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The interior and boundary regularity of weakly intrinsic biharmonic maps from 4-manifolds to spheres is proved.

1. Introduction

The regularity problem of harmonic maps has been intensively studied for a long time. Hélein [1991] proved that any weakly harmonic maps from a closed Riemannian surface to a compact Riemannian manifold without boundary is smooth. Later Qing [1993] proved the boundary regularity for weakly harmonic maps from compact Riemannian surface with boundary. However, when the domain dimension is greater than 2, Rivière [1995] constructed everywhere discontinuous weakly harmonic maps into spheres. This implies that there is no hope of getting any regularity results for weakly harmonic maps in higher dimensional cases. Therefore, it is of interest to study higher order energy functionals that enjoy better regularity properties.

Let M be a Riemmanian manifold and N be a compact Riemannian manifold without boundary that is isometrically embedded in $\mathbb{R}^K$. We say that u is a weakly intrinsic biharmonic map if it is a critical point of the functional $F(v) = \int_M |(\Delta v)^T|^2$ for $v \in W^{2,2}(M, N)$, where $(\Delta v)^T$ is the component of Δv in $\mathbb{R}^K$ that is tangent to N at $v(p) \in N$ for all $p \in M$. (Sometimes it is called the tension field $\tau(v)$ in the literature.) If the critical point u is smooth, we say u is an intrinsic biharmonic map. It is intrinsic in that the definition is independent of the choice of isometric embedding of the N into $\mathbb{R}^K$. If $u \in W^{2,2}(M, N)$ is a weakly harmonic map, then $(\Delta u)^T = 0$, and therefore u is obviously a minimizer of F. In other words, the class of all weakly intrinsic biharmonic maps can be regarded as an extension of the class of all weakly harmonic maps in $W^{2,2}(M, N)$. Another functional considered by Chang, Wang, and Yang [1999c] is $F_E(v) = \int_M |\Delta v|^2$, whose critical point is called a weakly extrinsic biharmonic map. Unlike a intrinsic biharmonic map, it depends on the choice of the embedding.

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The interior regularity of weakly intrinsic and extrinsic biharmonic maps from a bounded domain in $\mathbb{R}^4$ to a compact Riemannian manifold without boundary was established by C. Wang [2004]. And in recent paper of Lamm and Riviére [≥ 2008], they successfully rewrite the Euler–Lagrange equation of a weakly intrinsic and extrinsic biharmonic map into a conservation law, which simplifies the proof of interior regularity. However, it remains unclear whether this method can be used to prove the boundary regularity.

Here, we use the idea from [Chang et al. 1999c] to prove the interior and boundary regularity of weakly intrinsic biharmonic maps from four-dimensional Riemannian manifolds to S^n in $\mathbb{R}^{n+1}$, that is, if $u \in W^{2,2}(M, S^n)$ is weakly intrinsic biharmonic, then it is intrinsic biharmonic. Moreover, if u has smooth Dirichlet boundary data on ∂M, then it is smooth up to the boundary.

The paper is arranged as follows. In Section 2, we introduce necessary notations and derive the explicit Euler–Lagrange equations of a weakly intrinsic biharmonic map to S^k; the equations make up a fourth-order nonlinear elliptic system. As in [Chang et al. 1999c], by exploiting the special structure of the nonlinearity of these Euler–Lagrange equations, we are able to rewrite them as $\Delta^2 u =$ a linear combination of several special types of "divergence forms." From this, we can obtain the crucial L^p estimate which is key to the proof of interior Hölder regularity of u. In Section 3, we prove that if u is Hölder continuous, it must be smooth. The proof is based on an interesting observation in [Chang et al. 1999b] that if u is continuous, the coefficients of the nonlinear terms can be made very small by a specific scaling. Then by an iteration process, we prove that second derivatives of u are Hölder continuous. Now standard regularity theory implies that u is smooth, hence completing the proof of the interior regularity theorem. In Section 4, we prove the boundary regularity theorem by modifying the method of proof of interior regularity. For simplicity, we assume throughout the paper that the domain of the intrinsic biharmonic map is a flat Euclidean ball. The proof in the general case is essentially the same.

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2. Interior Hölder regularity

Here, we consider the interior Hölder regularity of a weakly intrinsic biharmonic map u. Since this is a local property, we may assume without loss of generality that $u : (B, g) \to S^n \subset \mathbb{R}^{n+1}$, where B is a 4-dimensional unit ball in $\mathbb{R}^4$ with Euclidean metric and S^n is canonically embedded in $\mathbb{R}^{n+1}$ with the induced standard metric. Δ, ∇, and div denote the Laplacian, gradient, and divergence.

2.1. ***The functional F.*** Let $u \in W^{2,2}(B, S^n)$ be a weakly intrinsic biharmonic map. Write $u(x) = (u^1(x), \dots, u^{n+1}(x)) \in \mathbb{R}^{n+1}$ for $x \in B$. It is well known that

$$((\Delta u)^T)^\alpha = \Delta u^\alpha + u^\alpha |\nabla u|^2 \quad \text{for } \alpha = 1, 2, \dots, n+1.$$

Therefore, by straightforward calculations, we have

$$F(u) = \int_B (|\Delta u|^2 - |\nabla u|^4).$$

And its Euler–Lagrange equation is

$$(1) \quad \Delta^2 u^\alpha = -\Big(\sum_\beta \big((\Delta u^\beta)^2 + \Delta(|\nabla u^\beta|^2) + 2\nabla\Delta u^\beta \cdot \nabla u^\beta\big) + 2|\nabla u|^4\Big) u^\alpha \\ - 2\,\mathrm{div}(|\nabla u|^2 \nabla u^\alpha) \quad \text{for } \alpha = 1, 2, \dots, n+1.$$

We say that $u \in W^{2,2}(B, S^n)$ is weakly intrinsic biharmonic if and only if it satisfies Equation (1) weakly.

2.2. ***Divergence forms.*** Now, we are going to write the right hand side of Equation (1) into a linear combination of certain types of "divergence forms." Using notations in [Chang et al. 1999c], we define

$$\begin{aligned} T_1 &\equiv \mathrm{div}(\nabla u^\alpha \Delta u^\beta (u^\beta - c^\beta)) \\ &\quad \text{or } \mathrm{div}((u^\beta - c^\beta)\langle \nabla\nabla u^\beta, \nabla u^\beta\rangle), \\ T_2 &\equiv \Delta((u^\alpha - c^\alpha)|\nabla u^\beta|^2) \\ &\quad \text{or } \Delta((u^\beta - c^\beta)\Delta u^\beta) \\ &\quad \text{or } \Delta(u^\alpha (u^\beta - c^\beta)\Delta u^\beta), \\ T_3 &\equiv \Delta(\mathrm{div}((u^\beta - c^\beta)\nabla u^\beta)), \end{aligned}$$

where c^β are constants and the β are summed from 1 to $n+1$.

In our case, we have to consider one more type, namely,

$$T_4 \equiv \mathrm{div}(|\nabla u|^2 (u^\alpha \nabla u^\beta - u^\beta \nabla u^\alpha)(u^\beta - c^\beta)).$$

Proposition 2.1. *The right hand side of Equation (1) can be written as a linear combination of T_l terms for $l = 1, 2, 3, 4$.*

Proof. At any point $p \in B$, we choose a normal coordinate $x = (x_1, \dots, x_4)$ at p and let u_i be the i-th covariant derivative of u. We name $S_1 = u^\alpha (\Delta u^\beta)^2$, $S_2 = 2u^\alpha u^\beta_j (\Delta u^\beta)_j$, and $S_3 = u^\alpha \Delta |\nabla u^\beta|^2$. Note that the j are summed from 1 to

4. Then

$$\begin{aligned}
\tfrac{1}{2}S_2 &= u^\alpha u^\beta_j(\Delta u^\beta)_j = \big(u^\alpha(\Delta u^\beta)_j - u^\beta(\Delta u^\alpha)_j\big)u^\beta_j\\
&= \big(u^\alpha(\Delta u^\beta)_j - u^\beta(\Delta u^\alpha)_j - u^\alpha_j(\Delta u^\beta) + u^\beta_j(\Delta u^\alpha)\big)u^\beta_j\\
&\quad + \big(u^\alpha_j(\Delta u^\beta) - u^\beta_j(\Delta u^\alpha)\big)u^\beta_j\\
&= \Big(\big(u^\alpha(\Delta u^\beta)_j - u^\beta(\Delta u^\alpha)_j - u^\alpha_j(\Delta u^\beta) + u^\beta_j(\Delta u^\alpha)\big)(u^\beta - c^\beta)\Big)_j\\
&\quad - \big(u^\alpha\Delta^2 u^\beta - u^\beta\Delta^2 u^\alpha\big)(u^\beta - c^\beta) + \big(u^\alpha_j(\Delta u^\beta) - u^\beta_j(\Delta u^\alpha)\big)u^\beta_j\\
&= \Big(\big(u^\alpha(\Delta u^\beta) - u^\beta(\Delta u^\alpha)\big)(u^\beta - c^\beta)\Big)_{jj}\\
&\quad - 2\big(\big(u^\alpha_j(\Delta u^\beta) - u^\beta_j(\Delta u^\alpha)\big)(u^\beta - c^\beta)\big)_j\\
&\quad - \big(\big(u^\alpha(\Delta u^\beta) - u^\beta(\Delta u^\alpha)\big)u^\beta_j\big)_j + \big(u^\alpha_j(\Delta u^\beta) - u^\beta_j(\Delta u^\alpha)\big)u^\beta_j\\
&\quad - \big(u^\alpha\Delta^2 u^\beta - u^\beta\Delta^2 u^\alpha\big)(u^\beta - c^\beta)\\
&= -\big(\big(u^\alpha(\Delta u^\beta) - u^\beta(\Delta u^\alpha)\big)u^\beta_j\big)_j + \big(u^\alpha_j(\Delta u^\beta) - u^\beta_j(\Delta u^\alpha)\big)u^\beta_j\\
&\quad - \big(u^\alpha\Delta^2 u^\beta - u^\beta\Delta^2 u^\alpha\big)(u^\beta - c^\beta) + (T_2 + T_1 \text{ terms}).
\end{aligned}$$

By [Chang et al. 1999c], we know that

$$\begin{aligned}
S_1 + S_3 &= (u^\alpha\Delta u^\beta - u^\beta\Delta u^\alpha)\Delta u^\beta + (T_\ell \text{ terms for } \ell = 1, 2, 3),\\
&= \big((u^\alpha\Delta u^\beta - u^\beta\Delta u^\alpha)u^\beta_j\big)_j - (u^\alpha_j\Delta u^\beta - u^\beta_j\Delta u^\alpha)u^\beta_j\\
&\quad - \big(u^\alpha(\Delta u^\beta)_j - u^\beta(\Delta u^\alpha)_j\big)u^\beta_j + (T_\ell \text{ terms}),\\
&= -\tfrac{1}{2}S_2 - \tfrac{1}{2}S_2 - \big(u^\alpha\Delta^2 u^\beta - u^\beta\Delta^2 u^\alpha\big)(u^\beta - c^\beta) + (T_\ell \text{ terms}),\\
S_1 + S_2 + S_3 &= (T_\ell \text{ terms}) - (u^\alpha\Delta^2 u^\beta - u^\beta\Delta^2 u^\alpha)(u^\beta - c^\beta)
\end{aligned}$$

But by (1), we get that

$$\begin{aligned}
u^\alpha\Delta^2 u^\beta - u^\beta\Delta^2 u^\alpha &= -2\operatorname{div}(|\nabla u|^2\nabla u^\beta)u^\alpha - \mu u^\alpha u^\beta\\
&\quad - \big(-2\operatorname{div}(|\nabla u|^2\nabla u^\alpha)u^\beta - \mu u^\alpha u^\beta\big)\\
&= -2\operatorname{div}(|\nabla u|^2\nabla u^\beta)u^\alpha + 2\operatorname{div}(|\nabla u|^2\nabla u^\alpha)u^\beta\\
&= 2\big(|\nabla u|^2(u^\beta u^\alpha_j - u^\alpha u^\beta_j)\big)_j
\end{aligned}$$

Hence we have

$$\begin{aligned}
\text{right side of (1)} &= -(\lambda + 2|\nabla u|^4)u^\alpha - 2\operatorname{div}(|\nabla u|^2\nabla u^\alpha)\\
&= (T_\ell \text{ terms}) + 2\big(|\nabla u|^2(u^\beta u^\alpha_j - u^\alpha u^\beta_j)\big)_j(u^\beta - c^\beta) - 2|\nabla u|^4 u^\alpha
\end{aligned}$$

$$
\begin{aligned}
&= (T_\ell \text{ terms}) + 2\big(|\nabla u|^2(u^\beta u^\alpha_j - u^\alpha u^\beta_j)\big)_j (u^\beta - c^\beta) \\
&\qquad\qquad\qquad\qquad - 2|\nabla u|^2(u^\alpha u^\beta_j - u^\beta u^\alpha_j)(u^\beta - c^\beta)_j \\
&= (T_\ell \text{ terms}) - 2\big(|\nabla u|^2(u^\alpha u^\beta_j - u^\beta u^\alpha_j)(u^\beta - c^\beta)\big)_j \\
&= (T_\ell \text{ terms for } \ell = 1, 2, 3, 4).
\end{aligned}
$$

The third equality follows from $u^\alpha |\nabla u|^2 = (u^\alpha u^\beta_j - u^\beta u^\alpha_j)(u^\beta - c^\beta)_j$. □

2.3. *Hölder continuity of u.*

Theorem 2.1. *If $u \in W^{2,2}(B, S^4)$ is weakly intrinsic biharmonic, then it is locally Hölder continuous on B with exponent β for some $\beta \in (0, 1)$.*

To prove this, we first need standard L^p elliptic estimates:

Lemma 2.1. *Suppose B_r is a Euclidean ball in $\mathbb{R}^4$ of radius $r > 0$ and $v \in W^{2,2}(B_r)$ is a weak solution on B_r of one of*

$$
\begin{aligned}
\Delta^2 v &= \operatorname{div}(F), \\
\Delta^2 v &= \Delta G, \\
\Delta^2 v &= \Delta(\operatorname{div} H),
\end{aligned}
$$

with $v = 0$ and $\partial v/\partial n = 0$ on ∂B_r. Then for any $1 < q < \infty$, the solution v satisfies the corresponding member of

$$
\begin{aligned}
\left\|\nabla^3 v\right\|_{L^q(B_r)} &\lesssim \|F\|_{L^q(B_r)}, \\
\left\|\nabla^2 v\right\|_{L^q(B_r)} &\lesssim \|G\|_{L^q(B_r)}, \\
\|\nabla v\|_{L^q(B_r)} &\lesssim \|H\|_{L^q(B_r)}.
\end{aligned}
$$

For any B_r and $p > 1$, we define

$$
\begin{aligned}
E(u)(B_r) &\equiv \Big(\int_{B_r} |\nabla^2 u|^2\Big)^{1/2} + \Big(\int_{B_r} |\nabla u|^4\Big)^{1/4}, \\
M_p(u)(B_r) &\equiv \Big(⨍_{B_r} |u - \overline{u}|^p\Big)^{1/p} \quad \text{where } \overline{u} = ⨍_{B_r} u, \\
D_p(u)(B_r) &\equiv \Big(r^p ⨍_{B_r} |\nabla u|^p\Big)^{1/p}.
\end{aligned}
$$

The following is the main technical lemma:

Lemma 2.2. *Let $u \in W^{2,2}(B, S^n)$ be a weakly intrinsic biharmonic map. Then for any p_1 such that $2 < p_1 < 4$ and $1/p_0 = 1/p_1 - 1/4$ and for any $0 < \beta < 1$, there exists $\tau < 1/4$ and $\varepsilon > 0$ such that if $E(u)(B) < \varepsilon$, then*

$$
(M_{p_0}(u) + D_{p_1}(u))(B_\tau) < \tau^\beta (M_{p_0}(u) + D_{p_1}(u))(B).
$$

Proof. We fix some $1/2 \le r < 1$ to be chosen later. Let $v = u - h$ where $\Delta^2 h^\alpha = 0$ on B_r and $h^\alpha = u^\alpha$ and $\partial h^\alpha/\partial n = \partial u^\alpha/\partial n$ on ∂B_r. Write $v = \sum_{i=1}^4 v_i$ such that $\Delta^2 v_i = (T_i \text{ terms})$ on B_r and $v_i = \partial v_i/\partial n = 0$ on ∂B_r for $i = 1, 2, 3, 4$. By Proposition 2.1 and Lemma 2.1, we get

$$\begin{aligned}
&\|\nabla^3 v_1\|_{L^{p_3}(B_r)} + \|\nabla^2 v_2\|_{L^{p_2}(B_r)} + \|\nabla v_3\|_{L^{p_1}(B_r)} + \|\nabla^3 v_4\|_{L^{p_3}(B_r)} \\
&\quad \lesssim \big\| |\nabla u||\nabla^2 u||u-c| \big\|_{L^{p_3}(B_r)} + \big\| |\nabla u|^2 |u-c| \big\|_{L^{p_2}(B_r)} + \big\| |\nabla u||u-c| \big\|_{L^{p_1}(B_r)} \\
&\qquad\qquad + \Big\| |\nabla u|^2 |u-c| \Big(\sum_{\alpha,\beta,j} (u^\alpha \nabla u^\beta - u^\beta \nabla u^\alpha)^2 \Big)^{1/2} \Big\|_{L^{p_3}(B_r)},
\end{aligned}$$

where $1/p_2 = 1/p_3 - 1/4$, $1/p_1 = 1/p_2 - 1/4$ and $c = (c^1, \dots, c^{n+1})$. Then by Sobolev imbedding theorem, we get

$$\begin{aligned}
\|\nabla v\|_{L^{p_1}(B_r)} &\lesssim \big\| |\nabla u||\nabla^2 u||u-c| \big\|_{L^{p_3}(B_r)} + \big\| |\nabla u|^2 |u-c| \big\|_{L^{p_2}(B_r)} \\
&\qquad + \big\| |\nabla u||u-c| \big\|_{L^{p_1}(B_r)} + \big\| |\nabla u|^2 |u-c||\nabla u| \big\|_{L^{p_3}(B_r)}.
\end{aligned}$$

Using the Hölder inequality, we have, for $1/p_0 = 1/p_1 - 1/4$,

$$\begin{aligned}
\|\nabla v\|_{L^{p_1}(B_r)} &\lesssim \|u-c\|_{L^{p_0}(B_r)} \|\nabla u\|_{L^4(B_r)} \|\nabla^2 u\|_{L^2(B_r)} \\
&\qquad + \|u-c\|_{L^{p_0}(B_r)} \|\nabla u\|^2_{L^4(B_r)} + \|u-c\|_{L^{p_0}(B_r)} \|\nabla u\|_{L^4(B_r)} \\
&\qquad + \|\nabla u\|^3_{L^4(B_r)} \|u-c\|_{L^{p_0}(B_r)}.
\end{aligned}$$

Applying the Sobolev imbedding theorem again to the left hand side, we get

$$\begin{aligned}
&\|v\|_{L^{p_0}(B_r)} + \|\nabla v\|_{L^{p_1}(B_r)} \\
&\quad \lesssim \big(\|\nabla^2 u\|^2_{L^2(B_r)} + \|\nabla u\|^2_{L^4(B_r)} + \|\nabla u\|^3_{L^4(B_r)} + \|\nabla u\|_{L^4(B_r)} \big) \times \|u-c\|_{L^{p_0}(B_r)} \\
&\quad \lesssim (E^3(u) + E^2(u) + E(u))(B) \, \|u-c\|_{L^{p_0}(B_r)} \, .
\end{aligned}$$

Now, with this key estimate, the proof proceeds exactly the same as in [Chang et al. 1999c]. But we write it down for the sake of completeness.

Set $c = \overline{u}$ and we choose r with $1/2 \le r < 1$ such that

$$\Big(\int_{\partial B_r} |u - \overline{u}|^{p_0} \Big)^{1/p_0} + \Big(\int_{\partial B_r} |\nabla u|^{p_1} \Big)^{1/p_1} \lesssim \Big(\int_B |u - \overline{u}|^{p_0} \Big)^{1/p_0} + \Big(\int_B |\nabla u|^{p_1} \Big)^{1/p_1}.$$

Then for any τ with $0 < \tau < 1/4$ and $x \in B_\tau$, the above justifies the second $\lesssim$ in

$$\begin{aligned}
|\nabla h(x)| \lesssim \int_{\partial B_r} |u - \overline{u}| + \int_{\partial B_r} \Big| \frac{\partial u}{\partial n} \Big| &\lesssim \Big(\int_B |u - \overline{u}|^{p_0} \Big)^{1/p_0} + \Big(\int_B |\nabla u|^{p_1} \Big)^{1/p_1} \\
&= (M_{p_0}(u) + D_{p_1}(u))(B).
\end{aligned}$$

For any $\tau < 1/4$, this in turn justifies the final step in

$$
\begin{aligned}
&(M_{p_0}(u)+D_{p_1}(u))(B_\tau)\\
&\quad= \Big(\tau^{-4}\int_{B_\tau}|u-\overline{u}|^{p_0}\Big)^{1/p_0}+\Big(\tau^{p_1-4}\int_{B_\tau}|\nabla u|^{p_1}\Big)^{1/p_1}\\
&\quad= \tau^{-4/p_0}\,\|u-\overline{u}\|_{L^{p_0}(B_\tau)}+\tau^{1-4/p_1}\,\|\nabla u\|_{L^{p_1}(B_\tau)}\\
&\quad\lesssim \tau^{-4/p_0}\,\|u-h(0)\|_{L^{p_0}(B_\tau)}+\tau^{1-4/p_1}\,\|\nabla u\|_{L^{p_1}(B_\tau)}\\
&\quad\lesssim \tau^{-4/p_0}\,\big(\|v\|_{L^{p_0}(B_\tau)}+\|h-h(0)\|_{L^{p_0}(B_\tau)}\big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad+\tau^{1-4/p_1}\,\big(\|\nabla v\|_{L^{p_1}(B_\tau)}+\|\nabla h\|_{L^{p_1}(B_\tau)}\big)\\
&\quad\lesssim \tau^{1-4/p_1}\,(E^3(u)+E^2(u)+E(u))(B)\,\|u-\overline{u}\|_{L^{p_0}(B)}+\tau\sup_{x\in B_\tau}|\nabla h(x)|\\
&\quad\lesssim \tau^{1-4/p_1}\,\varepsilon\,\|u-\overline{u}\|_{L^{p_0}(B)}+\tau(M_{p_0}(u)+D_{p_1}(u))(B),
\end{aligned}
$$

where $E(u)(B) < \varepsilon$.

If we choose τ sufficiently small, and then ε small, we get

$$(M_{p_0}(u)+D_{p_1}(u))(B_\tau) \le \tau^\gamma (M_{p_0}(u)+D_{p_1}(u))(B). \qquad \square$$

Proof of Theorem 2.1. Take any point $x \in B$. Suppose $B_\rho(x) \subset B$ is such that $E(u)(B_\rho(x)) < \varepsilon$. By Lemma 2.2, we know that

$$(M_{p_0}(u)+D_{p_1}(u))(B_{\tau\rho}(x)) < \tau^\gamma (M_{p_0}(u)+D_{p_1}(u))(B_{\tau\rho}(x)).$$

Note that $E(u)(B_s(x)) < \varepsilon$ for all $s < \rho$. So we can apply the Lemma 2.2 iteratively and get

$$(M_{p_0}(u)+D_{p_1}(u))(B_{\tau^j\rho}(x)) \le \tau^{\gamma j}(M_{p_0}(u)+D_{p_1}(u))(B_{\tau\rho}(x)) \quad \text{for } j \in \mathbb{N}.$$

From this, it can be shown that $D_{p_1}(B_s(y)) \le Cs^\gamma$ for some $C > 0$, for all y near x, and sufficiently small $s > 0$; see [Giaquinta 1983]. Then it follows that u is locally Hölder continuous with exponent $\beta = \gamma/4$ in B using Morrey's condition; again see [Giaquinta 1983]. $\square$

3. Higher interior regularity

Here we show that a weakly intrinsic biharmonic map u is smooth on B once it is continuous on B, hence completing the proof of interior regularity.

3.1. ***Two remarks.*** In fact, we consider a more general class of elliptic system and prove the following theorem:

Theorem 3.1. *If u is a weak continuous solution of the system*

$$\Delta^2 u^\alpha = f^\alpha(x, Du, D^2u) + \sum_{i=1}^{4}\frac{\partial g_i^\alpha}{\partial x_i}(x, Du, D^2u) \quad \text{on } B,$$

where

$$|f^\alpha(x, P, M)| \le \lambda_1(1 + |P|^4 + |M|^2),$$
$$|g_i^\alpha(x, P, M)| \le \lambda_2(1 + |P|^3 + |M|^{3/2}),$$

then $u \in C^{2,\beta}(B)$ *for some* $\beta \in (0, 1)$.

According to classical regularity theory, once the solution is $C^{2,\beta}(B)$, it is smooth on B. Since the Euler–Lagrange equation satisfied by u is included in this class, we have the following:

Corollary 3.1. *If u is a continuous weakly intrinsic biharmonic map on B, then it is smooth on B.*

Combining this with the result in Section 2, we finally get the main interior regularity theorem:

Theorem 3.2. *If $u \in W^{2,2}(B, S^n)$ is a weakly intrinsic biharmonic map, then $u \in C^\infty(B, S^n)$.*

Two remarks: First, to show that u is $C^{2,\beta}(B)$ we only need to show that $u_1(x) = (u(rx) - u(0))/c(r)$ belongs to $C^{2,\beta}(B)$, where $c(r) = \|u - u(0)\|_{L^\infty(B_r)} + r$. We may assume $u_1(x)$ to be small when u is continuous on B and r is sufficiently small. Then we get

$$\Delta^2 u_1 = \tilde{f}^\alpha(x, Du_1, D^2u_1) + \sum_{i=1}^{4} \frac{\partial \tilde{g}^\alpha}{\partial x_i}(x, Du_1, D^2u_1), \tag{2}$$

where

$$\tilde{f}^\alpha(x, P, M) = \frac{r^4}{c(r)} f^\alpha\Big(rx, \frac{c(r)}{r}P, \frac{c(r)}{r^2}M\Big),$$
$$\tilde{g}_i^\alpha(x, P, M) = \frac{r^3}{c(r)} g_i^\alpha\Big(rx, \frac{c(r)}{r}P, \frac{c(r)}{r^2}M\Big).$$

Thus u_1 is a weak continuous solution of the same type of equations with $\tilde{f}^\alpha$, $\tilde{g}_i^\alpha$ and $\tilde{a}^{cdst}$ satisfying the following growth conditions:

$$\begin{aligned} |\tilde{f}^\alpha(x, P, M)| &\le \tilde{\lambda}_1(1 + \mu_1|P|^4 + \mu_1|M|^2), \\ |\tilde{g}^\alpha(x, P, M)| &\le \tilde{\lambda}_2(1 + \mu_2|P|^3 + \mu_2|M|^{3/2}), \end{aligned} \tag{3}$$

where $\tilde{\lambda}_1 = c(r)^{1/2}\lambda_1$, $\mu_1 = c(r)^{1/2}$, $\tilde{\lambda}_2 = c(r)^{1/4}\lambda_2$ and $\mu_2 = c(r)^{1/4}$. So $\tilde{\lambda}_j$ and μ_j for $j = 1, 2$ can be made arbitrarily small as r is small. This important observation allows us to reduce the proof of Theorem 3.1 to a scaling argument.

Second, the theorem holds if we replace Δu by any elliptic systems.

3.2. ***Proof of Theorem 3.1.*** First of all, we need the following lemma:

Lemma 3.1. *Suppose v is a weak solution of the Equation (2) satisfying the growth conditions (3). And suppose*

$$\mu_1 \|v\|_{L^\infty} \le \delta \quad \textit{and} \qquad \int_B |D^2 v|^2 dx + \Big(\int_B |Dv|^4 dx\Big)^{1/2} + \int_B |v|^2 dx \le 1. \tag{4}$$

Then there exists an $r_0 > 0$ such that for $r < r_0$,

$$r^4 \int_{B_r} |D^2(v-h)|^2 dx + r^4 \Big(\int_{B_r} |D(v-h)|^4 dx\Big)^{1/2} + \int_{B_r} |v-h|^2 dx \\ \lesssim \tilde\lambda_1^2 + \tilde\lambda_2^2 + \delta, \tag{5}$$

where $h : B_{r_0} \to \mathbb{R}^K$ is such that $\Delta^2 h = 0$ in B_{r_0}, and $h = v$ and $\partial h/\partial n = \partial v/\partial n$ on B_{r_0}.

Proof. Using the Sobolev inequality and integration by parts, we have

$$\begin{aligned}
&r^4 \int_{B_r} |D^2(v-h)|^2 dx + r^4 \Big(\int_{B_r} |D(v-h)|^4 dx\Big)^{1/2} + \int_{B_r} |v-h|^2 dx \\
&\le {r_0}^4 \int_{B_{r_0}} |D^2(v-h)|^2 dx + {r_0}^4 \Big(\int_{B_{r_0}} |D(v-h)|^4 dx\Big)^{1/2} + \int_{B_{r_0}} |v-h|^2 dx \\
&\lesssim \int_{B_{r_0}} |D^2(v-h)|^2 dx \lesssim \int_{B_{r_0}} |\Delta(v-h)|^2 dx \\
&\lesssim \int_{B_{r_0}} \big(\tilde\lambda_1 |v-h| + \tilde\lambda_1 \mu_1 |v-h||Dv|^4 + \tilde\lambda_1 \mu_1 |v-h||D^2 v|^2\big) dx \\
&\quad + \int_{B_{r_0}} \big(\tilde\lambda_2 |D(v-h)| + \tilde\lambda_2 \mu_2 |D(v-h)||Dv|^3 + \tilde\lambda_2 \mu_2 |D(v-h)||D^2 v|^{3/2}\big) dx.
\end{aligned}$$

By [Chang et al. 1999a] we have the estimate

$$|(u-h)(x)| \lesssim \operatorname{osc}(u)(B_1) + \|Du\|_{L^4(\partial B_{r_0})} \le \|u\|_{L^\infty(B_1)} + 1$$

if we choose $r_0 > 1/2$ such that

$$\int_{\partial B_{r_0}} |Du|^4 d\sigma \lesssim \int_B |Du|^4 dx.$$

Using this estimate and the interpolation inequality, we get

$$\begin{aligned}\text{left side } \lesssim \int_{B_{r_0}} &\Big(\frac{1}{\varepsilon_1^2}\tilde{\lambda}_1^2+\varepsilon_1^2|v-h|^2+(\delta+1)\tilde{\lambda}_1(|Du|^4+|D^2u|^2)\Big)dx\\ &+\int_{B_{r_0}}\Big(\frac{1}{\varepsilon_2^2}\tilde{\lambda}_2^2+\varepsilon_2^2|D(v-h)|^2\Big)dx\\ &\quad+\tilde{\lambda}_2\mu_2\Big(\varepsilon_3^2\int_{B_{r_0}}|D(v-h)|^4dx+\frac{1}{\varepsilon_3^2}\Big(\int_{B_{r_0}}|Du|^4dx\Big)^{3/2}\Big)\\ &\qquad+\tilde{\lambda}_2\mu_2\Big(\varepsilon_3^2\int_{B_{r_0}}|D(v-h)|^4dx+\frac{1}{\varepsilon_3^2}\Big(\int_{B_{r_0}}|D^2u|^2dx\Big)^{3/2}\Big).\end{aligned}$$

From this, by choosing a suitable ε_j, we obtain the required estimate (5). □

Using Lemma 3.1, we can prove an important corollary:

Corollary 3.2. *For any* $0<\beta<1$ *and sufficiently small* $\tilde{\lambda}_i$ *with* $\mu_i>0$, *there exists* $0<\tau<1/4$ *such that if* v *is a weak solution of Equation (2) with growth conditions (3) that satisfies conditions (4), then there exists a second-order polynomial* $p(x)=\frac{1}{2}xAx+Bx+C$ *such that*

$$⨏_{B_\tau}|D^2(v-p)|^2dx+\Big(\frac{1}{\tau^4}⨏_{B_\tau}|D(v-p)|^4dx\Big)^{1/2}+\frac{1}{\tau^4}⨏_{B_\tau}|v-p|^2dx\le\tau^{2\beta}.$$

Also $|A|+|B|+|C|\le C_0$, *where* C_0 *is a universal constant.*

Proof. Let h be the biharmonic vector in the previous lemma, then

$$(6)\quad \|h\|_{C^3(B_{1/4})}\lesssim\int_{\partial B_{r_0}}(|u|+|Du|)d\sigma\lesssim\Big(\int_{B_1}|u|^2dx\Big)^{1/2}+\Big(\int_{B_1}|Du|^4dx\Big)^{1/4}\le C_0.$$

Let $p(x)$ be the second-order Taylor polynomial of h at 0, that is, let $p(x)=\frac{1}{2}xD^2h(0)x+Dh(0)x+h(0)$. By Lemma 3.1, we have, for $\tau<1/4$,

$$\begin{aligned}&⨏_{B_\tau}|D^2(v-p)|^2dx+\Big(\frac{1}{\tau^4}⨏_{B_\tau}|D(v-p)|^4dx\Big)^{1/2}+\frac{1}{\tau^4}⨏_{B_\tau}|v-p|^2dx\\ &\quad\le⨏_{B_\tau}|D^2(v-h)|^2dx+\Big(\frac{1}{\tau^4}⨏_{B_\tau}|D(v-h)|^4dx\Big)^{1/2}+\frac{1}{\tau^4}⨏_{B_\tau}|v-h|^2dx\\ &\qquad+⨏_{B_\tau}|D^2(h-p)|^2dx+\Big(\frac{1}{\tau^4}⨏_{B_\tau}|D(h-p)|^4dx\Big)^{1/2}+\frac{1}{\tau^4}⨏_{B_\tau}|h-p|^2dx\\ &\quad\le C\tau^{-8}(\tilde{\lambda}_1^2+\tilde{\lambda}_2^2+\delta)+\sup|D^3h|\tau^2\quad(\text{by Lemma 3.1})\\ &\quad\le C\tau^{-8}(\tilde{\lambda}_1^2+\tilde{\lambda}_2^2+\delta)+C_0\tau^2\quad(\text{by (6)}).\end{aligned}$$

Now, first take τ small such that the second term is less than or equal to $\tau^{2\beta}/2$, and then take $\tilde{\lambda}_j$, μ_j small (so that δ is also small) such that the rest is bounded by $\tau^{2\beta}/2$. Then the result follows. □

Proof of Theorem 3.1. First we prove this claim: There exists $\mathscr{C} > 0$, $0 < \tilde{\lambda}_i$, $\mu_i < 1$, and $\varepsilon_0 > 0$ such that if $|u| \le 1$ and u is a weak solution of Equation (2) with growth condition (3) and $\tilde{\lambda}_i$ has $\mu_i \le \varepsilon_0$, then for each $k \in \mathbb{N}$ there is a second order polynomial $p_k(x) = \frac{1}{2} x A_k x + B_k x + C_k$ such that

$$(7)\quad \fint_{B_{\tau^k}} |D^2(u-p_k)|^2 dx + \Big(\frac{1}{\tau^{4k}} \fint_{B_{\tau^k}} |D(u-p_k)|^4 dx\Big)^{1/2} + \frac{1}{\tau^{4k}} \fint_{B_{\tau^k}} |u-p_k|^2 dx \le \tau^{2\beta k}$$

and $|A_k| + |B_k| + |C_k| \le \mathscr{C}$, where $\mathscr{C}$ is a universal constant.

We prove this claim by induction on k. Using Corollary 3.2, the case $k = 1$ is true. To verify the inductive step, assume the claim is true for k and define

$$w_k(x) = \frac{(u-p_k)(r^k x)}{r^{(2+\beta)k}}.$$

Then we get

$$\Delta^2 w_k^\alpha = F^\alpha(x, Dw_k, D^2 w_k) + \sum_{i=1}^{4} \frac{\partial G^\alpha}{\partial x_i}(x, Dw_k, D^2 w_k),$$

where

$$F^\alpha(x, P, M) = \tau^{(2-\beta)k} \tilde{f}^\alpha(\tau^k x, Dp_k(\tau^k x) + \tau^{(1+\beta)k} P, D^2 p_k(\tau^k x) + \tau^{\beta k} M) + \tau^{(2-\beta)k} (D_{cd} \tilde{a}^{cdst})(D_{st} p_k)(\tau^k x),$$

$$G_i^\alpha(x, P, M) = \tau^{(1-\beta)k} \tilde{g}_i^\alpha(\tau^k x, Dp_k(\tau^k x) + \tau^{(1+\beta)k} P, D^2 p_k(\tau^k x) + \tau^{\beta k} M).$$

Next we check the growth conditions (3):

$$\begin{aligned}|F^\alpha(x, P, M)| &\le \tau^{(2-\beta)k}\big(\tilde{\lambda}_1\big(1 + 8\mu_1(\mathscr{C}^4 + \tau^{4(1+\beta)k}|P|^4) + 2\mu_1(\mathscr{C}^2 + \tau^{2\beta k}|M|^2)\big) + \varepsilon\mathscr{C}\big)\\ &\le \tilde{\lambda}(1 + 8\mu_1 \tau^{(6+3\beta)k}|P|^4 + 2\mu_1 r^{(2+\beta)k}|M|^2),\\ |G^\alpha(x, P, M)| &\le \tau^{(1-\beta)k}\tilde{\lambda}_2\big(1 + 4\mu_2(\mathscr{C}^3 + \tau^{3(1+\beta)k}|P|^3) + 2\mu_2(\mathscr{C}^{3/2} + \tau^{3\beta k/2}|M|^{3/2})\big)\\ &\le \tilde{\lambda}_2\big(1 + 4\mu_2 \tau^{(4+2\beta)k}|P|^3 + 2\mu_2 \tau^{(1+\beta k/2)}|M|^{3/2}\big).\end{aligned}$$

for $\tilde{\lambda}_j$, μ_j and τ sufficiently small. Now we verify the conditions (4) for w_k:

$$2\mu_1 \tau^{(2+\beta)k} \|w_k\|_{L^\infty(B_1)} = 2\mu_1 \|u - p_k\|_{L^\infty(B_{\tau^k})} \le 2\mu_1(1 + \mathscr{C}) \le \delta,$$

if μ_1 is initially chosen to be small. Also, we have

$$\begin{aligned}\int_B |D^2 w_k|^2 dx + \Big(\int_B |Dw_k|^4 dx\Big)^{1/2} + \int_B |w_k|^2 dx \\ = \int_B \Big| D^2 \frac{(u-p_k)(\tau^k x)}{\tau^{(2+\beta)k}} \Big|^2 dx + \Big(\int_B \Big| D \frac{(u-p_k)(\tau^k x)}{\tau^{(2+\beta)k}} \Big|^4 dx\Big)^{1/2} \\ + \int_B \Big| \frac{(u-p_k)(\tau^k x)}{\tau^{(2+\beta)k}} \Big|^2 dx \\ = \frac{1}{\tau^{2\beta k}} \Big(\fint_{B_{\tau^k}} |D^2(u-p_k)|^2 dx + \Big(\frac{1}{\tau^{4k}} \fint_{B_{\tau^k}} |D(u-p_k)|^4 dx\Big)^{1/2} \\ + \frac{1}{\tau^{4k}} \fint_{B_{\tau^k}} |u-p_k|^2 dx \Big) \\ \leq 1,\end{aligned}$$

by the induction hypothesis. So conditions (4) for w_k are satisfied.

Therefore, we can apply the Corollary 3.2 to w_k, that is, there exists a second order polynomial $q(x) = \frac{1}{2} x A x + B x + C$ such that

$$\fint_{B_\tau} |D^2(w_k - q)|^2 dx + \Big(\frac{1}{\tau^4} \fint_{B_\tau} |D(w_k - q)|^4 dx\Big)^{1/2} + \frac{1}{\tau^4} \fint_{B_\tau} |w_k - q|^2 dx \leq \tau^{2\beta}$$

and $|A| + |B| + |C| \leq C_0$. Then define $p_{k+1}(x) = p_k(x) + \tau^{(2+\beta)k} q(x/\tau^k)$. By a change of variable, we get

$$\begin{aligned}\fint_{B_{\tau^{k+1}}} |D^2(u - p_{k+1})|^2 + \Big(\frac{1}{\tau^{4(k+1)}} \fint_{B_{r^{k+1}}} |D(u - p_{k+1})|^4 dx\Big)^{1/2} \\ + \frac{1}{\tau^{4(k+1)}} \fint_{B_{r^{k+1}}} |u - p_{k+1}|^2 dx \leq \tau^{2(k+1)\beta}.\end{aligned}$$

This proves the inequality (7) for $k+1$. Now, it remains to show that $|A_{k+1}| + |B_{k+1}| + |C_{k+1}| \leq \mathscr{C}$. Initially, we set $\mathscr{C} = (3C_0)/(1 - 4^{-\beta})$. From the induction step, we know that for $j \leq k$, we have

$$\begin{aligned}|A_{j+1}| &\leq |A_j| + \tau^{\beta j} C_0, \\ |B_{j+1}| &\leq |B_j| + \tau^{(1+\beta) j} C_0, \\ |C_{j+1}| &\leq |C_j| + \tau^{(2+\beta) j} C_0.\end{aligned}$$

This implies

$$|A_{j+1}| + |B_{j+1}| + |C_{j+1}| \leq |A_j| + |B_j| + |C_j| + 3\tau^{\beta k} C_0.$$

Hence we have

$$|A_{k+1}| + |B_{k+1}| + |C_{k+1}| \leq \frac{3C_0}{1 - \tau^\beta} \leq \mathscr{C}.$$

This complete that proof of the claim for $k+1$.

Now, similarly to the proof of Theorem 2.1, this result implies that $u \in C^{2,\beta}(B)$, hence finishing the proof of Theorem 3.1. □

4. Boundary regularity

Here we will investigate the boundary regularity of weakly intrinsic biharmonic maps u. The main is this:

Theorem 4.1. *Suppose $u \in W^{2,2}(B^4, S^n)$ is a weakly intrinsic biharmonic map such that $u|_{\partial B} \in C^{l,\beta}(\partial B, S^n)$, and $\partial u/\partial n|_{\partial B} \in C^{l-1,\beta}(\partial B, S^n)$ for $l \in \mathbb{N}$ and $\beta \in (0,1)$. Then $u \in C^{l,\beta}(\overline{B}, S^n)$.*

Since the interior regularity has already been established in previous section, we concentrate on the neighborhood of the boundary ∂B. Without losing generality, we may assume that $u : (\Omega_1^+, g) \to S^n \subset \mathbb{R}^{k+1}$, where Ω_r^+ is the upper-half ball of radius r, that is, $\Omega_r^+ = \{(x,t) \in \mathbb{R}^4 \mid t \geq 0, |x|^2 + t^2 < r\}$. Then, the Dirichlet boundary conditions become

$$u(x,0) \in C^{l,\beta}(\Gamma_1, S^n) \quad \text{and} \quad \frac{\partial u}{\partial n}(x,0) \in C^{l-1,\beta}(\Gamma_1, S^n), \tag{8}$$

where Γ_1 is the flat part of $\partial\Omega_1^+$.

4.1. *Boundary $C^{0,\beta}$ regularity.*

To prove the main theorem, we first need to prove the boundary $C^{0,\beta}$ regularity of u, a consequence of this theorem:

Theorem 4.2. *Let $u \in W^{2,2}(\Omega_1^+, S^n)$ be a weakly intrinsic biharmonic map satisfying (8). Then $u \in C^{0,\beta}(\mathcal{U}, S^n)$, where $\mathcal{U}$ is a neighborhood of Γ_s for some $s \in (0,1)$ in Ω_1^+.*

Proof. First, for any $r > 0$ we define

$$M_p(u)(\Omega_r^+) = \Big(⨍_{\Omega_r^+} |u - u(0)|^p \Big)^{1/p} \quad \text{and} \quad D_p(u)(\Omega_r^+) = \Big(r^p ⨍_{\Omega_r^+} |\nabla u|^p \Big)^{1/p}.$$

Suppose $1/2 < r_1 < 1$ and $0 < \tau < r_1/4$, with both τ and r_1 to be chosen later. Let $h_1 : \Omega_{r_1}^+ \to \mathbb{R}^{n+1}$ be such that $\Delta^2 h_1 = 0$ in $\Omega_{r_1}^+$ and $h_1 = u$ and $\partial h_1/\partial n = \partial u/\partial n$ on $\partial\Omega_{r_1}^+$. For p_0 and p_1 as in Section 2, we have

$$\begin{aligned} M_{p_0}(u)(\Omega_\tau^+) + D_{p_1}(u)(\Omega_\tau^+) &= \Big(⨍_{\Omega_\tau^+} |u - u(0)|^{p_0} \Big)^{1/p_0} + \Big(\tau^{p_1} ⨍_{\Omega_\tau^+} |\nabla u|^{p_1} \Big)^{1/p_1} \\ &\leq \Big(⨍_{\Omega_\tau^+} |u - h_1|^{p_0} \Big)^{1/p_0} + \Big(\tau^{p_1} ⨍_{\Omega_\tau^+} |\nabla (u - h_1)|^{p_1} \Big)^{1/p_1} \\ &\quad + \Big(⨍_{\Omega_\tau^+} |h_1 - h_1(0)|^{p_0} \Big)^{1/p_0} + \Big(\tau^{p_1} ⨍_{\Omega_\tau^+} |\nabla h_1|^{p_1} \Big)^{1/p_1}. \end{aligned}$$

Similarly to Section 2, we get the key estimate on $\Omega_{r_1}^+$:

$$\int_{\Omega_{r_1}^+} |\nabla(u-h_1)|^{p_1} \lesssim (E^3+E^2+E)(\Omega_{r_1}^+)^{p_1}\Big(\int_{\Omega_{r_1}^+} |u-u(0)|^{p_0}\Big)^{p_1/p_0}.$$

Apply this and the Sobolev inequality, we get

$$M_{p_0}(u)(\Omega_\tau^+)+D_{p_1}(u)(\Omega_\tau^+) \lesssim +\Big(\fint_{\Omega_\tau^+}|h_1-h_1(0)|^{p_0}\Big)^{1/p_0}+\Big(\tau^{p_1}\fint_{\Omega_\tau^+}|\nabla h_1|^{p_1}\Big)^{1/p_1} \\ +\frac{1}{\tau^{4/p_0}}\,(E^3(u)+E^2(u)+E(u))(\Omega_1^+)\Big(\int_{\Omega_1^+}|u-u(0)|^{p_0}\Big)^{1/p_0}.$$

Now we apply the above inequality to $u(\tau^{k-1}x)$ for $k=2,3,\ldots$. Then by a change of variable, we get

$$M_{p_0}(u)(\Omega_{\tau^k}^+)+D_{p_1}(u)(\Omega_{\tau^k}^+) \lesssim +\Big(\fint_{\Omega_\tau^+}|h_k-h_k(0)|^{p_0}\Big)^{1/p_0}+\Big(\tau^{p_1}\fint_{\Omega_\tau^+}|\nabla h_k|^{p_1}\Big)^{1/p_1} \\ |+\tau^{-4/p_0}\,(E^3(u)+E^2(u)+E(u))(\Omega_1^+)\Big(\fint_{\Omega_{\tau^{k-1}}^+}|u-u(0)|^{p_0}\Big)^{1/p_0},$$

where $h_k:\Omega_{r_k}^+\to\mathbb{R}^{n+1}$ is such that $\Delta^2 h_k=0$ in $\Omega_{r_k}^+$, and $h_k(x)=u(\tau^{k-1}x)$ and $\partial h_k(x)/\partial n=\partial(u(\tau^{k-1}x))/\partial n$ on $\partial\Omega_{r_k}^+$, for some $r_k\in(r_1/2,r_1]$ to be chosen later. Now define $\tilde{h}_k(x)=h_1(\tau^{k-1}x)$. We have

$$\begin{aligned} &M_{p_0}(u)(\Omega_{\tau^k}^+)+D_{p_1}(u)(\Omega_{\tau^k}^+) \\ &\quad\lesssim \tau^{-4/p_0}\,(E^3(u)+E^2(u)+E(u))(\Omega_1^+)\Big(\fint_{\Omega_{\tau^{k-1}}^+}|u-u(0)|^{p_0}\Big)^{1/p_0} \\ &\qquad+\Big(\fint_{\Omega_\tau^+}|h_k-\tilde{h}_k|^{p_0}\Big)^{1/p_0}+\Big(\tau^{p_1}\fint_{\Omega_\tau^+}|\nabla(h_k-\tilde{h}_k)|^{p_1}\Big)^{1/p_1} \\ &\qquad+\Big(\fint_{\Omega_\tau^+}|\tilde{h}_k-\tilde{h}_k(0)|^{p_0}\Big)^{1/p_0}+\Big(\tau^{p_1}\fint_{\Omega_\tau^+}|\nabla\tilde{h}_k|^{p_1}\Big)^{1/p_1} \\ &\quad\lesssim \tau^{-4/p_0}\,(E^3(u)+E^2(u)+E(u))(\Omega_1^+)\Big(\fint_{\Omega_{\tau^{k-1}}^+}|u-u(0)|^{p_0}\Big)^{1/p_0} \\ &\qquad+\Big(\fint_{\Omega_{\tau^k}^+}|h_1-h_1(0)|^{p_0}\Big)^{1/p_0}+\Big(\tau^{kp_1}\fint_{\Omega_{\tau^k}^+}|\nabla h_1|^{p_1}\Big)^{1/p_1}+\tau\sup_{\Omega_\tau^+}|\nabla\phi_k| \\ &\quad\lesssim \tau^{-4/p_0}\,(E^3(u)+E^2(u)+E(u))(\Omega_1^+)M_{p_0}(u)(\Omega_{\tau^{k-1}}^+) \\ &\qquad\qquad+M_{p_0}(h_1)(\Omega_{\tau^k}^+)+D_{p_1}(h_1)(\Omega_{\tau^k}^+)+\tau\sup_{\Omega_\tau^+}|\nabla\phi_k|, \end{aligned}$$

where $\phi_k=h_k-\tilde{h}_k$. Note that $\Delta^2\phi_k=0$ in $\Omega_{r_k}^+$ and $\phi_k=\partial\phi_k/\partial n=0$ on Γ_{r_k}. Therefore, by Schauder theory (see [Agmon et al. 1959]), we know that ϕ_k is smooth on $\overline{\Omega^+}_\tau$. Moreover, let G be the Green function of Δ^2 on $\Omega^+_{r_k}$ satisfying Dirichlet boundary conditions. By Green's identity and that $\phi_k = \partial\phi_k/\partial n = 0$ on Γ_{r_k}, we get

$$\phi_k(x) = \int_{\partial\Omega^+_{r_k}\setminus\Gamma_{r_k}} \Big(\frac{\partial(\Delta G)}{\partial n}(x, y)\phi_k(y) - \Delta G(x, y)\frac{\partial\phi_k}{\partial n}(y)\Big) d\sigma(y).$$

So for $x \in \Omega^+_\tau$, we have the estimate

$$\begin{aligned}\sup_{\Omega^+_\tau}|\nabla\phi_k| &\lesssim \int_{\partial\Omega^+_{r_k}\setminus\Gamma_{r_k}} \Big(|\phi_k| + \Big|\frac{\partial\phi_k}{\partial n}\Big|\Big) d\sigma \\ &= \int_{\partial\Omega^+_{r_k}\setminus\Gamma_{r_k}} \big(|(u - h_1)(\tau^{k-1}x)| + |\nabla((u - h_1)(\tau^{k-1}x))|\big) d\sigma(x).\end{aligned}$$

Now we choose r_k such that

$$\int_{\partial\Omega^+_{r_k}\setminus\Gamma_{r_k}} |(u - h_1)(\tau^{k-1}x)| d\sigma(x) \lesssim \int_{\Omega^+_1} |(u - h_1)(\tau^{k-1}x)|,$$

$$\int_{\partial\Omega^+_{r_k}\setminus\Gamma_{r_k}} |\nabla((u - h_1)(\tau^{k-1}x))| d\sigma(x) \lesssim \int_{\Omega^+_1} |\nabla((u - h_1)(\tau^{k-1}x))|.$$

Applying these estimates and the Hölder inequality, we get

$$\begin{aligned}\sup_{\Omega^+_\tau}|\nabla\phi_k| &\lesssim \Big(\int_{\Omega^+_1} |(u - h_1)(\tau^{k-1}x)|^{p_0}\Big)^{1/p_0} + \Big(\int_{\Omega^+_1} |\nabla((u - h_1)(\tau^{k-1}x))|^{p_1}\Big)^{1/p_1} \\ &\lesssim \Big(\int_{\Omega^+_1} |u(\tau^{k-1}x) - u(0)|^{p_0}\Big)^{1/p_0} + \Big(\int_{\Omega^+_1} |\nabla(u(\tau^{k-1}x))|^{p_1}\Big)^{1/p_1} \\ &\quad + \Big(\int_{\Omega^+_1} |h_1(\tau^{k-1}x) - h_1(0)|^{p_0}\Big)^{1/p_0} + \Big(\int_{\Omega^+_1} |\nabla(h_1(\tau^{k-1}x))|^{p_1}\Big)^{1/p_1} \\ &\lesssim M_{p_0}(u)(\Omega^+_{\tau^{k-1}}) + D_{p_1}(u)(\Omega^+_{\tau^{k-1}}) \\ &\quad + M_{p_0}(h_1)(\Omega^+_{\tau^{k-1}}) + D_{p_1}(h_1)(\Omega^+_{\tau^{k-1}}).\end{aligned}$$

Therefore, we have

$$\begin{aligned}&M_{p_0}(u)(\Omega^+_{\tau^k}) + D_{p_1}(u)(\Omega^+_{\tau^k}) \\ &\qquad\lesssim \tau^{-4/p_0}\ (E^3(u) + E^2(u) + E(u))(\Omega^+_1) M_{p_0}(u)(\Omega^+_{\tau^k}) \\ &\qquad\quad + \tau\big(M_{p_0}(u)(\Omega^+_{\tau^{k-1}}) + D_{p_1}(u)(\Omega^+_{\tau^{k-1}})\big) \\ &\qquad\quad + M_{p_0}(h_1)(\Omega^+_{\tau^{k-1}}) + D_{p_1}(h_1)(\Omega^+_{\tau^{k-1}}) \\ &\qquad\quad + M_{p_0}(h_1)(\Omega^+_{\tau^k}) + D_{p_1}(h_1)(\Omega^+_{\tau^k}) \quad \text{for } k = 1, 2, \ldots.\end{aligned}$$

By the definition of h_1 and the boundary assumption on u, we can deduce from Schauder theory that $h_1 \in C^{1,\beta}(\overline{\Omega^+}_\tau, S^n)$, and so for $k \in \mathbb{N}$, we have

$$M_{p_0}(h_1)(\Omega^+_{\tau^k}) + D_{p_1}(h_1)(\Omega^+_{\tau^k}) \le C\tau^{k\beta}$$

for some constant $C > 0$ independent of k and some sufficiently small τ. Now first choose τ small. Then for sufficiently small $E(u)(\Omega^+_1)$ for $k \in \mathbb{N}$, we have

$$M_{p_0}(u)(\Omega^+_{\tau^k}) + D_{p_1}(u)(\Omega^+_{\tau^k}) \le \frac{\tau^\beta}{2}\big(M_{p_0}(u)(\Omega^+_{\tau^{k-1}}) + D_{p_1}(u)(\Omega^+_{\tau^{k-1}}) + C\tau^{(k-1)\beta}\big).$$

Now we can apply this inequality iteratively and get

$$M_{p_0}(u)(\Omega^+_{\tau^k}) + D_{p_1}(u)(\Omega^+_{\tau^k}) \le \tau^{k\beta}\big(M_{p_0}(u)(\Omega^+_1) + D_{p_1}(u)(\Omega^+_1) + \frac{C}{2} + \frac{C}{2^2} + \cdots\big) \lesssim \tau^{k\beta} \tag{9}$$

for all $k \in \mathbb{N}$. In fact, we can apply the argument to all $x \in \Gamma_s$ for some $s \in (0, 1)$ and obtain the estimate (9) for x. Then by a standard argument we can prove that $u \in C^{0,\beta}(\mathcal{U}, S^n)$, where $\mathcal{U}$ is a neighborhood of Γ_s in $\overline{\Omega^+}_1$ □

4.2. ***Proof of Theorem 4.1 for l = 1.*** This case is in fact a consequence of this theorem:

Theorem 4.3. *Let $u \in W^{2,2}(\Omega^+_1, S^n)$ be a weakly intrinsic biharmonic map satisfying (8) for $l = 1$. Then $u \in C^{1,\beta}(\mathcal{U}, S^n)$, where $\mathcal{U}$ is a neighborhood of Γ_s for some $s \in (0, 1)$ in Ω^+_1.*

First, for any $r > 0$, we define

$$M'_{p_0}(u)(\Omega^+_r) \equiv \Big(\frac{1}{r^{p_0}} \fint_{\Omega^+_r} |u - u(0) - \nabla u(0)x|^{p_0}\Big)^{1/p_0},$$
$$D'_{p_1}(u)(\Omega^+_r) \equiv \Big(\fint_{\Omega^+_r} |\nabla u - \nabla u(0)|^{p_1}\Big)^{1/p_1}.$$

We have to rewrite the right side of the Euler–Lagrange equation again so as to obtain the right estimate. First, from the proof of Proposition 2.1 and [Chang et al. 1999c], we observe that

$$\Delta^2 u^\alpha = T_l \text{ terms } + \Delta(u^\alpha|\nabla u|^2) \quad \text{for } l = 1, 2, 4.$$

Now we rewrite each of these terms in the following way:

Type T_1 terms:

$$\begin{aligned}(\mathrm{IA}) = \operatorname{div}(\nabla u^\alpha \Delta u^\beta (u^\beta - c^\beta) &= \operatorname{div}((\nabla u^\alpha - a^\alpha)\Delta u^\beta (u^\beta - c^\beta)\\ &\quad + \operatorname{div}(a^\alpha \Delta u^\beta (u^\beta - c^\beta)),\\ (\mathrm{IB}) = \operatorname{div}(\langle \nabla u^\beta, \nabla\nabla u^\beta\rangle (u^\beta - c^\beta)) &= \operatorname{div}((u^\beta - c^\beta)\langle(\nabla u^\alpha - a^\alpha), \nabla\nabla u^\beta\rangle)\\ &\quad + \operatorname{div}((u^\beta - c^\beta)\langle a^\beta, \nabla\nabla u^\beta\rangle).\end{aligned}$$

Type T_2 terms:

$$\begin{aligned}(\mathrm{IIA}) = \Delta((u^\beta - c^\beta)|\nabla u^\beta|^2) &= \Delta((u^\beta - c^\beta)\langle \nabla u^\beta, \nabla u^\alpha - a^\alpha\rangle)\\ &\quad + \Delta((u^\beta - c^\beta)\langle \nabla u^\beta, a^\beta\rangle),\\ (\mathrm{IIB}) = \Delta(u^\alpha (u^\beta - c^\beta)\Delta u^\beta) &= \Delta \operatorname{div}(u^\alpha (u^\beta - c^\beta)(\nabla u^\alpha - a^\alpha))\\ &\quad - \Delta(u^\alpha \langle \nabla u^\alpha - a^\alpha, \nabla u^\beta\rangle)\\ &\quad - \Delta((u^\beta - c^\beta)\langle \nabla u^\alpha, \nabla u^\beta - a^\beta\rangle).\end{aligned}$$

Term of the form $\Delta((u^\beta - c^\beta)\Delta u^\alpha)$ do not appear.

Type T_4 terms:

$$\begin{aligned}(\mathrm{IV}) &= \operatorname{div}(|\nabla u|^2 (u^\alpha \nabla u^\beta - u^\beta \nabla u^\alpha)(u^\beta - c^\beta))\\ &= \operatorname{div}(|\nabla u|^2 u^\alpha (\nabla u^\beta - a^\beta)(u^\beta - c^\beta)) + \operatorname{div}(a^\beta |\nabla u|^2 u^\alpha (u^\beta - c^\beta))\\ &\quad - \big(\operatorname{div}(|\nabla u|^2 u^\beta (\nabla u^\alpha - a^\alpha)(u^\beta - c^\beta)) + \operatorname{div}(a^\beta |\nabla u|^2 u^\alpha (u^\beta - c^\beta))\big).\end{aligned}$$

$\Delta(u^\alpha |\nabla u|^2)$ terms:

$$\begin{aligned}(\mathrm{V}) = \Delta(u^\alpha |\nabla u|^2) &= \operatorname{div}(\nabla(u^\alpha |\nabla u|^2))\\ &= \operatorname{div}(\nabla u^\alpha |\nabla u|^2) + 2\operatorname{div}(u^\alpha \langle \nabla\nabla u^\beta, \nabla u^\beta\rangle)\\ &= \Delta((u^\beta - c^\beta)|\nabla u^\beta|^2) - 2\operatorname{div}((u^\beta - c^\beta)\langle \nabla u^\beta, \nabla\nabla u^\beta\rangle)\\ &\quad + 2\Delta(u^\alpha \langle \nabla u^\beta - a^\beta, \nabla u^\beta\rangle) - 2\operatorname{div}(\nabla u^\alpha \langle \nabla u^\beta - a^\beta, \nabla u^\beta\rangle)\\ &\quad - 2\operatorname{div}(u^\alpha \langle \nabla u^\beta - a^\beta, \nabla\nabla u^\beta\rangle).\end{aligned}$$

Then

$$\begin{aligned}(\mathrm{V}) &= (\mathrm{IIA})\text{ term} + (\mathrm{IB})\text{ term} + 2\Delta(u^\alpha \langle \nabla u^\beta - a^\beta, \nabla u^\beta\rangle)\\ &\quad - 2\operatorname{div}(\nabla u^\alpha \langle \nabla u^\beta - a^\beta, \nabla u^\beta\rangle)\\ &\quad - 2\operatorname{div}(u^\alpha \langle \nabla u^\beta - a^\beta, \nabla\nabla u^\beta\rangle),\end{aligned}$$

where $a^\beta = \sum_{i=1}^4 a_i^\beta \partial/\partial x_i$ is any constant vector field and c^β is any constant.

Now we are ready to prove this technical lemma:

Lemma 4.1. *For any $r \in (0,1)$, the estimate*

$$\int_{\Omega_r^+} |\nabla(u-h)|^{p_1} \lesssim \big((E^2+E)(u)(\Omega_r^+)^{p_1} + \max_{\Omega_r^+}|u-c|^{p_1}\big)\Big(\int_{\Omega_r^+}|\nabla u - a|^{p_1} + |a|\int_{\Omega_r^+}|u-c|^{p_1}\Big)$$

holds on Ω_r^+, where $h : \Omega_r^+ \to \mathbb{R}^{k+1}$ is such that $\Delta^2 h = 0$ in Ω_r^+ and such that $h = u$ and $\partial h/\partial n = \partial u/\partial n$ on $\partial\Omega_r^+$.

Proof. Using Lemma 2.1 in Section 2 and the Hölder inequality, we get

$$\|\nabla(u-h)\|_{L^{p_1}(\Omega_r^+)} \lesssim (\mathrm{IA})' + (\mathrm{IB})' + (\mathrm{IIA})' + (\mathrm{IIB})' + (\mathrm{IV})' + (\mathrm{V})' \text{ terms},$$

where

$$\begin{aligned}
(\mathrm{IA})' &= \|\Delta u\|_{L^2(\Omega_r^+)}\|\nabla u - a\|_{L^{p_1}(\Omega_r^+)} + |a|\|\Delta u\|_{L^2(\Omega_r^+)}\|u-c\|_{L^{p_1}(\Omega_r^+)},\\
(\mathrm{IB})' &= \|\nabla^2 u\|_{L^2(\Omega_r^+)}\|\nabla u - a\|_{L^{p_1}(\Omega_r^+)} + |a|\|\nabla^2 u\|_{L^2(\Omega_r^+)}\|u-c\|_{L^{p_1}(\Omega_r^+)},\\
(\mathrm{IIA})' &= \|\nabla u\|_{L^4(\Omega_r^+)}\|\nabla u - a\|_{L^{p_1}(\Omega_r^+)} + |a|\|\nabla u\|_{L^4(\Omega_r^+)}\|u-c\|_{L^{p_1}(\Omega_r^+)},\\
(\mathrm{IIB})' &= (\max_{\Omega_r^+}|u-c|)\|\nabla u - a\|_{L^{p_1}(\Omega_r^+)} + \|\nabla u\|^2_{L^4(\Omega_r^+)}\|\nabla u - a\|_{L^{p_1}(\Omega_r^+)},\\
(\mathrm{IV})' &= \|\nabla u\|^2_{L^4(\Omega_r^+)}\|\nabla u - a\|_{L^{p_1}(\Omega_r^+)} + |a|\|\nabla u\|^2_{L^4(\Omega_r^+)}\|u-c\|_{L^{p_1}(\Omega_r^+)},\\
(\mathrm{V})' &= (\mathrm{IB})' + (\mathrm{IIA})'\\
&\quad + \big(\|\nabla u\|^2_{L^4(\Omega_r^+)} + \|\nabla u\|_{L^4(\Omega_r^+)} + \|\nabla^2 u\|_{L^2(\Omega_r^+)}\big)\|\nabla u - a\|_{L^{p_1}(\Omega_r^+)}.
\end{aligned}$$

After grouping terms, it is easy to obtain the required estimate. □

Proof of Theorem 4.3. Suppose $1/2 < r_1 < 1$ and $0 < \tau < r_1/4$, but both τ and r_1 are otherwise to be chosen later. Define h_1 as in previous section. By the Sobolev inequality, we have

$$\begin{aligned}
&M'_{p_0}(u)(\Omega_\tau^+) + D'_{p_1}(u)(\Omega_\tau^+)\\
&\quad\lesssim \tau^{-4/p_1}\Big(\fint_{\Omega_{r_1}^+}|\nabla(u-h_1)|^{p_1}\Big)^{1/p_1} + M'_{p_0}(h_1)(\Omega_\tau^+) + D'_{p_1}(h_1)(\Omega_\tau^+)\\
&\quad\lesssim \tau^{-4/p_1}\big((E^2+E)(u)(\Omega_1^+) + \max_{\Omega_1^+}|u-u(0)|\big)\\
&\qquad\times\big(D'_{p_1}(u)(\Omega_1^+) + |\nabla u(0)|M_{p_1}(u)(\Omega_1^+)\big) + M'_{p_0}(h_1)(\Omega_\tau^+) + D'_{p_1}(h_1)(\Omega_\tau^+).
\end{aligned}$$

The last inequality follows from Lemma 4.1 by setting $c^\alpha = u^\alpha(0)$ and $a^\beta = \nabla u^\beta(0)$. Now we apply the above inequality to $u(\tau^{k-1}x)$ for $k = 2, 3, \ldots$, and then by a change of variable, we get

$$\begin{aligned} M'_{p_0}(u)(\Omega^+_{\tau^k}) + D'_{p_1}(u)(\Omega^+_{\tau^k}) \\ \lesssim \tau^{-4/p_0}\big((E^2+E)(u)(\Omega^+_1) + \max_{\Omega^+_{\tau^{k-1}}}|u-u(0)|\big) \\ \times\big(D'_{p_1}(u)(\Omega^+_{\tau^{k-1}}) + |\nabla u(0)| M_{p_1}(u)(\Omega^+_{\tau^{k-1}})\big) \\ + \frac{1}{\tau^{k-1}}\big(M'_{p_0}(h_k)(\Omega^+_\tau) + D'_{p_1}(h_k)(\Omega^+_\tau)\big), \end{aligned}$$

where h_k is defined on $\Omega^+_{r_k}$ in previous section and r_k is to be chosen later. Repeating the proof method of Theorem 4.2, we consider $\tilde{h}_k(x) = h_1(\tau^{k-1}x)$. Then we have

$$\begin{aligned} M'_{p_0}(h_k)(\Omega^+_\tau) + D'_{p_1}(h_k)(\Omega^+_\tau) \\ \le M'_{p_0}(\phi_k)(\Omega^+_\tau) + D'_{p_1}(\phi_k)(\Omega^+_\tau) + M'_{p_0}(\tilde{h}_1)(\Omega^+_\tau) + D'_{p_1}(\tilde{h}_1)(\Omega^+_\tau) \\ \le \tau \sup_{\Omega^+_\tau}|\nabla^2\phi_k| + \big(M'_{p_0}(h_1)(\Omega^+_{\tau^k}) + D'_{p_1}(h_1)(\Omega^+_{\tau^k})\big)\tau^{k-1}, \end{aligned}$$

where $\phi_k = h_k - \tilde{h}_k$. Again note that by Schauder theory, we know that ϕ_k is smooth on $\overline{\Omega^+}_\tau$, and so $\nabla^2\phi_k$ is well defined. As before, by a Green function argument, we have the estimate

$$\begin{aligned} \frac{1}{\tau^{k-1}}\sup_{\Omega^+_\tau}|\nabla^2\phi_k| \lesssim M'_{p_0}(u)(\Omega^+_{\tau^{k-1}}) + D'_{p_1}(u)(\Omega^+_{\tau^{k-1}}) \\ + M'_{p_0}(h_1)(\Omega^+_{\tau^{k-1}}) + D'_{p_1}(h_1)(\Omega^+_{\tau^{k-1}}). \end{aligned}$$

Combining these results, we get

$$\begin{aligned} M'_{p_0}(u)(\Omega^+_{\tau^k}) + D'_{p_1}(u)(\Omega^+_{\tau^k}) \\ \lesssim \frac{1}{\tau^{4/p_0}}\big((E^2+E)(u)(\Omega^+_1) + \max_{\Omega^+_{\tau^{k-1}}}|u-u(0)|\big) \\ \times\big(D'_{p_1}(u)(\Omega^+_{\tau^{k-1}}) + |\nabla u(0)| M_{p_1}(u)(\Omega^+_{\tau^{k-1}})\big) \\ + \tau\big(M'_{p_0}(u)(\Omega^+_{\tau^{k-1}}) + D'_{p_1}(u)(\Omega^+_{\tau^{k-1}}) + M'_{p_0}(h_1)(\Omega^+_{\tau^{k-1}}) \\ + D'_{p_1}(h_1)(\Omega^+_{\tau^{k-1}})\big) + M'_{p_0}(h_1)(\Omega^+_{\tau^k}) + D'_{p_1}(h_1)(\Omega^+_{\tau^k}). \end{aligned}$$

By Schauder theory, we know that $h_1 \in C^{1,\beta}(\overline{\Omega^+}_\tau)$, and we know by Theorem 4.2 that $u \in C^{0,\beta}(\overline{\Omega^+}_\tau)$. Therefore, we have

$$\begin{aligned} M'_{p_0}(h_1)(\Omega^+_{\tau^k}) + D'_{p_1}(h_1)(\Omega^+_{\tau^k}) \lesssim \tau^{\beta k}, \\ M_{p_1}(u)(\Omega^+_{\tau^{k-1}}) \lesssim \tau^{\beta(k-1)}, \\ \max_{\Omega^+_{\tau^{k-1}}}|u-u(0)| \lesssim \tau^{\beta(k-1)} \end{aligned}$$

for $k = 2, 3, \ldots$ and τ sufficiently small. With these estimates, we first choose $k_0 \in \mathbb{N}$ such that $(k_0 - 1)\beta - 4/p_0 \geq 1$, then choose τ small, and finally, for $E(u)(\Omega_1^+)$ sufficiently small, we get

$$M'_{p_0}(u)(\Omega^+_{\tau^k}) + D'_{p_1}(u)(\Omega^+_{\tau^k}) \leq \frac{\tau^\beta}{2}\big(M'_{p_0}(u)(\Omega^+_{\tau^{k-1}}) + D'_{p_1}(u)(\Omega^+_{\tau^{k-1}}) + C\tau^{(k-1)\beta}\big)$$

for some constant $C > 0$ independent of k and $k \geq k_0$. Then iteratively applying the above inequality we get

$$\begin{aligned}(10)\quad M'_{p_0}(u)(\Omega^+_{\tau^k}) + D'_{p_1}(u)(\Omega^+_{\tau^k}) &\leq \frac{\tau^{k\beta}}{2}\big(M'_{p_0}(u)(\Omega^+_{\tau^{k_0}}) + D'_{p_1}(u)(\Omega^+_{\tau^{k_0}})\\ &\quad + C + \frac{C}{2} + \frac{C}{2^2} + \cdots\big) \lesssim \tau^{k\beta} \quad \text{for } k \geq k_0.\end{aligned}$$

Again, as in the proof of Theorem 2.1, we can apply the argument to all $x \in \Gamma_s$ and obtain the estimate (10) for x. Then by a standard argument it can be shown that $u \in C^{1,\beta}(\mathcal{U}, S^n)$. □

4.3. *Proof of Theorem 4.1 for $l \geq 2$.* Again, by standard regularity theory, it suffices to prove the case $l = 2$. As in Section 3, we consider a larger class of elliptic systems. In this section, we will prove this:

Theorem 4.4. *Suppose $u \in C^{1,\beta}(\overline{\Omega^+}_1, S^n)$ is a weak solution on Ω_1^+ of the elliptic system*

$$\Delta^2 u^\alpha = f^\alpha(x, Du, D^2u) + \sum_{i=1}^4 \frac{\partial g_i^\alpha}{\partial x_i}(x, Du, D^2u)$$

with growth conditions

$$(11)\quad \begin{aligned}|f^\alpha(x, P, M)| &\leq \lambda_1(1 + |P|^4 + |M|^2),\\ |g_i^\alpha(x, P, M)| &\leq \lambda_2(1 + |P|^3 + |M|^{3/2})\end{aligned}$$

and Dirichlet boundary data satisfying (8) for $l = 2$. Then $u \in C^{2,\beta}(\mathcal{U}, S^n)$, where $\mathcal{U}$ is a neighborhood of Γ_s in Ω_1^+ for some $s \in (0, 1)$.

Since the Euler–Lagrange equation of the intrinsic biharmonic map u belongs to this class of elliptic system and, by the previous section, we already know that $u \in C^{1,\beta}(\overline{\Omega^+}_1)$, we see Theorem 4.4 implies Theorem 4.1.

As in Section 3, to show that $u \in C^{2,\beta}(\mathcal{U})$, it suffices to show that $u_1(x) = (u(rx) - u(0))/c(r, K)$ belongs to $C^{2,\beta}(\mathcal{U})$, where $c(r, K) = K(\|u - u(0)\|_{L^\infty(B_r)} + r)$ for some $K > 1$ and $r > 0$. Since u is continuous, $c(r, K)$ becomes arbitrarily small as $r \to 0$. Therefore, by a computation in Section 3, we know u_1 satisfies the same type of elliptic system

$$(12)\qquad \Delta^2 u^\alpha = \tilde{f}^\alpha(x, Du, D^2u) + \sum_{i=1}^4 \frac{\partial \tilde{g}_i^\alpha}{\partial x_i}(x, Du, D^2u) \quad \text{in } \Omega_1^+$$

with growth conditions

$$
\begin{aligned}
|\tilde{f}^\alpha(x, P, M)| &\le \tilde{\lambda}_1(1 + \mu_1|P|^4 + \mu_1|M|^2), \\
|\tilde{g}^\alpha(x, P, M)| &\le \tilde{\lambda}_2(1 + \mu_2|P|^3 + \mu_2|M|^{3/2})
\end{aligned}
\tag{13}
$$

where $\tilde{\lambda}_1 = c(r, K)^{1/2}\lambda_1$, $\mu_1 = c(r, K)^{1/2}$, $\tilde{\lambda}_2 = c(r)^{1/4}\lambda_2$, and $\mu_2 = c(r, K)^{1/4}$. So $\tilde{\lambda}_j$ and μ_j for $j = 1, 2$ can be made arbitrarily small as r goes to zero. Also note we can assume $|u|_{C^{1,\beta}(\overline{\Omega^+}_1)}$, $|u|_{C^{2,\beta}(\Gamma_1)}$, and $|\partial u/\partial n|_{C^{1,\beta}(\Gamma_1)}$ to be very small if we fix a large enough K.

To prove Theorem 4.4, we need a lemma.

Lemma 4.2. *Suppose v is a weak solution of Equation (12) with growth conditions (13) and Dirichlet boundary data satisfying (8) for $l = 2$. Also suppose*

$$
\begin{aligned}
&\mu_1(|v - h|_{L^\infty(\Omega^+_{r_0})}) \le \delta, \\
&\int_{\Omega_1^+} |D^2 v|^2 dx + \Big(\int_{\Omega_1^+} |Dv|^4 dx\Big)^{1/2} + \int_{\Omega_1^+} |v|^2 dx \le 1
\end{aligned}
\tag{14}
$$

for some $r_0 \in (0, 1]$ and $h : \Omega^+_{r_0} \to \mathbb{R}^K$ such that $\Delta^2 h = 0$ in $\Omega^+_{r_0}$ and such that $h = v$ and $\partial h/\partial n = \partial v/\partial n$ on $\partial\Omega^+_{r_0}$. Then for $0 < r < r_0$,

$$
r^4 \int_{B_r} |D^2(v - k)|^2 dx + r^4\Big(\int_{B_r} |D(v - k)|^4 dx\Big)^{1/2} + \int_{B_r} |v - k|^2 dx \lesssim \tilde{\lambda}_1^2 + \tilde{\lambda}_2^2 + \delta.
$$

The proof of Lemma 4.2 is similar to that of Lemma 3.1 and is therefore omitted.

Proof of Theorem 4.4. First let $w_0 = u$, $p_0 = 0$, and $h_0 = h$ where $h : \Omega^+_{r_0} \to \mathbb{R}^{K+1}$ is such that $\Delta^2 h = 0$ in $\Omega^+_{r_0}$ and such that $h = u$ and $\partial h/\partial n = \partial u/\partial n$ on $\partial\Omega^+_{r_0}$ for some $r_0 \in (0, 1)$, to be chosen later. Let $\tau \in (0, r_0)$ also to be chosen later. For $k \in \mathbb{N}$, we define

$$
w_k = \frac{(u - p_k)(\tau^k x)}{\tau^{(2+\beta)k}},
$$

where $p_k(x) = p_{k-1}(x) + \tau^{(2+\beta)k} q_{k-1}(x/\tau^k)$ for $q_{k-1}(x) = \frac{1}{2} x D^2 h_{k-1}(0) x + Dh_{k-1}(0)x + h_{k-1}(0)$ and $h_{k-1} : \Omega^+_{r_{k-1}} \to \mathbb{R}^{n+1}$ such that $\Delta^2 h_{k-1} = 0$ in $\Omega^+_{r_{k-1}}$ and $h_{k-1} = w_{k-1}$, $\partial h_{k-1}/\partial n = \partial w_{k-1}/\partial n$ on $\partial\Omega^+_{r_{k-1}}$ for some $r_{k-1} \in (r_0/4, r_0/2)$, also to be chosen later.

Notice that by definition $h_k(0) = 0$ and $Dh_k(0) = 0$ for all $k \in \mathbb{N}$. So $p_1(x) = \frac{1}{2} x D^2 h(0) x + Dh(0)x + h(0)$ and $p_k(x) = p_{k-1}(x) + \frac{1}{2}\tau^\beta x D^2 h_{k-1}(0)x$ for $k \ge 2$. Also, it can be shown that $x D^2 h_{k-1}(0)x = 0$ and $D(\partial h_{k-1}/\partial n)(0)x = 0$ for all $x \in \Gamma_{r_{k-1}}$ for $k \ge 2$.

To prove Theorem 4.4, it suffices to prove that

$$
\fint_{\Omega^+_{\tau^k}} |D^2(u - p_k)|^2 dx + \Big(\frac{1}{\tau^{4k}} \fint_{\Omega^+_{\tau^k}} |D(u - p_k)|^4 dx\Big)^{1/2} + \frac{1}{\tau^{4k}} \fint_{\Omega^+_{\tau^k}} |u - p_k|^2 dx \le \tau^{2\beta k}
$$

for all $k \in \mathbb{N}$ and $|A_k| + |B_k| + |C_k| \leq \mathscr{C}$ for some constant $\mathscr{C}$ independent of k, where $p_k(x) = \frac{1}{2} x A_k x + B_k x + C_k$.

We prove this claim by induction on k. First consider when $k = 1$. By the discussion at the beginning of this section, we may assume without loss of generality that

$$\int_{\Omega_1^+} |u|^2 dx + \Big(\int_{\Omega_1^+} |Du|^4 dx\Big)^{1/2} + \int_{\Omega_1^+} |D^2 u|^2 dx \leq 1,$$

$$8\mu_1 \|u - h_0\|_{L^\infty(\Omega_{r_0}^+)} \leq \delta,$$

$$|u|_{C^{1,\beta}(\overline{\Omega^+}_1)} + |u|_{C^{2,\beta}(\Gamma_1)} + \Big|\frac{\partial u}{\partial n}\Big|_{C^{1,\beta}(\Gamma_1)} \leq \delta'$$

for small δ and δ' to be chosen later. Then we have

$$\begin{aligned}
&\fint_{\Omega_\tau^+} |D^2(u - p_1)|^2 dx + \Big(1/\tau^4 \fint_{\Omega_\tau^+} |D(u - p_1)|^4 dx\Big)^{1/2} + \frac{1}{\tau^4} \fint_{\Omega_\tau^+} |u - p_1|^2 dx \\
&\quad \leq \fint_{\Omega_\tau^+} |D^2(u - h_0)|^2 dx + \Big(\frac{1}{\tau^4} \fint_{\Omega_\tau^+} |D(u - h_0)|^4 dx\Big)^{1/2} + \frac{1}{\tau^4} \fint_{\Omega_\tau^+} |u - h_0|^2 dx \\
&\qquad + \fint_{\Omega_\tau^+} |D^2(h_0 - q_0)|^2 dx + \Big(\frac{1}{\tau^4} \fint_{\Omega_\tau^+} |D(h_0 - q_0)|^4 dx\Big)^{1/2} \\
&\qquad + \frac{1}{\tau^4} \fint_{\Omega_\tau^+} |h_0 - q_0|^2 dx \\
&\quad \leq C_1 \tau^{-8} (\tilde{\lambda}_1^2 + \tilde{\lambda}_2^2 + \delta) + 3[h_0]^2_{C^{2,\beta}(\overline{\Omega^+}_\tau)} \tau^{2\beta} \quad \text{(by Lemma 4.2)} \\
&\quad \leq C_1 \tau^{-8} (\tilde{\lambda}_1^2 + \tilde{\lambda}_2^2 + \delta) + C_2 \delta' \tau^{2\beta}.
\end{aligned}$$

Let $\tilde{\lambda}_i$, δ, and δ' be small enough that

$$C_1 \tau^{-8} (\tilde{\lambda}_1^2 + \tilde{\lambda}_2^2 + \delta) \leq \frac{\tau^{2\beta}}{2} \quad \text{and} \quad C_2 \delta' \tau^{2\beta} \leq \frac{\tau^{2\beta}}{4}. \tag{15}$$

Therefore, the claim is true for $k = 1$. Now assume the claim is true for k. Similarly, we have

$$\begin{aligned}
&\fint_{\Omega_\tau^+} |D^2(w_k - q_k)|^2 dx + \Big(\frac{1}{\tau^4} \fint_{\Omega_\tau^+} |D(w_k - q_k)|^4 dx\Big)^{1/2} + \frac{1}{\tau^4} \fint_{\Omega_\tau^+} |w_k - q_k|^2 dx \\
&\qquad \leq \fint_{\Omega_\tau^+} |D^2(w_k - h_k)|^2 dx + \Big(\frac{1}{\tau^4} \fint_{\Omega_\tau^+} |D(w_k - h_k)|^4 dx\Big)^{1/2} \\
&\qquad\quad + \fint_{\Omega_\tau^+} |D^2(h_k - q_k)|^2 dx + \Big(\frac{1}{\tau^4} \fint_{\Omega_\tau^+} |D(h_k - q_k)|^4 dx\Big)^{1/2} \\
&\qquad\quad + \frac{1}{\tau^4} \fint_{\Omega_\tau^+} |w_k - h_k|^2 dx + \frac{1}{\tau^4} \fint_{\Omega_\tau^+} |h_k - q_k|^2 dx
\end{aligned}$$

$$\leq \tau^{-8}\Big[\tau^4 \int_{\Omega_\tau^+} |D^2(w_k - h_k)|^2 dx + \tau^4 \Big(\int_{\Omega_\tau^+} |D(w_k - h_k)|^4 dx\Big)^{1/2} + \fint_{\Omega_\tau^+} |D^2(h_k - q_k)|^2 dx + \Big(\frac{1}{\tau^4} \fint_{\Omega_\tau^+} |D(h_k - q_k)|^4 dx\Big)^{1/2} + \int_{\Omega_\tau^+} |w_k - h_k|^2 dx\Big] + \frac{1}{\tau^4} \fint_{\Omega_\tau^+} |h_k - q_k|^2 dx.$$

To use Lemma 4.2, we must verify conditions (14) for w_k. First, by the induction hypothesis, we have

$$\int_{\Omega_1^+} |D^2 w_k|^2 dx + \Big(\int_{\Omega_1^+} |D w_k|^4 dx\Big)^{1/2} + \int_{\Omega_1^+} |w_k|^2 dx \leq 1.$$

Therefore, the second condition of (14) is satisfied. Second, by the computation in Section 3, the first condition of (14) for w_k becomes

$$8\mu_1 \tau^{(2+\beta)k} \|w_k - h_k\|_{L^\infty(\Omega_{r_k}^+)} \leq \delta.$$

It is easy to see that for $k \geq 0$,

$$h_k(x) = (\hat{h}_k(x) - p_k(\tau^k x))/\tau^{(2+\beta)k},$$

where $\hat{h}_k : \Omega_{r_k}^+ \to \mathbb{R}^{n+1}$ is such that $\Delta^2 \hat{h}_k = 0$ on $\Omega_{r_k}^+$ and such that $\hat{h}_k = u(\tau^k x)$ and $\partial \hat{h}_k / \partial n = \partial(u(\tau^k x))/\partial n$ on $\partial\Omega_{r_k}^+$. So the condition is equivalent to

$$8\mu_1 \|u(\tau^k x) - \hat{h}_k(x)\|_{L^\infty(\Omega_{r_k}^+)} \leq \delta.$$

By definition, $\|u\|_{L^\infty(\Omega_{r_k}^+)} \leq 1$ for any k. By the Schauder estimates, we have

$$\|\hat{h}_k(x)\|_{L^\infty(\Omega_{r_k}^+)} \lesssim |u|_{C^{1,\beta}(\overline{\Omega^+}_1)} \lesssim \delta'.$$

So by an initial choice of small μ_1 and δ', condition 4.3 is satisfied. Now we can apply Lemma 4.2 for w_k and get

$$\fint_{\Omega_\tau^+} |D^2(w_k - q_k)|^2 dx + \Big(\frac{1}{\tau^4} \fint_{\Omega_\tau^+} |D(w_k - q_k)|^4 dx\Big)^{1/2} + \frac{1}{\tau^4} \fint_{\Omega_\tau^+} |w_k - q_k|^2 dx \leq C_1 \tau^{-8}(\tilde{\lambda}_1^2 + \tilde{\lambda}_2^2 + \delta) + \fint_{\Omega_\tau^+} |D^2(h_k - q_k)|^2 dx + \Big(\frac{1}{\tau^4} \fint_{\Omega_\tau^+} |D(h_k - q_k)|^4 dx\Big)^{1/2} + \frac{1}{\tau^4} \fint_{\Omega_\tau^+} |h_k - q_k|^2 dx$$

$$
\begin{aligned}
&\leq \frac{\tau^{2\beta}}{2} + \fint_{\Omega_\tau^+} |D^2(h_k - \tilde{h}_k - q_k)|^2 dx \\
&\quad + \Big(\frac{1}{\tau^4} \fint_{\Omega_\tau^+} |D(h_k - \tilde{h}_k - q_k)|^4 dx\Big)^{1/2} + \frac{1}{\tau^4} \fint_{\Omega_\tau^+} |h_k - \tilde{h}_k - q_k|^2 dx \\
&\quad + \fint_{\Omega_\tau^+} |D^2\tilde{h}_k|^2 dx + \Big(\frac{1}{\tau^4} \fint_{\Omega_\tau^+} |D\tilde{h}_k|^4 dx\Big)^{1/2} + \frac{1}{\tau^4} \fint_{\Omega_\tau^+} |\tilde{h}_k|^2 dx,
\end{aligned}
$$

where $\tilde{h}_k(x) = ((h_0 - q_0)(\tau^k x))/\tau^{(2+\beta)k}$. Define $\phi_k = h_k - \tilde{h}_k$. Note that $\Delta^2 \phi_k = 0$ in $\Omega_{r_k}^+$ and $\phi_k = \partial\phi_k/\partial n = 0$ on Γ_{r_k}. Therefore by Schauder theory, ϕ_k is smooth on $\overline{\Omega^+}_\tau$, and so we have

$$
\begin{aligned}
&\fint_{\Omega_\tau^+} |D^2(w_k - q_k)|^2 dx + \frac{1}{\tau^4} \fint_{\Omega_\tau^+} |w_k - q_k|^2 dx + \Big(\frac{1}{\tau^4} \fint_{\Omega_\tau^+} |D(w_k - q_k)|^4 dx\Big)^{1/2} \\
&\qquad \leq C_1 \tau^{-8}(\tilde{\lambda}_1^2 + \tilde{\lambda}_2^2 + \delta) + 3[h_0]^2_{C^{2,\beta}(\overline{\Omega^+}_\tau)} \tau^{2\beta} + \tau^2 \sup_{\overline{\Omega^+}_\tau} |D^3\phi_k|^2 \\
&\qquad \leq \frac{\tau^{2\beta}}{2} + \frac{\tau^{2\beta}}{4} + \tau^2 \sup_{\overline{\Omega^+}_\tau} |D^3\phi_k|^2.
\end{aligned}
$$

The first and third term of the last inequality follow from Equation (15).

As before, we can estimate $|D^3\phi_k|^2$ as follows:

$$
\begin{aligned}
&\sup_{\overline{\Omega^+}_\tau} |D^3\phi_k|^2 \leq C_3 \Big(\int_{\partial\Omega_{r_k}^+ \setminus \Gamma_{r_k}} |\phi_k|^2 d\sigma + \Big(\int_{\partial\Omega_{r_k}^+ \setminus \Gamma_{r_k}} |\frac{\partial \phi_k}{\partial n}|^4 d\sigma\Big)^{1/2}\Big) \\
&\quad \leq C_4 \Big(\int_{\Omega_1^+} |w_k|^2 dx + \Big(\int_{\Omega_1^+} |Dw_k|^4 dx\Big)^{1/2} + \int_{\Omega_1^+} |\tilde{h}_k|^2 dx + \Big(\int_{\Omega_1^+} |D\tilde{h}_k|^4 dx\Big)^{1/2}\Big) \\
&\quad \leq C_4 \big(1 + 2[h_0]^2_{C^{2,\beta}(\overline{\Omega^+}_\tau)}\big) \\
&\quad \leq C_4(1 + 2C_2\delta').
\end{aligned}
$$

Then by an initial choice of small τ, we can assume that $\tau^2 C_4(1 + 2C_2\delta') \leq \tau^{2\beta}/4$. Therefore we get

$$
\fint_{\Omega_\tau^+} |D^2(w_k - q_k)|^2 dx + \Big(\frac{1}{\tau^4} \fint_{\Omega_\tau^+} |D(w_k - q_k)|^4 dx\Big)^{1/2} + \frac{1}{\tau^4} \fint_{\Omega_\tau^+} |w_k - q_k|^2 dx \leq \tau^{2\beta}
$$

By change of variable, we get

$$
\begin{aligned}
&\fint_{\Omega_{\tau^{k+1}}^+} |D^2(u - p_{k+1})|^2 dx + \Big(\frac{1}{\tau^{4(k+1)}} \fint_{\Omega_{\tau^{k+1}}^+} |D(u - p_{k+1})|^4 dx\Big)^{1/2} \\
&\qquad\qquad + \frac{1}{\tau^{4(k+1)}} \fint_{\Omega_{\tau^{k+1}}^+} |u - p_{k+1}|^2 dx \leq \tau^{2\beta(k+1)}.
\end{aligned}
$$

This finishes the proof for $k+1$. Finally, we need to show that $|A_k|+|B_k|+|C_k|$ has a bound that is independent of k. Note that $C_k = u(0)$ and $B_k = Du(0)$ for all k. So it suffices to consider A_k. First, we know that

$$|D^2 h_k(0)|^2 = |D^2\phi_k(0)|^2 \le C_4\biggl(\int_{\partial\Omega^+_{r_k}\setminus\Gamma_{r_k}} |\phi_k|^2 d\sigma + \Bigl(\int_{\partial\Omega^+_{r_k}\setminus\Gamma_{r_k}} \Bigl|\frac{\partial\phi_k}{\partial n}\Bigr|^4 d\sigma\Bigr)^{1/2}\biggr),$$

which is less than or equal to $C_4(1+2C_2\delta')$. So $|D^2 h_k(0)| \le C_5$ for some constant C_5 independent of k. The desired k-independence then follows by definition:

$$|A_k| = |D^2 h_0(0)| + \tau^\beta |D^2 h_1(0)| + \tau^{2\beta}|D^2 h_2(0)| + \cdots \le |D^2 h_0(0)| + \frac{C_5\tau^\beta}{1-\tau^\beta}. \qquad \square$$

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