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We present modules for the extended affine Lie algebra $\widetilde{\mathfrak{gl}_3(\mathbb{C}_q)}$ by using the idea of free fields. We give a necessary and sufficient condition for the modules to be unitary.

1. Introduction

Extended affine Lie algebras are a higher dimensional generalization of affine Kac–Moody Lie algebras introduced by Høegh-Krohn and Torrésani [1990] and systematically studied in [Allison et al. 1997; Berman et al. 1996]. It turns out that any extended affine Lie algebra of type A can be coordinatized by a quantum torus (or a nonassociative torus for some small rank cases). Representations for extended affine Lie algebras coordinatized by quantum tori and Lie algebras related to quantum tori have been studied in [Jakobsen and Kac 1989; Berman and Szmigielski 1999; Gao 2000a; 2000b; 2002; Eswara Rao 2004; 2003; Eswara Rao and Batra 2002; Gao and Zeng 2006; Eswara Rao and Zhao 2004; Lin and Tan 2004; 2006; Golenishcheva-Kutuzova and Kac 1998; Varagnolo and Vasserot 1998; Miki 2004; Zhang and Zhao 1996; Billig and Zhao 2004; Su and Zhu 2005; Lau 2005; Baranovsky et al. 2000] and elsewhere.

Wakimoto's free fields construction [1986] provides a remarkable way to realize affine Kac–Moody Lie algebras; see also [Feĭgin and Frenkel 1990; Etingof et al. 1998]. In [Gao and Zeng 2006], we used Wakimoto's idea to construct a class of representations for $\widetilde{\mathfrak{gl}_2(\mathbb{C}_q)}$ and found the necessary and sufficient condition for the representations to be unitary. Here, we will continue to construct representations for $\widetilde{\mathfrak{gl}_3(\mathbb{C}_q)}$. As witnessed in [Feĭgin and Frenkel 1990], the realization for $\widetilde{\mathfrak{gl}_3(\mathbb{C}_q)}$ is much more subtle and complicated than the one for $\widetilde{\mathfrak{gl}_2(\mathbb{C}_q)}$. This construction for $\widetilde{\mathfrak{gl}_3(\mathbb{C}_q)}$ might shed light on the general construction for $\widetilde{\mathfrak{gl}_n(\mathbb{C}_q)}$ with $n \geq 4$.

We then construct a hermitian form and determine when the form is positive definite (so the representations are unitary). Unlike [Gao and Zeng 2006], in which we defined the form on the monomial basis for the module (this idea goes back to [Wakimoto 1985]), we define the form directly on the basis consisting of certain

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iterated module actions on a "highest weight vector" 1. This facilitates verifying that the defined form is hermitian.

Let q be a nonzero complex number. A quantum 2-torus associated to q [Manin 1991] is the unital associative $\mathbb{C}$-algebra $\mathbb{C}_q[s^{\pm 1}, t^{\pm 1}]$ (or, simply $\mathbb{C}_q$) with generators $s^{\pm 1}, t^{\pm 1}$ and relations

$$ss^{-1} = s^{-1}s = tt^{-1} = t^{-1}t = 1 \quad \text{and} \quad ts = qst.$$

Define a $\mathbb{C}$-linear function $\kappa : \mathbb{C}_q \to \mathbb{C}$ by

$$\kappa(s^m t^n) = \delta_{(m,n),(0,0)}.$$

Let d_s, d_t be the degree operators on $\mathbb{C}_q$ defined for $m, n \in \mathbb{Z}$ by

$$d_s(s^m t^n) = m s^m t^n, \;\; d_t(s^m t^n) = n s^m t^n.$$

Let $\mathfrak{gl}_3(\mathbb{C}_q)$ be the Lie algebra of 3 by 3 matrices with entries in $\mathbb{C}_q$. We form a natural central extension of $\mathfrak{gl}_3(\mathbb{C}_q)$ as

$$\widehat{\mathfrak{gl}_3(\mathbb{C}_q)} = \mathfrak{gl}_3(\mathbb{C}_q) \oplus \mathbb{C}c_s \oplus \mathbb{C}c_t$$

with Lie bracket

$$\begin{aligned} (1\text{-}1) \quad & [E_{ij}(s^{m_1}t^{n_1}), E_{kl}(s^{m_2}t^{n_2})] \\ & = \delta_{jk} q^{n_1 m_2} E_{il}(s^{m_1+m_2}t^{n_1+n_2}) - \delta_{il} q^{n_2 m_1} E_{kj}(s^{m_1+m_2}t^{n_1+n_2}) \\ & \quad + m_1 q^{n_1 m_2} \delta_{jk}\delta_{il}\delta_{m_1+m_2,0}\delta_{n_1+n_2,0} c_s + n_1 q^{n_1 m_2} \delta_{jk}\delta_{il}\delta_{m_1+m_2,0}\delta_{n_1+n_2,0} c_t \end{aligned}$$

for $m_1, m_2, n_1, n_2 \in \mathbb{Z}$ and $1 \le i, j, k, l \le 3$, where $E_{ij}(s^m t^n)$ is the matrix whose only nonzero entry is $s^m t^n$ at the (i, j) position, and c_s and c_t are central elements of $\widehat{\mathfrak{gl}_3(\mathbb{C}_q)}$.

The derivations d_s and d_t can be extended to derivations on $\mathfrak{gl}_3(\mathbb{C}_q)$. Now we can define the semidirect product of the Lie algebra $\widehat{\mathfrak{gl}_3(\mathbb{C}_q)}$ and those derivations:

$$\widetilde{\mathfrak{gl}_3(\mathbb{C}_q)} = \widehat{\mathfrak{gl}_3(\mathbb{C}_q)} \oplus \mathbb{C}d_s \oplus \mathbb{C}d_t.$$

The Lie algebra $\widetilde{\mathfrak{gl}_3(\mathbb{C}_q)}$ is an extended affine Lie algebra of type A_2 with nullity 2. See [Allison et al. 1997] and [Berman et al. 1996] for definitions.

2. Module for $\widetilde{\mathfrak{gl}_3(\mathbb{C}_q)}$

In this section, we use Wakimoto's idea [1985] to construct $\widetilde{\mathfrak{gl}_3(\mathbb{C}_q)}$-modules as was done in [Gao and Zeng 2006].

Let $\mathbb{K}_1 = \{(3m+1, 3n+1) \mid m, n \in \mathbb{Z}\}$ and $\mathbb{K}_{-1} = \{(3m-1, 3n-1) \mid m, n \in \mathbb{Z}\}$ so that $\mathbb{K}_{-1} = -\mathbb{K}_1$. If $\boldsymbol{A} = (3m+1, 3n+1) \in \mathbb{K}_1$, we always write $\boldsymbol{A}_1 = m$ and

$\boldsymbol{A}_2 = n$, and similarly, if $\boldsymbol{B} = (3m-1, 3n-1) \in \mathbb{K}_{-1}$, then $\boldsymbol{B}_1 = m$ and $\boldsymbol{B}_2 = n$. Let

$$V = \mathbb{C}[x_{\boldsymbol{A}}, x_{\boldsymbol{B}} : \boldsymbol{A} \in \mathbb{K}_1, \boldsymbol{B} \in \mathbb{K}_{-1}]$$

be the (commutative) polynomial ring of infinitely many variables. The operators $x_{(m,n)}$ and $\partial/\partial x_{(m,n)}$ act on V by the usual multiplication and differentiation.

Form a family of 2×2 lower triangular matrices

$$X_{m,n} = \begin{pmatrix} a_{(m,n)} & 0 \\ c_{(m,n)} & d_{(m,n)} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$$

for $(m,n) \in \mathbb{K}_1 \cup \mathbb{K}_{-1}$ (so that $a_{(m,n)} d_{(m,n)} = 1$). Set

$$P_{(m,n)} = a_{(m,n)} \frac{\partial}{\partial x_{(m,n)}} \quad \text{and} \quad Q_{(m,n)} = c_{(m,n)} \frac{\partial}{\partial x_{(m,n)}} + d_{(m,n)} x_{(m,n)}$$

for $(m,n) \in \mathbb{K}_1 \cup \mathbb{K}_{-1}$. Then for $\boldsymbol{A}, \boldsymbol{A}' \in \mathbb{K}_1$ and $\boldsymbol{B}, \boldsymbol{B}' \in \mathbb{K}_{-1}$,

$$\begin{aligned}
&[P_{\boldsymbol{A}}, P_{\boldsymbol{A}'}] = [Q_{\boldsymbol{A}}, Q_{\boldsymbol{A}'}] = [P_{\boldsymbol{B}}, P_{\boldsymbol{B}'}] = [Q_{\boldsymbol{B}}, Q_{\boldsymbol{B}'}] = 0 \\
&[P_{\boldsymbol{A}}, P_{\boldsymbol{B}}] = [P_{\boldsymbol{A}}, Q_{\boldsymbol{B}}] = [Q_{\boldsymbol{A}}, Q_{\boldsymbol{B}}] = [P_{\boldsymbol{B}}, Q_{\boldsymbol{A}}] = 0 \\
&[P_{\boldsymbol{A}}, Q_{\boldsymbol{A}'}] = \delta_{\boldsymbol{A},\boldsymbol{A}'} \\
&[P_{\boldsymbol{B}}, Q_{\boldsymbol{B}'}] = \delta_{\boldsymbol{B},\boldsymbol{B}'}.
\end{aligned}$$

Fixing a complex number μ, we define operators on V. For the rest of this paper, sums involving $\boldsymbol{A}$ and $\boldsymbol{A}'$ (or any other decorated $\boldsymbol{A}$) will range over $\mathbb{K}_1$. Similarly, the $\boldsymbol{B}$'s range over $\mathbb{K}_{-1}$. That is, we write $\sum_{\boldsymbol{A} \in \mathbb{K}_1}$ simply as $\sum_{\boldsymbol{A}}$, and so on.

$$\begin{aligned}
e_{21}^{(\mu)}(m_1, n_1) = {} & -q^{-m_1 n_1} \mu P_{-(3m_1-1, 3n_1-1)} \\
& - \sum_{\boldsymbol{A},\boldsymbol{A}'} q^{n_1 \boldsymbol{A}'_1 + \boldsymbol{A}_2 m_1 + \boldsymbol{A}_2 \boldsymbol{A}'_1} Q_{\boldsymbol{A}+\boldsymbol{A}'+(3m_1-1,3n_1-1)} P_{\boldsymbol{A}} P_{\boldsymbol{A}'} \\
& - \sum_{\boldsymbol{A},\boldsymbol{B}} q^{n_1 \boldsymbol{A}_1 + \boldsymbol{B}_2 m_1 + \boldsymbol{B}_2 \boldsymbol{A}_1} Q_{\boldsymbol{A}+\boldsymbol{B}+(3m_1-1,3n_1-1)} P_{\boldsymbol{A}} P_{\boldsymbol{B}},
\end{aligned}$$

$$e_{12}^{(\mu)}(m_1, n_1) = Q_{(3m_1+1, 3n_1+1)},$$

$$e_{11}^{(\mu)}(m_1, n_1) = \tfrac{1}{2} \mu \delta_{(m_1,n_1),(0,0)} + \sum_{\boldsymbol{A}} q^{\boldsymbol{A}_1 n_1} Q_{\boldsymbol{A}+(3m_1,3n_1)} P_{\boldsymbol{A}},$$

$$\begin{aligned}
e_{22}^{(\mu)}(m_1, n_1) = {} & -\tfrac{1}{2} \mu \delta_{(m_1,n_1),(0,0)} \\
& - \sum_{\boldsymbol{A}} q^{\boldsymbol{A}_2 m_1} Q_{\boldsymbol{A}+(3m_1,3n_1)} P_{\boldsymbol{A}} - \sum_{\boldsymbol{B}} q^{\boldsymbol{B}_2 m_1} Q_{\boldsymbol{B}+(3m_1,3n_1)} P_{\boldsymbol{B}},
\end{aligned}$$

$$\begin{aligned}
e_{23}^{(\mu)}(m_1, n_1) = {} & -q^{-m_1 n_1} \mu P_{-(3m_1+1, 3n_1+1)} \\
& - \sum_{\boldsymbol{A},\boldsymbol{B}} q^{n_1 \boldsymbol{B}_1 + \boldsymbol{A}_2 m_1 + \boldsymbol{A}_2 \boldsymbol{B}_1} Q_{\boldsymbol{A}+\boldsymbol{B}+(3m_1+1,3n_1+1)} P_{\boldsymbol{A}} P_{\boldsymbol{B}} \\
& - \sum_{\boldsymbol{B},\boldsymbol{B}'} q^{n_1 \boldsymbol{B}'_1 + \boldsymbol{B}_2 m_1 + \boldsymbol{B}_2 \boldsymbol{B}'_1} Q_{\boldsymbol{B}+\boldsymbol{B}'+(3m_1+1,3n_1+1)} P_{\boldsymbol{B}} P_{\boldsymbol{B}'},
\end{aligned}$$

$$e_{32}^{(\mu)}(m_1,n_1)=Q_{(3m_1-1,3n_1-1)},$$
$$e_{31}^{(\mu)}(m_1,n_1)=\sum_{\boldsymbol{A}} q^{\boldsymbol{A}_1 n_1} Q_{\boldsymbol{A}+(3m_1-2,3n_1-2)} P_{\boldsymbol{A}},$$
$$e_{13}^{(\mu)}(m_1,n_1)=\sum_{\boldsymbol{B}} q^{\boldsymbol{B}_1 n_1} Q_{\boldsymbol{B}+(3m_1+2,3n_1+2)} P_{\boldsymbol{B}},$$
$$e_{33}^{(\mu)}(m_1,n_1)=\tfrac{1}{2}\mu\delta_{(m_1,n_1),(0,0)}+\sum_{\boldsymbol{B}} q^{\boldsymbol{B}_1 n_1} Q_{\boldsymbol{B}+(3m_1,3n_1)} P_{\boldsymbol{B}},$$
$$D_1^{(\mu)}=\sum_{\boldsymbol{A}} \boldsymbol{A}_1 Q_{\boldsymbol{A}} P_{\boldsymbol{A}}+\sum_{\boldsymbol{B}} \boldsymbol{B}_1 Q_{\boldsymbol{B}} P_{\boldsymbol{B}},$$
$$D_2^{(\mu)}=\sum_{\boldsymbol{A}} \boldsymbol{A}_2 Q_{\boldsymbol{A}} P_{\boldsymbol{A}}+\sum_{\boldsymbol{B}} \boldsymbol{B}_2 Q_{\boldsymbol{B}} P_{\boldsymbol{B}}.$$

Although the operators are infinite sums, they are well defined as operators on V. Now we have the following result:

Theorem 2.1. *The linear map* $\pi:\widetilde{\mathfrak{gl}_3(\mathbb{C}_q)}\to \operatorname{End}V$ *given by*

$$\pi(E_{ij}(s^{m_1}t^{n_1}))=e^{(\mu)}{}_{ij}(m_1,n_1),\qquad \begin{matrix}\pi(d_s)=D_1^{(\mu)}, & \pi(c_s)=0,\\ \pi(d_t)=D_2^{(\mu)}, & \pi(c_t)=0,\end{matrix}$$

for $m_1,n_1\in\mathbb{Z}$ *and* $1\le i,\ j\le 3$ *is a Lie algebra homomorphism.*

Proof. The proof is straightforward. However, we will provide a few details. It suffices to check the Lie bracket (1-1). We will do this systematically so that we won't miss any cases.

First, we have

$$\begin{aligned}
&\left[e^{(\mu)}{}_{11}(m_1,n_1),e^{(\mu)}{}_{11}(m_2,n_2)\right]\\
&\quad=\Big[\sum_{\boldsymbol{A}} q^{\boldsymbol{A}_1 n_1} Q_{(3m_1,3n_1)+\boldsymbol{A}} P_{\boldsymbol{A}},\sum_{\boldsymbol{A}'} q^{\boldsymbol{A}'_1 n_1} Q_{(3m_2,3n_2)+\boldsymbol{A}'} P_{\boldsymbol{A}'}\Big]\\
&\quad=\sum_{\boldsymbol{A}'} q^{(m_2+\boldsymbol{A}'_1)n_1+\boldsymbol{A}' n_2} Q_{(3m_1,3n_1)+(3m_2+3n_2)+\boldsymbol{A}'} P_{\boldsymbol{A}'}+\tfrac{1}{2}q^{m_2n_1}\mu\delta_{(m_1+m_2,n_1+n_2),(0,0)}\\
&\qquad-\sum_{\boldsymbol{A}} q^{\boldsymbol{A}_1 n_1+(m_1+\boldsymbol{A}_1)n_2} q_{(3m_1,3n_1)+(3m_2,3n_2)+\boldsymbol{A}} P_{\boldsymbol{A}}-\tfrac{1}{2}q^{m_1n_2}\mu\delta_{(m_1+m_2,n_1+n_2),(0,0)}\\
&\quad=q^{m_2n_1}e^{(\mu)}{}_{11}(m_1+m_2,n_1+n_2)-q^{m_1n_2}e^{(\mu)}{}_{11}(m_1+m_2,n_1+n_2).
\end{aligned}$$

The next two brackets are easy.

$$\left[e^{(\mu)}{}_{11}(m_1,n_1),e^{(\mu)}{}_{12}(m_2,n_2)\right]=q^{m_2n_1}e^{(\mu)}{}_{12}(m_1+m_2,n_1+n_2),$$
$$\left[e^{(\mu)}{}_{11}(m_1,n_1),e^{(\mu)}{}_{13}(m_2,n_2)\right]=q^{m_2n_1}e^{(\mu)}{}_{13}(m_1+m_2,n_1+n_2).$$

Next,

$$
\begin{aligned}
&\big[e^{(\mu)}{}_{11}(m_1,n_1), e^{(\mu)}{}_{21}(m_2,n_2)\big]\\
&\quad=\sum_{\boldsymbol{A}}(-\mu)q^{\boldsymbol{A}_1n_1-m_2n_2}\big[Q_{\boldsymbol{A}+(3m_1,3n_1)}P_{\boldsymbol{A}},\,P_{-(3m_2-1,3n_2-1)}\big]\\
&\qquad-\sum_{\boldsymbol{A},\bar{\boldsymbol{A}},\bar{\boldsymbol{A}}'}q^{\boldsymbol{A}_1n_1+\bar{\boldsymbol{A}}_1'n_2+\bar{\boldsymbol{A}}_2m_2+\bar{\boldsymbol{A}}_2\bar{\boldsymbol{A}}_1'}\big[Q_{\boldsymbol{A}+(3m_1,3n_1)}P_{\boldsymbol{A}},\,Q_{\bar{\boldsymbol{A}}+\bar{\boldsymbol{A}}'+(3m_2-1,3n_2-1)}P_{\bar{\boldsymbol{A}}}P_{\bar{\boldsymbol{A}}'}\big]\\
&\qquad-\sum_{\boldsymbol{A},\bar{\boldsymbol{A}},\boldsymbol{B}}q^{\boldsymbol{A}_1n_1+n_2\bar{\boldsymbol{A}}_1+\boldsymbol{B}_2m_2+\boldsymbol{B}_2\bar{\boldsymbol{A}}_1}\big[Q_{\boldsymbol{A}+(3m_1,3n_1)}P_{\boldsymbol{A}},\,Q_{\bar{\boldsymbol{A}}+\boldsymbol{B}+(3m_2-1,3n_2-1)}P_{\bar{\boldsymbol{A}}}P_{\boldsymbol{B}}\big]\\
&\quad=\mu q^{-(m_1+m_2)(n_1+n_2)+m_1n_2}P_{-(3(m_1+m_2)-1,3(n_1+n_2)-1)}\\
&\qquad-\sum_{\bar{\boldsymbol{A}},\bar{\boldsymbol{A}}'}q^{(\bar{\boldsymbol{A}}_1+\bar{\boldsymbol{A}}_1'+m_2)n_1+n_2\bar{\boldsymbol{A}}_1'+\bar{\boldsymbol{A}}_2m_2+\bar{\boldsymbol{A}}_2\bar{\boldsymbol{A}}_1'}Q_{(3m_1,3n_1)+\bar{\boldsymbol{A}}+\bar{\boldsymbol{A}}'+(3m_2-1,3n_2-1)}P_{\bar{\boldsymbol{A}}}P_{\bar{\boldsymbol{A}}'}\\
&\qquad+\sum_{\boldsymbol{A},\bar{\boldsymbol{A}}}q^{\boldsymbol{A}_1n_1+n_2(m_1+\boldsymbol{A}_1)+\bar{\boldsymbol{A}}_2m_2+\bar{\boldsymbol{A}}_2(m_1+\boldsymbol{A}_1)}Q_{\bar{\boldsymbol{A}}+(3m_1,3n_1)+\boldsymbol{A}+(3m_2-1,3n_2-1)}P_{\bar{\boldsymbol{A}}}P_{\boldsymbol{A}}\\
&\qquad+\sum_{\boldsymbol{A},\bar{\boldsymbol{A}}'}q^{\boldsymbol{A}_1n_1+n_2\boldsymbol{A}_1'+(n_1+\boldsymbol{A}_2)m_2+(n_1+\boldsymbol{A}_2)\bar{\boldsymbol{A}}_1'}Q_{\boldsymbol{A}+(3m_1,3n_1)+\bar{\boldsymbol{A}}'+(3m_2-1,3n_2-1)}P_{\bar{\boldsymbol{A}}'}P_{\boldsymbol{A}}\\
&\qquad+\sum_{\boldsymbol{A},\boldsymbol{B}}q^{\boldsymbol{A}_1n_1+n_2(m_1+\boldsymbol{A}_1)+bdb_2m_2+\boldsymbol{B}_2(m_1+\boldsymbol{A}_1)}Q_{\boldsymbol{A}+(3m_1,3n_1)+\boldsymbol{B}+(3m_2-1,3n_2-1)}P_{\boldsymbol{B}}P_{\boldsymbol{A}},
\end{aligned}
$$

(using that the second and fourth term cancel each other)

$$
\begin{aligned}
&=-q^{m_1n_2}(-\mu q^{-(m_1+m_2)(n_1+n_2)}P_{-(3(m_1+m_2)-1,3(n_1+n_2)-1)}\\
&\qquad-\sum_{\boldsymbol{A},\bar{\boldsymbol{A}}}q^{\boldsymbol{A}(n_1+n_2)+\bar{\boldsymbol{A}}_2(m_1+m_2)+\boldsymbol{A}_1\bar{\boldsymbol{A}}_2}Q_{(3(m_1+m_2)-1,3(n_1+n_2)-1)+\boldsymbol{A}+\bar{\boldsymbol{A}}}P_{\bar{\boldsymbol{A}}}P_{\boldsymbol{A}}\\
&\qquad-\sum_{\boldsymbol{A},\boldsymbol{B}}q^{\boldsymbol{A}_1(n_1+n_2)+\boldsymbol{B}_2(m_1+m_2)+\boldsymbol{B}_2+\boldsymbol{A}_1}Q_{\boldsymbol{A}+\boldsymbol{B}+(3(m_1+m_2)-1,3(n_1+n_2)-1)}P_{\boldsymbol{B}}P_{\boldsymbol{A}})\\
&=-q^{m_1n_2}e^{(\mu)}{}_{21}(m_1+m_2,n_1+n_2).
\end{aligned}
$$

We easily verify the next seven brackets.

$$
\begin{aligned}
&\big[e^{(\mu)}{}_{11}(m_1,n_1), e^{(\mu)}{}_{22}(m_2,n_2)\big]=0,\\
&\big[e^{(\mu)}{}_{11}(m_1,n_1), e^{(\mu)}{}_{23}(m_2,n_2)\big]=0,\\
&\big[e^{(\mu)}{}_{11}(m_1,n_1), e^{(\mu)}{}_{31}(m_2,n_2)\big]=-q^{m_1n_2}e^{(\mu)}{}_{31}(m_1+m_2,n_1+n_2),\\
&\big[e^{(\mu)}{}_{11}(m_1,n_1), e^{(\mu)}{}_{32}(m_2,n_2)\big]=0,\\
&\big[e^{(\mu)}{}_{11}(m_1,n_1), e^{(\mu)}{}_{33}(m_2,n_2)\big]=0,\\
&\big[e^{(\mu)}{}_{12}(m_1,n_1), e^{(\mu)}{}_{12}(m_2,n_2)\big]=0,\\
&\big[e^{(\mu)}{}_{12}(m_1,n_1), e^{(\mu)}{}_{13}(m_2,n_2)\big]=0.
\end{aligned}
$$

Next we have

$$
\begin{aligned}
&\big[e^{(\mu)}{}_{12}(m_1,n_1), e^{(\mu)}{}_{21}(m_2,n_2)\big]\\
&\quad = \mu q^{-m_2n_2}\delta_{(m_1,n_1),(-m_2,n_2)}\\
&\qquad + \sum_{\boldsymbol{A}'} q^{n_2\boldsymbol{A}'_1+n_1m_2+n_1\boldsymbol{A}'_1} Q_{(3m_1+1,3n_1+1)+\boldsymbol{A}'+(3m_2-1,3n_2-1)} P_{\boldsymbol{A}'}\\
&\qquad + \sum_{\boldsymbol{A}} q^{n_2m_1+\boldsymbol{A}_2m_2+\boldsymbol{A}_2m_1} Q_{\boldsymbol{A}+(3m_1+1,3n_1+1)+(3m_2-1,3n_2-1)} P_{\boldsymbol{A}}\\
&\qquad + \sum_{\boldsymbol{B}} q^{n_2m_1+\boldsymbol{B}_2m_2+\boldsymbol{B}_2m_1} Q_{(3m_1+1,3n_1+1)+\boldsymbol{B}+(3m_2-1,3n_2-1)} P_{\boldsymbol{B}}\\
&\quad = q^{n_1m_2}\big(\sum_{\boldsymbol{A}'} q^{(n_1+n_2)\boldsymbol{A}'_1} Q_{\boldsymbol{A}'+(3m_1+3m_2,3n_1+3n_2)} P_{\boldsymbol{A}'} + \tfrac{1}{2}\delta_{(m_1,n_1),(-m_2,n_2)}\big)\\
&\qquad - q^{n_2m_1}\big(-\sum_{\boldsymbol{A}} q^{\boldsymbol{A}_2(m_2+m_1)} Q_{\boldsymbol{A}+(3m_1+3m_2,3n_1+3n_2)} P_{\boldsymbol{A}}\\
&\qquad\qquad - \sum_{\boldsymbol{B}} q^{\boldsymbol{B}_2(m_2+m_1)} Q_{\boldsymbol{B}+(3m_1+3m_2,3n_1+3n_2)} P_{\boldsymbol{B}} - \tfrac{1}{2}\delta_{(m_1,n_1),(-m_2,n_2)}\big)\\
&\quad = q^{n_1m_2}e^{(\mu)}{}_{11}(m_1+m_2,n_1+n_2) - q^{n_2m_1}e^{(\mu)}{}_{22}(m_1+m_2,n_1+n_2).
\end{aligned}
$$

The following six brackets can be checked easily.

$$
\begin{aligned}
&[e^{(\mu)}{}_{12}(m_1,n_1), e^{(\mu)}{}_{22}(m_2,n_2)] = q^{n_1m_2}e^{(\mu)}{}_{12}(m_1+m_2,n_1+n_2),\\
&[e^{(\mu)}{}_{12}(m_1,n_1), e^{(\mu)}{}_{23}(m_2,n_2)] = q^{n_1m_2}e^{(\mu)}{}_{13}(m_1+m_2,n_1+n_2),\\
&[e^{(\mu)}{}_{12}(m_1,n_1), e^{(\mu)}{}_{31}(m_2,n_2)] = -q^{m_1n_2}e^{(\mu)}{}_{32}(m_1+m_2,n_1+n_2),\\
&[e^{(\mu)}{}_{12}(m_1,n_1), e^{(\mu)}{}_{32}(m_2,n_2)] = 0,\\
&[e^{(\mu)}{}_{12}(m_1,n_1), e^{(\mu)}{}_{33}(m_2,n_2)] = 0,\\
&[e^{(\mu)}{}_{13}(m_1,n_1), e^{(\mu)}{}_{13}(m_2,n_2)] = 0.
\end{aligned}
$$

Next,

$$
\begin{aligned}
&\big[e^{(\mu)}{}_{13}(m_1,n_1), e^{(\mu)}{}_{21}(m_2,n_2)\big]\\
&= \big[\sum_{\boldsymbol{B}} q^{\boldsymbol{B}_1n_1} Q_{\boldsymbol{B}+(3m_1+2,3n_1+2)} P_{\boldsymbol{B}}, -q^{-m_2n_2}\mu P_{-(3m_2-1,3n_2-1)}\\
&\qquad\qquad - \sum_{\boldsymbol{A},\boldsymbol{A}'} q^{n_2\boldsymbol{A}'_1+\boldsymbol{A}_2m_2+\boldsymbol{A}_2\boldsymbol{A}'_1} Q_{\boldsymbol{A}+\boldsymbol{A}'+(3m_2-1,3n_2-1)} P_{\boldsymbol{A}} P_{\boldsymbol{A}'}\\
&\qquad\qquad\qquad - \sum_{\boldsymbol{A},\boldsymbol{B}} q^{n_2\boldsymbol{A}_1+\boldsymbol{B}_2m_2+\boldsymbol{B}_2\boldsymbol{A}_1} Q_{\boldsymbol{A}+\boldsymbol{B}+(3m_2-1,3n_2-1)} P_{\boldsymbol{A}} P_{\boldsymbol{B}}\big]\\
&= -\sum_{\boldsymbol{A},\boldsymbol{B}} q^{(\boldsymbol{A}_1+\boldsymbol{B}_1+m_2)n_1+n_2+\boldsymbol{A}_1+\boldsymbol{B}_2m_2+\boldsymbol{B}_2\boldsymbol{A}_1} Q_{(3m_1+2,3n_1+2)+\boldsymbol{A}+\boldsymbol{B}+(3m_2-1,3n_2-1)} P_{\boldsymbol{A}} P_{\boldsymbol{B}}\\
&\quad + \sum_{\boldsymbol{A},\boldsymbol{B}} q^{n_2(m_1+\boldsymbol{B}_1)+\boldsymbol{A}_2m_2+\boldsymbol{A}_2(m_1+\boldsymbol{B}_1)+\boldsymbol{B}_1n_1} Q_{\boldsymbol{A}+(3m_1+2,3n_1+2)+\boldsymbol{B}+(3m_2-1,3n_2-1)} P_{\boldsymbol{A}} P_{\boldsymbol{B}}\\
&\qquad\qquad + q^{-m_2n_2+(-m_1-m_2)n_1}\mu P_{(-3m_1-3m_2-1,-3n_1-3n_2-1)}
\end{aligned}
$$

$$+\sum_{\boldsymbol{A}',\boldsymbol{B}} q^{n_2\boldsymbol{A}'_1+(n_1+\boldsymbol{B}_2)m_2+(n_1+\boldsymbol{B}_2)\boldsymbol{A}'_1+\boldsymbol{B}_1n_1} Q_{\boldsymbol{A}'+(3m_1+2,3n_1+2)+\boldsymbol{B}+(3m_2-1,3n_2-1)} P_{\boldsymbol{A}'} P_{\boldsymbol{B}}$$

$$+\sum_{\boldsymbol{B},\boldsymbol{B}'} q^{n_2(\boldsymbol{B}'_1+m_1)+\boldsymbol{B}_2m_2+\boldsymbol{B}_2(\boldsymbol{B}'_1+m_1)+\boldsymbol{B}'_1n_1} Q_{\boldsymbol{B}'+(3m_1+2,3n_1+2)+\boldsymbol{B}+(3m_2-1,3n_2-1)} P_{\boldsymbol{B}'} P_{\boldsymbol{B}}$$

(using that the first and fourth terms cancel each other)

$$= q^{n_2m_1}\Big(q^{-(m_1+m_2)(n_1+n_2)}\mu P_{-(3m_1+3m_2-1,3n_1+3n_2-1)}$$
$$+\sum_{\boldsymbol{A},\boldsymbol{A}'} q^{(n_1+n_2)\boldsymbol{A}'_1+\boldsymbol{A}_2(m_1+m_2)+\boldsymbol{A}_2\boldsymbol{A}'_1} Q_{\boldsymbol{A}+\boldsymbol{A}'+(3m_1+3m_2-1,3n_1+3n_2-1)} P_{\boldsymbol{A}} P_{\boldsymbol{A}'}$$
$$-\sum_{\boldsymbol{A},\boldsymbol{B}} q^{(n_1+n_2)\boldsymbol{A}_1+\boldsymbol{B}_2(m_1+m_2)+\boldsymbol{B}_2\boldsymbol{A}_1} Q_{\boldsymbol{A}+\boldsymbol{B}+(3m_1+3m_2-1,3n_1+3n_2-1)} P_{\boldsymbol{A}} P_{\boldsymbol{B}}\Big)$$
$$= -q^{n_2m_1} e^{(\mu)}{}_{23}(m_1+m_2,n_1+n_2).$$

The following two brackets are easy.

$$\big[e^{(\mu)}{}_{13}(m_1,n_1), e^{(\mu)}{}_{22}(m_2,n_2)\big]=0,$$
$$\big[e^{(\mu)}{}_{13}(m_1,n_1), e^{(\mu)}{}_{23}(m_2,n_2)\big]=0.$$

Next,

$$-\big[e^{(\mu)}{}_{13}(m_2,n_2), e^{(\mu)}{}_{31}(m_1,n_1)\big]=\big[e^{(\mu)}{}_{31}(m_1,n_1), e^{(\mu)}{}_{13}(m_2,n_2)\big]$$
$$=\sum_{\boldsymbol{A},\boldsymbol{B}} q^{n_1\boldsymbol{A}_1+n_2\boldsymbol{B}_1}\big[Q_{(3m_1-2,3n_1-2)+\boldsymbol{A}}P_{\boldsymbol{A}},\, Q_{(3m_2+2,3n_2+2)+\boldsymbol{B}}P_{\boldsymbol{B}}\big]$$
$$=\sum_{\boldsymbol{B}} q^{n_2\boldsymbol{B}_1+n_1(m_2+\boldsymbol{B}_1)} Q_{(3m_1+3m_2,3n_1+3n_2)+\boldsymbol{B}}P_{\boldsymbol{B}}$$
$$-\sum_{\boldsymbol{A}} q^{n_1\boldsymbol{A}_1+n_2(m_1+\boldsymbol{A}_1)} Q_{(3m_1+3m_2,3n_1+3n_2)+\boldsymbol{A}}P_{\boldsymbol{A}}$$
$$=q^{n_1m_2}\big(\sum_{\boldsymbol{B}} q^{n_2\boldsymbol{B}_1+n_1(m_2+\boldsymbol{B}_1)} Q_{(3m_1+3m_2,3n_1+3n_2)+\boldsymbol{B}}P_{\boldsymbol{B}}+\tfrac{1}{2}\mu\delta_{(m_1+m_2,n_1+n_2),(0,0)}\big)$$
$$-q^{n_2m_1}\big(\sum_{\boldsymbol{A}} q^{n_1\boldsymbol{A}_1+n_2(m_1+\boldsymbol{A}_1)} Q_{(3m_1+3m_2,3n_1+3n_2)+\boldsymbol{A}}P_{\boldsymbol{A}}+\tfrac{1}{2}\mu\delta_{(m_1+m_2,n_1+n_2),(0,0)}\big)$$
$$=q^{n_1m_2}e^{(\mu)}{}_{33}(m_1+m_2,n_1+n_2)-q^{n_2m_1}e^{(\mu)}{}_{11}(m_1+m_2,n_1+n_2).$$

The next two brackets are easy.

$$[e^{(\mu)}{}_{13}(m_1,n_1), e^{(\mu)}{}_{32}(m_2,n_2)]=q^{n_1m_2}e^{(\mu)}{}_{12}(m_1+m_2,n_1+n_2),$$
$$[e^{(\mu)}{}_{13}(m_1,n_1), e^{(\mu)}{}_{33}(m_2,n_2)]=q^{n_1m_2}e^{(\mu)}{}_{13}(m_1+m_2,n_1+n_2).$$

$$\big[e^{(\mu)}{}_{21}(m_1,n_1), e^{(\mu)}{}_{21}(m_2,n_2)\big]$$
$$=\mu q^{-m_1n_1}\sum_{\boldsymbol{A}} q^{(-m_1-m_2-\boldsymbol{A}_1)n_2+\boldsymbol{A}_2m_2+\boldsymbol{A}_2(-m_1-m_2-\boldsymbol{A}_1)} P_{\boldsymbol{A}}P_{(-3m_1-3m_2,-3n_1-3n_2)-\boldsymbol{A}}$$
$$-\mu q^{-m_2n_2}\sum_{\boldsymbol{A}'} q^{n_1\boldsymbol{A}'_1+(-n_1-n_2-\boldsymbol{A}'_2)m_1+(-n_1-n_2-\boldsymbol{A}'_2)\boldsymbol{A}'_1} P_{\boldsymbol{A}'}P_{(-3m_1-3m_2,-3n_1-3n_2)-\boldsymbol{A}'}$$

$$+\Big[\sum_{\boldsymbol{A},\boldsymbol{A}'} q^{n_1\boldsymbol{A}'_1+\boldsymbol{A}_2m_1+\boldsymbol{A}_2\boldsymbol{A}'_1} Q_{\boldsymbol{A}+\boldsymbol{A}'+(3m_1-1,3n_1-1)} P_{\boldsymbol{A}} P_{\boldsymbol{A}'},\ \sum_{\boldsymbol{A},\boldsymbol{A}'} q^{n_2\boldsymbol{A}'_1+\boldsymbol{A}_2m_2+\boldsymbol{A}_2\boldsymbol{A}'_1} Q_{\boldsymbol{A}+\boldsymbol{A}'+(3m_2-1,3n_2-1)} P_{\boldsymbol{A}} P_{\boldsymbol{A}'}\Big]$$

$$+\Big[\sum_{\boldsymbol{A},\boldsymbol{A}'} q^{n_1\boldsymbol{A}'_1+\boldsymbol{A}_2m_1+\boldsymbol{A}_2\boldsymbol{A}'_1} Q_{\boldsymbol{A}+\boldsymbol{A}'+(3m_1-1,3n_1-1)} P_{\boldsymbol{A}} P_{\boldsymbol{A}'},\ \sum_{\boldsymbol{A},\boldsymbol{B}} q^{n_2\boldsymbol{A}_1+\boldsymbol{B}_2m_2+\boldsymbol{B}_2\boldsymbol{A}_1} Q_{\boldsymbol{A}+\boldsymbol{B}+(3m_2-1,3n_2-1)} P_{\boldsymbol{A}} P_{\boldsymbol{B}}\Big]$$

$$+\Big[\sum_{\boldsymbol{A},\boldsymbol{B}} q^{n_1\boldsymbol{A}_1+\boldsymbol{B}_2m_1+\boldsymbol{B}_2\boldsymbol{A}_1} Q_{\boldsymbol{A}+\boldsymbol{B}+(3m_1-1,3n_1-1)} P_{\boldsymbol{A}} P_{\boldsymbol{B}},\ \sum_{\boldsymbol{A},\boldsymbol{A}'} q^{n_2\boldsymbol{A}'_1+\boldsymbol{A}_2m_2+\boldsymbol{A}_2\boldsymbol{A}'_1} Q_{\boldsymbol{A}+\boldsymbol{A}'+(3m_2-1,3n_2-1)} P_{\boldsymbol{A}} P_{\boldsymbol{A}'}\Big]$$

$$+\Big[\sum_{\boldsymbol{A},\boldsymbol{B}} q^{n_1\boldsymbol{A}_1+\boldsymbol{B}_2m_1+\boldsymbol{B}_2\boldsymbol{A}_1} Q_{\boldsymbol{A}+\boldsymbol{B}+(3m_1-1,3n_1-1)} P_{\boldsymbol{A}} P_{\boldsymbol{B}},\ \sum_{\boldsymbol{A},\boldsymbol{B}} q^{n_2\boldsymbol{A}_1+\boldsymbol{B}_2m_2+\boldsymbol{B}_2\boldsymbol{A}_1} Q_{\boldsymbol{A}+\boldsymbol{B}+(3m_2-1,3n_2-1)} P_{\boldsymbol{A}} P_{\boldsymbol{B}}\Big]$$

(the first and second terms cancel each other)

$$\begin{aligned}
&= \sum_{\boldsymbol{A},\bar{\boldsymbol{A}},\bar{\boldsymbol{A}}'} q^{n_1(\bar{\boldsymbol{A}}_1+\bar{\boldsymbol{A}}'_1+m_2)+\boldsymbol{A}_2m_1+\boldsymbol{A}_2(\bar{\boldsymbol{A}}_1+\bar{\boldsymbol{A}}'_1+m_2)+n_2\bar{\boldsymbol{A}}'_1+\bar{\boldsymbol{A}}_2m_2+\bar{\boldsymbol{A}}_2\bar{\boldsymbol{A}}'_1} \\
&\qquad\qquad \times Q_{\boldsymbol{A}+\bar{\boldsymbol{A}}+\bar{\boldsymbol{A}}'+(3m_1-1,3n_1-1)+(3m_2-1,3n_2-1)} P_{\boldsymbol{A}} P_{\bar{\boldsymbol{A}}} P_{\bar{\boldsymbol{A}}'} \\
&\quad + \sum_{\boldsymbol{A}',\bar{\boldsymbol{A}},\bar{\boldsymbol{A}}'} q^{n_1\boldsymbol{A}'_1+(\bar{\boldsymbol{A}}_2+\bar{\boldsymbol{A}}'_2+n_2)m_1+(\bar{\boldsymbol{A}}_2+\bar{\boldsymbol{A}}'_2+n_2)\boldsymbol{A}'_1+n_2\bar{\boldsymbol{A}}'_1+\bar{\boldsymbol{A}}_2m_2+\bar{\boldsymbol{A}}_2\bar{\boldsymbol{A}}'_1} \\
&\qquad\qquad \times Q_{\boldsymbol{A}'+\bar{\boldsymbol{A}}+\bar{\boldsymbol{A}}'+(3m_1-1,3n_1-1)+(3m_2-1,3n_2-1)} P_{\boldsymbol{A}'} P_{\bar{\boldsymbol{A}}} P_{\bar{\boldsymbol{A}}'} \\
&\quad - \sum_{\boldsymbol{A},\boldsymbol{A}',\bar{\boldsymbol{A}}} q^{n_2(\boldsymbol{A}_1+\boldsymbol{A}'_1+m_1)+\bar{\boldsymbol{A}}_2m_2+\bar{\boldsymbol{A}}_2(\boldsymbol{A}_1+\boldsymbol{A}'_1+m_1)+n_1\boldsymbol{A}'_1+\boldsymbol{A}_2m_1+\boldsymbol{A}_2\boldsymbol{A}'_1} \\
&\qquad\qquad \times Q_{\boldsymbol{A}+\boldsymbol{A}'+\bar{\boldsymbol{A}}+(3m_1-1,3n_1-1)+(3m_2-1,3n_2-1)} P_{\boldsymbol{A}} P_{\boldsymbol{A}'} P_{\bar{\boldsymbol{A}}} \\
&\quad - \sum_{\boldsymbol{A},\boldsymbol{A}',\bar{\boldsymbol{A}}'} q^{n_2\bar{\boldsymbol{A}}'_1+(\boldsymbol{A}_2+\boldsymbol{A}'_2+n_1)m_2+(\boldsymbol{A}_2+\boldsymbol{A}'_2+n_1)\bar{\boldsymbol{A}}'_1+n_1\boldsymbol{A}'_1+\boldsymbol{A}_2m_1+\boldsymbol{A}_2\boldsymbol{A}'_1} \\
&\qquad\qquad \times Q_{\boldsymbol{A}+\boldsymbol{A}'+\bar{\boldsymbol{A}}'+(3m_1-1,3n_1-1)+(3m_2-1,3n_2-1)} P_{\boldsymbol{A}} P_{\boldsymbol{A}'} P_{\bar{\boldsymbol{A}}'} \\
&\quad - \sum_{\boldsymbol{A},\boldsymbol{A}',\boldsymbol{B}} q^{n_2(\boldsymbol{A}_1+\boldsymbol{A}'_1+m_1)+\boldsymbol{B}_2m_2+\boldsymbol{B}_2(\boldsymbol{A}_1+\boldsymbol{A}'_1+m_1)+n_1\boldsymbol{A}'_1+\boldsymbol{A}_2m_1\boldsymbol{A}_2\boldsymbol{A}'_1} \\
&\qquad\qquad \times Q_{\boldsymbol{A}+\boldsymbol{A}'+\boldsymbol{B}+(3m_1-1,3n_1-1)+(3m_2-1,3n_2-1)} P_{\boldsymbol{A}} P_{\boldsymbol{A}'} P_{\boldsymbol{B}} \\
&\quad + \sum_{\boldsymbol{A},\boldsymbol{A}',\boldsymbol{B}} q^{n_1(\boldsymbol{A}_1+\boldsymbol{A}'_1+m_2)+\boldsymbol{B}_2m_1+\boldsymbol{B}_2(\boldsymbol{A}_1+\boldsymbol{A}'_1+m_2)+n_2\boldsymbol{A}'_1+\boldsymbol{A}_2m_2+\boldsymbol{A}_2\boldsymbol{A}'_1} \\
&\qquad\qquad \times Q_{\boldsymbol{A}+\boldsymbol{A}'+\boldsymbol{B}+(3m_1-1,3n_1-1)+(3m_2-1,3n_2-1)} P_{\boldsymbol{A}} P_{\boldsymbol{A}'} P_{\boldsymbol{B}} \\
&\quad + \sum_{\boldsymbol{A},\bar{\boldsymbol{A}},\boldsymbol{B}} q^{n_1\boldsymbol{A}_1+(\bar{\boldsymbol{A}}_2+\boldsymbol{B}_2+n_2)m_1+(\bar{\boldsymbol{A}}_2+\boldsymbol{B}_2+n_2)\boldsymbol{A}_1+n_2\bar{\boldsymbol{A}}_1+\boldsymbol{B}_2m_2+\boldsymbol{B}_2\bar{\boldsymbol{A}}_1} \\
&\qquad\qquad \times Q_{\boldsymbol{A}+\bar{\boldsymbol{A}}+\boldsymbol{B}+(3m_1-1,3n_1-1)+(3m_2-1,3n_2-1)} P_{\boldsymbol{A}} P_{\bar{\boldsymbol{A}}} P_{\boldsymbol{B}} \\
&\quad - \sum_{\boldsymbol{A},\bar{\boldsymbol{A}},\bar{\boldsymbol{B}}} q^{n_2\boldsymbol{A}_1+(\bar{\boldsymbol{A}}_2+\bar{\boldsymbol{B}}_2+n_1)m_2+(\bar{\boldsymbol{A}}_2+\bar{\boldsymbol{B}}_2+n_1)\boldsymbol{A}_1+n_1\bar{\boldsymbol{A}}_1+\bar{\boldsymbol{B}}_2m_1+\bar{\boldsymbol{B}}_2\bar{\boldsymbol{A}}_1} \\
&\qquad\qquad \times Q_{\boldsymbol{A}+\bar{\boldsymbol{A}}+\bar{\boldsymbol{B}}+(3m_1-1,3n_1-1)+(3m_2-1,3n_2-1)} P_{\boldsymbol{A}} P_{\bar{\boldsymbol{A}}} P_{\bar{\boldsymbol{B}}} \\
&= 0,
\end{aligned}$$

where we used that the first term cancels the fourth, the second term cancels the third, the fifth term cancels the seventh, and the sixth term cancels the eighth. Next,

$$
\begin{aligned}
&-\big[e^{(\mu)}{}_{21}(m_2,n_2),\,e^{(\mu)}{}_{22}(m_1,n_1)\big]=\big[e^{(\mu)}{}_{22}(m_1,n_1),\,e^{(\mu)}{}_{21}(m_2,n_2)\big]\\
&\quad=\mu\sum_{\boldsymbol{A}} q^{\boldsymbol{A}_2m_1-m_2n_2}\big[Q_{(3m_1,3n_1)+\boldsymbol{A}}P_{\boldsymbol{A}},\,P_{-(3m_2-1,3n_2-1)}\big]\\
&\qquad+\sum_{\boldsymbol{A},\boldsymbol{A}',\bar{\boldsymbol{A}}} q^{\boldsymbol{A}_2m_1+n_2\boldsymbol{A}'_1+\bar{\boldsymbol{A}}_2m_2+\bar{\boldsymbol{A}}_2\boldsymbol{A}'_1}\big[Q_{(3m_1,3n_1)+\boldsymbol{A}}P_{\boldsymbol{A}},\,Q_{\bar{\boldsymbol{A}}+\boldsymbol{A}'+(3m_2-1,3n_2-1)}P_{\boldsymbol{A}'}P_{\bar{\boldsymbol{A}}}\big]\\
&\qquad+\sum_{\boldsymbol{A},\bar{\boldsymbol{A}},\boldsymbol{B}} q^{\boldsymbol{A}_2m_1+n_2\bar{\boldsymbol{A}}_1+\boldsymbol{B}_2m_2+\boldsymbol{B}_2\bar{\boldsymbol{A}}_1}\big[Q_{(3m_1,3n_1)+\boldsymbol{A}}P_{\boldsymbol{A}},\,Q_{\bar{\boldsymbol{A}}+\boldsymbol{B}+(3m_2-1,3n_2-1)}P_{\boldsymbol{A}}P_{\boldsymbol{B}}\big]\\
&\qquad+\sum_{\boldsymbol{A},\boldsymbol{B},\bar{\boldsymbol{B}}} q^{\boldsymbol{B}_2m_1+n_2\boldsymbol{A}_1+\bar{\boldsymbol{B}}_2m_2+\bar{\boldsymbol{B}}_2\boldsymbol{A}_1}\big[Q_{(3m_1,3n_1)+\boldsymbol{B}}P_{\boldsymbol{B}},\,Q_{\boldsymbol{A}+\bar{\boldsymbol{B}}+(3m_2-1,3n_2-1)}P_{\boldsymbol{A}}P_{\bar{\boldsymbol{B}}}\big]\\
&\quad=-\mu q^{-m_2n_2+(-n_1-n_2)m_1}P_{-3m_1-3m_2+1,-3n_1-3n_2+1}\\
&\qquad+\sum_{\boldsymbol{A},\boldsymbol{A}'} q^{(\boldsymbol{A}_2+\boldsymbol{A}'_2+n_2)m_1+n_2\boldsymbol{A}'_1+\boldsymbol{A}_2m_2+\boldsymbol{A}_2\boldsymbol{A}'_1}Q_{(3m_1,3n_1)+\boldsymbol{A}+\boldsymbol{A}'+(3m_2-1,3n_2-1)}P_{\boldsymbol{A}}P_{\boldsymbol{A}'}\\
&\qquad-\sum_{\boldsymbol{A},\bar{\boldsymbol{A}}} q^{n_2(m_1+\bar{\boldsymbol{A}}_1)+\boldsymbol{A}_2m_2+\boldsymbol{A}_2(m_1+\bar{\boldsymbol{A}}_1)+\bar{\boldsymbol{A}}_2m_1}Q_{\boldsymbol{A}+\bar{\boldsymbol{A}}+(3m_1,3n_1)+(3m_2-1,3n_2-1)}P_{\boldsymbol{A}}P_{\bar{\boldsymbol{A}}}\\
&\qquad-\sum_{\boldsymbol{A}',\bar{\boldsymbol{A}}} q^{n_2\boldsymbol{A}'_1+(\bar{\boldsymbol{A}}_2+n_1)m_2+(\bar{\boldsymbol{A}}_2+n_1)\boldsymbol{A}'_1+\bar{\boldsymbol{A}}_2m_1}Q_{\boldsymbol{A}'+\bar{\boldsymbol{A}}+(3m_1,3n_1)+(3m_2-1,3n_2-1)}P_{\boldsymbol{A}'}P_{\bar{\boldsymbol{A}}}\\
&\qquad-\sum_{\boldsymbol{A},\boldsymbol{B}} q^{n_2(\boldsymbol{A}_1+m_1)+\boldsymbol{B}_2m_2+\boldsymbol{B}_2(\boldsymbol{A}_1+m_1)+\boldsymbol{A}_2m_1}Q_{\boldsymbol{A}+\boldsymbol{B}+(3m_1,3n_1)+(3m_2-1,3n_2-1)}P_{\boldsymbol{A}}P_{\boldsymbol{B}}\\
&\qquad+\sum_{\boldsymbol{A},\boldsymbol{B}} q^{(\boldsymbol{A}_2+\boldsymbol{B}_2+n_2)m_1+n_2\boldsymbol{A}_1+\boldsymbol{B}_2m_2+\boldsymbol{B}_2\boldsymbol{A}_1}Q_{\boldsymbol{A}+\boldsymbol{B}+(3m_1,3n_1)+(3m_2-1,3n_2-1)}P_{\boldsymbol{A}}P_{\boldsymbol{B}}\\
&\qquad-\sum_{\boldsymbol{A},\boldsymbol{B}} q^{n_2\boldsymbol{A}_1+(n_1+\boldsymbol{B}_2)m_2+(n_1+\boldsymbol{B}_2)\boldsymbol{A}_1+\boldsymbol{B}_2m_1}Q_{\boldsymbol{A}+\boldsymbol{B}+(3m_1,3n_1)+(3m_2-1,3n_2-1)}P_{\boldsymbol{A}}P_{\boldsymbol{B}}
\end{aligned}
$$

(using that the second and third terms cancel, as do the fifth and sixth)

$$
\begin{aligned}
&\quad=q^{n_1m_2}\big(-\mu q^{-(n_1+n_2)(m_1+m_2)}P_{-(3(m_1+m_2)-1,3(n_1+n_2)-1)}\\
&\qquad-\sum_{\bar{\boldsymbol{A}},\boldsymbol{A}'} q^{(n_1+n_2)\boldsymbol{A}'_1+(m_1+m_2)\bar{\boldsymbol{A}}_2+\bar{\boldsymbol{A}}_2\boldsymbol{A}'_1}Q_{(3m_1+3m_2-1,3n_1+3n_2-1)+\boldsymbol{A}'+\bar{\boldsymbol{A}}}P_{\boldsymbol{A}'}P_{\bar{\boldsymbol{A}}}\\
&\qquad-\sum_{\boldsymbol{A},\boldsymbol{B}} q^{(n_1+n_2)\boldsymbol{A}_1+(m_1+m_2)\boldsymbol{B}_2+\boldsymbol{B}_2\boldsymbol{A}_1}\big)Q_{(3m_1+3m_2-1,3n_1+3n_2-1)+\boldsymbol{A}+\boldsymbol{B}}P_{\boldsymbol{A}}P_{\boldsymbol{B}}\\
&\quad=q^{n_1m_2}e^{(\mu)}{}_{21}(m_1+m_2,n_1+n_2).
\end{aligned}
$$

Further,

$$
\begin{aligned}
&\big[e^{(\mu)}{}_{21}(m_1,n_1),\,e^{(\mu)}{}_{23}(m_2,n_2)\big]\\
&\quad=\mu q^{-m_1n_1}\sum_{\boldsymbol{A}} q^{n_2(-m_1-m_2-\boldsymbol{A}_1)+\boldsymbol{A}_2m_2+\boldsymbol{A}_2(-m_1-m_2-\boldsymbol{A}_1)}P_{\boldsymbol{A}}P_{(-3m_1-3m_2,-3n_1-3n_2)-\boldsymbol{A}}\\
&\qquad+\Big[\sum_{\boldsymbol{A},\boldsymbol{A}'} q^{n_1\boldsymbol{A}'_1+\boldsymbol{A}_2m_1+\boldsymbol{A}_2\boldsymbol{A}'_1}Q_{\boldsymbol{A}+\boldsymbol{A}'+(3m_1-1,3n_1-1)}P_{\boldsymbol{A}}P_{\boldsymbol{A}'},\\
&\qquad\qquad\sum_{\boldsymbol{A},\boldsymbol{B}} q^{n_2\boldsymbol{B}_1+\boldsymbol{A}_2m_2+\boldsymbol{A}_2\boldsymbol{B}_1}Q_{(3m_2+1,3n_2+1)+\boldsymbol{A}+\boldsymbol{B}}P_{\boldsymbol{A}}P_{\boldsymbol{B}}\Big]
\end{aligned}
$$

$$
\begin{aligned}
&-\mu q^{-m_2n_2}\sum_{\boldsymbol{A}} q^{n_1\boldsymbol{A}_1+(-n_1-n_2-\boldsymbol{A}_2)m_1+(-n_1-n_2-\boldsymbol{A}_2)\boldsymbol{A}_1} P_{\boldsymbol{A}} P_{(-3m_1-3m_2,-3n_1-3n_2)-\boldsymbol{A}}\\
&+\Big[\sum_{\boldsymbol{A},\boldsymbol{B}} q^{n_1\boldsymbol{A}_1+\boldsymbol{B}_2m_1+\boldsymbol{B}_2\boldsymbol{A}_1} Q_{\boldsymbol{A}+\boldsymbol{B}+(3m_1-1,3n_1-1)} P_{\boldsymbol{A}} P_{\boldsymbol{B}},\\
&\qquad\qquad\sum_{\boldsymbol{A},\boldsymbol{B}} q^{n_2\boldsymbol{B}_1+\boldsymbol{A}_2m_2+\boldsymbol{A}_2\boldsymbol{B}_1} Q_{(3m_2+1,3n_2+1)} P_{\boldsymbol{A}} P_{\boldsymbol{B}}\Big]\\
&+\Big[\sum_{\boldsymbol{A},\boldsymbol{B}} q^{n_1\boldsymbol{A}_1+\boldsymbol{B}_2m_1+\boldsymbol{B}_2\boldsymbol{A}_1} Q_{\boldsymbol{A}+\boldsymbol{B}+(3m_1-1,3n_1-1)} P_{\boldsymbol{A}} P_{\boldsymbol{B}},\\
&\qquad\qquad\sum_{\boldsymbol{B},\boldsymbol{B}'} q^{n_2\boldsymbol{B}'_1+\boldsymbol{B}_2m_2+\boldsymbol{B}_2\boldsymbol{B}'_1} Q_{(3m_2+1,3n_2+1)+\boldsymbol{B}+\boldsymbol{B}'} P_{\boldsymbol{B}} P_{\boldsymbol{B}'}\Big]
\end{aligned}
$$

(the first and third term cancel)

$$
\begin{aligned}
&=\sum_{\boldsymbol{A},\bar{\boldsymbol{A}},\boldsymbol{B}} q^{n_1(m_2+\bar{\boldsymbol{A}}_1+\boldsymbol{B}_1)+\boldsymbol{A}_2m_1+\boldsymbol{A}_2(m_1+\bar{\boldsymbol{A}}_1+\boldsymbol{B}_1)+n_2\boldsymbol{B}_1+\bar{\boldsymbol{A}}_2m_2+\bar{\boldsymbol{A}}_2\boldsymbol{B}_1}\\
&\qquad\qquad\times Q_{\boldsymbol{A}+\bar{\boldsymbol{A}}+\boldsymbol{B}+(3m_1+3m_2,3n_1+3n_2)} P_{\boldsymbol{A}} P_{\bar{\boldsymbol{A}}} P_{\boldsymbol{B}}\\
&\quad+\sum_{\boldsymbol{A}',\bar{\boldsymbol{A}},\boldsymbol{B}} q^{n_1\boldsymbol{A}'_1+(n_2+\bar{\boldsymbol{A}}_2+\boldsymbol{B}_2)m_1+(n_2+\bar{\boldsymbol{A}}_2+\boldsymbol{B}_2)\bar{\boldsymbol{A}}_1+n_2\boldsymbol{B}_1+\bar{\boldsymbol{A}}_2m_2+\bar{\boldsymbol{A}}_2\boldsymbol{B}_1}\\
&\qquad\qquad\times Q_{\boldsymbol{A}'+\bar{\boldsymbol{A}}+\boldsymbol{B}+(3m_1+3m_2,3n_1+3n_2)} P_{\boldsymbol{A}'} P_{\bar{\boldsymbol{A}}} P_{\boldsymbol{B}}\\
&\quad-\sum_{\boldsymbol{A},\boldsymbol{A}',\boldsymbol{B}} q^{n_2\boldsymbol{B}_1+(\boldsymbol{A}_2+\boldsymbol{A}'_2+n_1)m_2+(\boldsymbol{A}_2+\boldsymbol{A}'_2+n_1)\boldsymbol{B}_1+n_1\boldsymbol{A}'_1+\boldsymbol{A}_2m_1+\boldsymbol{A}_2\boldsymbol{A}'_1}\\
&\qquad\qquad\times Q_{\boldsymbol{A}'+\boldsymbol{A}+\boldsymbol{B}+(3m_1+3m_2,3n_1+3n_2)} P_{\boldsymbol{A}'} P_{\boldsymbol{A}} P_{\boldsymbol{B}}\\
&\quad+\sum_{\boldsymbol{A},\boldsymbol{B},\boldsymbol{B}'} q^{n_1(m_2+\boldsymbol{A}_1+\boldsymbol{B}'_1)+\boldsymbol{B}_2m_1+\boldsymbol{B}_2(m_2+\boldsymbol{A}_1+\boldsymbol{B}'_1)+n_2\boldsymbol{B}'_1+\boldsymbol{A}_2m_2+\boldsymbol{A}_2\boldsymbol{B}'_1}\\
&\qquad\qquad\times Q_{\boldsymbol{A}+\boldsymbol{B}+\boldsymbol{B}'+(3m_1+3m_2,3n_1+3n_2)} P_{\boldsymbol{A}} P_{\boldsymbol{B}} P_{\boldsymbol{B}'}\\
&\quad-\sum_{\boldsymbol{A},\bar{\boldsymbol{A}},\boldsymbol{B}} q^{n_2(\boldsymbol{A}_1+\boldsymbol{B}_1+m_1)+\bar{\boldsymbol{A}}_2m_2+\bar{\boldsymbol{A}}_2(\boldsymbol{A}_1+\boldsymbol{B}+m_1)+n_1\boldsymbol{A}_1+\boldsymbol{B}_2m_1+\boldsymbol{B}_2\boldsymbol{A}_1}\\
&\qquad\qquad\times Q_{\boldsymbol{A}+\bar{\boldsymbol{A}}+\boldsymbol{B}+(3m_1+3m_2,3n_1+3n_2)} P_{\boldsymbol{A}} P_{\bar{\boldsymbol{A}}} P_{\boldsymbol{B}}\\
&\quad+\sum_{\boldsymbol{A},\boldsymbol{B},\boldsymbol{B}'} q^{n_1\boldsymbol{A}_1+(n_2+\boldsymbol{B}_2+\boldsymbol{B}'_2)m_1+(n_2+\boldsymbol{B}_2+\boldsymbol{B}'_2)\boldsymbol{A}_1+n_2\boldsymbol{B}'_1+\boldsymbol{B}_2m_2+\boldsymbol{B}_2\boldsymbol{B}'_1}\\
&\qquad\qquad\times Q_{\boldsymbol{A}+\boldsymbol{B}+\boldsymbol{B}'+(3m_1+3m_2,3n_1+3n_2)} P_{\boldsymbol{A}} P_{\boldsymbol{B}} P_{\boldsymbol{B}'}\\
&\quad-\sum_{\boldsymbol{A},\boldsymbol{B},\bar{\boldsymbol{B}}} q^{n_2(\boldsymbol{A}_1+\bar{\boldsymbol{B}}_1+m_1)+\boldsymbol{B}_2m_2+\boldsymbol{B}_2(\boldsymbol{A}_1+\bar{\boldsymbol{B}}_1+m_1)+n_1\boldsymbol{A}_1+\bar{\boldsymbol{B}}_2m_1+\bar{\boldsymbol{B}}_2\boldsymbol{A}_1}\\
&\qquad\qquad\times Q_{\boldsymbol{A}+\boldsymbol{B}+\bar{\boldsymbol{B}}+(3m_1+3m_2,3n_1+3n_2)} P_{\boldsymbol{A}} P_{\boldsymbol{B}} P_{\bar{\boldsymbol{B}}}\\
&\quad-\sum_{\boldsymbol{A},\boldsymbol{B}',\bar{\boldsymbol{B}}} q^{n_2\boldsymbol{B}'_1+(\boldsymbol{A}_2+\bar{\boldsymbol{B}}_2+n_1)m_2+(\boldsymbol{A}_2+\bar{\boldsymbol{B}}_2+n_1)\boldsymbol{B}'_1+n_1\boldsymbol{A}_1+\bar{\boldsymbol{B}}_2m_1+\bar{\boldsymbol{B}}_2\boldsymbol{A}_1}\\
&\qquad\qquad\times Q_{\boldsymbol{A}+\boldsymbol{B}'+\bar{\boldsymbol{B}}+(3m_1+3m_2,3n_1+3n_2)} P_{\boldsymbol{A}} P_{\boldsymbol{B}'} P_{\bar{\boldsymbol{B}}}\\
&=0,
\end{aligned}
$$

where the first term and the third, the second and the fifth, the fourth and the eighth, and the sixth and the seventh cancel.

The following three brackets are easy.

$$
\begin{aligned}
&\big[e^{(\mu)}{}_{21}(m_1,n_1),\, e^{(\mu)}{}_{31}(m_2,n_2)\big]=0,\\
&\big[e^{(\mu)}{}_{21}(m_1,n_1),\, e^{(\mu)}{}_{32}(m_2,n_2)\big]=-q^{n_2m_1}e^{(\mu)}{}_{31}(m_1+m_2,n_1+n_2),\\
&\big[e^{(\mu)}{}_{21}(m_1,n_1),\, e^{(\mu)}{}_{33}(m_2,n_2)\big]=0.
\end{aligned}
$$

Next,

$$
\begin{aligned}
&\big[e^{(\mu)}{}_{22}(m_1,n_1), e^{(\mu)}{}_{22}(m_2,n_2)\big]\\
&\quad=\sum_{\boldsymbol{A},\boldsymbol{A}'} q^{\boldsymbol{A}_2 m_1+\boldsymbol{A}'_2 m_2}\big[Q_{(3m_1,3n_1)+\boldsymbol{A}}P_{\boldsymbol{A}},\, Q_{(3m_2,3n_2)+\boldsymbol{A}'}P_{\boldsymbol{A}'}\big]\\
&\qquad\qquad+\sum_{\boldsymbol{B},\boldsymbol{B}'} q^{\boldsymbol{B}_2 m_1+\boldsymbol{B}'_2 m_2}\big[Q_{(3m_1,3n_1)+\boldsymbol{B}}P_{\boldsymbol{B}},\, Q_{(3m_2,3n_2)+\boldsymbol{B}'}P_{\boldsymbol{B}'}\big]\\
&\quad=\sum_{\boldsymbol{A}'} q^{(n_2+\boldsymbol{A}'_2)m_1+\boldsymbol{A}'_2 m_2} Q_{(3m_1,3n_1)+(3m_2,3n_2)+\boldsymbol{A}'}P_{\boldsymbol{A}'}\\
&\qquad-\sum_{\boldsymbol{A}} q^{\boldsymbol{A}_2 m_1+(n_1+\boldsymbol{A}_2)m_2} Q_{(3m_1,3n_1)+(3m_2,3n_2)+\boldsymbol{A}}P_{\boldsymbol{A}}\\
&\qquad+\sum_{\boldsymbol{B}'} q^{(n_2+\boldsymbol{B}'_2)m_1+\boldsymbol{B}'_2 m_2} Q_{(3m_1,3n_1)+(3m_2,3n_2)+\boldsymbol{B}'}P_{\boldsymbol{B}'}\\
&\qquad-\sum_{\boldsymbol{B}} q^{\boldsymbol{B}_2 m_1+(n_1+\boldsymbol{B}_2)m_2} Q_{(3m_1,3n_1)+(3m_2,3n_2)+\boldsymbol{B}}P_{\boldsymbol{B}}\\
&\quad=q^{n_1m_2}\Big(-\sum_{\boldsymbol{A}} q^{\boldsymbol{A}_2(m_1+m_2)} Q_{(3m_1,3n_1)+(3m_2,3n_2)+\boldsymbol{A}}P_{\boldsymbol{A}}\\
&\qquad-\sum_{\boldsymbol{B}} q^{\boldsymbol{B}_2(m_1+m_2)} Q_{(3m_1,3n_1)+(3m_2,3n_2)+\boldsymbol{B}}P_{\boldsymbol{B}}-\tfrac{1}{2}\mu\delta_{(m_1+m_2,n_1+n_2),(0,0)}\Big)\\
&\qquad-q^{n_2m_1}\Big(-\sum_{\boldsymbol{A}} q^{\boldsymbol{A}_2(m_1+m_2)} Q_{(3m_1,3n_1)+(3m_2,3n_2)+\boldsymbol{A}}P_{\boldsymbol{A}}\\
&\qquad-\sum_{\boldsymbol{B}} q^{\boldsymbol{B}_2(m_1+m_2)} Q_{(3m_1,3n_1)+(3m_2,3n_2)+\boldsymbol{B}}P_{\boldsymbol{B}}-\tfrac{1}{2}\mu\delta_{(m_1+m_2,n_1+n_2),(0,0)}\Big)\\
&\quad=q^{n_1m_2}e^{(\mu)}{}_{22}(m_1+m_2,n_1+n_2)-q^{n_2m_1}e^{(\mu)}{}_{22}(m_1+m_2,n_1+n_2).
\end{aligned}
$$

$$
\begin{aligned}
&\big[e^{(\mu)}{}_{22}(m_1,n_1), e^{(\mu)}{}_{23}(m_2,n_2)\big]\\
&\quad=\Big[-\sum_{\boldsymbol{A}} q^{\boldsymbol{A}_2 m_1} Q_{(3m_1,3n_1)+\boldsymbol{A}}P_{\boldsymbol{A}}-\sum_{\boldsymbol{B}} q^{\boldsymbol{B}_2 m_1} Q_{(3m_1,3n_1)+\boldsymbol{B}}P_{\boldsymbol{B}},\\
&\qquad-q^{-m_2n_2}\mu P_{-(3m_2+1,3n_2+1)}-\sum_{\boldsymbol{A},\boldsymbol{B}} q^{n_2\boldsymbol{B}_1+\boldsymbol{A}_2m_2+\boldsymbol{A}_2\boldsymbol{B}_1} Q_{\boldsymbol{A}+\boldsymbol{B}+(3m_2+1,3n_2+1)}P_{\boldsymbol{A}}P_{\boldsymbol{B}}\\
&\qquad\qquad-\sum_{\boldsymbol{B},\boldsymbol{B}'} q^{n_2\boldsymbol{B}'_1+\boldsymbol{B}_2m_2+\boldsymbol{B}_2\boldsymbol{B}'_1} Q_{\boldsymbol{B}'+\boldsymbol{B}+(3m_2+1,3n_2+1)}P_{\boldsymbol{B}}P_{\boldsymbol{B}'}\Big]\\
&\quad=\sum_{\boldsymbol{A},\boldsymbol{B}} q^{(n_2+\boldsymbol{A}_2+\boldsymbol{B}_2)m_1+n_2\boldsymbol{B}_1+\boldsymbol{A}_2m_2+\boldsymbol{A}_2\boldsymbol{B}_1} Q_{(3m_1,3n_1)+\boldsymbol{A}+\boldsymbol{B}+(3m_2+1,3n_2+1)}P_{\boldsymbol{A}}P_{\boldsymbol{B}}\\
&\qquad-\sum_{\boldsymbol{A},\boldsymbol{B}} q^{n_2\boldsymbol{B}_1+(n_1+\boldsymbol{A}_2)m_2+(\boldsymbol{A}_2+n_1)\boldsymbol{B}_1+\boldsymbol{A}_2m_1} Q_{(3m_1,3n_1)+\boldsymbol{A}+\boldsymbol{B}+(3m_2+1,3n_2+1)}P_{\boldsymbol{A}}P_{\boldsymbol{B}}\\
&\qquad\qquad-q^{-m_2n_2+(-n_2-n_1)m_1}\mu P_{-(3m_2+1,3n_2+1)-(3m_1,3n_1)}\\
&\qquad-\sum_{\boldsymbol{A},\boldsymbol{B}} q^{n_2(m_1+\boldsymbol{B}_1)+\boldsymbol{A}_2m_2+\boldsymbol{A}_2(m_1+\boldsymbol{B}_1)+\boldsymbol{B}_2m_1} Q_{(3m_1,3n_1)+\boldsymbol{A}+\boldsymbol{B}+(3m_2+1,3n_2+1)}P_{\boldsymbol{A}}P_{\boldsymbol{B}}\\
&\qquad+\sum_{\boldsymbol{B},\boldsymbol{B}'} q^{(n_2+\boldsymbol{B}_2+\boldsymbol{B}'_2)m_1+n_2\boldsymbol{B}'_1+\boldsymbol{B}_2m_2+\boldsymbol{B}_2\boldsymbol{B}'_1} Q_{(3m_1,3n_1)+\boldsymbol{B}+\boldsymbol{B}'+(3m_2+1,3n_2+1)}P_{\boldsymbol{B}}P_{\boldsymbol{B}'}
\end{aligned}
$$

$$- \sum_{\boldsymbol{B},\boldsymbol{B}'} q^{n_2(\boldsymbol{B}'_1+m_1)+\boldsymbol{B}_2 m_2+\boldsymbol{B}_2(\boldsymbol{B}'_1+m_1)+\boldsymbol{B}'_2 m_1} Q_{(3m_1,3n_1)+\boldsymbol{B}+\boldsymbol{B}'+(3m_2+1,3n_2+1)} P_{\boldsymbol{B}} P_{\boldsymbol{B}'}$$
$$- \sum_{\boldsymbol{B},\boldsymbol{B}'} q^{n_2\boldsymbol{B}'_1+(n_1+\boldsymbol{B}_2)m_2+(\boldsymbol{B}_2+n_1)\boldsymbol{B}'_1+\boldsymbol{B}_2 m_1} Q_{(3m_1,3n_1)+\boldsymbol{B}+\boldsymbol{B}'+(3m_2+1,3n_2+1)} P_{\boldsymbol{B}} P_{\boldsymbol{B}'}$$

(the first term cancels the fourth, while the fifth cancels the sixth)

$$= q^{m_2 n_1}(-q^{-(m_1+m_2)(n_1+n_2)} \mu P_{-(3m_1+3m_2+1,3n_1+3n_2+1)}$$
$$- \sum_{\boldsymbol{A},\boldsymbol{B}} q^{\boldsymbol{A}_2(m_1+m_2)+(n_1+n_2)\boldsymbol{B}_1+\boldsymbol{A}_2\boldsymbol{B}_1} Q_{\boldsymbol{A}+\boldsymbol{B}+(3m_1+3m_2+1,3n_1+3n_2+1)} P_{\boldsymbol{A}} P_{\boldsymbol{B}}$$
$$- \sum_{\boldsymbol{B},\boldsymbol{B}'} q^{\boldsymbol{B}'_1(n_1+n_2)+\boldsymbol{B}_2(m_1+m_2)\boldsymbol{B}_2\boldsymbol{B}'_1}) Q_{\boldsymbol{B}+\boldsymbol{B}'+(3m_1+3m_2+1,3n_1+3n_2+1)} P_{\boldsymbol{B}} P_{\boldsymbol{B}'}$$
$$= q^{m_2 n_1} e^{(\mu)}{}_{23}(m_1+m_2, n_1+n_2).$$

The next three brackets are easy.

$$[e^{(\mu)}{}_{22}(m_1,n_1), e^{(\mu)}{}_{31}(m_2,n_2)] = 0,$$
$$[e^{(\mu)}{}_{22}(m_1,n_1), e^{(\mu)}{}_{32}(m_2,n_2)] = -q^{n_2 m_1} e^{(\mu)}{}_{32}(m_1+m_2, n_1+n_2),$$
$$[e^{(\mu)}{}_{22}(m_1,n_1), e^{(\mu)}{}_{33}(m_2,n_2)] = 0.$$

Next,

$$\left[e^{(\mu)}{}_{23}(m_1,n_1), e^{(\mu)}{}_{23}(m_2,n_2)\right]$$
$$= \Big[-q^{-m_1 n_1} \mu P_{-(3m_1+1,3n_1+1)} - \sum_{\boldsymbol{A},\boldsymbol{B}} q^{n_1\boldsymbol{B}_1+\boldsymbol{A}_2 m_1+\boldsymbol{A}_2\boldsymbol{B}_1} Q_{\boldsymbol{A}+\boldsymbol{B}+(3m_1+1,3n_1+1)} P_{\boldsymbol{A}} P_{\boldsymbol{B}}$$
$$- \sum_{\boldsymbol{B},\boldsymbol{B}'} q^{n_1\boldsymbol{B}'_1+\boldsymbol{B}_2 m_1+\boldsymbol{B}_2\boldsymbol{B}'_1} Q_{\boldsymbol{B}+\boldsymbol{B}'+(3m_1+1,3n_1+1)} P_{\boldsymbol{B}} P_{\boldsymbol{B}'},$$
$$-q^{-m_2 n_2} \mu P_{-(3m_2+1,3n_2+1)} - \sum_{\boldsymbol{A},\boldsymbol{B}} q^{n_2\boldsymbol{B}_1+\boldsymbol{A}_2 m_2+\boldsymbol{A}_2\boldsymbol{B}_1} Q_{\boldsymbol{A}+\boldsymbol{B}+(3m_2+1,3n_2+1)} P_{\boldsymbol{A}} P_{\boldsymbol{B}}$$
$$- \sum_{\boldsymbol{B},\boldsymbol{B}'} q^{n_2\boldsymbol{B}'_1+\boldsymbol{B}_2 m_2+\boldsymbol{B}_2\boldsymbol{B}'_1} Q_{\boldsymbol{B}+\boldsymbol{B}'+(3m_2+1,3n_2+1)} P_{\boldsymbol{B}} P_{\boldsymbol{B}'}\Big]$$
$$= \sum_{\boldsymbol{B}} \mu q^{-m_1 n_1+n_2(-m_1-m_2-\boldsymbol{B}_1)+\boldsymbol{B}_2 m_2+\boldsymbol{B}_2(-m_1-m_2-\boldsymbol{B}_1)} \times P_{\boldsymbol{B}} P_{-(3m_1+1,3n_1+1)-(3m_2+1,3n_2+1)-\boldsymbol{B}}$$
$$- \sum_{\boldsymbol{B}} \mu q^{-m_2 n_2+n_1\boldsymbol{B}_1+(-n_2-n_1-\boldsymbol{B}_2)m_1+(-n_2-n_1-\boldsymbol{B}_2)\boldsymbol{B}_1} \times P_{\boldsymbol{B}} P_{-(3m_1+1,3n_1+1)-(3m_2+1,3n_2+1)-\boldsymbol{B}}$$
$$+ \sum_{\boldsymbol{B},\bar{\boldsymbol{B}},\boldsymbol{B}'} q^{n_1(\boldsymbol{B}'_1+\bar{\boldsymbol{B}}_1+m_2)+\boldsymbol{B}_2 m_1+\boldsymbol{B}_2(\boldsymbol{B}'_1+\bar{\boldsymbol{B}}_1+m_2)+n_2\boldsymbol{B}'_1+\bar{\boldsymbol{B}}_2 m_2+\bar{\boldsymbol{B}}_2\boldsymbol{B}'_1} \times Q_{\boldsymbol{B}+\boldsymbol{B}'+\bar{\boldsymbol{B}}+(3m_1+1,3n_1+1)+(3m_2+1,3n_2+1)} P_{\boldsymbol{B}} P_{\boldsymbol{B}'} P_{\bar{\boldsymbol{B}}}$$
$$+ \sum_{\boldsymbol{B},\bar{\boldsymbol{B}},\boldsymbol{B}'} q^{n_1\boldsymbol{B}'_1+(\boldsymbol{B}_2+\bar{\boldsymbol{B}}_2+n_2)m_1+\boldsymbol{B}'_1(\boldsymbol{B}_2+\bar{\boldsymbol{B}}_2+n_2)+n_2\bar{\boldsymbol{B}}_1+\boldsymbol{B}_2 m_2+\boldsymbol{B}_2\bar{\boldsymbol{B}}_1} \times Q_{\boldsymbol{B}+\boldsymbol{B}'+\bar{\boldsymbol{B}}+(3m_1+1,3n_1+1)+(3m_2+1,3n_2+1)} P_{\boldsymbol{B}} P_{\boldsymbol{B}'} P_{\bar{\boldsymbol{B}}}$$

$$- \sum_{\boldsymbol{B},\bar{\boldsymbol{B}},\boldsymbol{B}'} q^{n_2(\boldsymbol{B}'_1+\bar{\boldsymbol{B}}_1+m_1)+\boldsymbol{B}_2 m_2+\boldsymbol{B}_2(\boldsymbol{B}'_1+\bar{\boldsymbol{B}}_1+m_1)+n_1\boldsymbol{B}'_1+\bar{\boldsymbol{B}}_2 m_1+\bar{\boldsymbol{B}}_2\boldsymbol{B}'_1} \times Q_{\boldsymbol{B}+\boldsymbol{B}'+\bar{\boldsymbol{B}}+(3m_1+1,3n_1+1)+(3m_2+1,3n_2+1)} P_{\boldsymbol{B}} P_{\boldsymbol{B}'} P_{\bar{\boldsymbol{B}}}$$

$$- \sum_{\boldsymbol{B},\bar{\boldsymbol{B}},\boldsymbol{B}'} q^{n_2\boldsymbol{B}'_1+(\boldsymbol{B}_2+\bar{\boldsymbol{B}}_2+n_1)m_2+(\boldsymbol{B}_2+\bar{\boldsymbol{B}}_2+n_1)\boldsymbol{B}'_1+n_1\bar{\boldsymbol{B}}_1+\boldsymbol{B}_2 m_1+\boldsymbol{B}_2\bar{\boldsymbol{B}}_1} \times Q_{\boldsymbol{B}+\boldsymbol{B}'+\bar{\boldsymbol{B}}+(3m_1+1,3n_1+1)+(3m_2+1,3n_2+1)} P_{\boldsymbol{B}} P_{\boldsymbol{B}'} P_{\bar{\boldsymbol{B}}}$$

$$+ \sum_{\boldsymbol{A},\boldsymbol{B},\boldsymbol{B}'} q^{n_1(\boldsymbol{B}_1+\boldsymbol{B}'_1+m_2)+\boldsymbol{A}_2 m_1+\boldsymbol{A}_2(\boldsymbol{B}_1+\boldsymbol{B}'_1+m_2)+n_2\boldsymbol{B}'_1+\boldsymbol{B}_2 m_2+\boldsymbol{B}_2\boldsymbol{B}'_1} \times Q_{\boldsymbol{A}+\boldsymbol{B}+\boldsymbol{B}'+(3m_1+1,3n_1+1)+(3m_2+1,3n_2+1)} P_{\boldsymbol{A}} P_{\boldsymbol{B}} P_{\boldsymbol{B}'}$$

$$+ \sum_{\boldsymbol{A},\boldsymbol{B},\boldsymbol{B}'} q^{n_1\boldsymbol{B}_1+(\boldsymbol{A}_2+\boldsymbol{B}'_2+n_2)m_1+(\boldsymbol{A}_2+\boldsymbol{B}'_2+n_2)\boldsymbol{B}_1+n_2\boldsymbol{B}'_1+\boldsymbol{A}_2 m_2+\boldsymbol{A}_2\boldsymbol{B}'_1} \times Q_{\boldsymbol{A}+\boldsymbol{B}+\boldsymbol{B}'+(3m_1+1,3n_1+1)+(3m_2+1,3n_2+1)} P_{\boldsymbol{A}} P_{\boldsymbol{B}} P_{\boldsymbol{B}'}$$

$$- \sum_{\boldsymbol{A},\boldsymbol{B},\boldsymbol{B}'} q^{n_2(\boldsymbol{B}_1+\boldsymbol{B}'_1+m_1)+\boldsymbol{A}_2 m_2+\boldsymbol{A}_2(\boldsymbol{B}_1+\boldsymbol{B}'_1+m_1)+n_1\boldsymbol{B}'_1+\boldsymbol{B}_2 m_1+\boldsymbol{B}_2\boldsymbol{B}'_1} \times Q_{\boldsymbol{A}+\boldsymbol{B}+\boldsymbol{B}'+(3m_1+1,3n_1+1)+(3m_2+1,3n_2+1)} P_{\boldsymbol{A}} P_{\boldsymbol{B}} P_{\boldsymbol{B}'}$$

$$- \sum_{\boldsymbol{A},\boldsymbol{B},\boldsymbol{B}'} q^{n_2\boldsymbol{B}_1+(\boldsymbol{A}_2+\boldsymbol{B}'_2+n_1)m_2+(\boldsymbol{A}_2+\boldsymbol{B}'_2+n_1)\boldsymbol{B}_1+n_1\boldsymbol{B}'_1+\boldsymbol{A}_2 m_1+\boldsymbol{A}_2\boldsymbol{B}'_1} \times Q_{\boldsymbol{A}+\boldsymbol{B}+\boldsymbol{B}'+(3m_1+1,3n_1+1)+(3m_2+1,3n_2+1)} P_{\boldsymbol{A}} P_{\boldsymbol{B}} P_{\boldsymbol{B}'}$$

$$= 0,$$

where the first two terms cancel, as do the third and the sixth, the fourth and the fifth, and the last two.

$$\big[e^{(\mu)}{}_{23}(m_1,n_1), e^{(\mu)}{}_{31}(m_2,n_2)\big]$$

$$= \Big[-q^{-m_1n_1}\mu P_{-(3m_1+1,3n_1+1)} - \sum_{\boldsymbol{A},\boldsymbol{B}} q^{n_1\boldsymbol{B}_1+\boldsymbol{A}_2 m_1+\boldsymbol{A}_2\boldsymbol{B}_1} Q_{\boldsymbol{A}+\boldsymbol{B}+(3m_1+1,3n_1+1)} P_{\boldsymbol{A}} P_{\boldsymbol{B}} - \sum_{\boldsymbol{B},\boldsymbol{B}'} q^{n_1\boldsymbol{B}'_1+\boldsymbol{B}_2 m_1+\boldsymbol{B}_2\boldsymbol{B}'_1} Q_{\boldsymbol{B}+\boldsymbol{B}'+(3m_1+1,3n_1+1)} P_{\boldsymbol{B}} P_{\boldsymbol{B}'}, \sum_{\boldsymbol{A}} q^{\boldsymbol{A}_1 n_2} Q_{(3m_2-2,3n_2-2)+\boldsymbol{A}} P_{\boldsymbol{A}}\Big]$$

$$= -q^{-m_1n_1+(-m_1-m_2)n_2}\mu P_{-(3m_1+1,3n_1+1)-(3m_2-2,3n_2-2)}$$

$$- \sum_{\boldsymbol{A},\boldsymbol{A}'} q^{n_1(m_2+\boldsymbol{A}'_1)+\boldsymbol{A}_2 m_1+\boldsymbol{A}_2(m_2+\boldsymbol{A}'_1)+\boldsymbol{A}'_1 n_2} Q_{\boldsymbol{A}+\boldsymbol{A}'+(3m_2-2,3n_2-2)+(3m_1+1,3n_1+1)} P_{\boldsymbol{A}} P_{\boldsymbol{A}'}$$

$$+ \sum_{\boldsymbol{A},\boldsymbol{B}} q^{(\boldsymbol{A}_1+\boldsymbol{B}_1+m_1)n_2+\boldsymbol{A}_2 m_1+\boldsymbol{A}_2\boldsymbol{B}_1+n_1\boldsymbol{B}_1} Q_{\boldsymbol{A}+\boldsymbol{B}+(3m_2-2,3n_2-2)+(3m_1+1,3n_1+1)} P_{\boldsymbol{A}} P_{\boldsymbol{B}}$$

$$- \sum_{\boldsymbol{A},\boldsymbol{B}} q^{n_1(m_2+\boldsymbol{A}_1)+\boldsymbol{B}_2 m_1+\boldsymbol{B}_2(m_2+\boldsymbol{A}_1)+\boldsymbol{A}_1 n_2} Q_{\boldsymbol{A}+\boldsymbol{B}+(3m_2-2,3n_2-2)+(3m_1+1,3n_1+1)} P_{\boldsymbol{A}} P_{\boldsymbol{B}}$$

$$- \sum_{\boldsymbol{A},\boldsymbol{B}} q^{n_1\boldsymbol{B}_1+(\boldsymbol{A}_2+n_2)m_1+(\boldsymbol{A}_2+n_2)\boldsymbol{B}_1+\boldsymbol{A}_1 n_2} Q_{\boldsymbol{A}+\boldsymbol{B}+(3m_2-2,3n_2-2)+(3m_1+1,3n_1+1)} P_{\boldsymbol{A}} P_{\boldsymbol{B}}$$

(the third term and the fifth cancel)

$$= q^{n_1m_2}\Big(-q^{-(m_1+m_2)(n_1+n_2)}\mu P_{-(3m_1+3m_2-1,3n_1+3n_2-1)} - \sum_{\boldsymbol{A},\boldsymbol{A}'} q^{(n_1+n_2)\boldsymbol{A}'_1+\boldsymbol{A}_2(m_1+m_2)+\boldsymbol{A}_2\boldsymbol{A}'_1} Q_{\boldsymbol{A}+\boldsymbol{A}'+(3m_2+3m_1-1,3n_2+3n_1-1)} P_{\boldsymbol{A}} P_{\boldsymbol{A}'}$$

$$- \sum_{\boldsymbol{A},\boldsymbol{B}} q^{(n_1+n_2)\boldsymbol{A}_1+\boldsymbol{B}_2(m_1+m_2)+\boldsymbol{B}_2\boldsymbol{A}_1} Q_{\boldsymbol{A}+\boldsymbol{B}+(3m_2+3m_1-1,3n_2+3n_1-1)} P_{\boldsymbol{A}} P_{\boldsymbol{B}} \Big)$$

$$= q^{n_1 m_2} e^{(\mu)}{}_{21}(m_1+m_2, n_1+n_2).$$

$$\big[e^{(\mu)}{}_{23}(m_1,n_1), e^{(\mu)}{}_{32}(m_2,n_2)\big]$$

$$= \Big[-\mu q^{-m_1 n_1} P_{-(3m_1+1,3n_1+1)} - \sum_{\boldsymbol{A},\boldsymbol{B}} q^{n_1\boldsymbol{B}_1+\boldsymbol{A}_2 m_1+\boldsymbol{A}_2\boldsymbol{B}_1} Q_{(3m_1+1,3n_1+1)+\boldsymbol{A}+\boldsymbol{B}} P_{\boldsymbol{A}} P_{\boldsymbol{B}}$$
$$- \sum_{\boldsymbol{B},\boldsymbol{B}'} q^{n_1\boldsymbol{B}'_1+\boldsymbol{B}_2 m_1+\boldsymbol{B}_2\boldsymbol{B}'_1} Q_{(3m_1+1,3n_1+1)+\boldsymbol{B}+\boldsymbol{B}'} P_{\boldsymbol{B}} P_{\boldsymbol{B}'},\ Q_{(3m_2-1,3n_2-1)}\Big]$$

$$= -\mu q^{-m_1 n_1} \delta_{(-m_1,-n_1),(m_2,n_2)}$$
$$- \sum_{\boldsymbol{A}} q^{n_1 m_2+\boldsymbol{A}_2 m_1+\boldsymbol{A}_2 m_2} Q_{(3m_1+1,3n_1+1)+\boldsymbol{A}+(3m_2-1,3n_2-1)} P_{\boldsymbol{A}}$$
$$- \sum_{\boldsymbol{B}} q^{n_1 m_2+\boldsymbol{B}_2 m_1+\boldsymbol{B}_2 m_2} Q_{(3m_1+1,3n_1+1)+\boldsymbol{B}+(3m_2-1,3n_2-1)} P_{\boldsymbol{B}}$$
$$- \sum_{\boldsymbol{B}'} q^{n_1\boldsymbol{B}'_1+n_2 m_1+n_2\boldsymbol{B}'_1} Q_{(3m_1+1,3n_1+1)+\boldsymbol{B}'+(3m_2-1,3n_2-1)} P_{\boldsymbol{B}'}$$

$$= q^{n_1 m_2}\Big(- \sum_{\boldsymbol{A}} q^{\boldsymbol{A}_2(m_1+m_2)} Q_{(3m_1+3m_2,3n_1+3n_2)+\boldsymbol{A}} P_{\boldsymbol{A}}$$
$$- \sum_{\boldsymbol{B}} q^{\boldsymbol{B}_2(m_1+m_2)} Q_{(3m_1+3m_2,3n_1+3n_2)+\boldsymbol{B}} P_{\boldsymbol{B}} - \tfrac{1}{2}\mu\delta_{(m_1+m_2,n_1+n_2),(0,0)}\Big)$$
$$- q^{n_2 m_1}\Big(\sum_{\boldsymbol{B}} q^{\boldsymbol{B}_1(n_1+n_2)} Q_{(3m_1+3m_2,3n_1+3n_2)+\boldsymbol{B}} P_{\boldsymbol{B}} + \tfrac{1}{2}\mu\delta_{(m_1+m_2,n_1+n_2),(0,0)}\Big)$$

$$= q^{n_1 m_2} e^{(\mu)}{}_{22}(m_1+m_2, n_1+n_2) - q^{n_2 m_1} e^{(\mu)}{}_{33}(m_1+m_2, n_1+n_2).$$

$$\big[e^{(\mu)}{}_{23}(m_1,n_1), e^{(\mu)}{}_{33}(m_2,n_2)\big]$$

$$= \Big[-\mu q^{-m_1 n_1} P_{-(3m_1+1,3n_1+1)} - \sum_{\boldsymbol{A},\boldsymbol{B}} q^{n_1\boldsymbol{B}_1+\boldsymbol{A}_2 m_1+\boldsymbol{A}_2\boldsymbol{B}_1} Q_{(3m_1+1,3n_1+1)+\boldsymbol{A}+\boldsymbol{B}} P_{\boldsymbol{A}} P_{\boldsymbol{B}}$$
$$- \sum_{\boldsymbol{B},\boldsymbol{B}'} q^{n_1\boldsymbol{B}'_1+\boldsymbol{B}_2 m_1+\boldsymbol{B}_2\boldsymbol{B}'_1} Q_{(3m_1+1,3n_1+1)+\boldsymbol{B}+\boldsymbol{B}'} P_{\boldsymbol{B}} P_{\boldsymbol{B}'},$$
$$\sum_{\boldsymbol{B}} q^{\boldsymbol{B}_1 n_2} Q_{(3m_2,3n_2)+\boldsymbol{B}} P_{\boldsymbol{B}}\Big]$$

$$= -q^{-m_1 n_1+(-m_1-m-2)n_2} \mu P_{-(3m_1+1,3n_1+1)-(3m_2,3n_2)}$$
$$- \sum_{\boldsymbol{A},\boldsymbol{B}} q^{n_1(m_2+\boldsymbol{B}_1)+\boldsymbol{A}_2 m_1+\boldsymbol{B}_2(m_2+\boldsymbol{B}_1)+\boldsymbol{B}_1 n_2} Q_{\boldsymbol{A}+\boldsymbol{B}+(3m_1+1,3n_1+1)+(3m_2,3n_2)} P_{\boldsymbol{A}} P_{\boldsymbol{B}}$$
$$- \sum_{\boldsymbol{B},\boldsymbol{B}'} q^{n_1(m_2+\boldsymbol{B}'_1)+\boldsymbol{B}_2 m_1+\boldsymbol{B}_2(m_2+\boldsymbol{B}'_1)+\boldsymbol{B}'_1 n_2} Q_{\boldsymbol{B}+(3m_2,3n_2)+\boldsymbol{B}'+(3m_1+1,3n_1+1)} P_{\boldsymbol{B}} P_{\boldsymbol{B}'}$$
$$- \sum_{\boldsymbol{B},\boldsymbol{B}'} q^{n_1\boldsymbol{B}'_1+(n_2+\boldsymbol{B}_2)m_1+(\boldsymbol{B}_2+n_2)\boldsymbol{B}'_1+\boldsymbol{B}_1 n_2} Q_{\boldsymbol{B}+(3m_2,3n_2)+\boldsymbol{B}'+(3m_1+1,3n_1+1)} P_{\boldsymbol{B}} P_{\boldsymbol{B}'}$$
$$+ \sum_{\boldsymbol{B},\boldsymbol{B}'} q^{(\boldsymbol{B}_1+\boldsymbol{B}'_1+m_1)n_2+n_1\boldsymbol{B}'_1+\boldsymbol{B}_2 m_1+\boldsymbol{B}_2\boldsymbol{B}'_1} Q_{\boldsymbol{B}+(3m_2,3n_2)+\boldsymbol{B}'+(3m_1+1,3n_1+1)} P_{\boldsymbol{B}} P_{\boldsymbol{B}'}$$

(the last two terms cancel)

$$= q^{n_1m_2}\Big(-q^{-(m_1+m_2)(n_1+n_2)}\mu P_{-(3m_1+3m_2+1,3n_1+3n_2+1)} \\ - \sum_{\boldsymbol{A},\boldsymbol{B}} q^{(n_1+n_2)\boldsymbol{B}_1+\boldsymbol{A}_2(m_1+m_2)+\boldsymbol{A}_2\boldsymbol{B}_1} Q_{\boldsymbol{A}+\boldsymbol{B}+(3m_1+3m_2+1,3n_1+3n_2+1)} P_{\boldsymbol{A}} P_{\boldsymbol{B}} \\ - \sum_{\boldsymbol{B},\boldsymbol{B}'} q^{(n_1+n_2)\boldsymbol{B}'_1+\boldsymbol{B}_2(m_1+m_2)+\boldsymbol{B}_2\boldsymbol{B}'_1} Q_{\boldsymbol{B}+\boldsymbol{B}'+(3m_1+3m_2+1,3n_1+3n_2+1)} P_{\boldsymbol{B}} P_{\boldsymbol{B}'}\Big)$$

$$= q^{n_1m_2} e^{(\mu)}{}_{23}(m_1+m_2, n_1+n_2).$$

The next five brackets are easy.

$$\big[e^{(\mu)}{}_{31}(m_1,n_1), e^{(\mu)}{}_{31}(m_2,n_2)\big] = 0,$$
$$\big[e^{(\mu)}{}_{31}(m_1,n_1), e^{(\mu)}{}_{32}(m_2,n_2)\big] = 0,$$
$$\big[e^{(\mu)}{}_{31}(m_1,n_1), e^{(\mu)}{}_{33}(m_2,n_2)\big] = -q^{m_1n_2} e^{(\mu)}{}_{31}(m_1+m_2, n_1+n_2),$$
$$\big[e^{(\mu)}{}_{32}(m_1,n_1), e^{(\mu)}{}_{32}(m_2,n_2)\big] = 0,$$
$$\big[e^{(\mu)}{}_{32}(m_1,n_1), e^{(\mu)}{}_{33}(m_2,n_2)\big] = -q^{m_1n_2} e^{(\mu)}{}_{32}(m_1+m_2, n_1+n_2).$$

$$\big[e^{(\mu)}{}_{33}(m_1,n_1), e^{(\mu)}{}_{33}(m_2,n_2)\big]$$
$$= \Big[\sum_{\boldsymbol{B}} q^{\boldsymbol{B}_1n_1} Q_{(3m_1,3n_1)+\boldsymbol{B}} P_{\boldsymbol{B}}, \sum_{\boldsymbol{B}} q^{\boldsymbol{B}_1n_2} Q_{(3m_2,3n_2)+\boldsymbol{B}} P_{\boldsymbol{B}}\Big]$$
$$= \sum_{\boldsymbol{B}} q^{(\boldsymbol{B}_1+m_2)n_1+\boldsymbol{B}_1n_2} Q_{(3m_1,3n_1)+(3m_2,3n_2)+\boldsymbol{B}} P_{\boldsymbol{B}} \\ - \sum_{\boldsymbol{B}} q^{(\boldsymbol{B}_1+m_1)n_2+\boldsymbol{B}_1n_1} Q_{(3m_1,3n_1)+(3m_2,3n_2)+\boldsymbol{B}} P_{\boldsymbol{B}}$$
$$= q^{n_1m_2} \sum_{\boldsymbol{B}} q^{\boldsymbol{B}_1(n_1+n_2} Q_{(3m_1,3n_1)+(3m_2,3n_2)+\boldsymbol{B}} P_{\boldsymbol{B}} \\ - q^{m_1n_2} \sum_{\boldsymbol{B}} q^{\boldsymbol{B}_1(n_1+n_2} Q_{(3m_1,3n_1)+(3m_2,3n_2)+\boldsymbol{B}} P_{\boldsymbol{B}}$$
$$= q^{n_1m_2} e^{(\mu)}{}_{33}(m_1+m_2, n_1+n_2) - q^{n_2m_1} e^{(\mu)}{}_{33}(m_1+m_2, n_1+n_2).$$

Next we check the brackets involving $D_1^{(\mu)}$ and $D_2^{(\mu)}$.

$$\big[D_1^{(\mu)}, e^{(\mu)}{}_{11}(m_1,n_1)\big]$$
$$= \Big[\sum_{\boldsymbol{A}} \boldsymbol{A}_1 Q_{\boldsymbol{A}} P_{\boldsymbol{A}} + \sum_{\boldsymbol{B}} \boldsymbol{B}_1 Q_{\boldsymbol{B}} P_{\boldsymbol{B}}, \sum_{\boldsymbol{A}} q^{\boldsymbol{A}_1n_1} Q_{(3m_1,3n_1)+\boldsymbol{A}} P_{\boldsymbol{A}} + \tfrac{1}{2}\mu\delta_{(m_1,n_1),(0,0)}\Big]$$
$$= \sum_{\boldsymbol{A}} (m_1+\boldsymbol{A}_1) q^{\boldsymbol{A}_1n_1} Q_{(3m_1,3n_1)+\boldsymbol{A}} P_{\boldsymbol{A}} - \sum_{\boldsymbol{A}} \boldsymbol{A}_1 q^{\boldsymbol{A}_1n_1} Q_{(3m_1,3n_1)+\boldsymbol{A}} P_{\boldsymbol{A}}$$
$$= m_1\Big(\sum_{\boldsymbol{A}} q^{\boldsymbol{A}_1n_1} Q_{(3m_1,3n_1)+\boldsymbol{A}} P_{\boldsymbol{A}} + \tfrac{1}{2}\mu\delta_{(m_1,n_1),(0,0)}\Big) = m_1 e^{(\mu)}{}_{11}(m_1,n_1).$$

The next two brackets are easy.

$$[D_1^{(\mu)}, e^{(\mu)}{}_{12}(m_1,n_1)] = m_1 e^{(\mu)}{}_{12}(m_1,n_1),$$
$$[D_1^{(\mu)}, e^{(\mu)}{}_{13}(m_1,n_1)] = m_1 e^{(\mu)}{}_{13}(m_1,n_1).$$

$$\begin{aligned}
&\big[D_1^{(\mu)}, e^{(\mu)}{}_{21}(m_1,n_1)\big] \\
&\quad = \big[\sum_{\boldsymbol{A}} \boldsymbol{A}_1 Q_{\boldsymbol{A}} P_{\boldsymbol{A}} + \sum_{\boldsymbol{B}} \boldsymbol{B}_1 Q_{\boldsymbol{B}} P_{\boldsymbol{B}}, \\
&\qquad -q^{-m_1n_1}\mu P_{-(3m_1-1,3n_1-1)} - \sum_{\boldsymbol{A},\boldsymbol{A}'} q^{n_1\boldsymbol{A}'_1+\boldsymbol{A}_2m_1+\boldsymbol{A}_2\boldsymbol{A}'_1} Q_{\boldsymbol{A}+\boldsymbol{A}'+(3m_1-1,3n_1-1)} P_{\boldsymbol{A}} P_{\boldsymbol{A}'} \\
&\qquad\qquad - \sum_{\boldsymbol{A},\boldsymbol{B}} q^{n_1\boldsymbol{A}_1+\boldsymbol{B}_2m_1+\boldsymbol{B}_2\boldsymbol{A}_1} Q_{\boldsymbol{A}+\boldsymbol{B}+(3m_1-1,3n_1-1)} P_{\boldsymbol{A}} P_{\boldsymbol{B}}\big] \\
&\quad = q^{-m_1n_1}\mu(-m_1) P_{-(3m_1-1,3n_1-1)} \\
&\qquad - \sum_{\boldsymbol{A},\boldsymbol{A}'} (\boldsymbol{A}_1+\boldsymbol{A}'_1+m_1) q^{n_1\boldsymbol{A}'_1+\boldsymbol{A}_2m_1+\boldsymbol{A}_2\boldsymbol{A}'_1} Q_{\boldsymbol{A}+\boldsymbol{A}'+(3m_1-1,3n_1-1)} P_{\boldsymbol{A}} P_{\boldsymbol{A}'} \\
&\qquad + \sum_{\boldsymbol{A},\boldsymbol{A}'} \boldsymbol{A}'_1 q^{n_1\boldsymbol{A}'_1+\boldsymbol{A}_2m_1+\boldsymbol{A}_2\boldsymbol{A}'_1} Q_{\boldsymbol{A}+\boldsymbol{A}'+(3m_1-1,3n_1-1)} P_{\boldsymbol{A}} P_{\boldsymbol{A}'} \\
&\qquad + \sum_{\boldsymbol{A},\boldsymbol{A}'} \boldsymbol{A}_1 q^{n_1\boldsymbol{A}'_1+\boldsymbol{A}_2m_1+\boldsymbol{A}_2\boldsymbol{A}'_1} Q_{\boldsymbol{A}+\boldsymbol{A}'+(3m_1-1,3n_1-1)} P_{\boldsymbol{A}} P_{\boldsymbol{A}'} \\
&\qquad - \sum_{\boldsymbol{A},\boldsymbol{B}} (\boldsymbol{A}_1+\boldsymbol{B}_1+m_1) q^{n_1\boldsymbol{A}_1+\boldsymbol{B}_2m_1+\boldsymbol{B}_2\boldsymbol{A}_1} Q_{\boldsymbol{A}+\boldsymbol{B}+(3m_1-1,3n_1-1)} P_{\boldsymbol{A}} P_{\boldsymbol{B}} \\
&\qquad + \sum_{\boldsymbol{A},\boldsymbol{B}} \boldsymbol{B}_1 q^{n_1\boldsymbol{A}_1+\boldsymbol{B}_2m_1+\boldsymbol{B}_2\boldsymbol{A}_1} Q_{\boldsymbol{A}+\boldsymbol{B}+(3m_1-1,3n_1-1)} P_{\boldsymbol{A}} P_{\boldsymbol{B}} \\
&\qquad + \sum_{\boldsymbol{A},\boldsymbol{B}} \boldsymbol{A}_1 q^{n_1\boldsymbol{A}_1+\boldsymbol{B}_2m_1+\boldsymbol{B}_2\boldsymbol{A}_1} Q_{\boldsymbol{A}+\boldsymbol{B}+(3m_1-1,3n_1-1)} P_{\boldsymbol{A}} P_{\boldsymbol{B}} \\
&\quad = m_1\big(-q^{-m_1n_1}\mu P_{-(3m_1-1,3n_1-1)} \\
&\qquad\qquad - \sum_{\boldsymbol{A},\boldsymbol{A}'} q^{n_1\boldsymbol{A}'_1+\boldsymbol{A}_2m_1+\boldsymbol{A}_2\boldsymbol{A}'_1} Q_{\boldsymbol{A}+\boldsymbol{A}'+(3m_1-1,3n_1-1)} P_{\boldsymbol{A}} P_{\boldsymbol{A}'} \\
&\qquad\qquad - \sum_{\boldsymbol{A},\boldsymbol{B}} q^{n_1\boldsymbol{A}_1+\boldsymbol{B}_2m_1+\boldsymbol{B}_2\boldsymbol{A}_1} Q_{\boldsymbol{A}+\boldsymbol{B}+(3m_1-1,3n_1-1)} P_{\boldsymbol{A}} P_{\boldsymbol{B}}\big) \\
&\quad = m_1 e^{(\mu)}{}_{21}(m_1,n_1).
\end{aligned}$$

$$\begin{aligned}
&\big[D_1^{(\mu)}, e^{(\mu)}{}_{22}(m_1,n_1)\big] \\
&\quad = \big[\sum_{\boldsymbol{A}} \boldsymbol{A}_1 Q_{\boldsymbol{A}} P_{\boldsymbol{A}} + \sum_{\boldsymbol{B}} \boldsymbol{B}_1 Q_{\boldsymbol{B}} P_{\boldsymbol{B}}, -\sum_{\boldsymbol{A}} q^{\boldsymbol{A}_2m_1} Q_{(3m_1,3n_1)+\boldsymbol{A}} P_{\boldsymbol{A}} \\
&\qquad\qquad - \sum_{\boldsymbol{B}} q^{\boldsymbol{B}_2m_1} Q_{(3m_1,3n_1)+\boldsymbol{B}} P_{\boldsymbol{B}} - \tfrac{1}{2}\mu\delta_{(m_1,n_1),(0,0)}\big] \\
&\quad = -\sum_{\boldsymbol{A}} (\boldsymbol{A}_1+m_1) q^{\boldsymbol{A}_2m_1} Q_{(3m_1,3n_1)+\boldsymbol{A}} P_{\boldsymbol{A}} + \sum_{\boldsymbol{A}} \boldsymbol{A}_1 q^{\boldsymbol{A}_2m_1} Q_{(3m_1,3n_1)+\boldsymbol{A}} P_{\boldsymbol{A}} \\
&\qquad - \sum_{\boldsymbol{B}} (m_1+\boldsymbol{B}_1) q^{\boldsymbol{B}_2m_1} Q_{(3m_1,3n_1)+\boldsymbol{B}} P_{\boldsymbol{B}} + \sum_{\boldsymbol{B}} \boldsymbol{B}_1 q^{\boldsymbol{B}_2m_1} Q_{(3m_1,3n_1)+\boldsymbol{B}} P_{\boldsymbol{B}}
\end{aligned}$$

$$
\begin{aligned}
&= m_1\Big(-\sum_{\boldsymbol{A}} q^{\boldsymbol{A}_2 m_1} Q_{(3m_1,3n_1)+\boldsymbol{A}} P_{\boldsymbol{A}} - \sum_{\boldsymbol{B}} q^{\boldsymbol{B}_2 m_1} Q_{(3m_1,3n_1)+\boldsymbol{B}} P_{\boldsymbol{B}} - \tfrac{1}{2}\mu\delta_{(m_1,n_1),(0,0)}\Big)\\
&= m_1 e^{(\mu)}{}_{22}(m_1,n_1),
\end{aligned}
$$

$$
\begin{aligned}
&\big[D_1^{(\mu)}, e^{(\mu)}{}_{23}(m_1,n_1)\big]\\
&\quad = \Big[\sum_{\boldsymbol{A}} \boldsymbol{A}_1 Q_{\boldsymbol{A}} P_{\boldsymbol{A}} + \sum_{\boldsymbol{B}} \boldsymbol{B}_1 Q_{\boldsymbol{B}} P_{\boldsymbol{B}},\\
&\qquad -q^{-m_1 n_1}\mu P_{-(3m_1+1,3n_1+1)} - \sum_{\boldsymbol{A},\boldsymbol{B}} q^{n_1\boldsymbol{B}_1+\boldsymbol{A}_2 m_1+\boldsymbol{A}_2\boldsymbol{B}_1} Q_{\boldsymbol{A}+\boldsymbol{B}+(3m_1+1,3n_1+1)} P_{\boldsymbol{A}} P_{\boldsymbol{B}}\\
&\qquad\qquad - \sum_{\boldsymbol{B},\boldsymbol{B}'} q^{n_1\boldsymbol{B}'_1+\boldsymbol{B}_2 m_1+\boldsymbol{B}_2\boldsymbol{B}'_1} Q_{\boldsymbol{B}+\boldsymbol{B}'+(3m_1+1,3n_1+1)} P_{\boldsymbol{B}} P_{\boldsymbol{B}'}\Big]\\
&\quad = q^{-m_1 n_1}\mu(-m_1) P_{-(3m_1+1,3n_1+1)}\\
&\qquad - \sum_{\boldsymbol{A},\boldsymbol{B}} (\boldsymbol{A}_1+\boldsymbol{B}_1+m_1) q^{n_1\boldsymbol{B}_1+\boldsymbol{A}_2 m_1+\boldsymbol{A}_2\boldsymbol{B}_1} Q_{\boldsymbol{A}+\boldsymbol{B}+(3m_1+1,3n_1+1)} P_{\boldsymbol{A}} P_{\boldsymbol{B}}\\
&\qquad + \sum_{\boldsymbol{A},\boldsymbol{B}} \boldsymbol{A}_1 q^{n_1\boldsymbol{B}_1+\boldsymbol{A}_2 m_1+\boldsymbol{A}_2\boldsymbol{B}_1} Q_{\boldsymbol{A}+\boldsymbol{B}+(3m_1+1,3n_1+1)} P_{\boldsymbol{A}} P_{\boldsymbol{B}}\\
&\qquad + \sum_{\boldsymbol{A},\boldsymbol{B}} \boldsymbol{B}_1 q^{n_1\boldsymbol{B}_1+\boldsymbol{A}_2 m_1+\boldsymbol{A}_2\boldsymbol{B}_1} Q_{\boldsymbol{A}+\boldsymbol{B}+(3m_1+1,3n_1+1)} P_{\boldsymbol{A}} P_{\boldsymbol{B}}\\
&\qquad - \sum_{\boldsymbol{B},\boldsymbol{B}'} (\boldsymbol{B}_1+\boldsymbol{B}'_1+m_1) q^{n_1\boldsymbol{B}'_1+\boldsymbol{B}_2 m_1+\boldsymbol{B}_2\boldsymbol{B}'_1} Q_{\boldsymbol{B}+\boldsymbol{B}'+(3m_1+1,3n_1+1)} P_{\boldsymbol{B}} P_{\boldsymbol{B}'}\\
&\qquad + \sum_{\boldsymbol{B},\boldsymbol{B}'} \boldsymbol{B}_1 q^{n_1\boldsymbol{B}'_1+\boldsymbol{B}_2 m_1+\boldsymbol{B}_2\boldsymbol{B}'_1} Q_{\boldsymbol{B}+\boldsymbol{B}'+(3m_1+1,3n_1+1)} P_{\boldsymbol{B}} P_{\boldsymbol{B}'}\\
&\qquad + \sum_{\boldsymbol{B},\boldsymbol{B}'} \boldsymbol{B}'_1 q^{n_1\boldsymbol{B}'_1+\boldsymbol{B}_2 m_1+\boldsymbol{B}_2\boldsymbol{B}'_1} Q_{\boldsymbol{B}+\boldsymbol{B}'+(3m_1+1,3n_1+1)} P_{\boldsymbol{B}} P_{\boldsymbol{B}'}\\
&\quad = m_1\Big(-q^{-m_1 n_1}\mu P_{-(3m_1+1,3n_1+1)}\\
&\qquad\qquad - \sum_{\boldsymbol{A},\boldsymbol{B}} q^{n_1\boldsymbol{B}_1+\boldsymbol{A}_2 m_1+\boldsymbol{A}_2\boldsymbol{B}_1} Q_{\boldsymbol{A}+\boldsymbol{B}+(3m_1+1,3n_1+1)} P_{\boldsymbol{A}} P_{\boldsymbol{B}}\\
&\qquad\qquad - \sum_{\boldsymbol{B},\boldsymbol{B}'} q^{n_1\boldsymbol{B}'_1+\boldsymbol{B}_2 m_1+\boldsymbol{B}_2\boldsymbol{B}'_1} Q_{\boldsymbol{B}+\boldsymbol{B}'+(3m_1+1,3n_1+1)} P_{\boldsymbol{B}} P_{\boldsymbol{B}'}\Big)\\
&\quad = m_1 e^{(\mu)}{}_{23}(m_1,n_1).
\end{aligned}
$$

The following two brackets are easy.

$$
\begin{aligned}
[D_1^{(\mu)}, e^{(\mu)}{}_{31}(m_1,n_1)] &= m_1 e^{(\mu)}{}_{31}(m_1,n_1),\\
[D_1^{(\mu)}, e^{(\mu)}{}_{32}(m_1,n_1)] &= m_1 e^{(\mu)}{}_{32}(m_1,n_1).
\end{aligned}
$$

$$
\begin{aligned}
&\big[D_1^{(\mu)}, e^{(\mu)}{}_{33}(m_1,n_1)\big]\\
&\quad = \Big[\sum_{\boldsymbol{A}} \boldsymbol{A}_1 Q_{\boldsymbol{A}} P_{\boldsymbol{A}} + \sum_{\boldsymbol{B}} \boldsymbol{B}_1 Q_{\boldsymbol{B}} P_{\boldsymbol{B}}, \sum_{\boldsymbol{B}} q^{\boldsymbol{B}_1 n_1} Q_{(3m_1,3n_1)+\boldsymbol{B}} P_{\boldsymbol{B}} + \tfrac{1}{2}\mu\delta_{(m_1,n_1),(0,0)}\Big]
\end{aligned}
$$

$$= \sum_{\boldsymbol{B}}(m_1 + \boldsymbol{B}_1)q^{\boldsymbol{B}_1 n_1} Q_{(3m_1,3n_1)+\boldsymbol{B}} P_{\boldsymbol{B}} - \sum_{\boldsymbol{B}} \boldsymbol{B}_1 q^{\boldsymbol{B}_1 n_1} Q_{(3m_1,3n_1)+\boldsymbol{B}} P_{\boldsymbol{B}}$$
$$= m_1\big(\sum_{\boldsymbol{B}} q^{\boldsymbol{B}_1 n_1} Q_{(3m_1,3n_1)+\boldsymbol{B}} P_{\boldsymbol{B}} + \tfrac{1}{2}\mu\delta_{(m_1,n_1),(0,0)}\big)$$
$$= m_1 e^{(\mu)}{}_{33}(m_1, n_1).$$

Similarly, we can get

$$\big[D_2^{(\mu)}, e^{(\mu)}{}_{ij}(m_1, n_1)\big] = n_1 e^{(\mu)}{}_{ij}(m_1, n_1)$$

for $1 \le i, j \le 3$. Finally,

$$[D_1^{(\mu)}, D_2^{(\mu)}] = \big[\sum_{\boldsymbol{A}} \boldsymbol{A}_1 Q_{\boldsymbol{A}} P_{\boldsymbol{A}} + \sum_{\boldsymbol{B}} \boldsymbol{B}_1 Q_{\boldsymbol{B}} P_{\boldsymbol{B}}, \sum_{\boldsymbol{A}} \boldsymbol{A}_2 Q_{\boldsymbol{A}} P_{\boldsymbol{A}} + \sum_{\boldsymbol{B}} \boldsymbol{B}_2 Q_{\boldsymbol{B}} P_{\boldsymbol{B}}\big] = 0.$$

Hence $\pi : \widetilde{\mathfrak{gl}_3(\mathbb{C}_q)} \to End(V)$ is a Lie algebra homomorphism. □

3. Hermitian form for $\widetilde{\mathfrak{gl}_3(\mathbb{C}_q)}$-module

From now on we need to assume that $|q| = 1$.

Define a $\mathbb{R}$-linear map $\omega : \widetilde{\mathfrak{gl}_3(\mathbb{C}_q)} \mapsto \widetilde{\mathfrak{gl}_3(\mathbb{C}_q)}$ by

$$\omega(\lambda x) = \bar{\lambda}\omega(x) \quad \text{for all } \lambda \in \mathbb{C} \text{ and } x \in \widetilde{\mathfrak{gl}_3(\mathbb{C}_q)},$$
$$\omega(E_{ij}(a)) = (-1)^{i+j} E_{ji}(\overline{a}) \quad \text{for } a \in \mathbb{C}_q,$$
$$\omega(d_s) = d_s, \qquad \omega(c_s) = c_s,$$
$$\omega(d_t) = d_t, \qquad \omega(c_t) = c_t.$$

Here, the $\mathbb{R}-$linear function $\overline{} : \mathbb{C}_q \to \mathbb{C}_q$ is defined as $\overline{\lambda s^m t^n} = \bar{\lambda} t^{-n} s^{-m} = \bar{\lambda} q^{mn} s^{-m} t^{-n}$, where $\bar{\lambda}$ is the complex conjugate for any $\lambda \in \mathbb{C}$ and $m, n \in \mathbb{Z}$.

From [Gao and Zeng 2006, Lemma 3.4], we have

Lemma 3.1. *ω is an antilinear antiinvolution of $\widetilde{\mathfrak{gl}_3(\mathbb{C}_q)}$.*

We write $\pi(E_{ij}(r)) \cdot v$ more simply as $E_{ij}(r) \cdot v$, for any $v \in V$, $r \in \mathbb{C}_q$.

In [Gao and Zeng 2006], we defined a hermitian form on the basis consisting of monomials and then used another basis consisting of iterated module actions on the "highest weight vector" 1 to determine the condition for the form being positive definite. Here we will use the second basis directly to define the hermitian form which is much simpler.

Lemma 3.2. *Consider, for $k, l \in \mathbb{Z}_+ \cup \{0\}$, the "vectors"*

$$E_{12}(\alpha_1)E_{12}(\alpha_2)\cdots E_{12}(\alpha_k)E_{32}(\beta_1)E_{32}(\beta_2)\cdots E_{32}(\beta_l) \cdot 1,$$

where

$$\alpha_i = s^{m_i} t^{n_i} \quad \textit{for } i = 1, \dots, k,$$
$$\beta_j = s^{u_j} t^{v_j} \quad \textit{for } j = 1, \dots, l,$$

and $m_i, n_i, u_j, v_j \in \mathbb{Z}$. *We say each is in level* (k, l) *in* W, *and together they form a basis for* V.

Proof. Define

$$f_{\boldsymbol{A},\boldsymbol{B}} = \prod_{(m,n)\in\mathbb{Z}^2} x_{(3m+1,3n+1)}^{\boldsymbol{A}_{(m,n)}} \cdot \prod_{(m',n')\in\mathbb{Z}^2} x_{(3m'-1,3n'-1)}^{\boldsymbol{B}_{(m',n')}},$$

for $\boldsymbol{A}_{(m,n)}, \boldsymbol{B}_{(m',n')} \in \mathbb{Z}_+ \cup \{0\}$, where only finitely many $\boldsymbol{A}_{(m,n)}, \boldsymbol{B}_{(m',n')}$ are nonzero. The $f_{\boldsymbol{A},\boldsymbol{B}}$ form a basis for V.

Let

$$g_{\boldsymbol{A}} = \prod_{(m,n)\in\mathbb{Z}^2} x_{(3m+1,3n+1)}^{\boldsymbol{A}_{(m,n)}} \quad \text{and} \quad h_{\boldsymbol{B}} = \prod_{(m',n')\in\mathbb{Z}^2} x_{(3m'-1,3n'-1)}^{\boldsymbol{B}_{(m',n')}}.$$

In a way similar to [Gao and Zeng 2006, Lemma 4.2], we can write $g_{\boldsymbol{A}}$ as a linear combination of $E_{12}(\alpha_1)\cdots E_{12}(\alpha_k)\cdot 1$ for $k \le \sum_{(m,n)} \boldsymbol{A}_{(m,n)}$, and $h_{\boldsymbol{B}}$ can be written as a linear combination of $E_{32}(\beta_1)\cdots E_{32}(\beta_l)\cdot 1$ for $l \le \sum_{(m',n')} \boldsymbol{B}_{(m',n')}$.

Since $E_{12}(\alpha)E_{32}(\beta)\cdot u = E_{32}(\beta)E_{12}(\alpha)\cdot u$ for any $u \in V$, we can write $f_{(\boldsymbol{A},\boldsymbol{B})}$ as a linear combination of $E_{12}(\alpha_1)\cdots E_{12}(\alpha_k)E_{32}(\beta_1), \dots, E_{32}(\beta_l)\cdot 1$. Hence the collection of $E_{12}(\alpha_1)\cdots E_{12}(\alpha_k)E_{32}(\beta_1), \dots, E_{32}(\beta_l)\cdot 1$ form a basis for V. □

Denote this basis in V by

$$\mathcal{B} = \left\{E_{12}(\alpha_1)\cdots E_{12}(\alpha_k)E_{32}(\beta_1)\cdots E_{32}(\beta_l)\cdot 1 \mid \text{for all } k, l \in \mathbb{N}, \alpha_i, \beta_j \in \mathbb{C}_q\right\}.$$

Lemma 3.3. *For any* $v \in V$,

- $\operatorname{lev}(v) = \operatorname{lev}(E_{ii}(a)\cdot v)$ *for* $i = 1, 2, 3$;
- $\operatorname{lev}(E_{12}(a)(v)) = \operatorname{lev}(v) + (1, 0)$;
- $\operatorname{lev}(E_{32}(a)\cdot v) = \operatorname{lev}(v) + (0, 1)$;
- $\operatorname{lev}(E_{21}(a)\cdot v) = \operatorname{lev}(v) - (1, 0)$ *or* $E_{21}(a)\cdot v = 0$ *if* $\operatorname{lev}(v) - (1, 0) \notin \mathbb{Z}_+^2$;
- $\operatorname{lev}(E_{23}(a)\cdot v) = \operatorname{lev}(v) - (0, 1)$ *or* $E_{23}(a)\cdot v = 0$ *if* $\operatorname{lev}(v) - (0, 1) \notin \mathbb{Z}_+^2$,

for any nonzero $a \in \mathbb{C}_q$.

Proof. We only check those v in the basis $\mathcal{B}$.

$$\begin{aligned}
&E_{22}(a)E_{12}(\alpha_1)E_{12}(\alpha_2)\cdots E_{12}(\alpha_k)E_{32}(\beta_1)E_{32}(\beta_2)\cdots E_{32}(\beta_l)\cdot 1\\
&\quad = E_{12}(\alpha_1)E_{22}(a)E_{12}(\alpha_2)\cdots E_{12}(\alpha_k)E_{32}(\beta_1)E_{32}(\beta_2)\cdots E_{32}(\beta_l)\cdot 1\\
&\qquad\quad - E_{12}(\alpha_1 a)E_{12}(\alpha_2)\cdots E_{12}(\alpha_k)E_{32}(\beta_1)E_{32}(\beta_2)\cdots E_{32}(\beta_l)\cdot 1\\
&\quad = E_{12}(\alpha_1)E_{12}(\alpha_2)\cdots E_{12}(\alpha_k)E_{32}(\beta_1)E_{32}(\beta_2)\cdots E_{32}(\beta_l)\cdot (\tfrac{1}{2}\mu)\kappa(a)\cdot 1
\end{aligned}$$

$$-\sum_{i=1}^{k} E_{12}(\alpha_1)\cdots E_{12}(\alpha_i a)\cdots E_{12}(\alpha_k)E_{32}(\beta_1)E_{32}(\beta_2)\cdots E_{32}(\beta_l)\cdot 1$$
$$+\sum_{i=1}^{l} E_{12}(\alpha_1)E_{12}(\alpha_2)\cdots E_{12}(\alpha_k)E_{32}(\beta_1)\cdots E_{32}(\beta_i a)\cdots E_{32}(\beta_l)\cdot 1,$$

so $\operatorname{lev}(v)=\operatorname{lev}(E_{22}(a).v)$. It is similar for $E_{11}(a)$, $E_{33}(a)$.

Further, $\operatorname{lev}(E_{12}(a)(v))=\operatorname{lev}(v)+(1,0)$ and $\operatorname{lev}(E_{32}(a).v)=\operatorname{lev}(v)+(0,1)$ are the definition of level.

For $E_{21}(a).v$, we prove by induction on the level of v: $E_{21}(a)\cdot v=0$ if $\operatorname{lev}(v)=(0,n)$, $n\in\mathbb{Z}_+\cup\{0\}$. If $n=0$, it is obvious that $E_{21}(a)\cdot 1=0$. Suppose it is true for n, then

$$\begin{aligned}
&E_{21}(a)E_{32}(\beta_1)E_{32}(\beta_2)\cdots E_{32}(\beta_{n+1})\cdot 1\\
&\quad=E_{32}(\beta_1)E_{21}(a)E_{32}(\beta_2)\cdots E_{32}(\beta_{n+1})\cdot 1-E_{31}(\beta_1 a)E_{32}(\beta_2)\cdots E_{32}(\beta_{n+1})\cdot 1\\
&\quad=-E_{32}(\beta_2)\cdots E_{32}(\beta_{n+1})E_{31}(\beta_1 a)\cdot 1\\
&\quad=0
\end{aligned}$$

by induction.

Supposing $\operatorname{lev}(E_{21}(a).v)=\operatorname{lev}(v)-(1,0)$ or $E_{21}(a)\cdot v=0$ is true for $\operatorname{lev}(v)=(m-1,n)$, then for $v=E_{12}(b)v'$ with $\operatorname{lev}(v')=(m-1,n)$ and $0\neq b\in\mathbb{C}_q$, we have

$$E_{21}(a)\cdot E_{12}(b)\cdot v'=E_{12}(b)E_{21}(a)\cdot v'+E_{22}(ab)\cdot v'-E_{11}(ba)\cdot v'.$$

Since $\operatorname{lev}(E_{21}(a)\cdot v')=(m-2,n)$, we have $\operatorname{lev}(E_{21}(a)\cdot E_{12}(b)\cdot v')=(m-1,n)$ or $E_{21}(a)\cdot E_{12}(b)\cdot v'=0$. It is similar for $E_{23}(a)$. □

We easily define a contravariant (with respect to π,ω) hermitian form on V by using the basis $\mathcal{B}$.

Assuming that μ is a real number, define the conjugate bilinear form on the elements in $\mathcal{B}$ by induction on the level:

$$(1,1)=1,\ (1,f)=0\quad\text{if}\quad\operatorname{lev}(f)\neq(0,0)$$

Suppose for any $v\in\mathcal{B}$, (u,v) is defined for any u such that $\operatorname{lev}(u)=(k',l')$ with $k'+l'=r-1$ if $\operatorname{lev}(u)=(k,l)$, with $k+l=r$. Then there exists a u' such that $\operatorname{lev}(u')=(k-1,l)$ or $\operatorname{lev}(u')=(k,l-1)$ and some $a\in\mathbb{C}_q$ such that $u=E_{12}(a)\cdot u'$ or $u=E_{32}(a)\cdot u'$.

Theorem 3.4. *The conjugate bilinear form defined through*

$$(E_{12}(a)\cdot u',v)=(u',\omega(E_{12}(a))\cdot v)\quad\textit{and}\quad(E_{32}(a)\cdot u',v)=(u',\omega(E_{32}(a))\cdot v)$$

is a hermitian form on V.

Proof. We must check that $(E_{ij}(a)\cdot u, v) = (u, \omega(E_{ij}(a))\cdot v)$ for $1 \le i, j \le 3$, and $a \in \mathbb{C}_q$. We must also check $(D_i \cdot u, v) = (u, \omega(D_i)\cdot v)$ for $i = 1, 2$.

By definition,

$$(E_{12}(a)u, v) = (u, \omega(E_{12}(a))v) \quad \text{and} \quad (E_{32}(a)u, v) = (u, \omega(E_{32}(a))v),$$

and so

$$\begin{aligned}(E_{13}(a)u, v) &= ([E_{12}(1), E_{23}(a)]u, v)\\ &= (E_{12}(1)E_{23}(a)u, v) - (E_{23}(a)E_{12}(1)u, v)\\ &= (u, \omega(E_{23}(a))\omega(E_{12}(1))v) - (u, \omega(E_{12}(1))\omega(E_{23}(a))v)\\ &= (u, -\omega([E_{23}(a), E_{12}(1)])v) = (u, \omega(E_{13}(a))v).\end{aligned}$$

We use induction on $\operatorname{lev}(u)$ to prove $(E_{11}(a)\cdot u, v) = (u, \omega(E_{11}(a))\cdot v)$. For any $v \in \mathcal{B}$,

$$(E_{11}(a)1, v) = \tfrac{1}{2}\mu\kappa(a)(1, v) = \tfrac{1}{2}\mu\kappa(a)\delta_{1,v}.$$

Since $\operatorname{lev}(E_{11}(a)\cdot v) = \operatorname{lev}(v)$ for any $v \in \mathcal{B}$,

$$(1, \omega(E_{11}(a))\cdot v) = (1, E_{11}(\bar{a})\cdot v) = \tfrac{1}{2}\mu\kappa(\bar{a})\delta_{1,v}.$$

Hence

$$(E_{11}(a)\cdot 1, v) = (1, \omega(E_{11}(a))\cdot v).$$

Suppose $(E_{11}(a)\cdot u, v) = (u, \omega(E_{11}(a))\cdot v)$ holds true for any $\operatorname{lev}(u) = (l, k)$ with $l+k = r-1$. Then for $\operatorname{lev}(u) = (l, k)$ with $l+k = r$, we have $u = E_{32}(b)\cdot u'$, with $\operatorname{lev}(u') = (l, k-1)$, and

$$\begin{aligned}(E_{11}(a)E_{32}(b)\cdot u', v) &= (E_{32}(b)E_{11}(a)\cdot u', v) = (E_{11}(a)\cdot u', \omega(E_{32}(b))\cdot v)\\ &= (u', \omega(E_{11}(a))\omega(E_{32}(b))\cdot v)\\ &= (u', \omega(E_{32}(b))\omega(E_{11}(a))\cdot v)\\ &= (E_{32}(b)\cdot u', \omega(E_{11}(a))\cdot v) = (u, \omega(E_{11}(a))\cdot v),\end{aligned}$$

or $u = E_{12}(b)\cdot u'$, with $\operatorname{lev}(u') = (l-1, k)$, and

$$\begin{aligned}&(E_{11}(a)E_{12}(b)\cdot u', v)\\ &\quad= (E_{12}(b)E_{11}(a)\cdot u', v) + ([E_{11}(a), E_{12}(b)]\cdot u', v)\\ &\quad= (E_{11}(a)\cdot u', \omega(E_{12}(b))\cdot v) + (u', \omega([E_{11}(a), E_{12}(b)])\cdot v)\\ &\quad= (u', \omega(E_{11}(a))\omega(E_{12}(b))\cdot v) - (u', [\omega(E_{11}(a)), \omega(E_{12}(b))]\cdot v)\\ &\quad= (u', \omega(E_{12}(b))\omega(E_{11}(a))\cdot v)\\ &\quad= (E_{12}(b)u', \omega(E_{11}(a))\cdot v) = (u, \omega(E_{11}(a))\cdot v).\end{aligned}$$

Thus $(E_{11}(a)\cdot u, v) = (u, \omega(E_{11}(a))\cdot v)$; and

$$\begin{aligned}(E_{22}(a)\cdot u, v) &= ([E_{21}(a), E_{12}(1)]\cdot u, v) + (E_{11}(a)u, v)\\ &= (E_{21}(a)E_{12}(1)\cdot u, v) - (E_{12}(1)E_{21}(a)\cdot u, v) + (E_{11}(a)u, v)\\ &= (u, \omega(E_{12}(1))\omega(E_{21}(a))\cdot v)\\ &\qquad - (u, \omega(E_{21}(a))\omega(E_{12}(1))\cdot v) + (u, \omega(E_{11}(a))\cdot v)\\ &= (u, \omega([E_{21}(a), E_{12}(1)]\cdot v)) + (u, \omega(E_{11}(a))\cdot v)\\ &= (u, \omega(E_{22}(a))\cdot v).\end{aligned}$$

It is similar for $(E_{33}(a)\cdot u, v) = (u, \omega(E_{33}(a))\cdot v)$.

For D_1, D_2, we also proceed by induction on the level of u. It is obvious that $(D_1\cdot 1, v) = 0$ for any $v\in\mathcal{B}$, and so $(D_1\cdot 1, 1) = (1, D_1\cdot 1) = 0$. Suppose $(1, D_1\cdot v) = 0$ is true for those $\operatorname{lev}(v) = (k', l')$ with $k'+l' = r > 0$. Then

$$\begin{aligned}(1, D_1E_{12}(s^mt^n)\cdot v) &= (1, E_{12}D_1\cdot v) + (1, m\cdot v) = 0,\\ (1, D_1E_{32}(s^mt^n)\cdot v) &= (1, E_{32}D_1\cdot v) + (1, m\cdot v) = 0.\end{aligned}$$

Thus $(D_1\cdot 1, v) = (1, D_1\cdot v)$.

Suppose for any $v\in\mathcal{B}$ that $(D_1\cdot u, v) = (u, D_1\cdot v)$ is true for all $\operatorname{lev}(u) = (k', l')$ such that $k'+l' = r$, then

$$\begin{aligned}(D_1\cdot E_{12}(s^mt^n)\cdot u, v) &= (E_{12}(s^mt^n)D_1\cdot u, v) + (m\cdot u, v)\\ &= (D_1\cdot u, \omega(E_{12}(s^mt^n))\cdot v) + (u, m\cdot v)\\ &= (u, D_1\omega(E_{12}(s^mt^n))\cdot v) + (u, m\cdot v)\\ &= (u, \omega(E_{12}(s^mt^n))D_1\cdot v)\\ &= (E_{12}(s^mt^n)u, D_1\cdot v).\end{aligned}$$

It is similar for $(D_1\cdot E_{32}(s^mt^n)\cdot u, v) = (E_{32}(s^mt^n)\cdot u, D_1\cdot v)$.

Hence $(D_1\cdot u, v) = (u, D_1\cdot v)$, and $(D_2\cdot u, v) = (u, D_2\cdot v)$. Note that $\omega(D_i) = D_i$ for $i = 1, 2$. □

4. Conditions for unitarity

In this section we will determine when the hermitian form given last section is positive definite.

Let $i\in\mathbb{N}$, $\gamma = (\gamma_1, \dots, \gamma_s)$ be the s-partition of i. We denote by $\operatorname{Par}_s(i)$ be the set of all s-partition of i.

Let $\gamma\in\operatorname{Par}_s(N)$. We say $pi'_1\times\pi'_2\in S_N\times S_N$ is equivalent to $\pi_1\times\pi_2\in S_N\times S_N$, where S_N is the permutation group of N letters, if, for all $z_1, \dots, z_N\in\mathbb{C}_q$ and

$w_1, \dots, w_N \in \mathbb{C}_q$,

$$\kappa(z_{\pi_1'(1)} w_{\pi_2'(1)}, \dots, z_{\pi_1'(\gamma_1)} w_{\pi_2'(\gamma_1)}) \cdots \kappa(z_{\pi_1'(\gamma_1+,\dots,\gamma_{s-1}+1)} w_{\pi_2'(\gamma_1+,\dots,+\gamma_{s-1}+1)}, \dots, z_{\pi_1'(N)} w_{\pi_2'(N)})$$

can be obtained from the analogous expression for $\pi_1 \times \pi_2$ by only rotating the variables. For example, $\kappa(z_1 w_1 z_2 w_2 z_3 w_3) = \kappa(z_3 w_3 z_1 w_1 z_2 w_2)$.

The following lemma is due to Jakobsen and Kac [1989].

Lemma 4.1. *Let* $z_1, z_2, \dots, z_N, w_1, w_2, \dots, w_N \in \mathbb{C}_q[s^{\pm 1}, t^{\pm 1}]$

$$\begin{pmatrix} 0 & z_1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & z_2 \\ 0 & 0 \end{pmatrix}\cdots\begin{pmatrix} 0 & z_N \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ w_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ w_2 & 0 \end{pmatrix}\cdots\begin{pmatrix} 0 & 0 \\ w_N & 0 \end{pmatrix}\cdot 1 \tag{4-2}$$

$$= \sum_{s=1}^{N} \sum_{\gamma \in \mathrm{Par}_s(N)} \sum_{[\pi_1 \times \pi_2] \in (S_N \times S_N)(\gamma)} (-1)^{\gamma_1 - 1}(-\mu)\kappa(z_{\pi_1(1)} w_{\pi_2(1)} \cdots z_{\pi_1(\gamma_1)} w_{\pi_2(\gamma_1)}) \cdot(-1)^{\gamma_2-1}(-\mu)\kappa(z_{\pi_1(\gamma_1+1)} w_{\pi_2(\gamma_1+1)} \cdots z_{\pi_1(\gamma_2)} w_{\pi_2(\gamma_2)})\cdot \cdots (-1)^{\gamma_s-1}(-\mu)\kappa(z_{\pi_1(\gamma_1+\cdots+\gamma_{s-1}+1)} w_{\pi_2(\gamma_1+\cdots+\gamma_{s-1}+1)} \cdots z_{\pi_1(N)} w_{\pi_2(N)}) \cdot 1$$

Lemma 4.2. *Let* $a_i, c_i, b_j, d_j \in \mathbb{C}_q$ *for* $i = 1, \dots, m$ *and* $j = 1, \dots, n$. *Let* $R = (a_i c_j)_{m \times m}$ *and* $U = (b_i d_j)_{n \times n}$, *and set*

$$\Lambda = \begin{pmatrix} R & 0 \\ 0 & U \end{pmatrix}_{(m+n)\times(m+n)} = (\lambda_{i,j})_{(m+n)\times(m+n)}.$$

Then

$$E_{21}(a_1)\cdots E_{21}(a_m) E_{23}(b_1)\cdots E_{23}(b_n) E_{12}(c_1)\cdots E_{12}(c_m) E_{32}(d_1)\cdots E_{32}(d_n)\cdot 1 \tag{4-3}$$

$$= \sum_{s=1}^{m+n} \sum_{\gamma \in \mathrm{Par}_s(m+n)} \sum_{[\pi_1 \times \pi_2] \in (S_{m+n} \times S_{m+n})(\gamma)} (-1)^{\gamma_1-1}(-\mu)\kappa(\lambda_{\pi_1(1),\pi_2(1)} \cdots \lambda_{\pi_1(\gamma_1)\pi_2(\gamma_1)}) \cdot(-1)^{\gamma_2-1}(-\mu)\kappa(\lambda_{\pi_1(\gamma_1+1),\pi_2(\gamma_1+1)} \cdots \lambda_{\pi_1(\gamma_2),\pi_2(\gamma_2)})\cdot \cdots (-1)^{\gamma_s-1}(-\mu)\kappa(\lambda_{\pi_1(\gamma_1+\cdots\gamma_{s-1}+1),\pi_2(\gamma_1+\cdots+\gamma_{s-1}+1)} \cdots \lambda_{\pi_1(N),\pi_2(N)}) \cdot 1.$$

Remark 4.3. It is easy to see that $\lambda_{i,j}$ in every summand should be from different rows and different columns of Λ. And if the summand of (4-3) contains some $\lambda_{i,j} = 0$, then this summand is 0. Hence (4-3) is in fact the sum of those $\lambda_{i,j}$ from R and U.

Proof. We proceed by induction on n. For $n = 0$, (4-3) is just (4-2). Next assume (4-3) is true up to $n - 1$,

$$E_{21}(a_1)\cdots E_{21}(a_m) E_{23}(b_1)\cdots E_{23}(b_n) E_{12}(c_1)\cdots E_{12}(c_m) E_{32}(d_1)\cdots E_{32}(d_n)\cdot 1$$

$$
\begin{aligned}
&= E_{21}(a_1)\cdots E_{21}(a_m)E_{23}(b_1)\cdots E_{23}(b_{n-1}) \\
&\qquad \cdot(E_{12}(c_1)E_{23}(b_n) - E_{13}(c_1b_n))E_{12}(c_2)\cdots E_{12}(c_m)E_{32}(d_1)\cdots E_{32}(d_n)\cdot 1 \\
&= E_{21}(a_1)\cdots E_{21}(a_m)E_{23}(b_1)\cdots E_{23}(b_{n-1})E_{12}(c_1)E_{23}(b_n)E_{12}(c_2) \\
&\qquad\qquad \cdots E_{12}(c_m)E_{32}(d_1)\cdots E_{32}(d_n)\cdot 1 \\
&\quad - E_{21}(a_1)\cdots E_{21}(a_m)E_{23}(b_1)\cdots E_{23}(b_{n-1})E_{12}(c_1)\cdots E_{12}(c_m)(E_{12}(c_1b_nd_1) \\
&\quad + E_{32}(d_1)E_{13}(c_1b_n))E_{32}(d_2)\cdots E_{32}(d_n)\cdot 1 \\
&= E_{21}(a_1)\cdots E_{21}(a_m)E_{23}(b_1)\cdots E_{23}(b_{n-1})E_{12}(c_1)E_{23}(b_n)E_{12}(c_2) \\
&\qquad\qquad \cdots E_{12}(c_m)E_{32}(d_1)\cdots E_{32}(d_n)\cdot 1 \\
&\quad + \sum_{i=1}^{n} E_{21}(a_1)\cdots E_{21}(a_m)E_{23}(b_1)\cdots E_{23}(b_{n-1}) \\
&\qquad\qquad \cdot E_{12}(-c_1b_nd_i)E_{12}(c_2)\cdots E_{12}(c_m)E_{32}(d_1)\cdots \widehat{E_{32}(d_i)}\cdots E_{32}(d_n)\cdot 1 \\
&= E_{21}(a_1)\cdots E_{21}(a_m)E_{23}(b_1)\cdots E_{23}(b_{n-1})E_{12}(c_1)E_{12}(c_2) \\
&\qquad\qquad \cdots E_{12}(c_m)E_{23}(b_n)E_{32}(d_1)\cdots E_{32}(d_n)\cdot 1 \\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{m} E_{21}(a_1)\cdots E_{21}(a_m)E_{23}(b_1)\cdots E_{23}(b_{n-1}) \\
&\qquad\qquad \cdot E_{12}(c_1)\cdots E_{12}(-c_jb_nd_i)\cdots E_{12}(c_m)E_{32}(d_1)\cdots \widehat{E_{32}(d_i)}\cdots E_{32}(d_n)\cdot 1 \\
&= \sum_{i=1}^{n}\sum_{j>i} E_{21}(a_1)\cdots E_{21}(a_m)E_{23}(b_1)\cdots E_{23}(b_{n-1})E_{12}(c_1)E_{12}(c_2)\cdots E_{12}(c_m) \\
&\qquad\qquad \cdot E_{32}(d_1)\cdots \widehat{E_{32}(d_i)}\cdots E_{32}(d_{j-1})E_{32}(-d_ib_nd_j - d_jb_nd_i)\cdots E_{32}(d_n)\cdot 1 \\
&\quad + \sum_{i=1}^{n} E_{21}(a_1)\cdots E_{21}(a_m)E_{23}(b_1)\cdots E_{23}(b_{n-1}) \\
&\qquad \cdot E_{12}(c_1)E_{12}(c_2)\cdots E_{12}(c_m)E_{32}(d_1)\cdots \widehat{E_{32}(d_i)}\cdots E_{32}(d_n)(-\mu)\kappa(b_nd_i)\cdot 1 \\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{m} E_{21}(a_1)\cdots E_{21}(a_m)E_{23}(b_1)\cdots E_{23}(b_{n-1}) \\
&\qquad\qquad \cdot E_{12}(c_1)\cdots E_{12}(-c_jb_nd_i)\cdots E_{12}(c_m)E_{32}(d_1)\cdots \widehat{E_{32}(d_i)}\cdots E_{32}(d_n)\cdot 1.
\end{aligned}
$$

Because (4-3) is true for $n-1$, we expand to see it is also true for n. □

Lemma 4.4. *The levels are orthogonal with respect to the the hermitian form; that is, the form vanishes when applied to two vectors from different levels.*

Proof. Only need to prove those elements in the basis $\mathcal{B}$. Let

$$
\begin{aligned}
u &= E_{12}(a_1)\cdots E_{12}(a_m)E_{32}(b_1)\cdots E_{32}(b_n)\cdot 1, \\
v &= E_{12}(c_1)\cdots E_{12}(c_k)E_{32}(d_1)\cdots E_{32}(d_l)\cdot 1,
\end{aligned}
$$

and suppose $(m, n) \neq (k, l)$.

First we prove $(u, v) = 0$ with $m = 0$. If $k = 0$, then, supposing $n > l$, we have

$$\begin{aligned}(u, v) =&(E_{32}(b_1)\cdots E_{32}(b_n)\cdot 1, E_{32}(d_1)\cdots E_{32}(d_l)\cdot 1)\\ =&((-1)^l E_{23}(\bar{d}_l)\cdots E_{23}(\bar{d}_1)E_{32}(b_1)\cdots E_{32}(b_n)\cdot 1, 1).\end{aligned}$$

Then, by Lemma 3.3, $\mathrm{lev}(E_{23}(\bar{d}_l)\cdots E_{23}(\bar{d}_1)E_{32}(b_1)\cdots E_{32}(b_n)\cdot 1) = (0, n-l)$, or, if $E_{23}(\bar{d}_l)\cdots E_{23}(\bar{d}_1)E_{32}(b_1)\cdots E_{32}(b_n)\cdot 1 = 0$, then $(u, v) = 0$.

For $k > 0$,

$$\begin{aligned}(u, v) =&(E_{32}(b_1)\cdots E_{32}(b_n)\cdot 1, E_{12}(c_1)\cdots E_{12}(c_k)E_{32}(d_1)\cdots E_{32}(d_l)\cdot 1)\\ =&(-E_{21}(\overline{c_1})E_{32}(b_1)\cdots E_{32}(b_n)\cdot 1,\\ &\qquad E_{12}(c_2)\cdots E_{12}(c_k)E_{32}(d_1)\cdots E_{32}(d_l)\cdot 1).\end{aligned}$$

Then from Lemma 3.3, we have $-E_{21}(\overline{c_1})E_{32}(b_1)\cdots E_{32}(b_n)\cdot 1=0$, and $(u, v)=0$.

Without loss of generality, we can assume that $m \le k$; then

$$\begin{aligned}(u, v) =& (E_{12}(a_1)\cdots E_{12}(a_m)E_{32}(b_1)\cdots E_{32}(b_n)\cdot 1,\\ &\qquad E_{12}(c_1)\cdots E_{12}(c_k)E_{32}(d_1)\cdots E_{32}(d_l)\cdot 1)\\ =& (E_{32}(b_1)\cdots E_{32}(b_n)\cdot 1,\\ &\quad (-1)^m E_{21}(\overline{a_m})\cdots E_{21}(\overline{a_1})\cdot E_{12}(c_1)\cdots E_{12}(c_k)E_{32}(d_1)\cdots E_{32}(d_l)\cdot 1).\end{aligned}$$

From Lemma 3.3,

$$\mathrm{lev}(E_{21}(\overline{a_m})\cdots E_{21}(\overline{a_1})\cdot E_{12}(c_1)\cdots E_{12}(c_k)E_{32}(d_1)\cdots E_{32}(d_l)\cdot 1) = (k-m, n)$$

or

$$E_{21}(\overline{a_m})\cdots E_{21}(\overline{a_1})\cdot E_{12}(c_1)\cdots E_{12}(c_k)E_{32}(d_1)\cdots E_{32}(d_l)\cdot 1 = 0.$$

Then, going back to the case $m = 0$, we get $(u, v) = 0$. □

Similarly to [Gao and Zeng 2006, Proposition 4.11] and together with Lemma 4.2, we have

Proposition 4.5. *The hermitian form on the element h in level (m, n) is a polynomial in μ, with leading term $c(-1)^{m+n}(-\mu)^{m+n} = c\mu^{m+n}$ for some constant $c > 0$.*

Theorem 4.6. (π, V) *can be made unitary if and only if $\mu > 0$.*

Proof. From [Gao and Zeng 2006, Theorem 4.12], the hermitian form in level $(0, n)$ and $(m, 0)$ is positive definite if and only if $\mu > 0$.

Define

$$T_{a,b}(s^{m_1}t^{n_1}s^{m_2}t^{n_2}\cdots s^{m_k}t^{n_k}) = s^{m_1+a}t^{n_1+b}s^{m_2+a}t^{n_2+b}\cdots s^{m_k+a}t^{n_k+b}$$

for $a, b \in \mathbb{Z}$. Extend this operator to the linear operator $\widetilde{T_{a,b}}$ on V by

$$\begin{aligned}\widetilde{T_{a,b}}&(E_{12}(\alpha_1)E_{12}(\alpha_2)\cdots E_{12}(\alpha_k)E_{32}(\beta_1)E_{32}(\beta_2)\cdots E_{32}(\beta_l)\cdot 1)\\ &= E_{12}(T_{a,b}\alpha_1)E_{12}(T_{a,b}\alpha_2)\cdots E_{12}(T_{a,b}\alpha_k)\\ &\qquad\qquad\qquad \cdot E_{32}(T_{a,b}\beta_1)E_{32}(T_{a,b}\beta_2)\cdots E_{32}(T_{a,b}\beta_l)\cdot 1.\end{aligned}$$

Following Lemma 4.2, $\widetilde{T_{a,b}}$ preserves the hermitian form on V. Denote

$$L_{l,r}(M,N) = \operatorname{span}\{E_{12}(s^{m_1}t^{n_1})\cdots E_{12}(s^{m_l}t^{n_l})E_{32}(s^{j_1}t^{k_1})\cdots E_{32}(s^{j_r}t^{k_r})\cdot 1$$

for $|m_i, n_i \geqq 0$ and $i = 1, \dots, l$, with $j_\iota, k_\iota \geqq 0$,

$$\sum_{i=1}^{l} m_i + \sum_{\iota=1}^{r} j_\iota \leqq M, \quad \text{and} \quad \sum_{i=1}^{r} n_i + \sum_{\iota=1}^{r} k_\iota \leqq N\}.$$

Since the hermitian form on two different levels is 0, we will prove the unitarity by induction on the level.

For any $\mu > 0$, the form is definite in level $(0, n)$; see [Gao and Zeng 2006, Theorem 4.12]. Suppose it is definite in level (r, n) for those $r < m$ and it is not definite in level (m, n).

From Proposition 4.5, we know that the hermitian form restricted to this level should be positive definite for μ big enough. Assuming it is not positive definite for some $\mu > 0$, there exist M, N such that the form restricted to $L_{m,n}(M, N)$ is not positive definite. From Proposition 4.5, the form on $L_{l,r}(M, N)$ varies smoothly with μ. Then we can find a μ_0 for which the form is not positive definite and, for all $\mu > \mu_0$, it is positive definite. We write $(\cdot\,,\,\cdot)_\mu$ for the hermitian form at μ.

Thus the radical of the form is nontrivial at μ_0, that is, there exists a nonzero $\tilde{h} \in L_{m,n}(M, N)$ such that, for any $h \in L_{m,n}(M, N)$, we have

$$(\tilde{h}, h)_{\mu_0} = 0.$$

Therefore for any arbitrary element $h_{m-1,n}$ in $L_{m-1,n}(M, N)$ and any $c \in \mathbb{C}$, we have

$$(E_{21}(c)\cdot\tilde{h}, h_{m-1,n})_{\mu_0} = 0.$$

Since the form is positive definite in level $(m-1, n)$, we have $E_{21}(c)\cdot\tilde{h} = 0$ for any $c \in \mathbb{C}$. Replacing $\tilde{h}$ by $\widetilde{T_{-a,-b}}(\tilde{h})$ if necessary, we can write

$$\tilde{h} = \sum_{i=1}^{m} a_i (E_{12}(1))^i x_i,$$

where $x_i = \sum E_{12}(\alpha_{i+1})\cdots E_{12}(\alpha_m)E_{32}(\beta_1)\cdots E_{32}(\beta_n)\cdot 1$ (here it is a finite sum), α_i, β_j is the of form $s^l t^k$, and l, k cannot both be 0.

Let i_0 be the smallest index such that $a_{i_0} \neq 0$; then $i_0 \geq 1$.

Since

$$
\begin{aligned}
&E_{21}(c)(E_{12}(1))^i E_{12}(\alpha_{i+1})\cdots E_{12}(\alpha_m)E_{32}(\beta_1)\cdots E_{32}(\beta_n)\cdot 1\\
&\quad= (E_{12}(1))^i E_{21}(c)E_{12}(\alpha_{i+1})\cdots E_{12}(\alpha_m)E_{32}(\beta_1)\cdots E_{32}(\beta_n)\cdot 1\\
&\qquad +i(E_{12}(1))^{i-1}(E_{22}(c)-E_{11}(c))E_{12}(\alpha_{i+1})\cdots E_{12}(\alpha_m)E_{32}(\beta_1)\cdots E_{32}(\beta_n)\cdot 1\\
&\qquad +(-2c)\tfrac{i(i-1)}{2}(E_{12}(1))^{i-1}E_{12}(\alpha_{i+1})\cdots E_{12}(\alpha_m)E_{32}(\beta_1)\cdots E_{32}(\beta_n)\cdot 1\\
&\quad= (E_{12}(1))^i E_{21}(c)E_{12}(\alpha_{i+1})\cdots E_{12}(\alpha_m)E_{32}(\beta_1)\cdots E_{32}(\beta_n)\cdot 1\\
&\qquad +i(E_{12}(1))^{i-1}((-2c)(m-i)-n)E_{12}(\alpha_{i+1})\cdots E_{12}(\alpha_m)E_{32}(\beta_1)\cdots E_{32}(\beta_n)\cdot 1\\
&\qquad +i(E_{12}(1))^{i-1}E_{12}(\alpha_{i+1})\cdots E_{12}(\alpha_m)E_{32}(\beta_1)\cdots E_{32}(\beta_n)(E_{22}(c)-E_{11}(c))\cdot 1\\
&\qquad +(-2c)\tfrac{i(i-1)}{2}(E_{12}(1))^{i-1}E_{12}(\alpha_{i+1})\cdots E_{12}(\alpha_m)E_{32}(\beta_1)\cdots E_{32}(\beta_n)\cdot 1\\
&\quad= (E_{12}(1))^i E_{21}(c)E_{12}(\alpha_{i+1})\cdots E_{12}(\alpha_m)E_{32}(\beta_1)\cdots E_{32}(\beta_n)\cdot 1\\
&\qquad +\big[ic(-\mu_0)+i((-2c)(m-i)-n)+(-2c)\tfrac{i(i-1)}{2}\big]\cdot\\
&\qquad (E_{12}(1))^{i-1}E_{12}(\alpha_{i+1})\cdots E_{12}(\alpha_m)E_{32}(\beta_1)\cdots E_{32}(\beta_n)\cdot 1,
\end{aligned}
$$

we have

$$E_{21}(c)\widetilde{h} = \gamma a_{i_0}(E_{12}(1))^{i_0-1}x_{i_0} + R,$$

where R contains those terms with powers of $E_{12}(1)$ greater than i_0-1 and

$$
\begin{aligned}
\gamma &= i_0c(-\mu_0)+i_0((-2c)(m-i_0)-n)+(-2c)\frac{i_0(i_0-1)}{2}\\
&= ci_0(-\mu_0-(m-i_0)-(m-1)).
\end{aligned}
$$

Since $m \ge i_0 \ge 1$, $\mu_0 \ge 0$, and $\gamma \ne 0$, this contradicts $E_{21}(c)\widetilde{h} = 0$. Thus, for any $\mu > 0$, the hermitian form is positive definite. □

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