Pacific Journal of Mathematics

FOCAL SETS IN TWO-DIMENSIONAL SPACE FORMS

Carlos Arturo Escudero, Agustí Reventós and Gil Solanes

Volume 233 No. 2 December 2007

PACIFIC JOURNAL OF MATHEMATICS
Vol. 233, No. 2, 2007

FOCAL SETS IN TWO-DIMENSIONAL SPACE FORMS

CARLOS ARTURO ESCUDERO, AGUSTÍ REVENTÓS AND GIL SOLANES

We relate the area of a convex set in a 2-dimensional space of constant curvature with some integrals over the curvature radius at its boundary.

1. Introduction

Let $M = \partial K$ be the boundary of a compact convex domain K in $\mathbb{R}^2$ of area F. Then we have the inequality

$$\int_M \frac{1}{k(s)} ds \geqslant 2F, \tag{1}$$

where ds is the arclength measure on M and $k = k(s) > 0$ is the curvature of M at the point of parameter s. Equality holds if and only if M is a circle. See for instance [Escudero and Rodríguez 1996] or [Zhou 2007].

Formula (1) is the 2-dimensional analogue of Heintze and Karcher's inequality:

$$\int_S \frac{1}{H} dA \geqslant 3V,$$

where H is the mean curvature of a compact embedded surface S in $\mathbb{R}^3$ bounding a domain of volume V. The inequality assumes $H > 0$, and equality holds if and only if S is a standard sphere; see [Ros 1988; Osserman 1990].

Escudero and Reventós [2007] improved equality (1), showing

$$\int_M \frac{1}{k(s)} ds = 2(F - F_e),$$

where $F_e \leq 0$ is the (algebraic) area of the domain bounded by the evolute of M.

Equivalently,

$$\int_M \frac{\rho(s)}{2} ds = F - F_e, \tag{2}$$

where $\rho(s) = 1/k(s)$ is the curvature radius of M at the point of parameter s.

MSC2000: primary 53C65, 53A04; secondary 52A10, 52A15, 52A55.

Keywords: curvature, focal sets, space forms, convex.

Work partially supported by DGICYT grant #BFM2003-03458 and Universidad Tecnológica de Pereira, project 3-05-2.

In this paper we generalize this equality to X_c^2, the 2-dimensional complete and simply connected riemannian manifold of constant curvature c, that is, for $c > 0$, the sphere $\mathbb{S}_c^2$ of radius $R = 1/\sqrt{c}$ for $c > 0$ or, for $c < 0$, the hyperbolic plane $\mathbb{H}_c^2$ (the sphere of imaginary radius $R = -i/\sqrt{c}$). We assume X_c^2 is oriented.

Using the same techniques as in [Gallego et al. 2005], we obtain a result that coincides, for $c = 0$, with formula (2). First, define

Assumption 1.1. Let K be a set in X_c^2 with smooth regular boundary M. Assume K is strongly convex if $c \geq 0$. If $c < 0$, assume it is strongly h-convex.

Theorem 1.2. *Under Assumption 1.1,*

$$\int_M \tan_c\left(\frac{\rho(s)}{2}\right) ds = F - F_e,$$

where ds is the arclength measure on M, F is the area of K, and F_e is the (*algebraic*) *area enclosed by the focal set $F(M)$ of M.*

The convexity notions used above as well as the generalized tangent function $\tan_c$ will be defined next.

2. Preliminaries

Definition 2.1. A domain $K \subset X_c^2$ is *regular* if its boundary M admits a *regular parametrization*. That is, there is an injective smooth map $\gamma : S^1(L) \to M$ such that $|\gamma'(s)| = 1$, where L is a constant, $S^1(L)$ is the euclidean circle of radius $L/2\pi$, and s is its arclength parameter.

Note that L is the perimeter of K. By choosing a regular parametrization γ, we make s the arclength parameter for M as well.

Definition 2.2. A regular domain $K \subset X_c^2$ is *convex* if the curvature at every point of $M = \partial K$ is nonnegative; if the curvature on M is always positive, K is *strongly convex*.

The sign of the curvature can be defined using the intrinsic covariant derivative ∇ of X_c^2 by the condition

$$\nabla_T T = k N,$$

where N is the inward normal vector field and T is a unit tangent vector.

Note that, if $c > 0$, then K lies in some half sphere of $\mathbb{S}_c^2$. If $c < 0$, we need a stronger convexity notion.

Definition 2.3. A regular domain $K \subset \mathbb{H}_c^2$ with smooth boundary M is said to be *h-convex* if the curvature at every point of M is greater than or equal to $\sqrt{-c}$. If the same curvature is always greater than $\sqrt{-c}$, the domain is *strongly h-convex*.

The hyperbolic disc is strongly h-convex because the curvature k of the boundary of a disc of radius r in $\mathbb{H}^2_c$ is given by

$$k = \sqrt{-c}\,\coth(\sqrt{-c}\,r),$$

and $\coth(t) \geq 1$ for all $t \in \mathbb{R}$.

The notion of convexity we give here is equivalent to the usual one of geodesic convexity. The h-convex sets are also called horocyclically convex sets, because in this case the arcs of horocycles joining points in K are contained in K.

To deal simultaneously with the euclidean plane, the sphere, and the hyperbolic plane, we use the functions

$$\operatorname{sn}_c(t) := \begin{cases} \dfrac{1}{\sqrt{-c}}\sinh(\sqrt{-c}\,t) & \text{for } c < 0, \\ t & \text{for } c = 0, \\ \dfrac{1}{\sqrt{c}}\sin(\sqrt{c}\,t) & \text{for } c > 0, \end{cases}$$

and

$$\operatorname{cn}_c(t) := \begin{cases} \cosh(\sqrt{-c}\,t) & \text{for } c < 0, \\ 1 & \text{for } c = 0, \\ \cos(\sqrt{c}\,t) & \text{for } c > 0. \end{cases}$$

Note the identities

$$c\,\operatorname{sn}_c^2(t) + \operatorname{cn}_c^2(t) = 1, \qquad \begin{aligned} \operatorname{cn}_c'(t) &= -c\,\operatorname{sn}_c(t), & \operatorname{cn}_c(2t) &= \operatorname{cn}_c^2(t) - c\,\operatorname{sn}_c^2(t), \\ \operatorname{sn}_c'(t) &= \operatorname{cn}_c(t), & \operatorname{sn}_c(2t) &= 2\operatorname{sn}_c(t)\operatorname{cn}_c(t). \end{aligned}$$

We shall use that the area and the perimeter of a disc in X^2_c of radius t are given respectively by

$$A(t) = \frac{2\pi}{c}(1 - \operatorname{cn}_c(t)) \quad \text{and} \quad L(t) = 2\pi\,\operatorname{sn}_c(t).$$

Definition 2.4. Let M be the boundary of a convex domain $K \subset X^2_c$ (make it h-convex if $c < 0$). For each point $x \in M$ we denote by $\rho(x)$ the curvature radius of M at x and define it through

$$k(x) = \cot_c \rho(x),$$

where $k(x)$ is the curvature of M at x.

Since $\coth(t) \geq 1$ for all $t \in \mathbb{R}$, the curvature radius when $c < 0$ is only defined if $k(x) \geq \sqrt{-c}$, that is, if K is h-convex.

Definition 2.5. Let M be the boundary of a convex domain $K \subset X^2_c$ (make it h-convex if $c < 0$). The *focal set* $F(M)$ of M is the set

$$F(M) = \{\exp_x(\rho(x)N(x));\, x \in M\} \subset X^2_c,$$

where $N(x)$ is the inward unit normal vector to M at $x \in M$.

Recall that $y = \exp_x(tv)$ with $|v| = 1$ means $y = \sigma(t)$ where $\sigma(s)$ is the unique geodesic such that $\sigma(0) = x$ and $\sigma'(0) = v$.

The focal set of M is also called the *evolute* of M. Note that $F(M)$ is locally smooth and that the normal geodesics to M are tangent to $F(M)$.

We will see that $F(M)$ is the set of critical values of $\phi(x,t) = \exp_x(tN(x))$ for $x \in M$ and $t \in \mathbb{R}$.

Definition 2.6. The *winding number* $\mathrm{wind}(\gamma, y)$ of a curve $\gamma : S^1(L) \to X_c^2$ with respect to a point $y \in X_c^2 \setminus \gamma(S^1(L))$ is the mapping degree of the map $\varphi : S^1(L) \to T_y X_c^2$ defined by the condition $\|\varphi(s)\| = 1$ and $\exp_y \lambda(s)\varphi(s) = \gamma(s)$ for some function $\lambda = \lambda(s) > 0$.

That is, to each point $\gamma(s)$ we associate the unit tangent vector at y that is tangent to the unique geodesic joining y and $\gamma(s)$. We say that φ is *the winding map* with respect to y associated to γ. Note that φ may be thought of as a map of $S^1(L)$ into S^1.

It can be seen that $\mathrm{wind}(\gamma, y)$ is equal to the algebraic intersection number of $\gamma(S^1(L))$ with an arbitrary geodesic ray emanating from y; see [Guillemin and Pollack 1974],

By moving y along an arc that does not meet $\gamma(S^1(L))$, we do not change the winding number. Hence, the winding number of γ with respect to y is constant when y stays in a connected component of $X_c^2 \setminus \gamma(S^1(L))$. See [do Carmo 1976, p. 392].

Definition 2.7. Let M be the boundary of a convex domain $K \subset X_c^2$ (make it h-convex if $c < 0$) and let $y \notin M$. We define

$$\mathrm{wind}(M, y) = \mathrm{wind}(\gamma, y),$$

where γ is a regular parametrization of M such that the basis $\{\gamma', N\}$ is positive.

We define the winding number of the focal set $F(M)$ by

$$\mathrm{wind}(F(M), y) = \mathrm{wind}(\tilde{\gamma}, y),$$

where $\tilde{\gamma}(s) = \exp_{\gamma(s)}(\rho(s)N(s))$ is the parametrization of $F(M)$ induced by the parametrization γ of M.

Once we fix the parametrization γ, we shall write $\rho(s)$ and $N(s)$ instead of $\rho(\gamma(s))$ and $N(\gamma(s))$.

The algebraic area of $F(M)$ is the area enclosed by $F(M)$, counted with sign and multiplicity. To be precise, we define the area F_e enclosed by $F(M)$ as

$$F_e = \int_{X_c^2} \mathrm{wind}(F(M), y)\, dy.$$

Remark 2.8. Let γ be a regular parametrization of the boundary M of a regular domain, and let φ be the winding map associated to γ with respect to $y \notin M$. Let $\psi = \varphi \circ \gamma^{-1}$. Since $\deg \psi = \deg \varphi$, and because the degree theorem gives

$$\int_M \psi^* \, dO_1 = \deg \psi \int_{S^1} dO_1,$$

where dO_1 is the arclength measure of S^1, we have

$$\operatorname{wind}(M, y) = \frac{1}{2\pi} \int_M \psi^* \, dO_1.$$

3. An integral involving the curvature radius

Let M be the boundary of a regular domain $K \subset X_c^2$. Consider the set

$$M_\rho = \cup_{x \in M} \left(\{x\} \times [0, \rho(x)]\right) \subset M \times \mathbb{R},$$

and the map $\phi : M_\rho \to X_c^2$ defined by $\phi(x, t) = \exp_x(tN(x))$. We say that ϕ is the *focal map* of M. Note that ϕ is a (possibly) noninjective local diffeomorphism in the interior of M_ρ.

Lemma 3.1. *Let M be the boundary of a regular domain $K \subset X_c^2$, and let $\phi : M_\rho \to X_c^2$ be the focal map. Then*

$$\phi^* dy = (cn_c(t) - k(s)\, sn_c(t))\, ds \wedge dt,$$

where dy is the area element of X_c^2, s is the arclength on M, and $k(s)$ is the curvature of M at $\gamma(s)$.

Proof. Recall that,

$$X_c^2 = \begin{cases} S^2\Big(\dfrac{1}{\sqrt{c}}\Big) = \big\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = \dfrac{1}{c}\big\} & \text{if } c > 0, \\ \mathbb{H}^2\Big(\dfrac{1}{\sqrt{c}}\Big) = \big\{(x, y, z) \in \mathbb{R}^{(2,1)} : x^2 + y^2 - z^2 = \dfrac{1}{c}, z > 0\big\} & \text{if } c < 0, \end{cases}$$

where $\mathbb{R}^{(2,1)}$ is the Lorentz–Minkowski space.

Using these models, the focal map $\phi : M_\rho \to X_c^2$ is given in coordinates by

$$\phi(s, t) = \mathrm{cn}_c(t)\gamma(s) + \mathrm{sn}_c(t)N(s) \qquad \text{for all } c \in \mathbb{R},$$

where $\gamma : S^1(L) \to X_c^2$ is a regular parametrization of M; see [Ratcliffe 1994].

On the other hand, since dy is a 2-form in X_c^2, there is a function $p = p(s, t)$ such that $\phi^* dy = p(s, t) ds \wedge dt$.

Let us compute $p(s, t)$. Recall

$$\nabla_{\gamma'}\gamma' = kN \quad \text{and} \quad \nabla_{\gamma'}N = -k\gamma',$$

where ∇ is the intrinsic covariant derivative of X_c^2 and N is the inward normal vector field.

We have

$$\begin{aligned} p(s,t) &= \phi^* dy\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right) = dy\left(\phi_*\left(\frac{\partial}{\partial s}\right)\phi_*\left(\frac{\partial}{\partial t}\right)\right) \\ &= dy\left((\mathrm{cn}_c(t) - k(s)\,\mathrm{sn}_c(t))\gamma'(s), -c\ \mathrm{sn}_c(t)\gamma(s) + \mathrm{cn}_c(t)N(s)\right). \end{aligned}$$

Let η be the volume element of $\mathbb{R}^3$ if $c > 0$ or of $\mathbb{R}^{2,1}$ if $c < 0$. Then dy is the contraction of η with the normal vector field to $S^2(1/\sqrt{c})$ or $\mathbb{H}^2(1/\sqrt{c})$, respectively. In both cases, the outward normal to X_c^2 at the point $\phi(s,t)$ is the vector $\phi(s,t)$. Hence $dy_{\phi(s,t)} = \sqrt{c}\, i_{\phi(s,t)}\eta$. Note that $\eta(\gamma, \gamma', N) = 1/\sqrt{c}$. Hence

$$\begin{aligned} p(s,t) &= \sqrt{c}\,\eta(\mathrm{cn}_c(t)\gamma(s) + \mathrm{sn}_c(t)N(s), (\mathrm{cn}_c(t) - k(s)\,\mathrm{sn}_c(t))\gamma'(s), \\ &\qquad -c\ \mathrm{sn}_c(t)\gamma(s) + \mathrm{cn}_c(t)N(s)) \\ &= \sqrt{c}\,\eta\left(\mathrm{cn}_c(t)\gamma(s), (\mathrm{cn}_c(t) - k(s)\,\mathrm{sn}_c(t))\gamma'(s), \mathrm{cn}_c(t)N(s)\right) + \\ &\qquad \sqrt{c}\,\eta\left(\mathrm{sn}_c(t)N(s), (\mathrm{cn}_c(t) - k(s)\,\mathrm{sn}_c(t))\gamma'(s), -c\ \mathrm{sn}_c(t)\gamma(s)\right) \\ &= \sqrt{c}\,(\mathrm{cn}_c(t) - k(s)\,\mathrm{sn}_c(t))(\mathrm{cn}_c^2(t) + c\,\mathrm{sn}_c^2(t))\eta(\gamma(s), \gamma'(s), N(s)) \\ &= (\mathrm{cn}_c(t) - k(s)\,\mathrm{sn}_c(t)). \end{aligned}$$

Finally, if $c = 0$, we have $p(s,t) = dy((1 - tk(s))\gamma', N) = (1 - tk(s))$. $\square$

Remark 3.2. Observe that $p(s,t) = cn_c(t) - k(s)\,sn_c(t) \geq 0$ if and only if $\cot_c \rho(s) = k(s) \leq \cot_c(t)$, that is, if and only if $t \leq \rho(s)$. This is the situation in the hypothesis of Lemma 3.1.

Definition 3.3. Let K be a convex set in X_c^2 with smooth regular boundary M, and let $y \in X_c^2$. Let $h_y : M \to \mathbb{R}$ be the distance function to y, that is, $h_y(x) = d(x,y)$. Let $x \in M$ be a critical point of h_y. We say that x is a *ρ-critical point* of h_y if $d(x,y) \leq \rho(x)$, where $\rho(x)$ is the curvature radius of M at x.

Note that if x is a *ρ-critical point* of h_y, then $y = \exp_x(tN(x))$ with $0 \leq t \leq \rho(x)$.

Theorem 3.4. *Under Assumption 1.1,*

$$\int_{X_c^2} \nu_\rho(y)\,dy = \int_M \tan_c\left(\frac{\rho(s)}{2}\right)ds,$$

where $\nu_\rho(y)$ is the number of ρ-critical points of the distance function h_y, s is the arclength of M, and $\rho(s)$ is the curvature radius of M at $\gamma(s)$.

Proof. Applying the coarea formula to the focal map ϕ, we have

$$\int_{\phi(M_\rho)} \#(\phi^{-1}(y))dy = \int_{M_\rho} |\phi^* dy|.$$

Because of its construction, ϕ catches each point $y \in \phi(M_\rho)$ exactly $\nu_\rho(y)$ times. Moreover, since $\#(\phi^{-1}(y)) = 0$ for $y \notin \phi(M_\rho)$, we have

$$\int_{X_c^2} \nu_\rho(y)dy = \int_{M_\rho} |\phi^* dy|. \tag{3}$$

By Remark 3.2 we have $|\phi^* dy| = \phi^* dy$, and hence

$$\begin{aligned}
\int_{X_c^2} \nu_\rho(y)dy &= \int_{M_\rho} \phi^* dy = \int_M \int_0^{\rho(s)} p(s,t)dt\, ds \\
&= \int_M \int_0^{\rho(s)} (\mathrm{cn}_c(t) - k(s)\,\mathrm{sn}_c(t))dt\, ds \\
&= -\frac{1}{c}\int_M (-c\,\mathrm{sn}_c(\rho(s)) + k(s)(1-\mathrm{cn}_c(\rho(s))))ds \\
&= -\frac{1}{c}\int_M (-c\,\mathrm{sn}_c(\rho(s)) + \cot_c(\rho(s))(1-\mathrm{cn}_c(\rho(s))))\, ds \\
&= -\frac{1}{c}\int_M \frac{\mathrm{cn}_c(\rho(s))-1}{\mathrm{sn}_c(\rho(s))}\, ds = \int_M \tan_c \frac{\rho(s)}{2}\, ds.
\end{aligned}$$

□

Remark 3.5. Note that $A(\rho(s))/L(\rho(s)) = \tan_c(\rho(s)/2)$, where $A(\rho(s))$ and $L(\rho(s))$ are respectively the area and the length of the disc of radius $\rho(s)$ in X_c^2.

Thus, we have proved

$$\int_{X_c^2} \nu_\rho(y)\, dy = \int_M \frac{A(\rho(s))}{L(\rho(s))}\, ds.$$

Lemma 3.6. *Adopt Assumption 1.1. Let $y \in K$, and let $x \in M$ be a minimum of the function h_y. Then x is a ρ-critical point of h_y.*

Proof. Let $\gamma(s)$ be an arclength parametrization of M. Consider $f(s) = h_y(\gamma(s))$. If s_0 is such that $\gamma(s_0) = x$, we have $f'(s_0) = 0$ and $f''(s_0) > 0$. Now $f'(s) = g(X, \gamma'(s))$, where g is the metric on X_c^2, and $X = \mathrm{grad}(d(\cdot, y))$ is the gradient field (over X_c^2) of the distance function to y. Then

$$\begin{aligned}
0 < f''(s_0) &= \gamma'(g(X, \gamma'(s))(s_0) \\
&= g(\nabla_{\gamma'}X, \gamma'(s))(s_0) + g(X, \nabla_{\gamma'}\gamma')(s_0) = k_f - k(s_0),
\end{aligned}$$

where $k_f = \cot_c(f(s_0))$ is the geodesic curvature of the circle through x with center y. Thus, we have $\cot_c(\rho(x)) < \cot_c(h_y(x))$, which implies $\rho(x) > d(x, y)$; thus x is a ρ-critical point. □

Lemma 3.7. *Under Assumption 1.1, we have*

$$\nu_\rho(y) = \mathrm{wind}(M, y) - \mathrm{wind}(F(M), y) \quad \text{for } y \notin M \cup F(M).$$

Proof. Let $\phi : M_\rho \to X_c^2$ be the focal map of M, that is, $\phi(x,t) = \exp_x(tN(x))$.

Following [White 1970], we put

$$I = \{n \in M_\rho;\, y = \phi(n)\} = \phi^{-1}(y),$$

for a fixed generic point $y \in X_c^2 \backslash (M \cup F(M))$; the last ensures I is finite. We define

$$e : M_\rho - I \to T_y^1 X_c^2$$

by the condition $||e(n)|| = 1$ and by $\exp_y \lambda(n) e(n) = \phi(n)$, for some function $\lambda(n)$. Let $I_\epsilon = \bigcup_{i \in I} C_i$, where C_i are small, disjoint discs surrounding the points $i \in I$.

Applying Stokes' theorem to the punctured manifold $M_\rho - I_\epsilon$, we obtain

$$0 = \int_{M_\rho - I_\epsilon} e^* d(dO_1) = \int_{\partial(M_\rho - I_\epsilon)} e^*\, dO_1.$$

Note that $\partial(M_\rho - I_\epsilon) = M \cup M_e \cup \bigcup_i \partial(C_i)$, where $M_e = \{(x, \rho(x));\, x \in M\}$. Note also that $\phi(M_e) = F(M)$.

Because e is an orientation-preserving local diffeomorphism, all the integrals $\int_{\partial(C_i)} e^*\, dO_1$ are equal to 2π. Hence, taking into account the orientations induced at the boundary, we have

$$\int_M e^*\big|_M\, dO_1 - \int_{M_e} e^*\big|_{M_e}\, dO_1 - 2\pi \#(I) = 0.$$

But $\#I = \nu_\rho(y)$, so

$$\frac{1}{2\pi}\int_M e^*\big|_M\, dO_1 - \frac{1}{2\pi}\int_{M_e} e^*\big|_{M_e}\, dO_1 = \nu_\rho(y). \tag{4}$$

Now we fix a regular parametrization $\gamma : S^1(L) \to M$. It is clear that $e^*\big|_M \circ \gamma$ is the winding map with respect to y associated to γ. It follows from Remark 2.8 that

$$\operatorname{wind}(M, y) = \frac{1}{2\pi}\int_M e^*\big|_M\, dO_1.$$

Analogously, let $j : S^1(L) \to M_\rho$ be the map $j(s) = (s, \rho(s))$, and let $\tilde{\gamma}$ be the parametrization of $F(M)$ induced by the parametrization of M. Then $\tilde{\varphi} = e^*\big|_{M_e} \circ j$ is the winding map with respect to y associated to $\tilde{\gamma}$. Note that $j(S^1(L)) = M_e$. Thus

$$\operatorname{wind}(F(M), y) = \operatorname{wind}(\tilde{\gamma}, y) = \deg \tilde{\varphi} = \deg e^*\big|_{M_e} = \frac{1}{2\pi}\int_{M_e} e^*\big|_{M_e}\, dO_1.$$

Hence, for each $y \in X_c^2 \backslash (M \cup F(M))$, equality (4) becomes

$$\operatorname{wind}(M, y) - \operatorname{wind}(F(M), y) = \nu_\rho(y). \qquad \square$$

Theorem 3.8. *Under Assumption 1.1, we have*

$$\int_M \tan_c\left(\frac{\rho(s)}{2}\right) ds = F - F_e,$$

where s is the arclength of M, F is the area of K, and F_e is the (algebraic) area of the focal set $F(M)$ of M.

Proof. From Theorem 3.4 and Lemma 3.7 we have

$$\int_M \tan_c\left(\frac{\rho(s)}{2}\right) ds = \int_{X_c^2} (\operatorname{wind}(M, y) - \operatorname{wind}(F(M), y))\, dy.$$

But $\operatorname{wind}(M, y) = 1$ if $y \in K$, and $\operatorname{wind}(M, y) = 0$ if $y \notin K$. The (algebraic) area of $F(M)$ is, by definition, the integral over X_c^2 of the winding number of $F(M)$ with respect to every $y \in X_c^2$. □

We also obtain a generalization of formula (1).

Corollary 3.9. *Under Assumption 1.1, we have*

$$\int_M \tan_c\left(\frac{\rho(s)}{2}\right) ds \geq F. \tag{5}$$

Equality holds if and only if M is a circle.

Proof. The inequality (5) is a consequence of $\nu_\rho(y) \geq \operatorname{wind}(M, y)$, which is evident because $y \notin K$ implies $\operatorname{wind}(M, y) = 0$, whereas $y \in K$ implies $\nu_\rho(y) \geq 1$. Note that this proves $\operatorname{wind}(F(M), y) \leq 0$ and $F_e \leq 0$. If M is a circle and $F(M)$ is its center, then $F_e = 0$, and we have equality in (5).

Finally, if equality holds in (5), we have

$$\int_Y \operatorname{wind}(F(M), y) dy = 0.$$

Since $\operatorname{wind}(F(M), y) \leq 0$, it must be that $\operatorname{wind}(F(M), y) = 0$ almost everywhere.

If $F(M)$ were not a point we could choose a small ball separated by $F(M)$ in two connected components. The winding number is a different integer in each of these parts, which gives a contradiction. □

Remark 3.10. If $c = 0$ we have $\int_M \rho(s) ds \geq 2F$.

This, together with Theorem 3.8 for $c=0$, gives $F_e \leq 0$, which is also a consequence of the Wirtinger inequality. Indeed, that inequality states that if $f : \mathbb{R} \to \mathbb{R}$ is a C^2-function of period 2π, then

$$\int_0^{2\pi} |f'|^2 d\phi \leq \int_0^{2\pi} |f''|^2 d\phi. \tag{6}$$

Equality holds if and only if $f(\phi) = a\cos\phi + b\sin\phi + c$ for constants a, b, and c. See, for instance, [Hopf 1983, p. 52].

It was seen in [Escudero and Reventós 2007] that

$$F_e = \frac{1}{2}\int_0^{2\pi} (p'^2 - p''^2)\,d\phi,$$

where $p(\phi)$ is the support function of the convex set; hence (6) implies $F_e \le 0$.

Conversely, $F_e \le 0$ for an arbitrary convex set implies (6). Indeed, given f we consider $p = f + c$ with c constant so that $p + p'' > 0$. Now we apply $F_e \le 0$ to the convex set with support function p.

Thus we have a geometrical interpretation of the Wirtinger inequality: every convex set K is covered by the geodesic segments joining each point of ∂K to the corresponding curvature center.

4. The integral of $\tan_c \rho(s)$

Note that, in the case $c = 0$, Theorem 3.8 gives $\int_M \rho(s)/2\,ds = F - F_e$, which is formula (2). It can also be written as

$$\frac{1}{2}\int_M \frac{1}{k(s)}ds = F - F_e,$$

and, since in X_c^2 the relation between the curvature $k(s)$ and the curvature radius $\rho(s)$ is given by $k(s) = \cot_c \rho(s)$, it seems interesting to estimate

$$\int_M \tan_c \rho(s)\,ds.$$

For this, we recall the Gauss–Bonnet theorem [Santaló 1976, p. 303]

$$\int_M k(s)\,ds + cF = 2\pi$$

and the isoperimetric inequality [p. 324]

$$L^2 + cF^2 - 4\pi F \ge 0.$$

We apply these to the convex set K ($M = \partial K$) of area F and perimeter L in X_c^2:

$$\begin{aligned}
4\pi F - cF^2 \le L^2 &= \Big(\int_M \sqrt{\cot_c(\rho(s))}\sqrt{\tan_c(\rho(s))}\,ds\Big)^2 \\
&\le \int_M \cot_c(\rho(s))\,ds \int_M \tan_c(\rho(s))\,ds \\
&= \int_M k(s)ds \int_M \tan_c(\rho(s))\,ds = (2\pi - cF)\int_M \tan_c(\rho(s))\,ds.
\end{aligned}$$

Hence we have

Theorem 4.1. *Under Assumption 1.1,*

$$\int_M \tan_c \rho(s)\, ds \geq F \frac{4\pi - cF}{2\pi - cF} \tag{7}$$

Equality holds if and only if M is a circle.

Remark 4.2. Since

$$1 < \frac{4\pi - cF}{2\pi - cF} \leq 2,$$

we have

$$\int_M \tan_c \rho(s)\, ds > F.$$

This also follows directly from formula (5).

Acknowledgments

We would like to thank professors E. Gallego and E. Teufel for many helpful conversations during the preparation of this work.

References

[do Carmo 1976] M. P. do Carmo, *Differential geometry of curves and surfaces*, Prentice-Hall, Englewood Cliffs, N.J., 1976. Translated from the Portuguese. MR 52 #15253 Zbl 1028.34017

[Escudero and Reventós 2007] C. A. Escudero and A. Reventós, "An interesting property of the evolute", *Amer. Math. Monthly* **114**:7 (2007), 623–628. MR 2341325

[Escudero and Rodríguez 1996] C. A. Escudero and C. Rodríguez, *Curvatura en un poligono y teorema de Ros para curvas planas*, Master's thesis, Universidad del Valle, Colombia, 1996.

[Gallego et al. 2005] E. Gallego, A. Reventós, G. Solanes, and E. Teufel, "Width of convex bodies in spaces of constant curvature", Preprint 34, Departament de Matemàtiques, Univ. Autònoma de Barcelona, 2005.

[Guillemin and Pollack 1974] V. Guillemin and A. Pollack, *Differential topology*, Prentice-Hall, Englewood Cliffs, N.J., 1974. MR 50 #1276 Zbl 0361.57001

[Hopf 1983] H. Hopf, *Differential geometry in the large*, Lecture Notes in Mathematics **1000**, Springer, Berlin, 1983. MR 85b:53001 Zbl 0526.53002

[Osserman 1990] R. Osserman, "Curvature in the eighties", *Amer. Math. Monthly* **97**:8 (1990), 731–756. MR 91i:53001 Zbl 0722.53001

[Ratcliffe 1994] J. G. Ratcliffe, *Foundations of hyperbolic manifolds*, Graduate Texts in Mathematics **149**, Springer, New York, 1994. MR 95j:57011 Zbl 0809.51001

[Ros 1988] A. Ros, "Compact hypersurfaces with constant scalar curvature and a congruence theorem", *J. Differential Geom.* **27**:2 (1988), 215–223. MR 89b:53096 Zbl 0638.53051

[Santaló 1976] L. A. Santaló, *Integral geometry and geometric probability*, Addison-Wesley, Reading, MA-London-Amsterdam, 1976. MR 55 #6340 Zbl 0342.53049

[White 1970] J. H. White, "Some differential invariants of submanifolds of Euclidean space", *J. Differential Geometry* **4** (1970), 207–223. MR 42 #1035 Zbl 0198.53501

[Zhou 2007] J. Zhou, "Curvature inequalities for curves", 2007. To appear in *Int. J. Math. Sci.* (India).

Received December 15, 2006. Revised June 26, 2007.

CARLOS ARTURO ESCUDERO
DEPARTAMENTO DE MATEMÁTICAS
UNIVERSIDAD TECNOLÓGICA DE PEREIRA
A.A.097 PEREIRA, RISARALDA
COLOMBIA

carlos10@utp.edu.co

AGUSTÍ REVENTÓS
DEPARTAMENT DE MATEMÀTIQUES
UNIVERSITAT AUTÒNOMA DE BARCELONA
08193 BELLATERRA (CERDANYOLA DEL VALLÈS)
SPAIN

agusti@mat.uab.es

GIL SOLANES
DEPARTAMENT D'ÀLGEBRA I GEOMETRIA
UNIVERSITAT DE BARCELONA
GRAN VIA DE LES CORTS CATALANES 585
08007 BARCELONA
SPAIN

solanes@ub.edu