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THEOREM OF REIFENBERG**

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## A GEOMETRIC APPROACH TO THE TOPOLOGICAL DISK THEOREM OF REIFENBERG

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**We offer a new approach to the classical topological disk theorem of Reifenberg in the codimension-1 case. Our proof, using  $\frac{1}{2}$ -level sets of the smoothing of the characteristic function of the domain  $\Omega$  as approximating surfaces of  $\partial\Omega$ , is much simpler than Reifenberg's original one (1960). We also extend the result to parabolic space.**

### 1. Introduction

We start by recalling the remarkable topological disk theorem in Chapter 4 of the celebrated paper [Reifenberg 1960], which investigated the higher-dimensional Plateau problem.

**Definition 1.1** (Reifenberg flat set). We say that a compact set  $K \subseteq B_1^n \subseteq \mathbb{R}^n$  is a  $m$ -dimensional  $(\delta, R)$ -Reifenberg flat set if for every  $a \in K$  and  $r \in (0, R]$  with  $|a| + r \leq 1$ , there exists an  $m$ -plane  $T_{a,r} \in \text{GL}(m, n)$  such that

$$\text{HD}(K \cap B_r^n(a), T_{a,r} \cap B_r^n(a)) \leq \delta r,$$

where

$$\text{HD}(A, B) = \sup\{\text{dist}(a, B) : a \in A\} + \sup\{\text{dist}(b, A) : b \in B\}.$$

is the Hausdorff distance.

The definition is only significant for small  $\delta > 0$ .

**Theorem 1.2** (Reifenberg's topological disc theorem). *If  $K$  is Reifenberg flat, then  $K \cap B_\theta^n$  is a  $C^\alpha$ -topological  $m$ -dimensional disk for some  $\theta(\delta, R) > 0$ .*

The statement just given is adopted from [Lin and Yang 2002, Chapter 2, p. 58]. Roughly speaking, the theorem shows that if in each ball  $B_r(a)$  with  $a \in K$   $K$  lie in a narrow strip of width less than  $2\delta r$  and moreover it is  $\delta r$ -dense in the narrow strip, then the surface is locally Euclidean, more precisely, it is locally a bi-Hölder image of the unit ball in  $\mathbb{R}^m$ . It should be noted that the strip is allowed to vary

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both with  $a$  and  $r$ . A typical example of Reifenberg flat set is the well-known Von Koch curve, which is a self-similar Jordan curve and a prototypical fractal set. In the last decade, important works related to this kind of set have appeared: see, for instance, [Capogna et al. 2005; David et al. 2001; David and Toro 1999; Kenig and Toro 1997; 1999; Toro 1997].

A complete proof of Theorem 1.2 can be found in Reifenberg's paper (a sketch is given in the introduction) or in [Morrey 1966, Chapter 10, p. 439]. In Reifenberg's own words, the proof *consists of constructing inductively a series of disks which converge to a piece of the surface and are such that each disk is the image of the previous one in such a manner that both the mapping function and its inverse are Lipschitz with constant  $k$  where  $k$  is fixed throughout the series. Once this series has been constructed the rest is easy.* Simply speaking, he constructed the approximate sets locally (mainly due to that the Reifenberg flat condition is a local property), the process *involves several averaging processes, each of which introduces complicated estimations, so the proof is in parts unavoidably messy.* However, in the codimension-1 case, we can construct the approximate sets globally, and the estimations are also straightforward. This makes us capable of writing down a direct proof to this theorem.

In the codimension-1 case, the surface can be seen as the boundary of some domain which we called the Reifenberg flat domain. We will prove this fact by a *locally separating theorem* in Section 4 of this paper.

**Definition 1.3** (Reifenberg flat domain). We say that a domain  $\Omega \subset \mathbb{R}^n$  is  $(\delta, R)$ -Reifenberg flat if for every  $x \in \partial\Omega$  and  $r \in (0, R]$ , there exists a unit vector  $\vec{n}(x, r) \in \mathbb{R}^n$ , such that

$$\begin{aligned} B_r(x) \cap \{y \in \mathbb{R}^n : \langle y - (x - r\delta\vec{n}(x, r)), \vec{n}(x, r) \rangle < 0\} \\ \subset B_r(x) \cap \Omega \\ \subset \{y \in \mathbb{R}^n : \langle y - (x + r\delta\vec{n}(x, r)), \vec{n}(x, r) \rangle < 0\}, \end{aligned}$$

where  $\langle \cdot, \cdot \rangle$  denotes the inner product in  $\mathbb{R}^n$ .

This kind of domain is geometrically invariant with respect to scaling. The study of elliptic partial differential equations on this domain is interesting [Byun and Wang 2004]. In the first part of this paper, we prove the following theorem.

**Theorem 1.4.** *Let  $\Omega \subset \mathbb{R}^n$  be a  $(\delta, R)$ -Reifenberg flat domain such that  $\partial\Omega$  contains the origin  $O$ , and suppose that*

$$B_2 \cap \{x \in \mathbb{R}^n : x_n < -\delta\} \subset B_2 \cap \Omega \subset \{x \in \mathbb{R}^n : x_n < \delta\}$$

*if  $\delta > 0$  is sufficiently small. Then there exist a map*

$$f : B_{1/2} \cap \{x_n = 0\} \rightarrow \partial\Omega \cap B_1$$

and numbers  $0 < \alpha < 1$  and  $\beta > 1$  such that

$$f(B_{1/2} \cap \{x_n = 0\}) \supset \partial\Omega \cap B_{1/4}$$

and

$$c_2|x - y|^\beta \leq |f(x) - f(y)| \leq c_1|x - y|^\alpha$$

for any  $x, y \in B_{1/2} \cap \{x_n = 0\}$ .

The base of our proof is the global construction of the approximating surfaces. We mollify the characteristic function of  $\Omega$  and use the  $\frac{1}{2}$ -level set of the mollified function as the approximate set of  $\partial\Omega$ . Varying  $\varepsilon$ , the mollifier scale, we get a series of approximating surfaces which converge to  $\partial\Omega$  as  $\varepsilon$  tends to zero. There exists a bi-Lipschitz mapping (normal projection operator of a tubular neighborhood) from each approximate set to the next one, and the Lipschitz constant is less than 2 (this estimation need some basic computation). Composing these bi-Lipschitz mappings, we will obtain a bi-Hölder mapping from the  $(n-1)$ -disk to  $\partial\Omega$ .

Our method is applicable to some degenerate metric spaces instead of the standard  $\mathbb{R}^n$ : for instance, the parabolic space  $\mathbb{R}^{n+1}$ . Section 5 treats the parabolic case. We prove that the boundary of a parabolic Reifenberg flat domain is locally a  $C^\alpha$ -topological disk (Theorem 5.3).

Our construction yields an approximation of Reifenberg flat domains by  $C^2$  Reifenberg flat domains, both elliptic and parabolic. This type of approximations allows to extend boundary regularity results for solutions of elliptic and parabolic equations to nonsmooth domains.

## 2. Approximating surfaces and bi-Lipschitz maps

**Mollifier.** Define

$$\eta(x) =: \begin{cases} C_1(1 - |x|^2)^3 & \text{if } |x| \leq 1, \\ 0 & \text{else,} \end{cases}$$

where the constant  $C_1$  is adjusted so  $\int_{\mathbb{R}^n} \eta(x) dx = 1$ . Next define

$$\eta_\varepsilon(x) =: \varepsilon^{-n} \eta(\varepsilon^{-1}x).$$

Thus

$$\eta_\varepsilon(x) \in C_0^2(B_{2\varepsilon}),$$

and

$$\int_{\mathbb{R}^n} \eta_\varepsilon(x) dx = 1.$$

**Level sets.** From now on, we will assume  $\Omega$  is as in Theorem 1.4. We define

$$\chi_\Omega(x) := \begin{cases} 1 & \text{if } x \in \Omega, \\ 0 & \text{otherwise,} \end{cases}$$

$$\chi_\varepsilon(x) := \chi_\Omega * \eta_\varepsilon, \quad L_\varepsilon := \{x \in \mathbb{R}^n : \chi_\varepsilon(x) = \frac{1}{2}\}.$$

**Curvature estimate of the  $\frac{1}{2}$ -level set  $L_\varepsilon$ .** We make some derivative estimates for  $\chi_\varepsilon$  near  $L_\varepsilon$ . Fix  $x \in L_\varepsilon$  and let  $y \in \partial\Omega$ , s.t.  $d =: |y - x| = \inf_{z \in \partial\Omega} |z - x|$ . Obviously  $d < \varepsilon$ . In fact, if  $\varepsilon \leq d$ , then  $B_\varepsilon(x) \in \Omega$  or  $B_\varepsilon(x) \in \overline{\Omega}^c$ , that implies  $\chi_\varepsilon(x) = 1$  or  $\chi_\varepsilon(x) = 0$ . Set

$$\begin{aligned} A_1 &:= \{z : \langle z - (y - 4\varepsilon\delta\vec{n}(y, 4\varepsilon)), \vec{n}(y, 4\varepsilon) \rangle < 0\} \cap B_{4\varepsilon}(y), \\ A_2 &:= \{z : \langle z - (y + 4\varepsilon\delta\vec{n}(y, 4\varepsilon)), \vec{n}(y, 4\varepsilon) \rangle \geq 0\} \cap B_{4\varepsilon}(y) - A_1, \\ A_3 &:= B_{4\varepsilon}(y) - (A_1 \cup A_2), \\ T &:= \partial A_1 - \partial B_{4\varepsilon}(y). \end{aligned}$$

**Lemma 2.1.**  $x \in A_2$  and  $d \leq 8\varepsilon\delta$ .

*Proof.* If  $x \in A_1$ , since  $d < \varepsilon$ ,

$$\chi_\varepsilon(x) = \int_{B_\varepsilon(x) \cap \Omega} \eta_\varepsilon(z - x) dz \geq \int_{B_\varepsilon(x) \cap A_1} \eta_\varepsilon(z - x) dz > \frac{1}{2}.$$

Similarly, if  $x \in A_3$ , then  $\chi_\varepsilon(x) < \frac{1}{2}$ . Thus  $x \in A_2$ .

$B_{8\varepsilon\delta}(x) \cap \partial\Omega \neq \emptyset$  implies  $d \leq 8\varepsilon\delta$ . □

From now on, we will assume that  $\vec{n}(y, 4\varepsilon) = -e_n = (0, \dots, 0, -1)$ . Let  $\hat{e}$  be any unit vector in  $\mathbb{R}^{n-1}$  and define  $e := (\hat{e}, 0) \in \mathbb{R}^n$ , so  $\langle e, e_n \rangle = 0$ .

**Lemma 2.2.** For any  $\bar{x} \in A_2 \cap B_\varepsilon(y)$ , we have

$$\begin{aligned} \frac{\partial}{\partial e_n} \chi_\varepsilon(\bar{x}) &\geq C_2 \varepsilon^{-1}, & \left| \frac{\partial}{\partial e} \chi_\varepsilon(\bar{x}) \right| &\leq C_3 \delta \varepsilon^{-1}, \\ \left| \frac{\partial^2}{\partial e^2} \chi_\varepsilon(\bar{x}) \right| &\leq C_4 \delta \varepsilon^{-2}, & \left| \frac{\partial^2}{\partial e \partial e_n} \chi_\varepsilon(\bar{x}) \right| &\leq C_4 \delta \varepsilon^{-2}, & \left| \frac{\partial^2}{\partial e_n^2} \chi_\varepsilon(\bar{x}) \right| &\leq C_5 \varepsilon^{-2}. \end{aligned}$$

*Proof.*

$$\begin{aligned} \frac{\partial}{\partial e_n} \chi_\varepsilon(\bar{x}) &= \left( \chi_\Omega * \frac{\partial}{\partial e_n} \eta_\varepsilon \right) (\bar{x}) = \int_\Omega \frac{\partial}{\partial e_n} \eta_\varepsilon(\bar{x} - z) dz \\ &= \int_{A_1 \cap B_\varepsilon(\bar{x})} \frac{\partial}{\partial e_n} \eta_\varepsilon(\bar{x} - z) dz + \int_{\Omega \cap A_2 \cap B_\varepsilon(\bar{x})} \frac{\partial}{\partial e_n} \eta_\varepsilon(\bar{x} - z) dz =: I_1 + I_2. \end{aligned}$$

By Green's Formula,

$$\begin{aligned} I_1 &= \int_{T \cap B_\varepsilon(\bar{x})} \eta_\varepsilon(\bar{x} - s) ds \geq \int_{T \cap B_{\varepsilon/2}(\bar{x})} \eta_\varepsilon(\bar{x} - s) ds \geq C'_2 \varepsilon^{-n} \cdot C''_2 \left(\frac{\varepsilon}{2}\right)^{n-1} \\ &= C''_2 \varepsilon^{-1}, \\ |I_2| &\leq \int_{A_2 \cap B_\varepsilon(\bar{x})} \left| \frac{\partial}{\partial e_n} \eta_\varepsilon(\bar{x} - z) \right| dz \leq C_2^{(4)} \varepsilon^{-n} \varepsilon^{-1} \cdot C_2^{(5)} \delta \varepsilon^n = C_2^{(6)} \delta \varepsilon^{-1}. \end{aligned}$$

If  $C_2^{(6)} \delta < C_2'''/2$ , set  $C_2 = C_2'''/2$ . We have

$$\begin{aligned} \frac{\partial}{\partial e} \chi_\varepsilon(\bar{x}) &= \int_\Omega \frac{\partial}{\partial e} \eta_\varepsilon(\bar{x} - z) dz \\ &= \int_{A_1 \cap B_\varepsilon(\bar{x})} \frac{\partial}{\partial e} \eta_\varepsilon(\bar{x} - z) dz + \int_{\Omega \cap A_2 \cap B_\varepsilon(\bar{x})} \frac{\partial}{\partial e} \eta_\varepsilon(\bar{x} - z) dz =: I_3 + I_4; \end{aligned}$$

but  $I_3 = 0$  by Green's formula, and

$$|I_4| \leq \int_{A_2 \cap B_\varepsilon(\bar{x})} \left| \frac{\partial}{\partial e} \eta_\varepsilon(\bar{x} - z) \right| dz \leq C_2^{(6)} \delta \varepsilon^{-1},$$

like  $I_2$ . Set  $C_3 = C_2^{(6)}$ . We have

$$\frac{\partial^2}{\partial e^2} \chi_\varepsilon(\bar{x}) = \int_{A_1 \cap B_\varepsilon(\bar{x})} \frac{\partial^2}{\partial e^2} \eta_\varepsilon(\bar{x} - z) dz + \int_{\Omega \cap A_2 \cap B_\varepsilon(\bar{x})} \frac{\partial^2}{\partial e^2} \eta_\varepsilon(\bar{x} - z) dz =: I_5 + I_6;$$

now  $I_5 = 0$  by Green's Formula, and

$$|I_6| \leq \int_{A_2 \cap B_\varepsilon(\bar{x})} \left| \frac{\partial^2}{\partial e^2} \eta_\varepsilon(\bar{x} - z) \right| dz \leq C'_4 \varepsilon^{-n} \varepsilon^{-2} \cdot C_2^{(5)} \delta \varepsilon^n = C_4 \delta \varepsilon^{-2}.$$

We omit the proofs of the last two inequalities, which are similar to ones above.  $\square$

Since  $\nabla \chi_\varepsilon(x) \neq 0$  for  $x \in L_\varepsilon \cap B_1$ ,  $L_\varepsilon \cap B_1$  is a  $C^2$   $(n - 1)$ -dimensional manifold and can be represented as the graph of some function from  $\mathbb{R}^{n-1}$  to  $\mathbb{R}$  locally by the Implicit Function Theorem. If you take a point  $\hat{x} \in L_\varepsilon \cap B_1$ , and  $\hat{y}$  is the nearest point from  $\partial\Omega$  to  $\hat{x}$ , then in  $A_2 \cap B_\varepsilon(\hat{y})$ , the function can be taken as

$$x_n = \varphi(x_1, \dots, x_{n-1}) =: \varphi(x_T).$$

**Lemma 2.3.**  $\left| \frac{\partial}{\partial \hat{e}} \varphi(x_T) \right| < \frac{C_3}{C_2} \delta$  and  $\left| \frac{\partial^2}{\partial \hat{e}^2} \varphi(x_T) \right| \leq C_6 \delta \varepsilon^{-1}$ .

*Proof.* By the Implicit Function Theorem, we have

$$\frac{\partial}{\partial \hat{e}} \varphi(x_T) = - \frac{\frac{\partial}{\partial e} \chi_\varepsilon(x_T, \varphi(x_T))}{\frac{\partial}{\partial e_n} \chi_\varepsilon(x_T, \varphi(x_T))},$$

so  $\left| \frac{\partial}{\partial \hat{e}} \varphi(x_T) \right| < \frac{C_3}{C_2} \delta$ ; and

$$\frac{\partial^2}{\partial \hat{e}^2} \varphi(x_T) = - \frac{\frac{\partial^2}{\partial e^2} \chi_\varepsilon(x_T, \varphi(x_T)) + \frac{\partial^2}{\partial e_n^2} \chi_\varepsilon \cdot \left( \frac{\partial}{\partial \hat{e}} \varphi(x_T) \right)^2 + 2 \frac{\partial^2}{\partial e \partial e_n} \chi_\varepsilon \cdot \left( \frac{\partial}{\partial \hat{e}} \varphi(x_T) \right)}{\frac{\partial}{\partial e_n} \chi_\varepsilon(x_T, \varphi(x_T))},$$

so

$$\left| \frac{\partial^2}{\partial \hat{e}^2} \varphi(x_T) \right| \leq \frac{C_4 \delta \varepsilon^{-2} + C_5 \varepsilon^{-2} \left( \frac{C_3}{C_2} \right)^2 \delta^2 + 2C_4 \delta \varepsilon^{-2} \frac{C_3}{C_2} \delta}{C_2 \varepsilon^{-1}} \leq \frac{2C_4 \delta \varepsilon^{-2}}{C_2 \varepsilon^{-1}} = C_6 \delta \varepsilon^{-1},$$

where the second inequality holds for  $\delta$  small enough. □

**Corollary 2.4.** *The principal curvature of  $L_\varepsilon$  in any tangent direction is less than  $C_6 \delta \varepsilon^{-1}$ . If  $C_6 \delta < 1$ , a tubular neighborhood with radii of  $\varepsilon$  exists.*

**Closeness of  $L_\varepsilon$  to  $\partial\Omega$ .** If  $E$  is an  $(n-1)$ -dimensional manifold in  $\mathbb{R}^n$  having a tubular neighborhood with radii of  $\tau$ , we denote by  $N(E, s)$  an  $s$ -tubular neighborhood of  $E$  for  $0 < s \leq \tau$ , and by  $\pi$  the normal projection map of the tubular neighborhood.

**Lemma 2.5.**  $\partial\Omega \cap B_{r-\varepsilon} \subset N(L_\varepsilon \cap B_r, 4\varepsilon\delta)$  for  $r \leq 1$ .

*Proof.* Given any  $\bar{x} \in \partial\Omega \cap B_{r-\varepsilon}$ , we have

$$\begin{aligned} B_{4\varepsilon}(\bar{x}) \cap \{x : \langle x - (\bar{x} - 4\varepsilon\delta\vec{n}(\bar{x}, 4\varepsilon)), \vec{n}(\bar{x}, 4\varepsilon) \rangle < 0\} \\ \subset B_{4\varepsilon}(\bar{x}) \cap \Omega \\ \subset \{x : \langle x - (\bar{x} + 4\varepsilon\delta\vec{n}(\bar{x}, 4\varepsilon)), \vec{n}(\bar{x}, 4\varepsilon) \rangle < 0\}. \end{aligned}$$

Thus  $\chi_\varepsilon(\bar{x} - 2\varepsilon\vec{n}) = 1$ ,  $\chi_\varepsilon(\bar{x} + 2\varepsilon\vec{n}) = 0$ . By the continuity of  $\chi_\varepsilon$ , there exists  $s \in (-1, 1)$  such that  $\chi_\varepsilon(\bar{x} + 2s\varepsilon\vec{n}) = \frac{1}{2}$ . In fact,  $|s| < 2\delta$  by the proof of Lemma 2.1. □

Taking  $\varepsilon = 4^{-k}$ ,  $k = 1, 2, \dots$ , we gain a series of approximating surfaces  $L_{4^{-k}}$ . Define  $\mathcal{S}_0 := \{x_n = 0\} \cap B_{1/2}$  and  $\mathcal{S}_k := L_{4^{-k}} \cap B_{r(k)}$ , where  $r(k) = \frac{1}{2} + \sum_{j=1}^k 4^{-j}$ .

**Corollary 2.6.**  $\mathcal{S}_k \in N(L_{4^{-k+1}}, 6 \cdot 4^{-k+1} \delta)$ .

*Proof.* For any  $x \in \mathcal{S}_k$ , there exists  $y \in \partial\Omega \cap B_{5/6}$  such that  $|x - y| < 8 \cdot 4^{-k} \delta$ , and  $z \in L_{4^{-k+1}} \cap B_1$  such that  $|y - z| < 4 \cdot 4^{-k+1} \delta$ , so  $|x - z| < 6 \cdot 4^{-k+1} \delta$ . □

**Bi-Lipschitz maps.**

**Lemma 2.7.**  $\pi : \mathcal{S}_k \rightarrow L_{4^{-k+1}}$  is inject, and if  $\delta$  is small enough, given any  $z^1, z^2 \in \mathcal{S}_k$ , satisfying  $|z^1 - z^2| \leq 4^{-k}$ , then

$$\frac{1}{2} |\pi(z^1) - \pi(z^2)| \leq |z^1 - z^2| \leq 2 |\pi(z^1) - \pi(z^2)|.$$



*Proof.* We write  $x^1 =: \pi(z^1)$ ,  $x^2 =: \pi(z^2) \in L_{4^{-k+1}}$ . For simplicity, we will still write  $4^{-k+1}$  as  $\varepsilon$  sometimes. By choosing an appropriate coordinate system, we can assume that original point  $O$  is the nearest point of  $\partial\Omega$  to  $z^1$ , and  $\vec{n}(0, 4\varepsilon) = -e_n$ . Define

$$\begin{aligned} A_1 &:= \{x : x_n > 4\varepsilon\delta\} \cap B_{4\varepsilon}; \\ A_2 &:= \{x : -4\varepsilon\delta \leq x_n \leq 4\varepsilon\delta\} \cap B_{4\varepsilon}; \\ A_3 &:= \{x : x_n < -4\varepsilon\delta\} \cap B_{4\varepsilon}. \end{aligned}$$

Then  $A_1 \subset \Omega$ ,  $A_3 \subset \bar{\Omega}^c$ , and  $x^1, x^2, z^1, z^2 \in A_2 \cap B_\varepsilon$ .

We assume  $L_\varepsilon \cap B_\varepsilon$  is the graph of function  $x_n = \varphi(x_T) = \varphi(x_1, \dots, x_{n-1})$  and  $L_{\varepsilon/4} \cap B_\varepsilon$  is the graph of function  $z_n = \psi(z_T)$ . Thus we can write

$$\begin{aligned} x^1 &= (x_T^1, \varphi(x_T^1)), & x^2 &= (x_T^2, \varphi(x_T^2)), \\ z^1 &= (z_T^1, \psi(z_T^1)), & z^2 &= (z_T^2, \psi(z_T^2)). \end{aligned}$$

We define  $l =: |x_T^1 - x_T^2|$ ,  $\hat{l} =: |z_T^1 - z_T^2|$ . By assumption,

$$\hat{l} \leq \varepsilon/4 \quad \text{and} \quad l \leq \varepsilon/2.$$

By Lemma 2.3 and the Differential Mean Value Theorem,

$$(2-1) \quad |\varphi(x_T^1) - \varphi(x_T^2)| \leq l \cdot \frac{C_3}{C_2} \delta, \quad |\psi(z_T^1) - \psi(z_T^2)| \leq \hat{l} \cdot \frac{C_3}{C_2} \delta.$$

Then

$$(2-2) \quad \begin{aligned} l &\leq |x^1 - x^2| \leq \sqrt{l^2 + 2\left(\frac{C_3}{C_2} \delta\right)^2 l^2} \leq \frac{11}{10} l \quad (\text{using } \delta \text{ small}), \\ \hat{l} &\leq |z^1 - z^2| \leq \frac{11}{10} \hat{l}. \end{aligned}$$

Since  $\pi(z^1) = x^1$  and  $\pi(z^2) = x^2$ , there exist  $\xi, \zeta \in \mathbb{R}$ , such that

$$(2-3) \quad z^1 = x^1 + \xi v(x^1) \quad \text{and} \quad z^2 = x^2 + \zeta v(x^2),$$

where  $v(x^1), v(x^2)$  are inward-pointing unit normal vectors  $L_\varepsilon$ , at  $x^1, x^2$ . Explicitly,

$$\begin{aligned} v_i(x^1) &= \frac{-D_i \varphi(x_T^1)}{\sqrt{1 + |\nabla \varphi(x_T^1)|^2}} \quad \text{for } i = 1, \dots, n-1, \\ v_n(x^1) &= \frac{1}{\sqrt{1 + |\nabla \varphi(x_T^1)|^2}}, \end{aligned}$$

and likewise for  $v(x^2)$ . By Corollary 2.6 we have

$$(2-4) \quad |\xi|, \quad |\zeta| \leq 6\varepsilon\delta;$$

moreover,  $|v_i(x^j)| < C_3/C_2\delta$  for  $i = 1, \dots, n-1$ , and

$$v_n(x^j) > \frac{1}{\sqrt{1 + (C_3/C_2\delta)^2}} \quad \text{for } j = 1, 2,$$

by Lemma 2.3. These two inequalities show that  $v(x^j)$  is very close to  $e_n$  as  $\delta$  is very small.

Subtracting the second equality in (2-3) from the first, we get

$$(2-5) \quad z_i^1 - z_i^2 = x_i^1 - x_i^2 + \xi(v_i(x^1) - v_i(x^2)) + (\xi - \zeta)v_i(x^2),$$

$$(2-6) \quad z_n^1 - z_n^2 = x_n^1 - x_n^2 + \xi(v_n(x^1) - v_n(x^2)) + (\xi - \zeta)v_n(x^2).$$

By Lemma 2.3 and the Differential Mean Value Theorem, we obtain

$$\begin{aligned} |v_i(x^1) - v_i(x^2)| &\leq |D_i\varphi(x_T^2) - D_i\varphi(x_T^1)| + |\nabla\varphi(x_T^1)|^2 \cdot |\nabla\varphi(x_T^2) - \nabla\varphi(x_T^1)| \\ &\leq 2l \cdot C_6\delta\varepsilon^{-1}, \end{aligned}$$

and likewise

$$(2-7) \quad |v_n(x^1) - v_n(x^2)| \leq (|\nabla\varphi(x_T^1)| + |\nabla\varphi(x_T^2)|) \cdot |\nabla\varphi(x_T^1) - \nabla\varphi(x_T^2)| \\ \leq l \cdot C_6\delta\varepsilon^{-1}.$$

Equality (2-6) implies

$$\begin{aligned} |\xi - \zeta| &\leq \sqrt{1 + \left(\frac{C_3}{C_2}\delta\right)^2} (|z_n^1 - z_n^2| + |x_n^1 - x_n^2| + |\xi| \cdot |v_n(x^1) - v_n(x^2)|) \\ &\leq \sqrt{1 + \left(\frac{C_3}{C_2}\delta\right)^2} \left(\hat{l} \cdot \frac{C_3}{C_2}\delta + l \cdot \frac{C_3}{C_2}\delta + 5\varepsilon\delta \cdot l \cdot C_6\delta\varepsilon^{-1}\right) \leq C_7\delta(l + \hat{l}); \end{aligned}$$

here the second inequality follows using (2-1), (2-4) and (2-7).

Using the triangle inequality, (2-5) implies

$$\hat{l} \leq l + |\xi| \cdot |v_i(x^1) - v_i(x^2)| + |\xi - \zeta| \cdot |v_i(x^2)| \leq (1 + C_8\delta^2)l + C_7\frac{C_3}{C_2}\delta^2\hat{l}.$$

Thus if  $\delta$  is small enough, we obtain  $\hat{l} \leq \frac{11}{10}l$ . Similarly, one can show that  $\hat{l} \geq \frac{9}{10}l$ . Combining these two bounds with (2-2), we conclude the proof.  $\square$

We define  $g_k : \mathcal{G}_{k-1} \rightarrow \mathcal{G}_k$  satisfying  $\pi(g_k(x)) = x$  for  $k = 1, 2, \dots$ , and  $h_k : \mathcal{G}_0 \rightarrow \mathcal{G}_k$  by setting  $h_k =: g_k \circ \dots \circ g_1$ .

**Corollary 2.8.**  $|h_k(x) - h_{k-1}(x)| = |g_k(h_{k-1}(x)) - h_{k-1}(x)| < 6 \cdot 4^{-k+1}\delta$ .

### 3. Proof of Theorem 1.4

Corollary 2.8 implies that  $\{x, h_1(x), \dots, h_k(x), \dots\}$  is a Cauchy sequence. Letting  $f(x)$  be the limit point, Lemma 2.1 indicates that  $f(x) \in \partial\Omega \cap B_1$ . Thus we get a map

$$f : B_{1/2} \cap \{x_n = 0\} = \mathcal{G}_0 \rightarrow \partial\Omega \cap B_1.$$

Given  $x$  and  $y \in \mathcal{G}_0$ , define  $d := |f(x) - f(y)|$ . If  $d < \frac{10}{3} 4^2 \delta$ , choose  $k$  satisfying

$$(3-1) \quad \frac{1}{3} 4^{-k+2} \delta \leq \frac{1}{10} d < \frac{1}{3} 4^{-k+3} \delta,$$

then by triangle inequality and Corollary 2.8,

$$(3-2) \quad \frac{8}{10} d < |h_k(x) - h_k(y)| < \frac{12}{10} d.$$

From Lemma 2.7,

$$(3-3) \quad 2^{-k} |x - y| \leq |h_k(x) - h_k(y)| \leq 2^k |x - y|.$$

However, (3-1) implies

$$\log_4 d^{-1} + 2 + \log_4 \frac{10}{3} + \log_4 \delta \leq k < \log_4 d^{-1} + 3 + \log_4 \frac{10}{3} + \log_4 \delta.$$

Combining this with (3-2) and (3-3), we obtain

$$c_2 |x - y|^\beta \leq d \leq c_1 |x - y|^\alpha,$$

where  $\alpha = 1/(1 + \lambda)$ ,  $\beta = 1/(1 - \lambda)$ , for  $\lambda = \log_4 2 = \frac{1}{2}$ .

Finally, the inclusion  $f(\mathcal{G}_0) \supset \partial\Omega \cap B_{1/4}$  follows from Lemma 2.5. This concludes the proof of Theorem 1.4.

**Remark 3.1.** From the argument it is easy to see that  $\alpha$  and  $\beta$  can be taken arbitrarily close to 1 as long as  $\delta$  is sufficiently small and  $c_1, c_2$  are allowed to be large.

### 4. The locally separating theorem

**Theorem 4.1.** *Let  $K$  be a set satisfying Definition 1.1 in the case  $m = n - 1$ . We assume that the original point  $O \in K$  and  $T_{O,1} = \{x \in \mathbb{R}^n : x_1 = 0\}$ . Set*

$$A_1 := B_1 \cap \{x : x_1 < -2\delta\},$$

$$A_2 := B_1 \cap \{x : -2\delta \leq x_1 \leq 2\delta\},$$

$$A_3 := B_1 \cap \{x : x_1 > 2\delta\}.$$

(I) *The set  $B_{3/4} - K$  has at least two connected components, one containing  $A_1 \cap B_{3/4}$  and the other  $A_3 \cap B_{3/4}$ . We denote by  $\Omega_1$  and  $\Omega_2$ , respectively, these two components.*

(II)  $B_{1/2} - K \subset \Omega_1 \cup \Omega_2$ .

(III) For any  $a \in K \cap B_{1/2}$  and  $r \in (0, R]$  satisfying  $|a| + r < 1/2$ ,  $B_r(a) \cap \Omega_i \neq \emptyset$  for  $i = 1, 2$ .

(IV)  $K \cap B_{1/2} = \partial\Omega_1 \cap B_{1/2} = \partial\Omega_2 \cap B_{1/2}$ .

*Proof.* We prove the disconnectedness of  $B_{3/4} - K$  by contradiction. If  $B_{3/4} - K$  is connected, it is arcwise connected. So we can choose a path which connect  $z_1 =: (-\frac{1}{4}, 0, \dots, 0)$  with  $z_2 =: (\frac{1}{4}, 0, \dots, 0)$  and is contained in  $B_{3/4} - K$ . We denote it as  $\gamma : [0, l] \rightarrow B_{3/4} - K$  (arclength parameter  $t$  is used), which satisfy  $\gamma(0) = z_1$  and  $\gamma(l) = z_2$ . Set  $d := \inf_{t \in [0, l]} \text{dist}(\gamma(t), K)$ . Any path connecting  $z_1$  with  $z_2$  and contained in  $B_1$  must traverse  $T_{O,1}$ . So the distance of the path to  $K$  is less than  $\delta$ . Hence, we assume  $d < \delta$  at first. We will move  $\gamma$  away from  $K$  constantly and enlarge the distance of the path to  $k$  till it is bigger than  $\delta$ . By this way, we will get a contradiction.

Let  $t_1$  be such that  $\gamma(t_1) \in \partial A_1$  and  $\gamma(t) \in A_2 \cup A_3$ , for  $t \geq t_1$ . We may as well assume that  $\gamma(t) \in A_1$  for  $0 \leq t < t_1$  (otherwise, we connect  $z_1$  with  $\gamma(t_1)$  by a straight line and take the new path as  $\gamma$ ). Let  $t^*$  be such that  $\gamma(t^*) \in \partial A_3$  and  $\gamma(t) \in A_1 \cup A_2$ , for  $t \leq t^*$ . For the same reason, we assume  $\gamma(t) \in A_3$ , for  $t^* < t \leq l$ .

Now define  $r_1 := \text{dist}(\gamma(t_1), K)$ ; obviously,  $d \leq r_1 < 3\delta$ . Letting  $y_1$  be a nearest point to  $\gamma(t_1)$  from  $K$ , we consider  $T_{y_1, 5r_1}$ . Because of

$$\text{dist}(\gamma(t_1), y_1) \leq \text{dist}(\gamma(t_1), y), \quad \text{for any } y \in K$$

and

$$\text{HD}(K \cap B_{5r_1}(y_1), T_{y_1, 5r_1} \cap B_{5r_1}(y_1)) < 5r_1\delta,$$

$\gamma(t_1) - y_1$  is almost vertical to  $T_{y_1, 5r_1}$ , more precisely,

$$\angle(\gamma(t_1) - y_1, T_{y_1, 5r_1}^\perp) < C(n)\delta.$$

Hence, if we move  $\gamma(t_1)$  to  $x_1 := \gamma(t_1) + d(\gamma(t_1) - y_1)$ , then

$$\text{dist}(x_1, K) > r_1 + 0.9d \quad (\text{use the smallness of } \delta).$$

If we have chosen  $t_k, r_k, y_k$  and  $x_k$ , then let  $t_{k+1}$  be such that

$$\gamma(t_{k+1}) \in \partial B_{r_k/3}(\gamma(t_k))$$

satisfying

$$\gamma(t) \in B_{r_k/3}(\gamma(t_k)) \quad \text{for } t_k \leq t < t_{k+1}.$$

Obviously,  $t_{k+1} - t_k \geq r_k/3 \geq d/3$ . Let  $r_{k+1}, y_{k+1}$  and  $x_{k+1}$  be defined as previous process. Thanks to the lower bound of step length, there exists  $j \in N$ , such that  $t_{j-1} < t^* \leq t_j$ .

Let  $\tilde{\gamma}$  be a path consisting of straight line segments  $x_k x_{k+1}$  ( $k = 0, \dots, j$ ), with  $x_0 = z_1$  and  $x_{j+1} = z_2$ . We claim that  $\text{dist}(\tilde{\gamma}, K) > \min\{1.8d, \delta\}$ . In fact,

The straight lines  $z_1x_1$  and  $x_jz_2$  are contained in  $A_1$  and  $A_3$  respectively, so their distances to  $K$  are greater than  $\delta$ . For any straight line  $x_kx_{k+1}$  ( $k = 1, \dots, j - 1$ ), we assume  $\text{dist}(x_kx_{k+1}, K) \leq \delta$ . The three relations

$$\begin{aligned} \angle(\gamma(t_k) - y_k, T_{y_k, 5r_k}^\perp) &< C(n)\delta, \\ \gamma(t_{k+1}) &\in \partial B_{r_k/3}(\gamma(t_k)), \\ \text{HD}(K \cap B_{5r_1}(y_1), T_{y_1, 5r_1} \cap B_{5r_1}(y_1)) &< 5r_1\delta \end{aligned}$$

together imply that

$$\angle(\gamma(t_{k+1}) - y_{k+1}, \gamma(t_k) - y_k) < C(n)\delta.$$

Therefore,  $\text{dist}(x_kx_{k+1}, K) > \min\{r_k, r_{k+1}\} + 0.8d \geq 1.8d$ .

If  $1.8d < \delta$ , we continue to move  $\tilde{\gamma}$ . After finitely many such moves, we obtain a path whose distance to  $K$  exceeds  $\delta$ . This proves (I).

For any point  $b \in A_2 \cap B_{1/2} - K$ , define  $d_b := \text{dist}(b, K)$ . We can move  $b$  (away from  $K$ ) to  $b_1$  such that  $\text{dist}(b_1, K) \geq 1.9d_b$  by the previous method. If necessary, we continue to move  $b_1$  to  $b_2, \dots$ . After finite (say  $i$ ) times, it will happen that  $\text{dist}(b_i, K) > 3\delta$ . So  $b_i \in A_1$ (or  $A_3$ ). However, the straight lines  $b_kb_{k+1}$  ( $k = 0, \dots, i - 1$ ), where  $b_0 = b$ , are contained in  $B_1 - K$ . So  $b \in \Omega_1$  or  $b \in \Omega_2$ . This proves (II).

For any  $a \in K \cap B_{1/2}$  and  $r \in (0, R]$  satisfying  $|a| + r < 1/2$ , let  $J \in \mathbb{N}$  be such that  $2^{J-1}r \leq 4\delta < 2^Jr$ . We consider  $B_{2^k r}(a)$  and  $T_{a, 2^k r}$  for  $k = 0, 1, \dots, J$ . For any  $k$ , the set  $B_{2^k r}(a) - N(T_{a, 2^k r}, 2^k r\delta)$  have just two parts. Because of the closeness of  $T_{a, 2^k r}$  and  $T_{a, 2^{k+1} r}$ , the two parts of  $B_{2^k r}(a) - N(T_{a, 2^k r}, 2^k r\delta)$  meet the two parts of  $B_{2^{k+1} r}(a) - N(T_{a, 2^{k+1} r}, 2^{k+1} r\delta)$  respectively. Moreover, due to that  $2^J r$  exceed the width of  $A_2$ , the two parts of  $B_{2^J r}(a) - N(T_{a, 2^J r}, 2^J r\delta)$  meet  $A_1 \cap B_{3/4}$  and  $A_3 \cap B_{3/4}$  respectively. Note that if two connected subset meet, then they belong to the same connected component. Therefore, the two parts of  $B_r(a) - N(T_{a, r}, r\delta)$  belong to  $\Omega_1$  and  $\Omega_2$  respectively. This proves (III).

(II) and (III) imply (IV). □

**Remark 4.2.** Together, Theorems 4.1 and 1.4 imply Theorem 1.2 in the case  $m = n - 1$ .

### 5. Extension to parabolic space

We now turn to the extension of the theorem of parabolic space. We start with some notations and definitions:

- (1)  $y = (y_1, \dots, y_n) = (y', y_n)$  is a typical point of  $\mathbb{R}^n$ , and  $y' \in \mathbb{R}^{n-1}$ .

- (2)  $|y^1 - y^2| =: (\sum_{i=1}^n (y_i^1 - y_i^2)^2)^{1/2}$  and  $\langle y^1, y^2 \rangle =: \sum_{i=1}^n y_i^1 y_i^2$  are the standard Euclidean metric and inner product.
- (3)  $x = (y, t) \in \mathbb{R}^n \times \mathbb{R}$  is a point of  $\mathbb{R}^{n+1}$ .
- (4) The parabolic distance between  $x^1$  and  $x^2$  is defined as
 
$$|x^1 - x^2|_p := (|y^1 - y^2|^2 + |t^1 - t^2|)^{1/2},$$
 and  $\mathbb{R}^{n+1}$  with this distance is called *parabolic space*.
- (5)  $B_r(a) := \{y \in \mathbb{R}^n : |y - a| < r\}$ , where  $a \in \mathbb{R}^n$ ,  $B_r =: B_r(0)$ .
- (6)  $T_r(a) := \{s \in \mathbb{R}^{n-1} : |s - a| < r\}$ , where  $a \in \mathbb{R}^{n-1}$ .
- (7) The *parabolic ball* is  $\mathfrak{p}B_r(a) := \{x \in \mathbb{R}^{n+1} : |x - a|_p < r\}$ , where  $a \in \mathbb{R}^{n+1}$ .
- (8)  $\mathfrak{p}T_r(a) := \{z \in \mathbb{R}^{(n-1)+1} : |z - a|_p < r\}$ , where  $a \in \mathbb{R}^{(n-1)+1}$ .
- (9) A *hyperplane* in parabolic space  $\mathbb{R}^{n+1}$  has the form  $\tilde{E} = E \times \mathbb{R}$ , namely,  $\tilde{E} =: \{(y, t) \in \mathbb{R}^{n+1} : y \in E\}$ , where  $E$  is a hyperplane in Euclidean  $\mathbb{R}^n$ .
- (10) For  $y^0 \in \mathbb{R}^n$  and  $\vec{n} \in \mathbb{R}^n$  with  $|\vec{n}| = 1$ , we denote by  $E(y^0, \vec{n}) := \{y \in \mathbb{R}^n : \langle y - y^0, \vec{n} \rangle = 0\}$  the hyperplane passing through  $y^0$  with normal vector  $\vec{n}$  in  $\mathbb{R}^n$ .

In the codimension-1 case, if we substitute parabolic ball, parabolic hyperplane and parabolic distance for Euclidean ball, hyperplane and distance in Definition 1.1, we get the definition of a *Reifenberg flat set in parabolic space*. We give two simple examples demonstrating what such set looks like. It is easy to verify them.

**Example 5.1.** In parabolic  $\mathbb{R} \times \mathbb{R}$ , the line with slope  $k$ , namely, the graph of  $t = ky + b$  is  $(\delta, R)$ -Reifenberg flat if  $R < \frac{1}{2}\delta k$ .

**Example 5.2.** In parabolic  $\mathbb{R} \times \mathbb{R}$ , the graph of  $t = \text{sign}(y)k|y|^{1+\alpha}$  is  $(\delta, R)$ -Reifenberg flat if  $R < (k(\delta/2)^{1+\alpha})^{1/(1-\alpha)}$ , where  $0 \leq \alpha < 1$ .

Now we state our second main theorem.

**Theorem 5.3.** Let  $\Omega \subset \mathbb{R}^{n+1}$  be a domain such that the origin  $O = (0, 0)$  lies in  $\partial\Omega$ , with normal vector  $\vec{n}(O, 2) = e_n$ . Let  $\delta > 0$  be a (small) constant. Assume that, for every  $x^1 = (y^1, t^1) \in \mathfrak{p}B_2 \cap \partial\Omega$  and every  $r > 0$  satisfying  $\mathfrak{p}B_r(x^1) \subset \mathfrak{p}B_2$ , there exists a unit vector  $\vec{n}(x^1, r) \in \mathbb{R}^n$  such that

$$\begin{aligned} \mathfrak{p}B_r(x^1) \cap \{(y, t) \in \mathbb{R}^{n+1} : \langle y - (y^1 - r\delta\vec{n}(x^1, r)), \vec{n}(x^1, r) \rangle < 0\} \\ \subset \mathfrak{p}B_r(x^1) \cap \Omega \\ \subset \{(y, t) \in \mathbb{R}^{n+1} : \langle y - (y^1 + r\delta\vec{n}(x^1, r)), \vec{n}(x^1, r) \rangle < 0\}. \end{aligned}$$

Then there exist a map

$$f : \mathfrak{p}T_{1/2} \subset \mathbb{R}^{(n-1)+1} \rightarrow \partial\Omega \cap \mathfrak{p}B_1$$

and numbers  $0 < \alpha < 1$  and  $\beta > 1$  such that

$$f(pT_{1/2}) \supset \partial\Omega \cap pB_{1/4}$$

and

$$c_2|x^1 - x^2|_p^\beta \leq |f(x^1) - f(x^2)|_p \leq c_1|x^1 - x^2|_p^\alpha$$

for any  $x^1, x^2 \in pT_{1/2}$ .

Domains satisfying the “for every...” condition of Theorem 5.3 are called *parabolic Reifenberg flat*. Such domains are geometrically invariant with respect to parabolic scaling. For the study of parabolic equations on parabolic Reifenberg flat domains, see [Byun and Wang 2005].

The proof of Theorem 5.3 parallels that of Theorem 1.4, so we omit most details. Attention should be paid to the degenerate direction  $t$ .

**The parabolic mollifier.** We define

$$\eta(y, t) := \begin{cases} C_1(\frac{1}{2} - |y|^4 - |t|^2)^3 & \text{if } |y|^4 + |t|^2 \leq \frac{1}{2}, \\ 0 & \text{else,} \end{cases}$$

where the constant  $C_1$  is so adjusted that  $\int_{\mathbb{R}^{n+1}} \eta(y, t) dy dt = 1$ . Next define

$$\eta_\varepsilon(y, t) := \varepsilon^{-(n+2)}\eta(\varepsilon^{-1}y, \varepsilon^{-2}t)$$

Thus

$$\eta_\varepsilon(y, t) \in C_0^2(pB_\varepsilon),$$

and

$$\int_{\mathbb{R}^{n+1}} \eta_\varepsilon(y, t) dy dt = 1.$$

**Level sets.** From now on, we assume  $\Omega$  is as in Theorem 5.3. Let  $\chi_\Omega$  be the characteristic function of  $\Omega$ , and define

$$\begin{aligned} \chi_\varepsilon &:= \chi_\Omega * \eta_\varepsilon, \\ L_\varepsilon &:= \{x \in \mathbb{R}^{n+1} : \chi_\varepsilon(x) = \frac{1}{2}\}, \quad \Omega_t := \{y \in \mathbb{R}^n, (y, t) \in \Omega\}, \\ L_{\varepsilon,t} &:= \{y \in \mathbb{R}^n : (y, t) \in L_\varepsilon\}, \quad \partial\Omega_t := \{y \in \mathbb{R}^n, (y, t) \in \partial\Omega\}. \end{aligned}$$

**Curvature estimate of the  $\frac{1}{2}$ -level set  $L_{\varepsilon,t}$ .** Fix  $y^0 \in L_\varepsilon, t^0$ , and let  $y^1 \in \partial\Omega_{t^0}$  be such that  $|y^1 - y^0| = \inf_{y \in \partial\Omega_{t^0}} |y - y^0| =: d$ . Obviously,  $d < \varepsilon$ . Define

$$\begin{aligned} A_1 &:= \{(y, t) : \langle y - (y^1 - 4\varepsilon\delta\vec{n}((y^1, t^0), 4\varepsilon)), \vec{n}((y^1, t^0), 4\varepsilon) \rangle < 0\} \cap pB_{4\varepsilon}(y^1, t^0), \\ A_2 &:= \{(y, t) : \langle y - (y^1 + 4\varepsilon\delta\vec{n}((y^1, t^0), 4\varepsilon)), \vec{n}((y^1, t^0), 4\varepsilon) \rangle \geq 0\} \cap pB_{4\varepsilon}(y^1, t^0) \\ A_3 &:= pB_{4\varepsilon}(y^1, t^0) - (A_1 \cup A_2), \\ T &:= \partial A_1 - \partial pB_{4\varepsilon}(y^1, t^0). \end{aligned}$$

**Lemma 5.4.**  $(y^0, t^0) \in A_2$ , and  $d \leq 8\varepsilon\delta$ .

*Proof.* The proof is the same as for Lemma 2.1. □

We assume that  $e_n = -\vec{n}((y^1, t^0), 4\varepsilon)$ . Let  $\hat{e}$  be any unit vector in  $\mathbb{R}^{n-1}$  and set  $e := (\hat{e}, 0) \in \mathbb{R}^n$ .

**Lemma 5.5.** For any  $\bar{x} = (\bar{y}, \bar{t}) \in A_2 \cap pB_\varepsilon(y^1, t^0)$  we have

$$\begin{aligned} \frac{\partial}{\partial e_n} \chi_\varepsilon(\bar{x}) &\geq C_2 \varepsilon^{-1}, & \left| \frac{\partial}{\partial e} \chi_\varepsilon(\bar{x}) \right| &\leq C_3 \delta \varepsilon^{-1}, & \left| \frac{\partial}{\partial \bar{t}} \chi_\varepsilon(\bar{x}) \right| &\leq C_4 \delta \varepsilon^{-2}, \\ \left| \frac{\partial^2}{\partial e^2} \chi_\varepsilon(\bar{x}) \right| &\leq C_5 \delta \varepsilon^{-2}, & \left| \frac{\partial^2}{\partial e \partial e_n} \chi_\varepsilon(\bar{x}) \right| &\leq C_5 \delta \varepsilon^{-2}, & \left| \frac{\partial^2}{\partial e_n^2} \chi_\varepsilon(\bar{x}) \right| &\leq C_6 \delta \varepsilon^{-2}, \\ \left| \frac{\partial^2}{\partial e \partial \bar{t}} \chi_\varepsilon(\bar{x}) \right| &\leq C_7 \delta \varepsilon^{-3} & \left| \frac{\partial^2}{\partial e_n \partial \bar{t}} \chi_\varepsilon(\bar{x}) \right| &\leq C_8 \delta \varepsilon^{-3}. \end{aligned}$$

*Proof of the third inequality.*

$$\begin{aligned} \frac{\partial}{\partial \bar{t}} \chi_\varepsilon(\bar{x}) &= \int_\Omega \frac{\partial}{\partial \bar{t}} \eta_\varepsilon(\bar{y}-y, \bar{t}-t) dy dt \\ &= \int_{A_1 \cap pB_\varepsilon(\bar{y}, \bar{t})} \frac{\partial}{\partial \bar{t}} \eta_\varepsilon(\bar{y}-y, \bar{t}-t) dy dt \\ &\quad + \int_{\Omega \cap A_2 \cap pB_\varepsilon(\bar{y}, \bar{t})} \frac{\partial}{\partial \bar{t}} \eta_\varepsilon(\bar{y}-y, \bar{t}-t) dy dt \\ &=: I_5 + I_6; \end{aligned}$$

but  $I_5 = 0$  by Green's formula, and

$$\begin{aligned} |I_6| &\leq \int_{A_2 \cap pB_\varepsilon(\bar{y}, \bar{t})} \left| \frac{\partial}{\partial \bar{t}} \eta_\varepsilon(\bar{y}-y, \bar{t}-t) \right| dy dt \\ &\leq C'_4 \varepsilon^{-(n+2)} \varepsilon^{-2} \cdot C''_4 \delta \varepsilon^{n+2} = C_4 \delta \varepsilon^{-2}. \end{aligned} \quad \square$$

From the preceding lemma, we know that  $\nabla \chi_\varepsilon(x) \neq 0$  for  $x \in L_\varepsilon \cap pB_1$ ; thus  $L_\varepsilon \cap pB_1$  is a  $C^2$   $n$ -dimensional manifold and can be represent as the graph of some function from  $\mathbb{R}^{n-1} \times \mathbb{R}$  to  $\mathbb{R}$  locally. Take  $x \in L_\varepsilon \cap pB_1$  of the form  $(y^0, t^0)$ ; then in  $A_2 \cap pB_\varepsilon(y^1, t^0)$ , the function can be taken as

$$y_n = \varphi(y_1, \dots, y_{n-1}, t) =: \varphi(y_T, t).$$

**Lemma 5.6.**

$$\begin{aligned} \left| \frac{\partial}{\partial \hat{e}} \varphi(y_T, t) \right| &< \frac{C_3}{C_2} \delta, & \left| \frac{\partial^2}{\partial \hat{e}^2} \varphi(y_T, t) \right| &\leq C_9 \delta \varepsilon^{-1}, \\ \left| \frac{\partial}{\partial t} \varphi(y_T, t) \right| &< \frac{C_4}{C_2} \delta \varepsilon^{-1}, & \left| \frac{\partial^2}{\partial \hat{e} \partial t} \varphi(y_T, t) \right| &\leq C_{10} \delta \varepsilon^{-2}. \end{aligned}$$

*Proof.* The proof is the same as for Lemma 2.3. □



**Corollary 5.7.** *For any  $t \in (-1, 1)$ ,  $L_{\varepsilon,t} \cap B_1$  is a  $C^2$   $(n - 1)$ -dimensional manifold in  $\mathbb{R}^n$ . The principal curvature of  $L_{\varepsilon,t}$  in any tangent direction is less than  $C_9\delta\varepsilon^{-1}$ . So a tubular neighborhood with radii  $\varepsilon$  exists as long as  $C_9\delta < 1$ .*

**Closeness of  $L_\varepsilon$  to  $\partial\Omega$ .** Recall from page 326 the notation  $N(E, s)$  for an  $s$ -tubular neighborhood of  $E$  and  $\pi$  for the corresponding normal projection map. Further, for  $F \in \mathbb{R}^{n+1}$ , let  $F_t =: \{y \in \mathbb{R}^n : (y, t) \in F\}$ . If each  $F_t$  has a tubular neighborhood of radius  $\tau$ , we define the *parabolic tubular neighborhood*

$$pN(F, \tau) := \bigcup_t N(F_t, \tau) \times \{t\},$$

and the corresponding *projection map*  $\tilde{\pi}: pN(F, \tau) \rightarrow F$  such that  $\tilde{\pi}(y, t) = (\pi_t y, t)$ , where  $\pi_t$  is the normal projection operator of  $N(F_t, \tau)$ .

**Lemma 5.8.**  $\partial\Omega \cap pB_{r-\varepsilon} \subset pN(L_\varepsilon \cap pB_r, 4\varepsilon\delta)$ , for  $r \leq 1$ .

*Proof.* Same as for Lemma 2.5. □

Taking  $\varepsilon = 4^{-k}$ ,  $k = 1, 2, \dots$ , we obtain a sequence of approximate sets  $L_{4^{-k}}$ . Denote  $\mathcal{S}_0 =: \{y_n = 0\} \cap pB_{1/2}$ ,  $\mathcal{S}_k =: L_{4^{-k}} \cap pB_{r(k)}$ , where  $r(k) = \frac{1}{2} + \sum_{j=1}^k 4^{-j}$ .

**Corollary 5.9.**  $\mathcal{S}_k \in pN(L_{4^{-k+1}}, 6 \cdot 4^{-k+1}\delta)$ .

*Proof.* Same as for Corollary 2.6. □

**Bi-Lipschitz maps.**

**Lemma 5.10.**  $\tilde{\pi}: \mathcal{S}_k \rightarrow L_{4^{-k+1}}$  is injective, and if  $\delta$  is small enough,  $|z^1 - z^2|_p \leq 4^{-k}$ , then

$$\frac{1}{2}|\tilde{\pi}(z^1) - \tilde{\pi}(z^2)|_p \leq |z^1 - z^2|_p \leq 2|\tilde{\pi}(z^1) - \tilde{\pi}(z^2)|_p.$$

*Proof.* We write  $x^1 =: \tilde{\pi}(z^1) = (y^1, t^1)$ ,  $x^2 =: \tilde{\pi}(z^2) = (y^2, t^2) \in L_{4^{-k+1}}$ , and  $z^1 = (y^3, t^1)$ ,  $z^2 = (y^4, t^2) \in \mathcal{S}_k$ . For simplicity, we still write  $4^{-k+1}$  as  $\varepsilon$  sometimes. We assume that  $O$  is the nearest point of  $\partial\Omega \cap \{t = t^1\}$  to  $z^1$ , by choosing an appropriate coordinate system, we can let  $O = (0, 0)$  (that means  $t^1 = 0$ ) and  $\vec{n}(0, 4\varepsilon) = -e_n$ . Define

$$\begin{aligned} A_1 &:= \{(y, t) : y_n > 4\varepsilon\delta\} \cap pB_{4\varepsilon}, \\ A_2 &:= \{(y, t) : -4\varepsilon\delta \leq y_n \leq 4\varepsilon\delta\} \cap pB_{4\varepsilon}, \\ A_3 &:= \{(y, t) : y_n < -4\varepsilon\delta\} \cap pB_{4\varepsilon}. \end{aligned}$$

Then  $A_1 \subset \Omega$ ,  $A_3 \subset \bar{\Omega}^c$ , and  $x^1, x^2, z^1, z^2 \in A_2 \cap pB_\varepsilon$ .

We assume  $L_\varepsilon \cap pB_\varepsilon$  is the graph of a function  $y_n = \varphi(y_T, t) = \varphi(y_1, \dots, y_{n-1}, t)$ , and  $L_{\varepsilon/4} \cap pB_\varepsilon$  is the graph of a function  $y_n = \psi(y_T, t)$ . Thus, we can write

$$\begin{aligned} x^1 &= (y_T^1, \varphi(y_T^1, 0), 0), & x^2 &= (y_T^2, \varphi(y_T^2, t^2), t^2), \\ z^1 &= (y_T^3, \psi(y_T^3, 0), 0), & z^2 &= (y_T^4, \psi(y_T^4, t^2), t^2). \end{aligned}$$

Set  $l := |y_T^1 - y_T^2|$  and  $\hat{l} := |y_T^3 - y_T^4|$ ; by assumption,

$$(5-1) \quad \hat{l} \leq \varepsilon/4, \quad l \leq \varepsilon/2, \quad |t^2| \leq \varepsilon^2.$$

By Lemma 5.6 and the Differential Mean Value Theorem,

$$(5-2) \quad \begin{aligned} |\varphi(y_T^1, 0) - \varphi(y_T^2, t^2)| &\leq l \frac{C_3}{C_2} \delta + |t^2| \frac{C_4}{C_2} \delta \varepsilon^{-1}, \\ |\psi(y_T^3, 0) - \varphi(y_T^4, t^2)| &\leq \hat{l} \frac{C_3}{C_2} \delta + |t^2| \frac{C_4}{C_2} \delta \varepsilon^{-1}. \end{aligned}$$

Then

$$(5-3) \quad \begin{aligned} \sqrt{l^2 + |t^2|} \leq |x^1 - x^2|_p &\leq \sqrt{l^2 + 2\left(\frac{C_3}{C_2}\delta\right)^2 l^2 + 2\left(\frac{C_4}{C_2}\delta\right)^2 \varepsilon^{-2} |t^2|^2 + |t^2|} \\ &\leq \frac{11}{10} \sqrt{l^2 + |t^2|} \quad (\text{using } \delta \text{ small and (5-1)}); \end{aligned}$$

and

$$(5-4) \quad \sqrt{\hat{l}^2 + |t^2|} \leq |z^1 - z^2|_p \leq \frac{11}{10} \sqrt{\hat{l}^2 + |t^2|}.$$

Since  $\tilde{\pi}(z^1) = x^1$ ,  $\tilde{\pi}(z^2) = x^2$ , there exist  $\xi$  and  $\zeta \in \mathbb{R}$  such that

$$(5-5) \quad y^3 = y^1 + \xi v(x^1), \quad y^4 = y^2 + \zeta v(x^2),$$

where  $v(x^1)$ ,  $v(x^2)$  are inward-pointing unit normal vectors of  $L_\varepsilon$  at  $x^1$ ,  $x^2$  (with expressions similar to those near the bottom of page 327). By Corollary 5.9,

$$(5-6) \quad |\xi|, |\zeta| \leq 6 \cdot \varepsilon \delta;$$

moreover,  $|v_i(x^j)| < C_3/C_2\delta$  for  $i = 1, \dots, n-1$ , and, by Lemma 5.6,

$$v_n(x^j) > \frac{1}{\sqrt{1 + (C_3/C_2\delta)^2}} \quad \text{for } j = 1, 2.$$

These two inequalities show that  $v(x^j)$  is very close to  $e_n$  when  $\delta$  is very small.

Subtracting the second equality in (5-5) from the first, we get

$$(5-7) \quad y_i^3 - y_i^4 = y_i^1 - y_i^2 + \xi(v_i(x^1) - v_i(x^2)) + (\xi - \zeta)v_i(x^2),$$

$$(5-8) \quad y_n^3 - y_n^4 = y_n^1 - y_n^2 + \xi(v_n(x^1) - v_n(x^2)) + (\xi - \zeta)v_n(x^2).$$

By Lemma 5.6 and the Differential Mean Value Theorem,

$$\begin{aligned} |v_i(x^1) - v_i(x^2)| &\leq |D_i\varphi(y_T^2, t^2) - D_i\varphi(y_T^1, 0)| + |\nabla_{y_T}\varphi(y_T^1, 0)|^2 \cdot |\nabla_{y_T}\varphi(y_T^2, t^2) - \nabla_{y_T}\varphi(y_T^1, 0)| \\ &\leq 2(l \cdot C_9\delta\varepsilon^{-1} + |t^2| \cdot C_{10}\delta\varepsilon^{-2}) \end{aligned}$$

and

$$\begin{aligned}
 (5-9) \quad |v_n(x^1) - v_n(x^2)| &\leq (|\nabla_{y_T} \varphi(y_T^1, 0)| + |\nabla_{y_T} \varphi(y_T^2, t^2)|) \cdot |\nabla_{y_T} \varphi(y_T^1, 0) - \nabla_{y_T} \varphi(y_T^2, t^2)| \\
 &\leq l C_9 \delta \varepsilon^{-1} + |t^2| C_{10} \delta \varepsilon^{-2}.
 \end{aligned}$$

Equality (5-8) implies

$$\begin{aligned}
 |\xi - \zeta| &\leq \sqrt{1 + \left(\frac{C_3}{C_2} \delta\right)^2} (|y_n^3 - y_n^4| + |y_n^1 - y_n^2| + |\xi| |v_n(x^1) - v_n(x^2)|) \\
 &\leq \sqrt{1 + \left(\frac{C_3}{C_2} \delta\right)^2} \left( \hat{l} \frac{C_3}{C_2} \delta + |t^2| \frac{C_4}{C_2} \delta \varepsilon^{-1} + l \frac{C_3}{C_2} \delta + |t^2| \frac{C_4}{C_2} \delta \varepsilon^{-1} \right. \\
 &\quad \left. + 5\varepsilon \delta (l C_9 \delta \varepsilon^{-1} + |t^2| C_{10} \delta \varepsilon^{-2}) \right) \\
 &\leq C_{11} (\delta l + \delta \hat{l} + \delta \varepsilon^{-1} |t^2|),
 \end{aligned}$$

where we used (5-2), (5-6) and (5-9) in the second inequality and the third holds for  $\delta$  small.

Using the triangle inequality, (5-7) implies

$$\begin{aligned}
 \hat{l} &\leq l + |\xi| |v_i(x^1) - v_i(x^2)| + |\xi - \zeta| |v_i(x^2)| \\
 &\leq (1 + C_{12} \delta^2) l + C_{11} \frac{C_3}{C_2} \delta^2 \hat{l} + C_{12} \delta^2 \varepsilon^{-1} |t^2|.
 \end{aligned}$$

Thus, if  $\delta$  is small enough,

$$\hat{l} \leq \frac{11}{10} l + \delta \varepsilon^{-1} |t^2|.$$

Similarly, we can show that

$$(5-10) \quad \hat{l} \geq \frac{9}{10} l - \delta \varepsilon^{-1} |t^2|.$$

Therefore,

$$(5-11) \quad \sqrt{\hat{l}^2 + |t^2|} \leq \sqrt{2\left(\frac{11}{10}\right)^2 l^2 + 2\delta^2 \varepsilon^{-2} |t^2|^2 + |t^2|} \leq \sqrt{2} \frac{11}{10} \sqrt{l^2 + |t^2|},$$

where we have used (5-1). On the other hand, if  $l^2 \leq |t^2|$ , we have

$$(5-12) \quad \sqrt{\hat{l}^2 + |t^2|} \geq \sqrt{|t^2|} \geq \frac{1}{\sqrt{2}} \sqrt{l^2 + |t^2|},$$

and if  $l^2 > |t^2|$ , then (5-10) becomes

$$\hat{l} > \frac{9}{10} l - \delta \varepsilon^{-1} l^2 > \left(\frac{9}{10} - \delta\right) l > \frac{4}{5} l$$

(using (3-1)), so we also have

$$(5-13) \quad \sqrt{\hat{l}^2 + |t^2|} \geq \frac{4}{5} \sqrt{l^2 + |t^2|}.$$

Combining (5-3), (5-4), (5-11), (5-12) and (5-13), we conclude the proof.  $\square$

We define  $g_k : \mathcal{S}_{k-1} \rightarrow \mathcal{S}_k$  satisfying  $\tilde{\pi}(g_k(x)) = x$  for  $k = 1, 2, \dots$ , and  $h_k : \mathcal{S}_0 \rightarrow \mathcal{S}_k$  by setting  $h_k := g_k \circ \dots \circ g_1$ .

**Corollary 5.11.**  $|h_k(x) - h_{k-1}(x)| = |g_k(h_{k-1}(x)) - h_{k-1}(x)| < 6 \cdot 4^{-k+1} \delta$ .

*Proof of Theorem 5.3.* The proof is assembled from Corollary 5.11 and Lemmas 5.8 and 5.10 in precisely the same way as the proof of Theorem 1.4 (Section 3), with obvious modifications to the notation. The same constants can be used.  $\square$

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