, we have $1 - C/q_0 \le a(s_0) \le C$.*

Proof. After noting

$$a(s_0) = \exp\left\{\int_{b_0}^{s_0}\left(\tilde{Q}_0(s) + \frac{D_1}{D_1 + (s - P_1)(\tilde{\lambda}(s) - P_1)}k(s)\right)ds\right\},$$

a direct computation yields

$$\tilde{Q}_0(s) = \frac{1}{\frac{A(1+b_0^2)^3}{q_0^2} + O(q_0^{-3})}$$
$$\times\left\{-\frac{1}{s_0 + \eta_0 - b_0}\left(\frac{A^{3/2}b_0(1+b_0^2)^{7/2}}{q_0^3} + O(q_0^{-4})\right) + O(q_0^{-2}) + O(1)\right\},$$

then, together with (i) of Lemma 2.3, we have

$$\int_{b_0}^{s_0}\tilde{Q}_0(s)ds = O(q_0^{-1}).$$

In addition, by Lemmas A.3, A.5, and A.6, and from $A_2(s) = -1$, we have

$$0 < k(s) = \frac{2}{\sqrt{A_1^2(s) - 4A_0(s)} - A_1(s)} \le Ce^{m_0q_0^2}.$$

Since

$$\frac{D_1}{D_1 + (s - P_1)(\tilde{\lambda}(s) - P_1)} = 1 + O(q_0^{-1}) + O(e^{-m_0q_0^2}),$$

the lemma follows from

$$\int_{b_0}^{s_0}\frac{D_1}{D_1 + (s - P_1)(\tilde{\lambda}(s) - P_1)}k(s)ds \le C.$$ □

Lemma A.8.

$$
\begin{aligned}
\beta_0-\mu_1\beta_1+\mu_1^2\beta_2&=\frac{A(1+b_0^2)^4}{4b_0q_0^2}+O(q_0^{-3})+O(e^{-m_0q_0^2}),\\
\beta_1-2\mu_1\beta_2&=-\frac{\sqrt{A}b_0(1+b_0^2)^{5/2}}{2q_0^3}+O(q_0^{-4})+O(e^{-m_0q_0^2}),\\
\beta_2&=-\frac{Ab_0(1+b_0^2)^2}{q_0^2}+O(q_0^{-3})+O(e^{-m_0q_0^2}).
\end{aligned}
$$

Proof. We note that $\mu_1=((1-b_0^2)/(2b_0))(1+O(e^{-m_0q_0^2}))$. Then by Lemma 3.2, Lemma A.2, and a direct computation, we have

$$
\begin{aligned}
\beta_0-\mu_1\beta_1+\mu_1^2\beta_2&=\Big(P_1(s_0)-s_0-\mu_1(P_2(s_0)-s_0^2)+\mu_1^2(s_0P_2(s_0)-s_0^2P_1(s_0))\Big)\\
&\qquad+\Big(s_0-\tilde{\lambda}(s_0)\Big)\Big(\tfrac{1}{2}-\mu_1s_0-\mu_1^2\big(\tfrac{1}{2}P_2(s_0)-s_0P_1(s_0)\big)\Big)\\
&=\Big(\frac{A(1+b_0^2)^4}{4b_0q_0^2}+O(q_0^{-4})\Big)+\Big(\frac{(1+b_0^2)^2}{8}+O(q_0^{-2})\Big)O(q_0^{-3})+O(e^{-m_0q_0^2})\\
&=\frac{A(1+b_0^2)^4}{4b_0q_0^2}+O(q_0^{-3})+O(e^{-m_0q_0^2}),
\end{aligned}
$$

and

$$
\begin{aligned}
\beta_2&=s_0P_2(s_0)-s_0^2P_1(s_0)-\big(1/2P_2(s_0)-s_0P_1(s_0)\big)\big(s_0-\tilde{\lambda}(s_0)\big)\\
&=\Big(-\frac{Ab_0(1+b_0^2)^2}{q_0^2}+O(q_0^{-4})\Big)+\Big(\frac{b_0^2}{2}+O(q_0^{-2})\Big)O(q_0^{-3})+O(e^{-m_0q_0^2})\\
&=-\frac{Ab_0(1+b_0^2)^2}{q_0^2}+O(q_0^{-3})+O(e^{-m_0q_0^2}).
\end{aligned}
$$

Also, it follows from Lemmas 2.3 and 3.2 that

$$
\begin{aligned}
\beta_1-2\mu_1\beta_2&=P_2(s_0)-s_0^2-2\mu_1\big(s_0P_2(s_0)-s_0^2P_1(s_0)\big)\\
&\qquad+\big(s_0+2\mu_1(\tfrac{1}{2}P_2(s_0)-s_0P_1(s_0))\big)\big(s_0-\tilde{\lambda}(s_0)\big)\\
&=O(e^{-m_0q_0^2})+\Big(\frac{b_0(1+b_0^2)}{2}+O(q_0^{-2})\Big)\Big(-\frac{\sqrt{A}(1+b_0^2)^{3/2}}{q_0^3}+O(q_0^{-4})\Big)\\
&=-\frac{\sqrt{A}b_0(1+b_0^2)^{5/2}}{2q_0^3}+O(q_0^{-4})+O(e^{-m_0q_0^2}),
\end{aligned}
$$

where we have used second relation from Lemma A.2 to derive the order of q_0^{-3} in $\beta_1-2\mu_1\beta_2$. This yields the constants C_3 and C_5 in (4-1). □

References

[Bertin 1994] J. J. Bertin, *Hypersonic aerothermodynamics*, American Institute of Aeronautics and Astronautics, Washington, DC, 1994.

[Chen et al. 2002] S. Chen, Z. Xin, and H. Yin, "Global shock waves for the supersonic flow past a perturbed cone", *Comm. Math. Phys.* **228**:1 (2002), 47–84. MR 2003c:76076 Zbl 1006.76080

[Chen et al. 2003] S. Chen, Z. Geng, and D. Li, "The existence and stability of conic shock waves", *J. Math. Anal. Appl.* **277**:2 (2003), 512–532. MR 2004b:76124 Zbl 1036.76028

[Courant and Friedrichs 1948] R. Courant and K. O. Friedrichs, *Supersonic Flow and Shock Waves*, Interscience Publishers, New York, N. Y., 1948. MR 10,637c Zbl 0041.11302

[Cox and Crabtree 1965] R. N. Cox and L. F. Crabtree, *Elements of hypersonic aerodynamics*, Academic Press, New York, 1965.

[Cui and Yin 2006] D. Cui and H. Yin, *Global conic shock wave for the steady supersonic flow past a cone: polytropic case*, Preprint, Institute of Mathematical Science at Nanjing University, 2006.

[Godin 1997] P. Godin, "Global shock waves in some domains for the isentropic irrotational potential flow equations", *Comm. Partial Differential Equations* **22** (1997), 1929–1997. MR 99i:35097 Zbl 0956.35086

[Godin 2005] P. Godin, "The lifespan of a class of smooth spherically symmetric solutions of the compressible Euler equations with variable entropy in three space dimensions", *Arch. Ration. Mech. Anal.* **177**:3 (2005), 479–511. MR 2006i:76092 Zbl 1075.76052

[Keyfitz and Warnecke 1991] B. L. Keyfitz and G. G. Warnecke, "The existence of viscous profiles and admissibility for transonic shocks", *Comm. Partial Differential Equations* **16**:6-7 (1991), 1197–1221. MR 92i:76063 Zbl 0741.35037

[Klainerman and Sideris 1996] S. Klainerman and T. C. Sideris, "On almost global existence for nonrelativistic wave equations in 3D", *Comm. Pure Appl. Math.* **49**:3 (1996), 307–321. MR 96m:35231 Zbl 0867.35064

[Li and Yu 1985] T. T. Li and W. C. Yu, *Boundary value problems for quasilinear hyperbolic systems*, Duke University Mathematics Series, V, Duke University Mathematics Department, Durham, NC, 1985. MR 88g:35115 Zbl 0627.35001

[Lien and Liu 1999] W.-C. Lien and T.-P. Liu, "Nonlinear stability of a self-similar 3-dimensional gas flow", *Comm. Math. Phys.* **204**:3 (1999), 525–549. MR 2000f:76106 Zbl 0945.76033

[Majda 1984] A. Majda, *Compressible fluid flow and systems of conservation laws in several space variables*, Applied Mathematical Sciences **53**, Springer, New York, 1984. MR 85e:35077 Zbl 0867.35064

[Majda 1991] A. J. Majda, "One perspective on open problems in multi-dimensional conservation laws", pp. 217–238 in *Multidimensional hyperbolic problems and computations* (Minneapolis, MN, 1989), edited by J. Glimm and A. J. Majda, IMA Vol. Math. Appl. **29**, Springer, New York, 1991. MR 91m:35142 Zbl 0769.35038

[Majda and Thomann 1987] A. Majda and E. Thomann, "Multidimensional shock fronts for second order wave equations", *Comm. Partial Differential Equations* **12**:7 (1987), 777–828. MR 88k:35130 Zbl 0632.35047

[Sideris 1985] T. C. Sideris, "Formation of singularities in three-dimensional compressible fluids", *Comm. Math. Phys.* **101**:4 (1985), 475–485. MR 87d:35127 Zbl 0606.76088

[Smoller 1983] J. Smoller, *Shock waves and reaction-diffusion equations*, vol. 258, Grundlehren der Mathematischen Wissenschaften, Springer, New York, 1983. MR 84d:35002 Zbl 0508.35002

[Tsien 1946] H.-s. Tsien, "Similarity laws of hypersonic flows", *J. Math. Phys. Mass. Inst. Tech.* **25** (1946), 247–251. MR 8,237d Zbl 0063.07867

[Xin and Yin 2006] Z. Xin and H. Yin, "Global multidimensional shock wave for the steady supersonic flow past a three-dimensional curved cone", *Anal. Appl.* (*Singap.*) **4**:2 (2006), 101–132. MR 2007e:35193 Zbl 05022342

[Yin 2002] H. Yin, "Long shock for supersonic flow past a curved cone", pp. 207–215 in *Geometry and nonlinear partial differential equations* (Hangzhou, 2001), edited by S. Chen and S.-T. Yau, AMS/IP Stud. Adv. Math. **29**, Amer. Math. Soc., Providence, RI, 2002. MR 2003m:35163 Zbl 1008.35056

[Yin 2006] H. C. Yin, "Global existence of a shock for the supersonic flow past a curved wedge", *Acta Math. Sin.* (*Engl. Ser.*) **22**:5 (2006), 1425–1432. MR 2007g:35151

[Zheng 2001] Y. Zheng, *Systems of conservation laws*, Progress in Nonlinear Differential Equations and their Applications, 38, Birkhäuser, Boston, 2001. Two-dimensional Riemann problems. MR 2002e:35155 Zbl 0971.35002

Received January 19, 2007. Revised May 27, 2007.

DACHENG CUI
DEPARTMENT OF MATHEMATICS
NANJING UNIVERSITY
NANJING, 210093
CHINA

dacheng@sina.com

HUICHENG YIN
DEPARTMENT OF MATHEMATICS
NANJING UNIVERSITY
NANJING, 210093
CHINA

huicheng@nju.edu.cn