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The exterior square L-function attached to an automorphic cuspidal representation of GL_{2n} has a pole if and only if a certain period integral does not vanish on the space of the representation. We conjecture, in the "if" direction, a similar result is true for representations of $\mathrm{GL}_2(D)$, where D is a division algebra. We prove a partial result which provides evidence for the conjecture. The proof is based on a relative trace formula.

1. Introduction

Let F be a number field, $\mathbb{A}$ its ring of adeles, and D a division algebra of rank m^2 over F. We regard $D^\times$, GL_m, $G = \mathrm{GL}_2(D)$, and $G' = \mathrm{GL}_{2m}$ as algebraic groups defined over F. The multiplicative group $F^\times$ of F is identified with the center Z of each one of these groups. For an algebraic group H over F and a place v of F, we will denote the group of its F_v-points by H_v. By an automorphic representation of $H(\mathbb{A})$, we will mean a subrepresentation of $L^2(Z(\mathbb{A})H(F)\backslash H(\mathbb{A}))$.

The Jacquet–Langlands correspondence associates to each automorphic representation π of $D^\times(\mathbb{A})$ an automorphic representation π' of $\mathrm{GL}_m(\mathbb{A})$ such that $\pi_v \simeq \pi'_v$ at all places where $D_v^\times \simeq \mathrm{GL}_m(F_v)$; see [Harris and Taylor 2001]. It is conjectured that there is a similar Jacquet–Langlands correspondence between representations π of $G(\mathbb{A})$ and π' of $G'(\mathbb{A})$ (or more generally between any inner forms of GL_n) such that $\pi_v \simeq \pi'_v$ when $G_v \simeq G'_v$. A consequence of this conjecture is that multiplicity one and strong multiplicity one theorems should hold for G. Such a correspondence has been established when D is split at each infinite place [Badulescu 2007].

Suppose π and π' are cuspidal representations of $G(\mathbb{A})$ and $G'(\mathbb{A})$, respectively, that satisfy the Jacquet–Langlands correspondence. Assume that π satisfies multiplicity one and strong multiplicity one, that is, if π^0 is an automorphic representation of $G(\mathbb{A})$ with $\pi_v \simeq \pi_v^0$ for almost all v, then $\pi = \pi^0$. The purpose of this

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note is to illustrate how the relative trace formula may be used to compare *Shalika periods* on π with those on π'.

Let S be the subgroup of G of elements of the form

$$\left\{\begin{pmatrix} A & X \\ 0 & A \end{pmatrix}\right\} = \left\{\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}\begin{pmatrix} I & X \\ 0 & I \end{pmatrix}\right\} = \left\{\begin{pmatrix} I & X \\ 0 & I \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}\right\},$$

where $A \in D^\times$ and $X \in D$. Similarly, let S' be the subgroup of G' of elements of the same form, but with $A \in \mathrm{GL}_m$ and $X \in M_{m\times m}$. This is called the *Shalika subgroup* of G (or G'). Let ψ be a nontrivial additive character of $\mathbb{A}/F$. Define a character θ on $S(\mathbb{A})$ (or on $S'(\mathbb{A})$) by

$$\theta\left(\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}\begin{pmatrix} I & X \\ 0 & I \end{pmatrix}\right) = \psi(\mathrm{tr}(X)),$$

where tr denotes the reduced trace of D (or the trace on $M_{m\times m}$). The Shalika subgroup is unimodular, and a Haar measure on $S(\mathbb{A})$ (or on $S'(\mathbb{A})$) is given by a product $dA\, dX$ where dA and dX are both Haar measures on the appropriate spaces.

We say that an automorphic representation π of $G(\mathbb{A})$ is *distinguished* (by θ), if it has trivial central character and the period integral

$$\lambda(\phi) := \int_{Z(\mathbb{A})S(F)\backslash S(\mathbb{A})} \phi(s)\theta^{-1}(s)ds$$

is nonzero for some smooth function ϕ in the space of π. Note that the quotient $Z(\mathbb{A})S(F)\backslash S(\mathbb{A})$ is compact, and so the integral converges. Conjugating by a matrix of the form

$$\gamma = \begin{pmatrix} \alpha\cdot I & 0 \\ 0 & I \end{pmatrix} \quad \text{for } \alpha \in F^\times,$$

we see that if the condition of distinction is satisfied for π, it is also satisfied with ψ replaced by the character $x \mapsto \psi(\alpha x)$. Thus the condition is independent of the choice of ψ.

We make the same definitions for S' *mutatis mutandis*. We define a character θ on $S'(\mathbb{A})$ by

$$\theta\left(\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}\begin{pmatrix} I & X \\ 0 & I \end{pmatrix}\right) = \psi(\mathrm{tr}(X)).$$

Thus a cuspidal automorphic representation π' is *distinguished* if and only if its central character is trivial and the period integral

$$\lambda'(\phi) := \int_{Z(\mathbb{A})S'(F)\backslash S'(\mathbb{A})} \phi(s)\theta^{-1}(s)ds$$

is nonzero for at least one smooth element ϕ in the space of π'. Note that the quotient $Z(\mathbb{A})S'(F)\backslash S'(\mathbb{A})$ has finite volume — it is the product of the volumes of $Z(\mathbb{A})\,\mathrm{GL}_m(F)\backslash\,\mathrm{GL}_m(\mathbb{A})$ and $M_{m\times m}(F)\backslash M_{m\times m}(\mathbb{A})$. Since the cusp form ϕ is bounded, the integral converges.

We conjecture that if π is distinguished, then π' also is. The significance of this is as follows. Recall that π' is distinguished if and only if the exterior square L-function $L(s,\pi';\Lambda^2)$ attached to π' has a pole at $s=1$; see [Bump and Friedberg 1990; Jacquet and Shalika 1990; Jiang 2006]. This is proved using an integral representation of $L(s,\pi';\Lambda^2)$. On the other hand, there is no integral representation for the exterior square L-function $L(s,\pi;\Lambda^2)$ attached to π. Nonetheless, according to the conjecture, the nonvanishing of the period integral λ should imply the existence of a pole for $L(s,\pi;\Lambda^2)$.

Here we use a relative trace formula to establish the following partial result in the case $m=2$.

Theorem 1.1. *Let D be a quaternion algebra that ramifies at at least one infinite place. Suppose π_{v_0} is supercuspidal for some finite place v_0 where D splits. If π is distinguished by θ, then the automorphic cuspidal representation π' of G' corresponding to π is distinguished by θ'.*

The assumptions on D at infinity and π_{v_0} are purely technical assumptions made to keep the trace formulas as simple as possible; specifically, we avoid the need for truncation, whose details are presently unclear in this setting.

Subsequently, our conjecture has been proven for $m=2$ by Gan and Takeda using the theta correspondence [2007]. The authors remark, however, that their method will not apply to higher rank. On the other hand, it is expected that the trace formula approach we use here can be made completely general (with considerable work). Separately, Jiang, Nien and Qin have proved our conjecture, under some restrictions, for general n by yet another method [2007].

It is natural to also ask if the conjecture's converse is true, for this would make equivalent the nonvanishing of a Shalika period on π with the existence of a pole for $L(s,\pi;\Lambda^2)$. For now, we will not discuss this question but refer the reader to [Gan and Takeda 2007].

Introducing some more notations, we write a matrix g of G (or G') in the form

$$g=\begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Then we denote by P the parabolic subgroup of matrices for which $C=0$, by U the subgroup of those for which $C=0$ and $A=B=I$, by M the subgroup of those for which $C=0$ and $B=0$, and by H the subgroup of those for which $C=B=0$ and $A=D$. Thus S (or S') is the semidirect product of H and U.

Henceforth, we restrict ourselves to the situation $m = 2$, that is, D is a quaternion algebra, $G = \mathrm{GL}_2(D)$, and $G' = \mathrm{GL}_4$.

2. Local Orbital Integrals for GL_4

2.1. ***Relevant Double Cosets.*** Let F be any field and S be the Shalika subgroup of $G = \mathrm{GL}_4$ over F. The group $S(F) \times S(F)$ operates on $G(F)$ by $g \mapsto s_1 g s_2^{-1}$. Denote by σ the algebraic additive character $\sigma : S \to F$ defined by

$$\sigma\left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}\begin{pmatrix} I & X \\ 0 & I \end{pmatrix}\right) = \mathrm{tr}(X)\,.$$

We say that an element ξ is *relevant* if the algebraic character of $S \times S$,

$$(u_1, u_2) \mapsto \sigma(u_1) - \sigma(u_2),$$

is trivial on the stabilizer of ξ. It amounts to the same to require that

$$\sigma(\xi s \xi^{-1}) = \sigma(s)$$

on the group $S^\xi := S \cap \xi^{-1} S \xi$.

Let us write the elements of GL_4 in the form

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Recall P is the parabolic subgroup of G of matrices for which $C = 0$. Any double coset of S is contained in a double coset of P. There are 3 double cosets of P. The rank of C determines the double coset PgP. If $C = 0$, then g is in P. If C is invertible, then g lies in the double coset of

$$w := \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Finally, if C has rank 1, then g is in the double coset of

$$w_0 := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Now we come back to the double S-coset of an element g. If $C = 0$, then the double coset of g contains an element of the form

$$\xi = \begin{pmatrix} I & 0 \\ 0 & \gamma \end{pmatrix}.$$

Then S^{ξ} is the group of matrices of the form

$$\begin{pmatrix} A & X \\ 0 & A \end{pmatrix}, \ A = \gamma^{-1} A \gamma.$$

The element ξ is relevant if and only if $\operatorname{tr}(X) = \operatorname{tr}(X\gamma^{-1})$ for all X. This is so only if $\gamma = 1$. Thus the only relevant double coset of S contained in P is S itself.

If C is invertible then the double coset of g contains an element of the form

$$\xi_\gamma := \begin{pmatrix} 0 & \gamma \\ I & 0 \end{pmatrix}.$$

Then S^{ξ_γ} is the group of matrices of the form

$$\begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \quad \text{for } g \in T_\gamma ,$$

where T_γ is the centralizer of γ in $\mathrm{GL}_2(F)$. Such ξ_γ is relevant. Two elements ξ_{γ_1} and ξ_{γ_2} are in the same double coset if and only if γ_1 and γ_2 are conjugate in $\mathrm{GL}_2(F)$.

Next, we make the preliminary observation that for $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ to be relevant it is necessary that for $Y_1, Y_2 \in M_{2\times 2}(F)$ the relations

$$Y_1 C = 0, \qquad C Y_2 = 0, \qquad Y_1 D = A Y_2$$

imply $\operatorname{tr} Y_1 = \operatorname{tr} Y_2$.

If C has rank 1, then simple computations show that the double coset of g always contains an element of the form

$$\begin{pmatrix} 0 & 0 & a & b \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c & d \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c & d \end{pmatrix},$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is invertible.

In the first case, the preliminary observation implies that if g is relevant then $c = 1$. Further computations show that the double coset contains a unique element of the form

$$\eta_r := \begin{pmatrix} 0 & 0 & 0 & r \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

with $r \neq 0$. The intersection $S^{\eta_r} = S \cap \eta_r^{-1} S \eta_r$ is the group of matrices of the form

$$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & x \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and η_r is relevant.

In the second case, the preliminary observation implies that if g is relevant then $c = 0$. Further computations show that the double coset contains a unique element of the form

$$\epsilon_r := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

with $r \neq 0$. The group S^{ϵ_r} is the group of elements of the form

$$z \begin{pmatrix} 1 & x & u & y \\ 0 & 1 & 0 & ru \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

with z in Z. It is easily checked that such an element is relevant if and only if $r = -1$. Thus we have another relevant double coset, namely, the double coset of

$$\epsilon := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with S^{ϵ}, the group of matrices of the form

$$z \begin{pmatrix} 1 & x & -u & y \\ 0 & 1 & 0 & u \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

An important observation is that the antiautomorphism

$$g \mapsto w_1 \, {}^{t}g w_1 \quad \text{for } w_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

leaves invariant P, its unipotent radical U, the subgroup S and the character σ of S, and fixes any relevant double coset. Indeed, this antiautomorphism sends ξ_γ to

$\xi_{w_2\,{}^t\gamma w_2}$, with

$$w_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Since γ is conjugate to ${}^t\gamma$, the elements ξ_γ and $\xi_{w_2\,{}^t\gamma w_2}$ are in the same double coset. For the other double cosets, the given representative is actually invariant under the antiautomorphism. The observation follows. Altogether, we have:

Lemma 2.1. *The relevant S-double cosets for* $\mathrm{GL}_4(F)$ *are S, $S\xi_\gamma S$ (where γ is determined up to* $\mathrm{GL}_2(F)$*-conjugacy), $S\eta_r S$ with $r \neq 0$, and $S\epsilon S$.*

2.2. ***Local orbital integrals for* $\mathbf{GL_4(F)}$.** Let F be a local field. By abuse of notations, we often write $G = \mathrm{GL}_4(F)$. Let S denote the Shalika subgroup of G. Let ψ be a nontrivial additive character of F. We endow the vector space $M_{2\times 2}(F)$ with the self-dual Haar measure for the character $\psi \circ \mathrm{tr}$. On the other hand, we choose a Haar measure on $\mathrm{GL}_2(F)$ and $T_\gamma(F)$, the centralizer of γ in $\mathrm{GL}_2(F)$, in the usual way (as in the ordinary trace formula computations). We use the isomorphisms

$$X \leftrightarrow \begin{pmatrix} 1_2 & X \\ 0 & 1_2 \end{pmatrix} \quad \text{and} \quad g \leftrightarrow \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}$$

to transport these measures to U and H, respectively. The product of these measures is then a Haar measure on S.

As in the global case, we define a character $\theta : S(F) \to \mathbb{C}^\times$ by

$$\theta\left(\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}\begin{pmatrix} I & X \\ 0 & I \end{pmatrix}\right) = \psi(\mathrm{tr}(X)).$$

To say that an element g is relevant amounts to saying that the character

$$(u_1, u_2) \mapsto \theta(u_1)\theta(u_2)^{-1}$$

is trivial on the stabilizer of G in $S(F) \times S(F)$, that is,

$$\theta(gsg^{-1}) = \theta(g) \quad \text{for all } g \in S^g(F) := S(F) \cap g^{-1}S(F)g.$$

Assuming g is relevant, we study the *orbital integrals* of a function $f \in C_c^\infty(G)$, that is, the integrals

$$\Xi(f, g) := \int f(s_1 g s_2^{-1})\theta(s_1)\theta(s_2)^{-1} ds_1 ds_2.$$

The integral is over the quotient of $S(F) \times S(F)$ by the stabilizer of g in $S(F) \times S(F)$. We can also write this integral as

$$\Xi(f, g) := \int_{S^g(F)\backslash S(F)} \left(\int_{S(F)} f(s_1 g s_2)\theta(s_1) ds_1\right) \theta(s_2) ds_2. \tag{2-1}$$

If $g=1$, then

$$\Xi(f,1)=\int_S f(s)\theta(s)ds$$

and convergence is evident for smooth f of compact support.

If $g=\xi_\gamma$, then computing formally at first we define

$$F_f(h):=\int f\left(\begin{pmatrix} I & X\\ 0 & I\end{pmatrix}\begin{pmatrix} g & 0\\ 0 & g\end{pmatrix}\begin{pmatrix} 0 & h\\ I & 0\end{pmatrix}\begin{pmatrix} I & Y\\ 0 & I\end{pmatrix}\right)\psi(\mathrm{tr}(X+Y))dXdYdg\,.$$

Then

$$\Xi(f,\xi_\gamma)=O(F_f,\gamma)\,, \tag{2-2}$$

where, for any smooth function of compact support ϕ on $\mathrm{GL}_2(F)$, we denote by $O(\phi,\gamma)$ the orbital integral of ϕ on γ, that is,

$$O(\phi,\gamma)=\int_{\mathrm{GL}_2(F)/T_\gamma(F)}\phi(h\gamma h^{-1})dh\,.$$

To justify our computation we prove a lemma.

Lemma 2.2. *The integral defining F_f converges. If ϕ is any smooth function of compact support on $F^\times$, then the function $F_f(h)\phi(\det h)$ is a smooth function of compact support on* $\mathrm{GL}_2(F)$.

Proof. Indeed

$$\begin{pmatrix} I & X\\ 0 & I\end{pmatrix}\begin{pmatrix} 0 & gh\\ g & 0\end{pmatrix}\begin{pmatrix} I & Y\\ 0 & I\end{pmatrix}=\begin{pmatrix} Xg & gh+XgY\\ g & gY\end{pmatrix}$$

has determinant equal to $\det g^2\cdot\det h$. If this matrix belongs to a compact set and $\det h$ is also in a compact set, then $\det g$ is in a compact set of $F^\times$. By inspection, g is in a compact set of $M_{2\times 2}(F)$ and thus in fact in a compact set of $G(F)$. Now Xg and gY belong to compact sets of $M_{2\times 2}(F)$, and therefore X and Y do as well. Next, gh is in a compact set of $M_{2\times 2}(F)$, and hence so is h. Since $\det h$ is in a compact set of $F^\times$, we have finally that h is in a compact set of $G(F)$. □

For a given γ, we have $\det h\gamma h^{-1}=\det\gamma$. Thus our computation (2-2) is justified; the orbital integral converges. More precisely, for a given γ, it is equal to the orbital integral of a smooth function (which depends on γ) of compact support on $G(F)$.

In particular, assume F is non-Archimedean, ψ is unramified — that is, the largest ideal of F on which ψ is trivial is $\mathbb{O}$ — and f is the characteristic function of $\mathrm{GL}_4(\mathbb{O})$. Let ϕ_0 be the characteristic function of $\mathbb{O}^\times$. Then $F_f(h)\phi_0(\det h)$ is the characteristic function Φ_0 of $\mathrm{GL}_2(\mathbb{O})$. In other words, for $|\det\gamma|=1$, the orbital integral of Φ_0 at γ is $\Xi(f,\xi_\gamma)$.

We briefly discuss the convergence of the other orbital integrals. For η_r it suffices to prove that if the product

$$\begin{pmatrix} a & u & x & y \\ 0 & b & z & t \\ 0 & 0 & a & u \\ 0 & 0 & 0 & b \end{pmatrix} \eta_r \begin{pmatrix} 1 & u_1 & 0 & y_1 \\ 0 & 1 & z_1 & t_1 \\ 0 & 0 & 1 & u_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} u & uu_1+x & y+xz_1 & ar+t_1x+u_1y+uy_1 \\ b & bu_1+z & t+zz_1 & tu_1+by_1+t_1z \\ 0 & a & u+az_1 & at_1+uu_1 \\ 0 & 0 & b & bu_1 \end{pmatrix}$$

belongs to a compact set of GL_4, then a and b lie in compact sets of $F^\times$ and the other variables in a compact set of F. This is immediate.

For the element ϵ, it suffices to prove that if the product

$$\begin{pmatrix} a & u & x & y \\ 0 & b & z & t \\ 0 & 0 & a & u \\ 0 & 0 & 0 & b \end{pmatrix} \epsilon \begin{pmatrix} a_1 & 0 & x_1 & 0 \\ 0 & 1 & z_1 & 0 \\ 0 & 0 & a_1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} aa_1 & x & -a_1u+ax_1+xz_1 & 0 \\ 0 & z & -a_1b+zz_1 & t \\ 0 & a & az_1 & u \\ 0 & 0 & 0 & b \end{pmatrix}$$

belongs to a compact set of GL_4, then a, b, a_1 are in compact sets of $F^\times$ and the other variables are in a compact set of F. Again, this is immediate.

3. Local orbital integrals for $\mathrm{GL}_2(D)$

3.1. ***Local orbital integrals.*** Let F be any field and D a quaternion (or even any division) algebra of center F. Again, let S be the Shalika subgroup of $G=\mathrm{GL}_2(D)$ of matrices of the form

$$\begin{pmatrix} A & B \\ 0 & A \end{pmatrix},$$

and let σ be the algebraic character

$$\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \mapsto \mathrm{tr}(X),$$

where tr is the reduced trace. We say that an element ξ is *relevant* if the algebraic character

$$(u_1, u_2) \mapsto \sigma(u_1) - \sigma(u_2)$$

is trivial on the stabilizer of ξ in $S \times S$, or, what amounts to the same,

$$\sigma(\xi s \xi^{-1}) = \sigma(s)$$

on the group $S^\xi := S \cap \xi^{-1} S \xi$. Describing the relevant elements is similar to the previous case but simpler. The only relevant elements (up to double cosets) are the identity and those of the form

$$\xi_\gamma := \begin{pmatrix} 0 & \gamma \\ I & 0 \end{pmatrix} \quad \text{for } \gamma \in D^\times.$$

The subgroup S^{ξ_γ} is the group of matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$$

with A belonging to T_γ, the centralizer of γ in $D^\times$.

Let P (the parabolic subgroup), M, U (the unipotent radical), and H be subgroups given respectively by matrices of the form

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}, \qquad \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \qquad \begin{pmatrix} I & B \\ 0 & I \end{pmatrix}, \qquad \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}.$$

Let us denote by $g \mapsto {}^\iota g$ an antiautomorphism of D such that $g + {}^\iota g = \mathrm{tr}(g) I$ and $g {}^\iota g = \det g I$. Then g and ${}^\iota g$ have the same characteristic polynomial and are thus conjugate in $D^\times$. We set then

$$\tau\left[\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right] = w \begin{pmatrix} {}^\iota A & {}^\iota C \\ {}^\iota B & {}^\iota D \end{pmatrix} w, \quad \text{where } w = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Again, this is an antiautomorphism leaving the relevant double cosets invariant.

Now suppose F is a local field and $G = \mathrm{GL}_2(D)$. Let ψ be a nontrivial additive character of F. We endow the vector space D with self-dual Haar measure for the character $\psi \circ \mathrm{tr}$, where tr is the reduced trace. Let ψ be a nontrivial additive character of F. We endow the vector space $M_{2\times 2}(F)$ with the self-dual Haar measure for the character $\psi \circ \mathrm{tr}$, where tr is the reduced trace. On the other hand, we choose a Haar measure on $D^\times$ and T_γ in the usual way (as in the ordinary trace formula computations). We use the isomorphisms

$$X \leftrightarrow \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \qquad \text{and} \qquad g \leftrightarrow \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}$$

to transport these measures to U and H, respectively. The product of these measures is then a Haar measure on S.

Define $\theta : S(F) \to \mathbb{C}^\times$ by

$$\theta\left(\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}\begin{pmatrix} I & X \\ 0 & I \end{pmatrix}\right) = \psi(\mathrm{tr}(X)).$$

We consider the orbital integral of a relevant element g,

$$\Xi(f, g) = \int_{S^g(F)\backslash S(F)} \left(\int_{S(F)} f(s_1 g s_2)\theta(s_1) ds_1\right)\theta(s_2) ds_2 .$$

Then

$$\Xi(f, I) = \int_S f(s)\theta(s) ds .$$

If $g = \xi_\gamma$, then we define

$$F_f(h) := \int f\left(\begin{pmatrix} I & X \\ 0 & I \end{pmatrix}\begin{pmatrix} 0 & gh \\ g & 0 \end{pmatrix}\begin{pmatrix} I & Y \\ 0 & I \end{pmatrix}\right)\psi(\mathrm{tr}(X+Y))dXdYdg$$

and

$$\Xi(f, \xi_\gamma) = O(F_f, \gamma), \quad \text{where } O(F_f, \gamma) = \int_{D^\times/T_\gamma} F_f(h\gamma h^{-1})dh\,.$$

The product $F_f(h)\phi(\det h)$, for ϕ smooth of compact support on $F^\times$, is a function of compact support on $D^\times$, and $\Xi(f, \xi_\gamma)$ is the orbital integral of a smooth function of compact support on $D^\times$ (depending on $\det\gamma$).

3.2. *Matching orbital integrals.* Now let F be a local field. Let f be a function on $\mathrm{GL}_2(D)$ with support contained in the set Ω_D of elements

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

such that $\det C \neq 0$. Then, on the support of f, $\det C$ remains in a compact set of $F^\times$. In the formula for computing $F_f(h)$,

$$F_f(h) = \int f\begin{pmatrix} Xg & gh + XgY \\ g & gY \end{pmatrix}\psi(\mathrm{tr}(X+Y))dXdYdg,$$

we see that $\det g^2 \cdot \det h$ and $\det g$ remain in a compact set of $F^\times$. Thus $\det h$ is in a compact set of $F^\times$, and the function F_f is a smooth function of compact support on $D^\times$. Any smooth function of compact support can be obtained this way.

The same discussion applies to $G' = \mathrm{GL}_4(F)$. We let Ω_4 be the subset

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

of G' such that $\det C \neq 0$. Then for any smooth function f' of compact support contained in Ω_4, the function $F_{f'}$ is a smooth function of compact support.

Recall the notion of matching orbital integrals for smooth functions of compact support on G and G'. If γ and γ' are semisimple noncentral elements of G and G' with the same characteristic polynomials, we write $\gamma \sim \gamma'$. We say ϕ and ϕ', functions on $D^\times$ and $\mathrm{GL}_2(F)$ respectively, *have matching orbital integrals* if $O(\phi, \gamma) = O(\phi', \gamma')$ whenever γ and γ' are semisimple noncentral elements such that $\gamma \sim \gamma'$, and $O(\phi, \gamma') = 0$ when γ' is an element of G' with distinct eigenvalues in $F^\times$. Recall that for every ϕ there is a function ϕ' with matching orbital integrals; see for example [Rogawski 1983, Section 2].

Thus for any f with support contained in Ω_D, there is a function f' with support contained in Ω_4 such that $\Xi(f, \xi_\gamma) = \Xi(f', \xi'_\gamma)$ whenever γ and γ' are semisimple noncentral elements with $\gamma \sim \gamma'$, and $\Xi(f', \xi'_\gamma) = 0$ each time γ' is an element with distinct eigenvalues in $F^\times$.

In the Archimedean case we will denote by $\Omega_{D,e}$ the set of matrices of the form

$$\begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & gh \\ g & 0 \end{pmatrix} \begin{pmatrix} I & Y \\ 0 & I \end{pmatrix} \quad \text{for } h \in D^\times - Z(F),$$

and we will assume that the support of f is contained in $\Omega_{D,e}$. Similarly, we will denote by $\Omega_{4,e}$ the set of matrices of the form

$$\begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & gh \\ g & 0 \end{pmatrix} \begin{pmatrix} I & Y \\ 0 & I \end{pmatrix},$$

where h is elliptic regular, that is, has distinct eigenvalues not in F. We will assume the support of f' is contained in $\Omega_{4,e}$. Any f with support contained in $\Omega_{D,e}$ matches a function f' with support contained in $\Omega_{4,e}$.

4. A simple trace formula for $\mathbf{GL_2}(D)$

Let $G = \mathrm{GL}_2(D)$, where D is a quaternion algebra over a number field F. An element $\xi \in G(F)$ is relevant if and only if

$$\theta(\xi u \xi^{-1}) = \theta(u)$$

for all $u \in S^\xi(\mathbb{A}) = S(\mathbb{A}) \cap \xi^{-1} S(\mathbb{A}) \xi$. We consider then the orbital integral

$$\Xi(f, \xi) := \int \left(\int f(s_1 \xi s_2) \theta(s_1) ds_1 \right) \theta(s_2) ds_2$$

with $s_1 \in S(\mathbb{A})$ and $s_2 \in S^\xi(\mathbb{A}) \backslash S(\mathbb{A})$. Suppose $\xi = \xi_\gamma$. Computing formally, we set

$$F_f(h) = \int f\left(\begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & gh \\ g & 0 \end{pmatrix} \begin{pmatrix} I & Y \\ 0 & I \end{pmatrix} \right) dg \psi(\mathrm{tr}(X+Y)) dX dY .$$

Then $\Xi(f, \xi_\gamma) = O(F_f, \gamma)$, where $O(\phi, \gamma)$ is the global orbital integral

$$O(\phi, \gamma) = \int_{D^\times(\mathbb{A})/T_\gamma(\mathbb{A})} \phi(h \gamma h^{-1}) dh$$

and T_γ denotes the centralizer of γ in $D^\times$. To justify our computations we prove the following lemma.

Lemma 4.1. *If the matrix*

$$\begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & gh \\ g & 0 \end{pmatrix} \begin{pmatrix} I & Y \\ 0 & I \end{pmatrix} = \begin{pmatrix} Xg & gh + XgY \\ g & gY \end{pmatrix}$$

is in a compact set of $\mathrm{GL}_2(D(\mathbb{A}))$ *and the reduced determinant* $\det h$ *is in* $F^\times$, *then* $\det h$ *takes only finitely many values*, g *and* h *remain in a compact set of* $D^\times(\mathbb{A})$, *and* X *and* Y *lie in a compact set of* $D(\mathbb{A})$.

Proof. Indeed g is in a compact set of $D(\mathbb{A})$ and $\det g^2 \cdot \det h$ in a compact set of $\mathbb{A}^\times$. A fortiori, $\det h^{-1}$ remains in a compact set of $\mathbb{A}$. Since it is in $F^\times$, it remains in a finite set. Hence $\det g^2$ — and thus $\det g$ — remains in a compact set of $\mathbb{A}^\times$. Hence g is in fact in a compact set of $D^\times(\mathbb{A})$. Now Xg and Yg — and thus X and Y — are in compact sets of $D(\mathbb{A})$. □

The lemma shows in particular that there is a smooth function of compact support ϕ on $D^\times(\mathbb{A})$ such that $F_f(h) = \phi(h)$ when $\det h$ is in $F^\times$. Then

$$\Xi(f, \xi_\gamma) = O(\phi, \gamma)\,.$$

For each finite place v, let α_i for $1 \le i \le 4$ be an F_v-basis of D_v. In an obvious way, it gives a basis $\alpha_{i,j}$ for $1 \le i, j \le 4$ of $M_{2\times 2}(D_v)$. Let α_v and $\alpha_{2,v}$ be the $\mathbb{O}_v$-modules generated respectively by $\{\alpha_i\}$ and $\left\{\alpha_{i,j}\right\}$. Then for almost all v, D_v is split, α_v is a maximal compact subring of D_v, and $\alpha_{2,v}$ is a maximal compact subring of $M_{2\times 2}(D_v)$. The groups $K_v^1 := \alpha_v^\times$ and $K_v := \alpha_{2,v}^\times$ are then maximal compact subgroups of $D_v^\times$ and $\mathrm{GL}_2(D_v)$, respectively. We choose maximal compact subgroups K_v at the remaining places and set $K = \prod K_v$.

We assume that $f = \prod f_v$, where, for almost all v, the function f_v is the characteristic function of K_v. Then, for a given γ and almost all v, if $\det h = \det \gamma$, then $F_{f_v}(h_v) = \phi_{0,v}(h_v)$, where $\phi_{0,v}$ is the characteristic function of $\alpha_v^\times$. Furthermore, for a given γ, almost all orbital integrals are equal to 1, for a suitable choice of the measures.

Consider the kernel function

$$K(x, y) = K_f(x, y) = \sum_{Z(F)\backslash G(F)} \int_{Z(\mathbb{A})} f(zx^{-1}\xi y)dz. \tag{4-3}$$

We assume that at every place v where D is ramified the support of the function f_v is contained in the set Ω_{D_v}. Furthermore, if v is Archimedean we assume that the support of f_v is contained in $\Omega_{D_v,e}$. Then the intersection $\operatorname{supp} f \cap S(\mathbb{A})\xi S(\mathbb{A})$, where $\xi \in S(F)$, is empty unless ξ is in the double coset of an element of the form ξ_γ with γ in a *finite* union of conjugacy classes of $D^\times - F^\times$. We now compute

$$\int_{Z(\mathbb{A})S(F)\backslash S(\mathbb{A})} \int_{Z(\mathbb{A})S(F)\backslash S(\mathbb{A})} K(s_1, s_2)\theta(s_1)^{-1} ds_1 \theta(s_2) ds_2\,.$$

This can be computed as

$$\int_{Z(\mathbb{A})S(F)\backslash S(\mathbb{A})}\Bigg(\sum_{\xi\in S(F)\backslash G(F)}\int_{S(\mathbb{A})} f(s_1\xi s_2)\theta(s_1)ds_1\Bigg)\theta(s_2)ds_2$$

$$=\int_{Z(\mathbb{A})S(F)\backslash S(\mathbb{A})}\Bigg(\sum_{\gamma,\sigma\in S^{\xi_\gamma}(F)\backslash S(F)}\int_{S(\mathbb{A})} f(s_1\xi_\gamma\sigma s_2)\theta(s_1)ds_1\Bigg)\theta(s_2)ds_2$$

$$=\sum_\gamma\int_{Z(\mathbb{A})S^{\xi_\gamma}(F)\backslash S(\mathbb{A})}\Bigg(\int_{S(\mathbb{A})} f(s_1\xi_\gamma s_2)\theta(s_1)ds_1\Bigg)\theta(s_2)ds_2\,,$$

where γ runs over a set of representatives for the conjugacy classes in $D^\times - Z$. We now use the fact that

$$s_2\mapsto\int_{S(\mathbb{A})} f(s_1\xi_\gamma s_2)\theta(s_1)ds_1\ \theta(s_2)$$

is invariant on the left under $S^{\xi_\gamma}(\mathbb{A})$ to write this as

$$\sum_\gamma \mathrm{Vol}(Z(\mathbb{A})S^{\xi_\gamma}(F))\backslash S^{\xi_\gamma}(\mathbb{A}))\int_{S^{\xi_\gamma}(A)\backslash S(\mathbb{A})}\Bigg(\int_{S(\mathbb{A})} f(s_1\xi s_2)\theta(s_1)ds_1\Bigg)\theta(s_2)ds_2$$

Recall that S^{ξ_γ} is the group of matrices of the form

$$\begin{pmatrix} g & 0\\ 0 & g\end{pmatrix}\quad\text{for } g\in T_\gamma,$$

where T_γ is the centralizer of γ in $D^\times$. Thus our integral has the final expression

$$\sum_\gamma \mathrm{Vol}(F^\times(\mathbb{A})T_\gamma(F)\backslash T_\gamma(\mathbb{A}))\ \Xi(f,\gamma)\,.$$

We will assume furthermore that there is a place v_0 where D is split and the function f_{v_0} is supercuspidal, that is,

$$\int_{U(F_{v_0})} f_{v_0}(u)du=0$$

each time U is the unipotent radical of a parabolic subgroup of G_{v_0} defined over F_{v_0}. We then have

$$K_f(x,y)=\sum_\pi K_{\pi,f}(x,y)\,,$$

where the sum is over all cuspidal representations π of $G(\mathbb{A})$ with trivial central character that have a supercuspidal component at v_0. Here,

$$K_{\pi,f}(x,y)=\sum \pi(f)\phi_i(x)\overline{\phi}_i(y)\,,$$

where ϕ_i is an orthonormal basis of the space of π. We set

$$B_\pi(f)=\int_{S(F)Z(\mathbb{A})\backslash S(\mathbb{A})}\int_{S(F)Z(\mathbb{A})\backslash S(\mathbb{A})}K_{\pi,f}(s_1,s_2)\theta^{-1}(s_1)\theta(s_2)ds_1ds_2\,.$$

This is the *global Bessel distribution* attached to π. At least, if each f_v is K_v finite and we take the ϕ_i to be K-finite, then we can write

$$B_\pi(f)=\sum_i\lambda(\pi(f)\phi_i)\overline{\lambda(\phi_i)}\,.$$

In fact, one can show that the series converges absolutely. At any rate, we denote by $\mathcal{H}_\pi$ the Hilbert space of the representation π, by $\mathcal{V}_\pi$ the space of smooth vectors, and by $\mathcal{V}_\pi^*$ its topological dual. Then

$$\mathcal{V}_\pi\subseteq\mathcal{H}_\pi\subseteq\mathcal{V}_\pi^*\,.$$

We still denote by π the natural representation of π on $\mathcal{V}_\pi$ and $\mathcal{V}_\pi^*$. The scalar product $(\cdot,\,\cdot)$ on $\mathcal{H}_\pi\times\mathcal{H}_\pi$ extends to $\mathcal{V}_\pi\times\mathcal{V}_\pi^*$ or $\mathcal{V}_\pi^*\times\mathcal{V}_\pi$. Finally, for f smooth of compact support and $\lambda\in\mathcal{V}_\pi^*$, the vector $\pi(f)\lambda$ is in $\mathcal{V}_\pi$. The period integral λ is in $\mathcal{V}_\pi^*$. Then, at least under the assumption of K-finiteness,

$$B_\pi(f)=\sum_i(\pi(f)(\phi_i),\lambda)(\phi_i,\lambda)=(\pi(f)\lambda,\lambda)\,.$$

This expression still holds for f smooth of compact support; compare with [Shalika 1974, page 184]. Then

$$\int_{S(F)Z(\mathbb{A})\backslash S(\mathbb{A})}\int_{S(F)Z(\mathbb{A})\backslash S(\mathbb{A})}K_f(s_1,s_2)\theta^{-1}(s_1)\theta(s_2)ds_1ds_2=\sum_\pi B_\pi(f)\,.$$

Of course, π is distinguished if and only if the distribution B_π is not identically 0.

Finally, we get

$$\sum_\pi B_\pi(f)=\sum_\gamma\mathrm{Vol}(F^\times(\mathbb{A})T_\gamma(F)\backslash T_\gamma(\mathbb{A}))\;\Xi(f,\gamma)\,.$$

5. A simple trace formula for GL_4

Now we consider the group $G'=\mathrm{GL}_4$ with Shalika subgroup S'. We choose maximal compact subgroups K'_v in the usual way. Thus $K'_v=\mathrm{GL}_4(\mathbb{O}_v)$ if v is finite. We set $K'=\prod_v K'_v$. We let f' be a smooth function of compact support on $G'(\mathbb{A})$. We assume that $f'=\prod_v f'_v$, where f'_v is the characteristic function of K'_v for almost all v. We assume that, at each place v where D is ramified, the support of f'_v is contained in the set $\Omega_{4,v}$. Furthermore if v is an Archimedean place where D is ramified, we assume that the support of f'_v is contained in the set $\Omega_{4,v,e}$. Then there are only finitely many cosets $S'(F)\xi S'(F)$ for $\xi\in G'(F)$

such that the support of f' intersects $S'(\mathbb{A})\xi S'(\mathbb{A})$. Furthermore they are cosets of the form $S'(F)\xi_\gamma S'(F)$, where $\gamma \in \mathrm{GL}_2(F)$ is elliptic at any Archimedean place where D splits.

We introduce the kernel function

$$K'(x, y) = K_{f'}(x, y) = \sum_{Z(F)\backslash G(F)} \int_{Z(\mathbb{A})} f'(zx^{-1}\xi y)dz.$$

We find, as before,

$$\iint K'(s_1, s_2)\theta(s_1)^{-1}\theta(s_2)ds_1ds_2 = \sum_\gamma \mathrm{Vol}(F^\times(\mathbb{A})\backslash T_\gamma(\mathbb{A}))\,\Xi(f', \xi_\gamma)\,,$$

where γ runs through a set of representatives for the conjugacy classes of elliptic elements of $\mathrm{GL}_2(F)$ — in fact, the elements elliptic at each Archimedean place where D is ramified. Next, we assume that f'_{v_0} is supercuspidal at a finite place v_0 where D splits. We have then an identity

$$K'(x, y) = \sum_{\pi'} K_{\pi', f'}(x, y)\,,$$

where the sum is over all cuspidal representations π' of $G'(\mathbb{A})$ with trivial central character which have a supercuspidal component at v_0. As before, we have set

$$K_{\pi', f'}(x, y) = \sum_i \pi'(f')\phi_i(x)\overline{\phi_i(y)}\,,$$

where the sum is over an orthonormal basis ϕ_i of π'. We set

$$B_{\pi'}(f') = \iint K_{\pi', f'}(s_1, s_2)\theta(s_1)^{-1}\theta(s_2)ds_1ds_2\,.$$

This is the global Bessel distribution attached to π'. The representation π' is distinguished if and only if the distribution $B_{\pi'}$ is not identically 0.

The sum over π' converges in the space of smooth rapidly decreasing functions on $G'(F)\backslash G'(\mathbb{A})$. Now we can integrate over the space of finite volume $(Z(\mathbb{A})S'(F)\backslash S'(\mathbb{A}))^2$ to get

$$\iint K'(s_1, s_2)\theta(s_1)^{-1}\theta(s_2)ds_1ds_2 = \sum_{\pi'} B_{\pi'}(f')\,.$$

On the other hand, at least when f' is K'-finite,

$$B_{\pi'}(f') = \sum_i \lambda'(\pi'(f')\phi_i)\overline{\lambda'(\phi_i)}\,.$$

As before, we can think of λ' as a continuous linear form on the space of smooth vectors and write

$$B_\pi(f) = (\pi'(f')\lambda', \lambda')\,.$$

6. Comparison

Now we compare the two expressions we have just obtained. We choose matching f and f'. At a place v where D is ramified, we demand that f_v and f'_v match in the sense given above. At a place v where D splits, we have an isomorphism $D_v \simeq \mathrm{GL}_2(F_v)$ and thus an isomorphism $G_v \simeq G'_v$ which takes S_v to S'_v. Then the isomorphism is unique up to inner automorphisms defined by elements of S_v and S'_v. Furthermore, at almost all places we may assume that the isomorphism $D_v \simeq M_{2\times 2}(F_v)$ takes α_v to $M_{2\times 2}(\mathbb{O}_v)$. Then the isomorphism $G_v \simeq G'_v$ takes the maximal compact subgroup K_v to K'_v. We then assume that f_v and f'_v correspond to one another by this isomorphism. (Since $G_v \simeq G'_v$ almost everywhere, there is no issue of a fundamental lemma.) Then $\Xi(f, \xi'_\gamma) = 0$ unless there exists $\gamma \in D^\times - F^\times$ with $\gamma \sim \gamma'$, in which case $\Xi(f, \xi_\gamma) = \Xi(f', \xi'_\gamma)$. Moreover, T_γ and $T_{\gamma'}$ are then isomorphic. In particular, $T_\gamma(\mathbb{A})/T_\gamma(F)F^\times(\mathbb{A})$ and $T_{\gamma'}(\mathbb{A})/T_\gamma(F)F^\times(\mathbb{A})$ have the same volume. We conclude that

$$\sum \mathrm{Vol}(T_\gamma(\mathbb{A})/F^\times(\mathbb{A}))\,\Xi(f, \xi_\gamma) = \sum \mathrm{Vol}(T_{\gamma'}(\mathbb{A})/F^\times(\mathbb{A}))\,\Xi(f', \xi_{\gamma'})\,,$$

and thus

$$\sum_\pi B_\pi(f) = \sum_{\pi'} B_{\pi'}(f')\,.$$

If π is distinguished, the distribution B_π is nonzero. Our next task is to prove that if π is distinguished, then we can choose f as above such that $B_\pi(f) \neq 0$. This requires local preliminaries.

7. Local periods: non-Archimedean case

Let F be a local non-Archimedean field and $G = \mathrm{GL}_2(D)$ where D is a quaternion algebra over F. Let π be an irreducible admissible unitary representation of G with trivial central character. Let V be the space of smooth vectors of π. Let V^* be the dual space of V. We define the space of *Shalika functionals* of π to be

$$\mathcal{S}(\pi) = \left\{\lambda \in V^* \mid \lambda(\pi(s)v) = \theta(s)\lambda(v),\ v \in V,\ s \in S\right\}.$$

We say that π is *distinguished* if $\mathcal{S}(\pi) \neq 0$.

If λ and μ are in $\mathcal{S}(\pi)$, we can define a distribution

$$B(f) = \sum_i \lambda(\pi(\phi)\phi_i)\overline{\mu(\phi_i)}\,,$$

where ϕ_i is an orthonormal basis of smooth vectors in π. As in the global case we have inclusion $V \subseteq \mathcal{H} \subseteq V^*$, and we can write the distribution in the form $B(f) = (\pi(f)\lambda, \mu)$. For every $s_1, s_2 \in S$, we have $B(L_{s_1} R_{s_2} f) = \theta(s_1 s_2^{-1}) B(f)$.

We first recall the following result.

Proposition 7.1 [Prasad and Raghuram 2000]. *Let π be an irreducible, admissible, unitary representation of $\mathrm{GL}_2(D)$. Then $\dim_{\mathbb{C}} \mathcal{S}(\pi)$ is at most one.*

We briefly give the argument here. The involution τ introduced previously leaves P, U, H and S invariant. In addition $\theta(\tau(s)) = \theta(s)$. For any function f we set $f^\tau(x) = f(\tau(x))$, and for any distribution T we define T^τ by $T^\tau(f) = T(f^\tau)$. A standard argument shows that the proposition above follows from:

Proposition 7.2. *Let Λ be a distribution on G such that, for all $s_1, s_2 \in S(F)$ and all functions f, $\Lambda(R_{s_1} L_{s_2} f) = \theta(s_1)\theta(s_2)^{-1}\Lambda(f)$. Then $\Lambda^\tau = \Lambda$.*

Proof. As we have observed, the double cosets are invariant under τ. Thus the orbital integrals satisfy the hypotheses of the proposition and the conclusion. One concludes by using an argument of density. See [Prasad and Raghuram 2000] for details. $\square$

Now suppose that $\mathcal{S}(\pi) \neq 0$. Choose $\lambda \neq 0$ in $\mathcal{S}(\pi)$ and set

$$B_\pi(f) = \sum_i \lambda(\pi(f)\phi_i)\overline{\lambda(\phi_i)}\,.$$

This is the (local) Bessel distribution associated to π (and λ). Recall the open set $\Omega = \Omega_D$ of matrices of the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with C invertible.

Proposition 7.3. *The restriction of B_π to the open set Ω is nonzero.*

Proof. Assume that this restriction is 0. Then B_π is supported on P. Thus it is in fact a distribution on P. We show that the restriction of B_π to $P - S$ is 0. To that end, we use:

Lemma 7.4 [Raghuram ≥ 2008, Lemma 6.7]. *If T is a distribution on $P - S$ such that $T(L_{u_1} R_{u_2} f) = \theta(u_1 u_2^{-1}) T(f)$ for any $u_1, u_2 \in U$, then $T = 0$.*

Proof. Since the author leaves the details to the reader there, we give a proof here. Let M be the group of diagonal matrices and M_0 the open subset of matrices of the form $\mathrm{diag}(a, b)$ with $a \neq b$. Thus $P - S = M_0 U = U M_0$. The property of invariance of T on the right implies there is a distribution μ on M_0 such that

$$T(f) = \int \alpha_f(a, b)\, d\mu(a, b), \quad \text{where } \alpha_f(a, b) := \int f\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} u\right) \theta(u)\, du\,.$$

The function α_f is an arbitrary smooth function of compact support on M_0. The property of invariance on the left implies that μ satisfies that, for every $X \in D$,

$$\psi(\mathrm{tr}(a^{-1}Xb - X))d\mu(a,b) = d\mu(a,b)\,.$$

This can also be written as

$$\psi(\mathrm{tr}((ba^{-1} - I)X))d\mu(a,b) = d\mu(a,b)\,.$$

Let ϕ be a smooth function of compact support on D. The above identity implies

$$\widehat{\phi}(ba^{-1} - I)d\mu(a,b) = d\mu(a,b)\,,$$

where $\widehat{\phi}$ is the Fourier transform of ϕ. If α is any smooth function of compact support on M_0, the difference $ba^{-1} - I$ remains in a compact set on the support of α. We can choose $\widehat{\phi}$ so that $\widehat{\phi}(ba^{-1} - I) = 0$ on the support of α. Thus $\alpha(a,b)\widehat{\phi}(ba^{-1} - I) = 0$ and $\int \alpha(a,b)d\mu(a,b) = 0$. So $T = 0$. □

At this point we are reduced to the case where B_π is supported on S and hence is a distribution on S. Thus

$$B_\pi(f) = c\int_S f(s)\theta(s)ds \tag{7-4}$$

for some constant c; see[Bernšteĭn and Zelevinskiĭ 1976] or [Bump 1997, Proposition 4.3.2].

Recall that a distribution μ is said to be of *positive type* if $\mu(f * f^*) \geq 0$ for all f, where $f^*(x) = \overline{f(x^{-1})}$. Then the completion of $\langle f_1, f_2\rangle := \mu(f_1 * f_2^*)$ modulo its kernel is a unitary representation of G which is said to be associated with π.

By definition, $B_\pi(f)$ is a distribution of positive type. Indeed,

$$B_\pi(f^* * f) = (\pi(f)\lambda, \pi(f)\lambda)\,. \tag{7-5}$$

Because π is irreducible, the representation associated to the distribution B_π is π itself.

On the other hand, $\int f(s)\theta(s)ds$ is clearly of positive type. Thus $c \geq 0$. The representation associated to the distribution $\int f(s)\theta(s)$ is the unitary representation σ of G induced by the character θ. If $c > 0$ then $\sigma \simeq \pi$ [Dixmier 1977, p. 40]. Thus σ is admissible. But this is a contradiction. Indeed, let $\mathcal{U}$ be a small enough open subring of D. Denote by K_0 the subgroup of matrices congruent to I modulo $\mathcal{U}$. Then

$$K_0 = (K_0 \cap P)\,.\,(\overline{U} \cap K_0)\,,$$

where $\overline{U}$ is the transpose of U. Consider the subspace V_0 of σ of functions f supported on PK_0 and invariant under K_0 on the right. Such a function f is uniquely determined by the function ϕ on $D^\times$ determined by

$$\phi(h) = f\begin{pmatrix} h & 0 \\ 0 & I \end{pmatrix}.$$

Now ϕ is any function such that $\phi(k_1 h k_2) = \phi(h)$ for k_1, k_2 congruent to 1 modulo $\mathcal{U}$. Thus V_0 is infinite dimensional, which contradicts the fact that σ is admissible. Thus $c = 0$ and $B_\pi = 0$, a contradiction. □

8. An argument of Shalika

Before we proceed to the Archimedean case we review an argument of Shalika [1974] on the transversal order of distributions supported by a manifold. Let G be a real Lie group and G_1, G_2 be closed subgroups. We do not assume that the groups are connected. Let $\mathfrak{g}$, $\mathfrak{g}_1$, $\mathfrak{g}_2$ be their Lie algebras, and $\mathcal{U}(\mathfrak{g})$, $\mathcal{U}(\mathfrak{g}_1)$, $\mathcal{U}(\mathfrak{g}_2)$ the enveloping algebras. For every $X \in \mathfrak{g}$, let $\rho(X)$ be the corresponding left invariant vector field. Similarly, for every $X \in \mathcal{U}(\mathfrak{g})$, let $\rho(X)$ be the corresponding differential operator. Thus, for $X \in \mathfrak{g}$,

$$\rho(X) f(g) = \frac{d}{dt}\bigg|_{t=0} f(g \exp(tX)).$$

We denote by $X \mapsto \check{X}$ the involution of $\mathcal{U}(\mathfrak{g})$ such that $\check{X} = -X$ for $X \in \mathfrak{g}$. If T is a distribution and $X \in \mathfrak{g}$ we define $\rho(X)T$ by $\rho(X)T(f) = T(\rho(\check{X})f)$.

We assume that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$. Then at any point $x \in G$, $T_x(G) = \rho(\mathfrak{g}_1)_x \oplus \rho(\mathfrak{g}_2)_x$. Here $T_x(G)$ is the tangent space at x, and L_x is the evaluation of a vector field L at x. In particular, if $x \in G_1$ then $T_x(G) = T_x(G_1) \oplus \rho(\mathfrak{g}_2)_x$. We denote by $\mathcal{U}(\mathfrak{g})_n$ and $\mathcal{U}(\mathfrak{g}_i)_n$ the canonical filtrations of the enveloping algebras. We choose a basis of $\mathfrak{g}_2$ and then use it to construct a basis of standard monomials X_q of $\mathcal{U}(\mathfrak{g}_i)$. We let $|q|$ be the degree of the monomial. Thus $X_q \in \mathcal{U}(\mathfrak{g}_2)_{|q|}$ and $X_q \notin \mathcal{U}(\mathfrak{g}_2)_{|q|-1}$. Let T be a distribution on G that is supported on G_1. Then, if x is a point of G_1, there is a relatively compact open neighborhood U of x in G such that the restriction of T to U has the form

$$T|U = \sum_q \rho(X_q) T_q,$$

where the T_q are uniquely determined distributions on $U \cap G_1$ (almost all 0). We say that $T|U$ has transversal order $\leq n$ if $|q| \leq n$ for all q with $T_q \neq 0$; if, in addition, there is at least one q such that $|q| = n$ and $T_q \neq 0$, we say that T has transversal order n on $U \cap G_1$. This notion is independent of the choice of the basis. Shalika observes that if X is in $\mathcal{U}(\mathfrak{g}_1)$ and T has transversal order $\leq n$, then $\rho(X)T$ has transversal order $\leq n$. Similarly, if ϕ is a smooth function on G and T has transversal order $\leq n$, then ϕT has transversal order $\leq n$.

Often this can be used to show that if the distribution T satisfies a suitable differential equation then, in fact, it is 0. For instance, suppose G_2 has dimension 3 and let X_1, X_2, X_3 be a basis of $\mathfrak{g}_2$. Suppose that

$$\rho(X_1^2+X_2^2+X_3^2)T=\rho(D)T+\phi T\,,$$

where $D\in\mathcal{U}(\mathfrak{g}_1)$ and ϕ is a smooth function on G. Suppose T is nonzero. Then we can find U as above such that $T|U\neq 0$. Then let $n\geq 0$ be the transversal order of $T|U$. The right hand side has transversal order $\leq n$. On the other hand, we claim the transversal order of the left hand side is $n+2$; this gives a contradiction and proves our assertion. To check our claim, we can take for basis of $\mathcal{U}(\mathfrak{g}_i)$ the monomials $X_1^aX_2^bX_3^c=X_q$ for $q=(a,b,c)$; we let $|q|=a+b+c$. Let us order lexicographically the multiindices q with $|q|=n$. Let $q=(a,b,c)$ be the largest index with $|q|=n$ and T_q nonzero. Then (writing XT for $\rho(X)T$),

$$T|U=X_{a,b,c}T_{a,b,c}+\sum_{\substack{a'+b'+c'=n\\(a',b',c')<(a,b,c)}}X_{a',b',c'}T_{a',b',c'}+\sum_{|q'|<n}X_{q'}T_{q'}\,.$$

Then

$$\begin{aligned}(X_1^2+X_2^2+X_3^2)T|U&=X_{a+2,b,c}T_{a,b,c}+X_{a,b+2,c}T_{a,b,c}+X_{a,b,c+2}T_{a,b,c}\\&+\sum_{\substack{a'+b'+c'=n\\(a',b',c')<(a,b,c)}}(X_{a'+2,b',c'}T_{a',b',c'}+X_{a',b'+2,c'}T_{a',b',c'}+X_{a',b',c'+2}T_{a',b',c'})+\sum_{|q'|<n}X_{q'}S_{q'}\,.\end{aligned}$$

Now $(a+2,b,c)$ is larger than all the monomials q' with $|q'|=n$ that appear in this formula. Our claim follows.

Similarly, suppose that $X\in\mathfrak{g}_2$, $X\neq 0$ and $\mathfrak{g}_2$ has an arbitrary dimension. Suppose further that $XT=\rho(D)T+\phi T$, where D and ϕ are as above. Then again $T=0$. The proof is similar but simpler.

9. Local periods: Archimedean case

Let $F=\mathbb{R}$ and $G=\mathrm{GL}_2(\mathbb{H})$, where $\mathbb{H}$ is the Hamilton quaternion algebra over $\mathbb{R}$. Let π be an irreducible admissible unitary representation of G. Let V be the space of smooth vectors of π equipped with its usual topology. Let V^* be the topological dual space of V. We define the space of *Shalika functionals* of π to be

$$\mathcal{S}(\pi)=\left\{\lambda\in V^*\mid\lambda(\pi(s)v)=\theta(s)\lambda(v),\ v\in V,\ s\in S\right\}.$$

If λ and μ are in $\mathcal{S}(\pi)$ we can define a corresponding distribution B by

$$B(f)=(\pi(f)\lambda,\mu)\,.$$

Our first goal in this section is to establish the following proposition.

Proposition 9.1. *Let π be an irreducible, admissible, unitary representation of* $\mathrm{GL}_2(\mathbb{H})$. *Then* $\dim_{\mathbb{C}} \mathcal{S}(\pi)$ *is at most one.*

To state the Archimedean analogue of Proposition 7.2, we need to introduce an element of the center $\mathfrak{Z}(\mathfrak{g})$ of the enveloping algebra of the Lie algebra $\mathfrak{g}$ of $G = \mathrm{GL}_2(\mathbb{H})$. Let $\{1, i, j, k\}$ denote the usual basis for $\mathbb{H}$ over $\mathbb{R}$. Thus $ij = k$ and $i^2 = j^2 = k^2 = -1$. Also $\operatorname{tr} i = \operatorname{tr} j = \operatorname{tr} k = 0$ and we take ι to be the involution that takes i, j, k to $-i, -j, -k$. Then the involution τ of $\mathrm{GL}(2, \mathbb{H})$ is defined by

$$\tau \begin{pmatrix} A & B \\ C & D \end{pmatrix} = w \begin{pmatrix} {}^{\iota}A & {}^{\iota}C \\ {}^{\iota}B & {}^{\iota}D \end{pmatrix} w\,, \quad \text{where } w = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

We may identify $\mathfrak{g}$ with $M_{2\times 2}(\mathbb{H})$. We let $\mathfrak{g}_0$ be the subspace of $X \in \mathfrak{g}$ with $\operatorname{tr} X = 0$ and B the $\mathbb{R}$-bilinear invariant form defined by $B(X, Y) = \operatorname{tr}(XY)$. Thus $\mathfrak{g}_0$ is a 15-dimensional Lie algebra over $\mathbb{R}$ with basis

$$E_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad E_{1,a} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \qquad E_{2,b} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix},$$
$$E_{3,b} = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}, \qquad E_{4,a} = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix},$$

where $a \in \{i, j, k\}$ and $b \in \{1, i, j, k\}$. For an element E_α of the above basis, let E^α be the corresponding dual basis element with respect to $B(X, Y)$, i.e, $B(E_\alpha, E^\beta) = \delta_{\alpha,\beta}$ for all α, β in the index set. One may compute

$$E^0 = \tfrac{1}{4} E_0, \qquad \begin{aligned} E^{1,a} &= -\tfrac{1}{2} E_{1,a}, & E^{2,a} &= -\tfrac{1}{2} E_{3,a}, & E^{2,1} &= \tfrac{1}{2} E_{3,1}, \\ E^{4,a} &= -\tfrac{1}{2} E_{4,a}, & E^{3,a} &= -\tfrac{1}{2} E_{2,a}, & E^{3,1} &= \tfrac{1}{2} E_{2,1}, \end{aligned}$$

where a runs through $\{i, j, k\}$. We set

$$\begin{aligned} \Delta &= \sum E_\alpha E^\alpha \\ &= \frac{1}{4} E_0^2 - \frac{1}{2} \sum_{\{i,j,k\}} \left(E_{1,a}^2 + E_{4,a}^2\right) + \frac{1}{2} \sum_{\{1,i,j,k\}} a^2 \left(E_{2,a} E_{3,a} + E_{3,a} E_{2,a}\right). \end{aligned}$$

This element is invariant under $\mathrm{Ad}(G)$ because tr is. In particular, it is in $\mathfrak{Z}(\mathfrak{g})$. Thus the element Δ acts on V by a scalar. Also $\tau(\Delta) = \Delta$ and $\check{\Delta} = \Delta$.

Proposition 9.2. *Suppose $F = \mathbb{R}$. Let Λ be a distribution on G such that, for all $s_1, s_2 \in S(F)$ and all functions f,*

$$\Lambda(L_{s_1} R_{s_2} f) = \theta(s_1 s_2^{-1}) \Lambda(f)\,.$$

Suppose furthermore that $\Delta\Lambda = k\Delta$ for some $k \in \mathbb{C}$. Then $\Lambda^\tau = \Lambda$.

It is a standard argument that this implies the previous proposition [Gel'fand and Kajdan 1975; Shalika 1974]. The proof of Proposition 9.2 will follow from two subsequent propositions.

Let P, U, M, and H respectively be the subgroups of matrices of the form

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}, \quad \begin{pmatrix} I & B \\ 0 & I \end{pmatrix}, \quad \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \quad \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}.$$

We denote the Lie algebra of one of these groups by the corresponding lower case gothic letter. Let $\Omega = \Omega_{\mathbb{H}}$ be the open subset of matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{with } C \text{ invertible.}$$

Proposition 9.3. *Suppose Λ is a distribution on Ω such that $\Lambda(L_{s_1} R_{s_2} f) = \theta(s_1 s_2^{-1})\Lambda(f)$ for all $s_1, s_2 \in S$ and all f. Then $\Lambda^\tau = \Lambda$.*

Proof. Given $f \in C_c^\infty(\Omega)$, we have defined

$$F_f(h) = \int_{U \times U} f\left[u_1 \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \begin{pmatrix} 0 & h \\ I & 0 \end{pmatrix} u_2\right] \theta(u_1 u_2) du_1 du_2 dg.$$

Let Λ be a distribution on Ω such that $\Lambda(L_s R(u) f) = \theta(s)\theta(u)^{-1} f$. There is a unique distribution Λ^* on $\mathbb{H}^\times$ such that $\Lambda^*(F_f) = \Lambda(f)$. In other words,

$$\Lambda(f) = \int \left(\int f\left(u_1 \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \begin{pmatrix} 0 & h \\ I & 0 \end{pmatrix} u_2\right) \theta(u_1)\theta(u_2) du_1 du_2 dg \right) d\Lambda^*(h).$$

Moreover, Λ satisfies $\Lambda(R(m) f) = \Lambda(f)$ for all $m \in H$ if and only Λ^* is an invariant distribution. Assuming this is the case, we have

$$\begin{aligned} \Lambda^\tau(f) &= \Lambda(f^\tau) \\ &= \int \left(\int f\left(u_1 \begin{pmatrix} 0 & {}^\iota h \\ I & 0 \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} u_2\right) \theta(u_1)\theta(u_2) du_1 du_2 dg \right) d\Lambda^*(h) \\ &= \int \left(\int f\left(u_1 \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \begin{pmatrix} 0 & {}^\iota h \\ I & 0 \end{pmatrix} u_2\right) \theta(u_1)\theta(u_2) du_1 du_2 dg \right) d\Lambda^*(h). \end{aligned}$$

Hence $(\Lambda^\tau)^* = (\Lambda^*)^\iota$. Now we appeal to a well-known result.

Lemma 9.4. *Let Ξ be an invariant distribution on $\mathbb{H}^\times$. Then $\Xi^\iota = \Xi$.*

For the convenience of the reader we provide a proof.

Proof. Let T be a torus of $\mathbb{H}^\times$ that is stable under ι; for instance, we can take T to be the stabilizer of i. Then ι induces on T conjugation by an element of the normalizer of T. Now any conjugacy class intersects T. Thus if f is an invariant function, then $f^\iota = f$. Since Ξ is invariant and $\mathbb{H}^\times/Z$ is compact we have, for any function f,

$$\Xi(f) = \Xi(f_0), \quad \text{where } f_0(g) = \int_{\mathbb{H}^\times/Z} f(hgh^{-1})dh.$$

Then $\Xi^\iota(f) = \Xi^\iota(f_0) = \Xi(f_0^\iota) = \Xi(f_0) = \Xi(f)$. The lemma follows. □

Applying the above lemma to Λ^* establishes Proposition 9.3. □

Coming back to the proof of Proposition 9.2, we see that the restriction of $\Lambda - \Lambda^\tau$ to Ω vanishes. Thus $\Lambda - \Lambda^\tau$ is supported on P. Since $\Delta(\Lambda - \Lambda^\tau) = k\Delta$, it will suffice to prove:

Proposition 9.5. *Suppose Λ is a distribution on G supported on P such that $\Delta\Lambda = \kappa\Lambda$ for some $\kappa \in \mathbb{C}$ and $\Lambda(R_u f) = \theta(u)^{-1}\Lambda(f)$ for all $u \in U$ and all f. Then $\Lambda = 0$.*

Proof. We can write the element $\Delta \in \mathcal{U}(\mathfrak{g})$ in the form

$$\Delta = D + \sum_{a \in \{1,i,j,k\}} a^2 E_{3,a} E_{2,a},$$

where $D \in \mathcal{U}(\mathfrak{m})$. First observe that, for $a \in \{1, i, j, k\}$,

$$E_{2,a}\Lambda = 2i\pi \operatorname{tr}(a)\Lambda = 2i\pi\delta_{1,a}\Lambda.$$

Thus $2i\pi E_{3,1}\Lambda = \Delta\Lambda - D\Lambda = \kappa\Lambda - D\Lambda$, and

$$E_{3,1} \in \mathcal{U}(\bar{\mathfrak{u}}), \quad \text{where } \bar{\mathfrak{u}} = \left\{ \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix} \right\},$$

is the Lie algebra of the subgroup $\overline{U}$, the transpose of U. Certainly $\mathfrak{g} = \mathfrak{p} \oplus \bar{\mathfrak{u}}$. Thus by Shalika's argument $\Lambda = 0$. This finishes proving all the above propositions. □

Now let $\lambda \neq 0$ in $\mathcal{S}(\pi)$. We define the *local Bessel distribution*

$$B_\pi(f) = \sum_i \lambda(\pi(f)v_i)\,\overline{\lambda(v_i)}\,.$$

Let K be a maximal compact subgroup. As before, this is well defined, at least if f is K-finite, and the ϕ_i a basis of the K-finite vectors. For general f, we may take $B_\pi(f)$ to be

$$B_\pi(f) = (\pi(f)\lambda, \lambda)\,.$$

Since π is irreducible, $\Delta B_\pi = k B_\pi$ for some k. We already know that the restriction of B_π to Ω is nonzero.

We want to show something more precise.

Proposition 9.6. *The restriction of B_π to Ω_e is nonzero.*

Proof. As before, the distribution B_π descends to a distribution on $\mathbb{H}^\times$. We use a slightly different notation from before. For $g \in \Omega$, let $\Phi_f \in C^\infty(\Omega)$ be given by

$$\Phi_f(g) = \int f\left(\begin{pmatrix} I & X \\ 0 & I \end{pmatrix} g \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} I & Y \\ 0 & I \end{pmatrix}\right) \psi(\mathrm{tr}(X+Y))\, dA\, dX\, dY\,.$$

For any function Φ on Ω denote by $r\Phi$ the function on $\mathbb{H}^\times$ defined by

$$r\Phi(\gamma) = \Phi\begin{pmatrix} 0 & \gamma \\ I & 0 \end{pmatrix}.$$

There is a unique distribution T on $\mathbb{H}^\times$ such that $B_\pi(f) = T(r\Phi_f)$. We have to show that T is not supported on the center $\mathbb{R}^\times$ of $\mathbb{H}^\times$.

To that end we show that T satisfies a certain partial differential equation. Recall we took

$$\Delta = \sum E_\alpha E^\alpha = \frac{1}{4}E_0^2 - \frac{1}{2}\sum_{\{i,j,k\}}\left(E_{1,a}^2 + E_{4,a}^2\right) + \frac{1}{2}\sum_{\{1,i,j,k\}} b^2\left(E_{2,b}E_{3,b} + E_{3,b}E_{2,b}\right)$$

in $\mathfrak{Z}(\mathfrak{g})$. We compute

$$E_{3,1}E_{2,1} = E_{2,1}E_{3,1} - E_0 \qquad \text{and} \qquad E_{3,a}E_{2,a} = E_{2,a}E_{3,a} + E_0$$

for $a \in \{i, j, k\}$. Thus we may rewrite

$$\Delta = \frac{1}{4}E_0^2 - \frac{1}{2}\sum_{\{i,j,k\}}(E_{1,a}^2 + E_{4,a}^2) + \sum_{(1,i,j,k)} b^2 E_{3,b}E_{2,b} + 2E_0.$$

Since Δ is $\mathrm{Ad}(G)$ invariant, $\rho(\Delta)\Phi_f = \Phi_{\rho(\Delta)f}$. Since $B_\pi(\rho(\Delta)f) = \kappa B_\pi(f)$, we see that $T(r\Delta\Phi_f) = \kappa T(r\Phi_f)$. To understand what this means, we need to know what $r\Delta\Phi_f$ is. By linearity, we may write

$$\Delta\Phi_f = \frac{1}{4}\rho(E_0^2)\Phi_f - \frac{1}{2}\sum_{\{i,j,k\}}\left(\rho(E_{1,a}^2)\Phi_f + \rho(E_{4,a}^2)\Phi_f\right) + \sum_{\{1,i,j,k\}} b^2\rho(E_{3,b})\rho(E_{2,b})\Phi_f + 2\rho(E_0)\Phi_f. \tag{9-6}$$

By definition,

$$\begin{aligned}
\rho(E_0)\Phi_f\begin{pmatrix}0&\gamma\\ I&0\end{pmatrix} &= \frac{d}{dt}\bigg|_{t=0}\int f\left(\begin{pmatrix}I&X\\0&I\end{pmatrix}\begin{pmatrix}0&\gamma\\I&0\end{pmatrix}\begin{pmatrix}e^t&0\\0&e^{-t}\end{pmatrix}\begin{pmatrix}A&0\\0&A\end{pmatrix}\begin{pmatrix}I&Y\\0&I\end{pmatrix}\right)\psi\\
&= \frac{d}{dt}\bigg|_{t=0}\int f\left(\begin{pmatrix}I&X\\0&I\end{pmatrix}\begin{pmatrix}0&\gamma e^{-2t}\\I&0\end{pmatrix}\begin{pmatrix}e^t&0\\0&e^{t}\end{pmatrix}\begin{pmatrix}A&0\\0&A\end{pmatrix}\begin{pmatrix}I&Y\\0&I\end{pmatrix}\right)\psi\\
&= \frac{d}{dt}\bigg|_{t=0}\int f\left(\begin{pmatrix}I&X\\0&I\end{pmatrix}\begin{pmatrix}0&\gamma e^{-2t}\\I&0\end{pmatrix}\begin{pmatrix}A&0\\0&A\end{pmatrix}\begin{pmatrix}I&Y\\0&I\end{pmatrix}\right)\psi\\
&= \frac{d}{dt}\bigg|_{t=0} r\Phi_f(\gamma e^{-2t}) = \rho(-2X_0)r\Phi_f,
\end{aligned}$$

where the integrals on the first three lines are taken over $dA\,dX\,dY$, ψ is short for $\psi(\mathrm{tr}(X+Y))$, and $X_0 = I \in \mathbb{H} = \mathrm{Lie}(\mathbb{H}^\times)$. Similarly,

$$\begin{aligned}
\rho(E_0^2)\Phi_f\begin{pmatrix}0&\gamma\\ I&0\end{pmatrix} &= \frac{d^2}{dtds}\bigg|_{t=0,\,s=0}\int f\Bigg(\begin{pmatrix}I&X\\0&I\end{pmatrix}\begin{pmatrix}0&\gamma\\I&0\end{pmatrix}\\
&\qquad\times\begin{pmatrix}e^{t+s}&0\\0&e^{-t-s}\end{pmatrix}\begin{pmatrix}A&0\\0&A\end{pmatrix}\begin{pmatrix}I&Y\\0&I\end{pmatrix}\Bigg)\psi\\
&= \frac{d^2}{dtds}\bigg|_{t=0,\,s=0} r\Phi_f(\gamma e^{-2(t+s)}).
\end{aligned}$$

Thus $(\rho(E_0)\Phi_f) = \rho(-2X_0)r\Phi_f$ and $r(\rho(E_0^2)\Phi_f) = \rho(4X_0^2)r\Phi_f$.

Let $a \in \{i, j, k\}$. Computing as above, we see

$$\begin{aligned}
\rho(E_{1,a})\Phi_f\begin{pmatrix}0&\gamma\\ 1&0\end{pmatrix} &= \frac{d}{dt}\bigg|_{t=0}\int f\left(\begin{pmatrix}I&X\\0&I\end{pmatrix}\begin{pmatrix}0&\gamma\\I&0\end{pmatrix}\begin{pmatrix}e^{at}&0\\0&I\end{pmatrix}\begin{pmatrix}A&0\\0&A\end{pmatrix}\begin{pmatrix}I&Y\\0&I\end{pmatrix}\right)\psi\\
&= \frac{d}{dt}\bigg|_{t=0}\int f\left(\begin{pmatrix}I&X\\0&I\end{pmatrix}\begin{pmatrix}0&\gamma e^{-at}\\I&0\end{pmatrix}\begin{pmatrix}e^{at}&0\\0&e^{at}\end{pmatrix}\begin{pmatrix}A&0\\0&A\end{pmatrix}\begin{pmatrix}I&Y\\0&I\end{pmatrix}\right)\psi\\
&= \frac{d}{dt}\bigg|_{t=0} r\Phi_f(\gamma e^{-at}) = \rho(-X_a)r\Phi_f,
\end{aligned}$$

where $X_a = a \in \mathbb{H} = \mathrm{Lie}(\mathbb{H}^\times)$. Thus $r(\rho(E_{1,a}^2)\Phi_f) = \rho(X_a^2)r\Phi_f$.

Similarly

$$\begin{aligned}
\rho(E_{4,a})\Phi_f\begin{pmatrix}0&\gamma\\ I&0\end{pmatrix} &= \frac{d}{dt}\bigg|_{t=0}\int f\left(\begin{pmatrix}I&X\\0&I\end{pmatrix}\begin{pmatrix}0&\gamma\\I&0\end{pmatrix}\begin{pmatrix}I&0\\0&e^{at}\end{pmatrix}\begin{pmatrix}A&0\\0&A\end{pmatrix}\begin{pmatrix}I&Y\\0&I\end{pmatrix}\right)\psi\\
&= \frac{d}{dt}\bigg|_{t=0}\int f\left(\begin{pmatrix}I&X\\0&I\end{pmatrix}\begin{pmatrix}0&\gamma e^{at}\\I&0\end{pmatrix}\begin{pmatrix}A&0\\0&A\end{pmatrix}\begin{pmatrix}I&Y\\0&I\end{pmatrix}\right)\psi\\
&= \frac{d}{dt}\bigg|_{t=0} r\Phi_f(\gamma e^{at}) = \rho(X_a)\Phi_f.
\end{aligned}$$

Thus $r(\rho(X_{4,a}^2)\Phi_f)=\rho(X_a^2)r\Phi_f$.

Next, for $b\in\{1,i,j,k\}$,

$$\begin{aligned}\rho(E_{2,b}\Phi_f)(g)&=\frac{d}{dt}\bigg|_{t=0}\int f\left(\begin{pmatrix}I&X\\0&I\end{pmatrix}g\begin{pmatrix}I&tb\\0&I\end{pmatrix}\begin{pmatrix}A&0\\0&A\end{pmatrix}\begin{pmatrix}I&Y\\0&I\end{pmatrix}\right)\psi\\&=\frac{d}{dt}\bigg|_{t=0}\int f\left(\begin{pmatrix}I&X\\0&I\end{pmatrix}g\begin{pmatrix}A&0\\0&A\end{pmatrix}\begin{pmatrix}I&A^{-1}tbA\\0&I\end{pmatrix}\begin{pmatrix}I&Y\\0&I\end{pmatrix}\right)\psi\\&=\frac{d}{dt}\bigg|_{t=0}\psi(-\operatorname{tr}(tb))\Phi_f(g).\end{aligned}$$

Note that $\operatorname{tr}b=0$ if $b=i$, j or k, so the above quantity will vanish. On the other hand, if $b=1$, then

$$\frac{d}{dt}\bigg|_{t=0}\psi(-\operatorname{tr}(tb))=\frac{d}{dt}\bigg|_{t=0}e^{-4\pi i\eta t}=-4\pi i\eta\,,$$

if $\psi(x)=2i\pi\eta$. Hence $\rho(E_{2,b})\Phi_f=-4\pi i\eta\delta_{1,b}\Phi_f$.

Thus we only need to compute

$$\begin{aligned}\rho(E_{3,1})\Phi_f\begin{pmatrix}0&\gamma\\1&0\end{pmatrix}&=\frac{d}{dt}\bigg|_{t=0}\int f\left(\begin{pmatrix}I&X\\0&I\end{pmatrix}\begin{pmatrix}0&\gamma\\I&0\end{pmatrix}\begin{pmatrix}I&0\\t&I\end{pmatrix}\begin{pmatrix}A&0\\0&A\end{pmatrix}\begin{pmatrix}I&Y\\0&I\end{pmatrix}\right)\psi\\&=\frac{d}{dt}\bigg|_{t=0}\int f\left(\begin{pmatrix}I&X\\0&I\end{pmatrix}\begin{pmatrix}I&t\gamma\\0&I\end{pmatrix}\begin{pmatrix}0&\gamma\\I&0\end{pmatrix}\begin{pmatrix}A&0\\0&A\end{pmatrix}\begin{pmatrix}I&Y\\0&I\end{pmatrix}\right)\psi\\&=\frac{d}{dt}\bigg|_{t=0}\psi(-t\operatorname{tr}(\gamma))r\Phi_f(\gamma)=-2\pi i\eta\operatorname{tr}\gamma r\Phi_f(\gamma).\end{aligned}$$

Summing up, we have

$$r\rho(\Delta)\Phi_f=\rho\big(X_0^2-X_i^2-X_j^2-X_k^2-4X_0\big)r\Phi_f(\gamma)-8\pi^2\eta^2\operatorname{tr}\gamma r\Phi_f(\gamma).$$

Hence $\kappa T(\phi)=T(D\phi)$, where D is the differential operator with variable coefficients given by

$$D\phi(\gamma)=\rho\big(X_0^2-X_i^2-X_j^2-X_k^2-4X_0\big)\phi(\gamma)-8\pi^2\operatorname{tr}(\gamma)\phi(\gamma).$$

We want to show T is not supported on $\mathbb{R}^\times$. We apply Shalika's argument to the groups $G_1=\mathbb{R}^\times$ and $G_2=\mathbb{H}_1=\{h:\det h=1\}$. The Lie algebra of G_1 is $\mathbb{R}X_0$ and the Lie algebra of G_2 is $\mathbb{H}_0=\{h:\operatorname{tr}h=0\}$ with basis X_i,X_j,X_k. If T is supported on $\mathbb{R}^\times$, we get $T=0$, a contradiction. □

10. Local periods for GL_4

Let π a unitary irreducible representation of $G'=\mathrm{GL}_4$. As before, we define $\mathcal{S}(\pi)$. Then $\dim(\mathcal{S}(\pi))\leq 1$. This is established in [Jacquet and Rallis 1996] in the non-Archimedean case (in the context of GL_{2n}) and in [Ash and Ginzburg 1994, Lemma

5.4.2] in the Archimedean case. We remark that, at least in the non-Archimedean case, this would follow from the fact that the relevant orbits are invariant under an involution of GL_4.

In addition, if F is non-Archimedean, the character ψ is unramified, and the representation π admits a $\mathrm{GL}_2(\mathbb{O}_F)$-invariant vector $v_0 \neq 0$, then $\lambda(v_0) \neq 0$ for any $\lambda \neq 0$ in $\mathcal{S}(\pi)$. This follows from the discussion in [Bump and Friedberg 1990] or [Jacquet and Shalika 1990]. If $\lambda \neq 0$ is in $\mathcal{S}(\pi)$, we define the Bessel distribution $B_\pi(f)$ as before.

11. Proof of Theorem 1.1

Let F be a number field and $G = \mathrm{GL}_2(D)$, where D is a quaternion algebra over F. Suppose π_1 is an automorphic cuspidal representation of $G(\mathbb{A})$ distinguished by θ. Let λ be the linear form

$$\lambda(\phi) = \int_{S(F)Z(\mathbb{A})\backslash S(\mathbb{A})} \phi(s)\overline{\theta}(s)ds\,.$$

As we have observed, the quotient is compact so that the integral is absolutely convergent and defines a continuous linear form on the space of smooth vectors of π. Recall the Bessel distribution

$$B_{\pi_1}(f) = (\pi_1(f)\lambda, \lambda)\,.$$

It follows that every local component π_{1v} of π_1 is distinguished by θ_v. Furthermore, one can choose the local linear forms $\lambda_v \in \mathcal{S}(\pi_{1v})$ so that, if $f = \prod_v f_v$, then

$$B_{\pi_1}(f) = \prod_v B_{\pi_{1v}}(f_v)\,.$$

This factorization into local distributions follows from the uniqueness of local distributions established in [Prasad and Raghuram 2000] and Proposition 9.1 above. Of course, the local λ_v are so chosen that in this product almost all factors are 1. We assume D ramifies at some infinite place v and π_{1v_0} is supercuspidal for some finite place v_0 which splits D. We choose the functions at a place v where D_v ramifies as in the previous sections. Thus f_v is supported on the open set Ω_{D_v} if v is finite and D ramifies at v and the set $\Omega_{D_v,e}$ is v is infinite and D ramifies at v. As we have seen, if D ramifies at v, then we have $B_{\pi_{1v}}(f_v) \neq 0$ for at least one choice of f_v. It is elementary that there is a choice of f_{v_0} supercuspidal such that $B_{\pi_{1v_0}}(f_{v_0}) \neq 0$

We choose a matching function f' on $G'(\mathbb{A})$ as explained above. Then we have the identity

$$\sum_\pi B_\pi(f) = \sum_{\pi'} B_{\pi'}(f')\,.$$

On the left and right, the sums are respectively over all cuspidal automorphic representations π of $G(\mathbb{A})$ and π' of $G'(\mathbb{A})$ that are distinguished and supercuspidal at the place v_0.

Let $U(\pi_1)$ (or $U'(\pi_1)$) be the set of all cuspidal representations π of G (or G') such that $\pi_v \simeq \pi_{1v}$ at almost all places where π_1 is unramified and π_{v_0} is supercuspidal. By the assumption that π_1 has a Jacquet-Langlands lift to GL_4 and the strong multiplicity one assumption for GL_4, we find $U'(\pi_1)$ contains precisely one element, π_1'. Then the principle of infinite linear independence of characters of the Hecke algebra [Langlands 1980, Section 11] gives

$$\sum_{\pi\in U(\pi_1)} B_\pi(f) = \sum_{\pi'\in U'(\pi_1)} B_{\pi'}(f') = B_{\pi_1'}(f').$$

By our strong multiplicity one assumption on π_1', that is, $U(\pi_1)=\{\pi_1\}$, we see $B_{\pi_1'}(f') = B_{\pi_1}(f)$ is not identically zero. Thus π_1' is distinguished as claimed.

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References

[Ash and Ginzburg 1994] A. Ash and D. Ginzburg, "p-adic L-functions for $\mathrm{GL}(2n)$", *Invent. Math.* **116**:1-3 (1994), 27–73. MR 95f:11038 Zbl 0807.11029

[Badulescu 2007] A. I. Badulescu, "Global Jacquet–Langlands correspondence, multiplicity one and classification of automorphic representations", preprint, 2007, Available at http://arxiv.org/abs/0704.2920. With an appendix by N. Grbac. To appear in *Inv. Math.*

[Bernšteĭn and Zelevinskiĭ 1976] I. N. Bernšteĭn and A. V. Zelevinskiĭ, "Representations of the group $\mathrm{GL}(n, F)$, where F is a local non-Archimedean field", *Uspehi Mat. Nauk* **31**:3(189) (1976), 5–70. In Russian; translated in *Russian Math. Surveys* **31**:3 (1976), 1–68. MR 54 #12988 Zbl 0348.43007

[Bump 1997] D. Bump, *Automorphic forms and representations*, Cambridge Studies in Advanced Mathematics **55**, Cambridge University Press, Cambridge, 1997. MR 97k:11080 Zbl 0868.11022

[Bump and Friedberg 1990] D. Bump and S. Friedberg, "The exterior square automorphic L-functions on $\mathrm{GL}(n)$", pp. 47–65 in *Festschrift in honor of I. I. Piatetski-Shapiro on the occasion of his sixtieth birthday, Part II* (Ramat Aviv, Israel, 1989), edited by S. Gelbart et al., Israel Math. Conf. Proc. **3**, Weizmann, Jerusalem, 1990. MR 93d:11050 Zbl 0712.11030

[Dixmier 1977] J. Dixmier, *C^*-algebras*, North-Holland Publishing Co., Amsterdam, 1977. MR 56 #16388 Zbl 0372.46058

[Gan and Takeda 2007] W. T. Gan and S. Takeda, preprint, 2007, Available at http://arxiv.org/abs/0705.1576.

[Gel'fand and Kajdan 1975] I. M. Gel'fand, D. A. Kajdan, "Representations of the group $\mathrm{GL}(n, K)$ where K is a local field", pp. 95–118 in *Lie groups and their representations* (Budapest, 1971), edited by I. M. Gel'fand, Halsted, New York, 1975. MR 53 #8334 Zbl 0348.22011

[Harris and Taylor 2001] M. Harris and R. Taylor, *The geometry and cohomology of some simple Shimura varieties*, Annals of Mathematics Studies **151**, Princeton University Press, Princeton, NJ, 2001. MR 2002m:11050 Zbl 1036.11027

[Jacquet and Rallis 1996] H. Jacquet and S. Rallis, "Uniqueness of linear periods", *Compositio Math.* **102**:1 (1996), 65–123. MR 97k:22025 Zbl 0855.22018

[Jacquet and Shalika 1990] H. Jacquet and J. Shalika, "Exterior square L-functions", pp. 143–226 in *Automorphic forms, Shimura varieties, and L-functions* (Ann Arbor, MI, 1988), vol. II, edited by L. Clozel and J. S. Milne, Perspect. Math. **11**, Academic Press, Boston, 1990. MR 91g:11050 Zbl 0695.10025

[Jiang 2006] D. Jiang, "On the fundamental automorphic L-functions of $\mathrm{SO}(2n+1)$", *Int. Math. Res. Not.* (2006), Art. ID 64069, 26. MR 2007c:11063 Zbl 05039693

[Jiang et al. 2007] D. Jiang, C. Nien, and Y. Qin, "Local Shalika models and functoriality", preprint, 2007.

[Langlands 1980] R. P. Langlands, *Base change for* GL(2), Annals of Mathematics Studies **96**, Princeton University Press, Princeton, N.J., 1980. MR 82a:10032 Zbl 0444.22007

[Prasad and Raghuram 2000] D. Prasad and A. Raghuram, "Kirillov theory for $\mathrm{GL}_2(\mathcal{D})$ where $\mathcal{D}$ is a division algebra over a non-Archimedean local field", *Duke Math. J.* **104**:1 (2000), 19–44. MR 2001i:22024 Zbl 0958.22010

[Raghuram $\geq$ 2008] A. Raghuram, "On the restriction of to $D^* \times D^*$ of representations of p-adic $\mathrm{GL}_2(D)$", *Canad. J. Math.*. To appear.

[Rogawski 1983] J. D. Rogawski, "Representations of $\mathrm{GL}(n)$ and division algebras over a p-adic field", *Duke Math. J.* **50**:1 (1983), 161–196. MR 84j:12018 Zbl 0523.22015

[Shalika 1974] J. A. Shalika, "The multiplicity one theorem for GL_n", *Ann. of Math.* (2) **100** (1974), 171–193. MR 50 #545 Zbl 0316.12010

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