CMC CAPILLARY SURFACES AT REENTRANT CORNERS

MARIA ATHANASSENAS AND KIRK LANCASTER

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For a capillary graph in a vertical cylinder $\Omega \times \mathbb{R} \subset \mathbb{R}^3$, the existence of a reentrant corner $P \in \partial\Omega$ makes the determination of the continuity at P (or the behavior of the radial limits at P) of the solution problematic. Since continuity is the necessary consequence of the existence of a "central fan" of radial limits under certain conditions, the determination of necessary and sufficient conditions for the existence of a central fan is a very important open question in the mathematical theory of capillarity. Examples by Finn and Shi suggest that "central fans" may be very rare in the sense that arbitrarily small perturbations can eliminate them. In this note we obtain examples of capillary graphs (with zero mean curvature), each of which is continuous or has a central fan at a reentrant corner.

1. Introduction

The nonparametric capillary problem in a cylinder $\Omega \times \mathbb{R}$ over a piecewise smooth domain Ω with corners at points $A_1, \ldots, A_n \in \partial\Omega$ and gravitational constant κ consists of finding the solution(s) $h \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus \{A_1, \ldots, A_n\})$ of the Dirichlet problem

(1a) $$\operatorname{div}(Th) = \kappa h + \lambda \quad \text{in } \Omega ,$$

(1b) $$Th \cdot \nu = \cos(\gamma) \quad \text{on } \partial\Omega \setminus \{A_1, \ldots, A_n\},$$

where ν is the outer unit normal to $\partial\Omega$ and

$$Th = \frac{\nabla h}{\sqrt{1 + |\nabla h|^2}}.$$

An extremely interesting and important question about nonparametric capillary surfaces is that of determining necessary and sufficient conditions for the continuity at a corner P of a solution h of (1). As a solution of a boundary value problem for a quasilinear elliptic equation with positive genre ($g = 2$), the possibility of obtaining such conditions should seem remote; solutions of equations like (1a) can behave near a point on the boundary in significantly different ways than can solutions of

equations such as Laplace's equation or a Poisson equation, as illustrated by [Jin and Lancaster 1999] and [Serrin 1969].

Let us assume Ω is a domain in $\mathbb{R}^2$ whose boundary has a corner of size 2α at a point, which we may temporarily take to be the origin $O = (0, 0)$ for convenience; we may assume the domain is oriented so that the rays $\theta = \alpha$ and $\theta = -\alpha$ are tangent to $\partial\Omega$ at O and so that the directions $\theta \in (-\alpha, \alpha)$ are the interior directions to Ω from O, where (r, θ) denotes polar coordinates about O. We may parametrize $\partial\Omega$ near O by $(x(s), y(s))$ so that $(x(0), y(0)) = (0, 0)$ and $y(s) > 0$ if $s > 0$ but $y(s) < 0$ if $s < 0$. Let us also suppose that the contact angle $\gamma = \gamma(x, y)$ might depend on $(x, y) \in \partial\Omega$ and assume the limits

$$(2) \qquad \gamma_1 = \lim_{s \downarrow 0} \gamma(x(s), y(s)) \quad \text{and} \quad \gamma_2 = \lim_{s \uparrow 0} \gamma(x(s), y(s))$$

both exist.

The study of capillary surfaces in wedge domains can be tricky, as illustrated by [Keller et al. 1991] and [Vreeburg 1990]. Concus and Finn [1991] mentioned that [Vreeburg 1990] was incorrect; we note that its argument is invalid since it does not distinguish between mean curvature and Gauss curvature and since it ignores [Tam 1986]. Vreeburg fails to consider the possibility of a jump discontinuity in the trace of the capillary surface on the boundary cylinder; see Section 2 and [Lancaster and Siegel 1996b]. Concus and Finn [1996b] noted that [Keller et al. 1991] was incorrect. Keller, King and Merchant [1991] argued that the only ruled minimal surfaces in $\mathbb{R}^3$ were planes, ignoring helicoids, and seemed to be confused about the notion of principal curvature. As with Vreeburg, they failed to consider the possibility of bounded, discontinuous (at the corner) capillary surfaces when $\gamma_1 \neq \gamma_2$. Concus and Finn did consider this important possibility; as part of their investigation of these capillary surfaces, they discussed the question of continuity at a corner and in 1992 formulated the following conjecture for convex corners; see for example [Shi 2006].

Concus–Finn Conjecture. *Suppose $\alpha \in (0, \pi/2)$, $\gamma_1, \gamma_2 \in (0, \pi)$, and $|\gamma_1 - \gamma_2| > \pi - 2\alpha$. Then a solution of* (1) *must be discontinuous at O.*

Concus and Finn [1996a] and Lancaster and Siegel [1996b; 1996a] studied the behavior near O of capillary surfaces over domains with a corner at O whose contact angles might have differing limits at O; see also [Concus and Finn 1994; Concus et al. 1992].

For convex corners ($2\alpha \leq \pi$), the continuity or discontinuity of a solution of (1) at such a corner is completely determined by "local" geometric information, that is, γ_1, γ_2, and α. At nonconvex corners ($2\alpha > \pi$), the task of finding necessary and sufficient conditions for continuity is much more difficult, and local information (that is, γ_1, γ_2 and α) is not necessarily sufficient to determine continuity.

Our first goal is to provide an overview of results related to the continuity of a solution h of (1) and to introduce a problem, the "central fan question", which we consider to be the most important open problem in the mathematical theory of capillary surfaces; we do this in Section 2. Our second goal is to demonstrate that for a range of piecewise constant contact angle data γ, there exists a domain Ω with a reentrant corner at P such that *either the solution h of the capillary problem with $\kappa = 0$ is continuous at P or the radial limits of h at P exhibit a central fan.*

Our examples complement [Shi 2006, Example 6.6], which is a slight modification of [Lancaster and Siegel 1996b, Example 2]. One motivation for seeking examples containing central fans (or continuity) is the insight into the central fan question these examples might provide. The cited examples use comparison arguments which require $\kappa > 0$; in contrast, our examples illustrate the limiting case of $\kappa = 0$. Our solutions are minimal surfaces satisfying contact angle boundary conditions. Such solutions have qualitatively different behavior than solutions of (1a) with $\kappa > 0$, and, for example, the a priori height bounds, which play a critical role in the cited examples, do not hold when $\kappa = 0$. Our examples are obtained by solving Riemann–Hilbert problems, representing minimal surfaces in terms of Weierstrass f and g representations, and proving that these surfaces behave as claimed in Theorems 3.1, 3.2, 7.1, and 7.2. In principle, our solutions are constructive; computing one of our surfaces would require computing a conformal map and a certain number of complex contour integrals.

2. Radial limits, central fans and continuity at corners

Consider $\Omega \subset \mathbb{R}^2$ whose boundary has a corner of size 2α at the origin $O = (0, 0)$ as described above, and assume the limits in (2) both exist.

Definition. For a function h with domain $\Omega \subset \mathbb{R}^2$ and $P \in \partial\Omega$, the *radial limit* of h at P in the direction θ is

$$Rh(\theta) = Rh(\theta; P) = \lim_{r \to 0^+} h(P + r\omega(\theta)), \quad \text{where } \omega(\theta) = (\cos(\theta), \sin(\theta));$$

here θ is an angle for which $P + r\omega(\theta) \in \Omega$ when $r > 0$ is small. (The existence of each limit above follows from [Lancaster and Siegel 1996b, Theorem 1] when h satisfies (1), γ is piecewise continuous with $\gamma \in [\epsilon, \pi - \epsilon]$ for some $\epsilon > 0$, and $|\gamma_1 + \gamma_2 - \pi| < 2\alpha$.)

2.1. *Convex corners.* Lancaster and Siegel [1996b, Corollary 4] proved that a solution h of (1) must be continuous at O if $\alpha \leq \frac{\pi}{2}$, $|\gamma_1 - \gamma_2| \leq \pi - 2\alpha$, and $|\gamma_1 + \gamma_2 - \pi| < 2\alpha$. The proof follows from lower bounds on the sizes of "fans" of constant radial limits obtained in [Lancaster and Siegel 1996b, Theorem 2]. These bounds implied the "fans" touch or overlap and hence Rh is a constant

function on $[-\alpha, \alpha]$, where $Rh(\pm\alpha)$ are defined as the one-sided limits at O of the trace of h. These limits are proven to exist as part of the proof of [Theorem 1]; if h is discontinuous at O, [Theorem 1] implies Rh is not constant. When $|\gamma_1 + \gamma_2 - \pi| > 2\alpha$, we know that no solution of (1) can remain bounded near O; see [Concus and Finn 1974b; Concus and Finn 1974a]. When $|\gamma_1 + \gamma_2 - \pi| = 2\alpha$, a variation of Tam's argument [1986] should establish the continuity of a solution at P. The Concus–Finn conjecture was recently proven by Lancaster [≥ 2008]; hence h is discontinuous at the corner if $|\gamma_1 - \gamma_2| > \pi - 2\alpha$. Thus, assuming the successful application of Tam's arguments, the continuity or discontinuity of a solution of (1) at a convex corner is completely determined by "local" geometric information, that is, γ_1, γ_2, and α. When $\alpha \leq \pi/2$, h is continuous at O if and only if $|\gamma_1 + \gamma_2 - \pi| \leq 2\alpha$ and $|\gamma_1 - \gamma_2| \leq \pi - 2\alpha$.

2.2. Nonconvex corners. We now discuss the results of [Lancaster and Siegel 1996b]. The conclusion of [Theorem 1] is that the radial limits of h behave in one of the following ways:

(i) *There exist α_1 and α_2 such that $-\alpha \leq \alpha_1 < \alpha_2 \leq \alpha$ and such that Rh is constant on $[-\alpha, \alpha_1]$ and $[\alpha_2, \alpha]$ and strictly increasing or strictly decreasing on $[\alpha_1, \alpha_2]$. Label these case* (I) *and case* (D), *respectively.*

(ii) *There exist $\alpha_1, \alpha_L, \alpha_R, \alpha_2$ such that $-\alpha \leq \alpha_1 < \alpha_L < \alpha_R < \alpha_2 \leq \alpha$, where $\alpha_R = \alpha_L + \pi$ and such that Rh is constant on $[-\alpha, \alpha_1]$, $[\alpha_L, \alpha_R]$, and $[\alpha_2, \alpha]$ and either strictly increasing on $[\alpha_1, \alpha_L]$ and strictly decreasing on $[\alpha_R, \alpha_2]$ or strictly decreasing on $[\alpha_1, \alpha_L]$ and strictly increasing on $[\alpha_R, \alpha_2]$. Label these case* (ID) *and case* (DI), *respectively.*

We note that the language in (ii) is a little ambiguous since the word "strictly" is not present; an examination of [Parts 2 and 3] of the proof of [Theorem 1] demonstrates that the statement above is correct and the size of the "central fan" $[\alpha_L, \alpha_R]$, in directions for which Rh is constant, is exactly π. When $\kappa = \lambda = 0$ in (1a), that the size of the central fan is exactly π was obtained in [Lancaster 1988, Theorem 2]. The conclusion of [Lancaster and Siegel 1996b, Theorem 2] is that there exist lower bounds on the sizes of the "side fans" $[-\alpha, \alpha_1]$ and $[\alpha_2, \alpha]$ depending on which of the cases (I), (D), (ID), or (DI) holds; these bounds are

- $\alpha_1 + \alpha \geq \gamma_2$ for (D) and (DI);
- $\alpha_1 + \alpha \geq \pi - \gamma_2$ for (I) and (ID);
- $\alpha - \alpha_2 \geq \gamma_1$ for (I) and (DI);
- $\alpha - \alpha_2 \geq \pi - \gamma_1$ for (D) and (ID).

Central fans will be of particular interest to us. [Lancaster and Siegel 1996b, Example 2] is a nonparametric capillary surface, perhaps the first in which a central

fan can be seen to exist. Shi [2006] assumes the Concus–Finn conjecture holds and proves that if $\kappa > 0$ and $\alpha > \pi/2$ and either $|\gamma_1 - \gamma_2| > 2\alpha - \pi$ or $|\gamma_1 + \gamma_2 - \pi| > 2\pi - 2\alpha$, then a solution h of (1) must be discontinuous at O. Since the Concus–Finn conjecture has been proven, this means that the question of continuity at a nonconvex corner reduces to the situation where the size 2α of the opening angle and the limiting contact angles γ_1 and γ_2 are related by

(3) $$|\gamma_1 - \gamma_2| \leq 2\alpha - \pi \quad \text{and} \quad |\gamma_1 + \gamma_2 - \pi| \leq 2\pi - 2\alpha.$$

When $|\gamma_1 + \gamma_2 - \pi| \leq 2\pi - 2\alpha$, we claim that the only possibilities are that h is discontinuous at the corner O and (i) holds at O or that h is continuous at O. To see this, suppose h is discontinuous at O and either case (ID) or (DI) holds. If (ID) holds, then $\alpha - \alpha_2 \geq \pi - \gamma_1$ and $\alpha_1 - (-\alpha) \geq \pi - \gamma_2$, and (using the fact that central fans are of size π) the sum of the sizes of the three fans is at least $\pi - \gamma_1 + \pi + \pi - \gamma_2$. If (DI) holds, then $\alpha - \alpha_2 \geq \gamma_1$ and $\alpha_1 - (-\alpha) \geq \gamma_2$, and the sum of the sizes of the three fans is at least $\gamma_1 + \pi + \gamma_2$. Now $|\gamma_1 + \gamma_2 - \pi| \leq 2\pi - 2\alpha$ implies $2\alpha \leq \pi + \gamma_1 + \gamma_2$ and $2\alpha \leq 3\pi - \gamma_1 - \gamma_2$, and these imply that the sum of the sizes of the fans is at least 2α; since Rh is constant on each fan, Rh must be constant, and our assumption that h is discontinuous at O is false. We see therefore that a *necessary and sufficient condition* for the continuity of h at O is that $|\gamma_1 + \gamma_2 - \pi| \leq 2\pi - 2\alpha$ and the radial limits of h cannot be monotonic, that is, [Lancaster and Siegel 1996b, Theorem 1(i)] cannot hold. Consider the following:

Definition (symmetry condition). We say that a capillary problem (1) satisfies an "(even) symmetry condition at a point $P \in \partial\Omega$" if there is a rigid motion $L : \mathbb{R}^2 \to \mathbb{R}^2$ such that $L(P) = O$, $L(\Omega)$ is symmetric with respect to the x-axis and $\gamma \circ L^{-1}(x, -y) = \gamma \circ L^{-1}(x, y)$ for all $(x, y) \in L(\partial\Omega)$.

[Lancaster and Siegel 1996b, Corollary 2] is the first result which provides sufficient conditions for the existence of a central fan; it implies this: *Suppose $\kappa \geq 0$, Ω has a reentrant (that is, nonconvex) corner at $P \in \partial\Omega$, (1) satisfies the symmetry condition at P, and h satisfies (1). Then $h^* = h \circ L^{-1}$ is even in y and $Rh^*(\theta; O)$ is constant for $\theta \in [-\pi/2, \pi/2]$.*

Shi and Finn [2004] examined [Lancaster and Siegel 1996b, Example 2] and modified its construction to obtain a $\kappa > 0$ capillary surface in a nonsymmetric domain; their solution satisfied the angle requirements of [Lancaster and Siegel 1996b, Corollary 2], that is, (3) with $\gamma_1 = \gamma_2$, but was discontinuous at O. The Shi–Finn construction made, in a sense they explained, an arbitrarily small perturbation in the domain which left a neighborhood of O unaltered.

We define the *central fan question* to be the problem of finding necessary and sufficient conditions for excluding [Lancaster and Siegel 1996b, Theorem 1(i)], that is, case (I) and case (D), at a reentrant corner; if cases (I) and (D) are excluded,

then one of case (ID), case (DI), or Rh is constant (and h is continuous at the corner) must hold, and so there exists a central fan of constant radial limits (of size π in cases (ID) and (DI) or of size 2α otherwise) at the corner. We consider this question to be the most important open problem in the mathematical theory of capillarity for these reasons:

 (i) The (unresolved aspect of the) *continuity question* — that is, "Is a solution h of (1) continuous or discontinuous at a corner?" — is a special case of the central fan question.

 (ii) Any sufficient condition for the existence of central fans will probably be "unstable", as illustrated in [Shi and Finn 2004].

(iii) It seems likely that this question will be very challenging and may require the development of new mathematical techniques or the modification of existing techniques.

(iv) The central fan question has inherent geometric and analytic appeal. The question was first raised by Finn in [1998], and [Shi and Finn 2004] is the first investigation of this question; see also [Finn 1999; Finn 2002].

 (v) A solution of this problem offers the possibility of obtaining insight about boundary value problems for general quasilinear elliptic partial differential equations.

3. Construction of the domains Ω and main theorems

We will construct an appropriate "capillary graph" (for example, [Huff and Mc-Cuan 2006]) using a Weierstrass representation of minimal surfaces. We assume the domain Ω is symmetric about the coordinate axes and starshaped as indicated in Figure 1. We assume $\cos(\gamma(x, y))$ is an odd function of x and y. In each quadrant, $\partial\Omega$ has a reentrant corner and our principal interest is in the behavior of the solution at these corners. We note that the symmetry condition does not hold at P (nor at Q, R, and S) if $\gamma_1 \neq \gamma_2$; specifically, h will not be symmetric with respect to the line through P that bisects angle APB in Figure 1 if $\gamma_1 \neq \gamma_2$. We will restrict our attention to Ω_0, which is the quarter of the domain in the second quadrant and require the Gauss map of the surface $z = h$ over Ω_0 to be injective. This requirement that the Gauss map be injective is critical for our construction using the Weierstrass (f, g)-representation of minimal surfaces; however, minimal surfaces with injective Gauss maps over nonconvex domains are in some sense rare and this requirement will "pick out" (depending on γ) the particular domain Ω in which we will work.

Specifically, to create our domain, we let $\delta \in (0, \pi/4)$ and, for $B = (x_0, 0)$ with $x_0 \in (-\cot(\delta), -\tan(\delta))$, consider the domain $\Omega(B) \subset \mathbb{R}^2$ illustrated in Figure 1
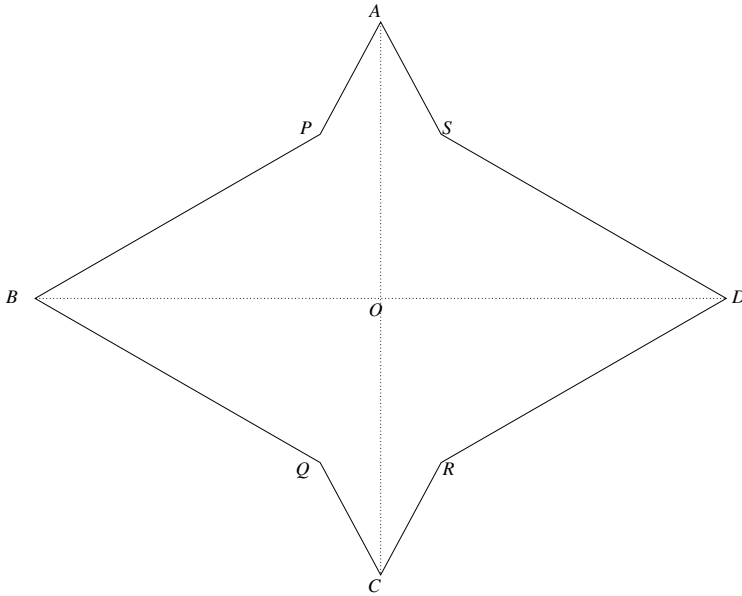
Figure 1. The domain Ω.

whose boundary is a polygon with eight sides that is symmetric with respect to the horizontal and vertical axes and whose two sides in the second quadrant have vertices at $(0, 1)$, P, and B, where $P = P(x_0)$ is the point of intersection of the lines $y = \cot(\delta)x + 1$ and $y = \tan(\delta)(x - x_0)$. We have labeled the vertices of $\partial\Omega$, starting at $(0, 1)$ and moving in a counterclockwise direction, as A, P, B, Q, C, R, D, and S; we labeled the origin as O. The convex angles OAP and OBP have measure δ, and the nonconvex angle APB has measure $3\pi/2 - 2\delta$. Let $\Omega_0(B)$ be the (open) region whose boundary is the nonconvex quadrilateral $APBO$.

Consider $\Omega = \Omega(B)$ as the cross section of a vertical cylinder Λ. We need to specify our contact angle boundary condition on Λ. Let $\gamma_1, \gamma_2 \in (0, \pi/2)$ satisfy the condition

$$(4) \qquad \tan(\delta) < \frac{\cos(\gamma_2)}{\cos(\gamma_1)} < \cot(\delta)$$

and define the function $\gamma : \partial\Omega \to \mathbb{R}$ by

$$\gamma(x, y) = \begin{cases} \gamma_1 & \text{if } (x, y) \in AP \cup CR, \\ \gamma_2 & \text{if } (x, y) \in PB \cup RD, \\ \pi - \gamma_2 & \text{if } (x, y) \in BQ \cup DS, \\ \pi - \gamma_1 & \text{if } (x, y) \in QC \cup SA. \end{cases}$$

Consider the capillary problem (1) in the cylinder Λ with contact angle boundary data γ, gravitational constant $\kappa \geq 0$, and Lagrange multiplier λ. We are interested in the behavior of the solution in zero gravity. In this case, $\kappa = 0$ and the divergence theorem together with (1) implies

$$\lambda|\Omega| \; = \; \int_\Omega \mathrm{div}(Th)\, dA \; = \; \int_{\partial\Omega} \cos(\gamma)\, ds.$$

Since $\cos(\gamma(x, y))$ is an odd function of x and of y, we see that $\lambda = 0$. This means the solution h will be a minimal surface.

First we will consider pairs (γ_1, γ_2) of contact angles that satisfy the additional restrictions $\gamma_1 \geq \delta$ and $\gamma_2 \geq \delta$. It turns out that the simplest case to state, Theorem 3.1, is the most difficult to prove. In Section 7, we will discuss the situation in which one or both of the assumptions $\gamma_1 \geq \delta$ and $\gamma_2 \geq \delta$ is removed. This results in a number of cases, illustrated in Figures 5 and 6, which affect the geometry of the stereographic projection of the Gauss map; however each case follows rather easily from the proofs of Theorems 3.1 and 3.2.

Theorem 3.1. *Let $\delta \in (0, \pi/4)$ and $\gamma_1, \gamma_2 \in (0, \pi/2)$ satisfy (4),*

$$\gamma_1 + \gamma_2 \geq \pi/2 - 2\delta, \qquad \gamma_1 \geq \delta, \quad \text{and} \quad \gamma_2 \geq \delta.$$

There exists a unique $x_0 \in (-\cot(\delta), -\tan(\delta))$ such that if $B = (x_0, 0)$, $\Omega = \Omega(B)$, and $\Omega_0 = \Omega_0(B)$, then the boundary value problem (1) with $\kappa = \lambda = 0$ and γ as defined above has a (unique) solution $h \in C^2(\Omega) \cap C^0(\overline{\Omega})$ which satisfies

  (i) $h(0, 0) = 0$ *and*

 (ii) *the Gauss map $\tilde{G}$ of the surface $\Gamma_0 = \{(x, y, h(x, y)) \; : \; (x, y) \in \Omega_0\}$ defined by*

$$(5) \qquad \tilde{G}(x, y, h(x, y)) = \frac{(\nabla h(x, y), -1)}{\sqrt{1 + |\nabla h(x, y)|^2}} \quad \text{for } (x, y) \in \Omega_0,$$

    extends to an injective map $\tilde{G} \in C^0(\overline{\Gamma_0} : \mathbf{S}^2)$.

In addition, if $\gamma_1 + \gamma_2 > \pi/2 - 2\delta$, $\gamma_1 > \delta$ and $\gamma_2 > \delta$, then $h \in C^1(\overline{\Omega})$.

Theorem 3.2. *Let $\delta \in (0, \pi/4)$ and $\gamma_1, \gamma_2 \in (0, \pi/2)$ satisfy (4),*

$$\gamma_1 + \gamma_2 < \pi/2 - 2\delta, \qquad \gamma_1 \geq \delta, \quad \text{and} \quad \gamma_2 \geq \delta.$$

For each $\theta_1 \in [\delta + \gamma_2 - \pi, -\pi/2 - \delta - \gamma_1]$, there exists a unique $x_0 = x_0(\theta_1) \in (-\cot(\delta), -\tan(\delta))$ such that if $B = (x_0, 0)$, $\Omega = \Omega(B)$, and $\Omega_0 = \Omega_0(B)$, then the boundary value problem (1) with $\kappa = \lambda = 0$ and γ as defined above has a (unique) solution $h \in C^2(\Omega) \cap C^0(\overline{\Omega} \setminus \{P, Q, R, S\})$ which satisfies

  (i) $h(0, 0) = 0$ *and*

(ii) *the Gauss map $\tilde{G}$ over Ω_0, given in (5), extends to an injective map $\tilde{G} \in C^0(\overline{\Gamma_0} : \mathbf{S}^2)$, where $\Gamma_0 = \{(x, y, h(x, y)) : (x, y) \in \Omega_0\}$.*

This solution is discontinuous at P, Q, R, and S and

$$Rh(\theta; P) = \begin{cases} \text{constant} & \text{if } \theta \in [\delta - \pi, \delta + \gamma_2 - \pi], \\ \text{strictly decreasing} & \text{if } \theta \in [\delta + \gamma_2 - \pi, \theta_1], \\ \text{constant} & \text{if } \theta \in [\theta_1, \theta_1 + \pi], \\ \text{strictly increasing} & \text{if } \theta \in [\theta_1 + \pi, \pi/2 - \delta - \gamma_1], \\ \text{constant} & \text{if } \theta \in [\pi/2 - \delta - \gamma_1, \pi/2 - \delta]. \end{cases}$$

$Rh(\theta; Q)$, $Rh(\theta; R)$, and $Rh(\theta; S)$ behave in appropriately similar manners. In particular, θ_1 determines the position of the central fan and, through the determination of the point B, the geometry of the domain.

4. Proof preliminaries

We are interested in the stereographic projection E of the image of the Gauss map of our solution over $\overline{\Omega_0(B)}$ for an appropriate B. In this section, we assume $\gamma_i \geq \delta$ for $i = 1, 2$. To determine E, let $B_1 = \{w \in \mathbb{C} : |w| < 1\}$, $Q_1 = \{w \in B_1 : \text{Re}(w) < 0, \text{Im}(w) > 0\}$ and set

$$E_0 = \{w \in Q_1 : |w - w_1| > \tan(\gamma_1), |w - w_2| > \tan(\gamma_2)\},$$

where $w_1 = u_1 + iv_1 = -\cos(\delta)\sec(\gamma_1) + i\sin(\delta)\sec(\gamma_1)$ and $w_2 = u_2 + iv_2 = -\sin(\delta)\sec(\gamma_2) + i\cos(\delta)\sec(\gamma_2)$. The constant contact angle condition on the vertical plane $\overline{AP} \times \mathbb{R}$ corresponds to requiring that the Gauss map lie on the circle that is the intersection of S^2 and a cone with opening angle γ_1. The stereographic projection of this circle is the circle $C_1 = \{w : |w - w_1| = \tan(\gamma_1)\}$ centered at w_1. In the same way, we see that the constant contact angle condition on the vertical plane $\overline{PB} \times \mathbb{R}$ corresponds to requiring that the Gauss map lie on the circle that is the intersection of S^2 and a cone with opening angle γ_2. The stereographic projection of this second circle is $C_2 = \{w : |w - w_2| = \tan(\gamma_2)\}$. Then $E = \overline{E_0}$.

Let us examine the set E, which is a simply connected subset of the closed unit disk. The boundary of E consists of portions of the circles C_1 and C_2 which are orthogonal to the unit circle ∂B_1, the positive imaginary axis, the negative real axis and (possibly) the unit quarter-circle $\{w \in \partial B_1 : \text{Re}(w) \leq 0, \text{Im}(w) \geq 0\}$. If the circles C_1 and C_2 intersect in $|w| \leq 1$, then let $w_0 = u_0 + iv_0$ be this point of intersection; (4) implies $w_0 \in Q_1$, and by reflecting about ∂B_1, we see that

$$(v_2 - v_1)v_0 = (u_1 - u_2)u_0.$$

Note that C_1 lies entirely in the open set $\mathrm{Re}(w) < 0$ if $\gamma_1 < \pi/2 - \delta$, and C_2 lies entirely in the open set $\mathrm{Im}(w) > 0$ if $\gamma_2 < \pi/2 - \delta$. Let us write

$$\sigma_1 = \partial E \cap C_1,$$
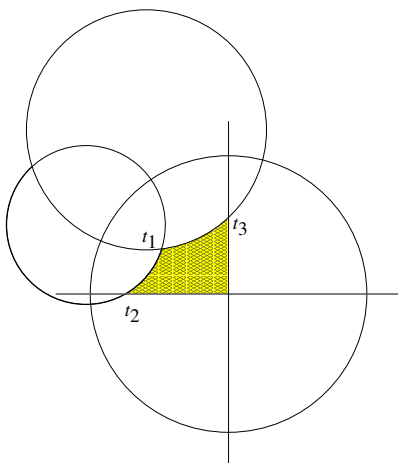$$\sigma_2 = \partial E \cap C_2,$$

$$\sigma_3 = \{w \in \partial E \ : \ \mathrm{Im}(w) = 0\},$$
$$\sigma_4 = \{w \in \partial E \ : \ \mathrm{Re}(w) = 0\},$$
$$\sigma_5 = \{w \in \partial E \ : \ |w| = 1\}.$$

We will set $t_0 = 0$. We denote the remaining corners of ∂E as $t_1, \ldots, t_L$ for $3 \le L \le 4$. The t_i have these properties:

(i) $t_1, t_2 \in \sigma_1$ and $\mathrm{Im}(t_2) = 0$; that is, t_2 is the intersection of C_1 with the negative real axis, and t_1 is the intersection of C_1 with either C_2 (see (iii)) or ∂B_1 (see (iv)).

(ii) $t_3 \in \sigma_2$ and $\mathrm{Re}(t_3) = 0$; that is, t_3 is the intersection of C_2 with the positive imaginary axis.

(iii) $\gamma_1 + \gamma_2 \ge \pi/2 - 2\delta$ implies $L = 3$ and $t_1 \in \sigma_2$.

(iv) $\gamma_1 + \gamma_2 < \pi/2 - 2\delta$ implies $L = 4$, $t_4 \in \sigma_2$ and $t_1, t_4 \in \partial B_1$.

When condition (iii) holds, Figure 2 illustrates the shape of E, while Figure 3 illustrates the shape of E when condition (iv) holds. An example of a listing of the corners in counterclockwise order is t_1, t_2, t_0, t_3, and, if $L = 4$, t_4. Notice that when condition (iv) holds, $t_1 = \exp((\pi - \delta - \gamma_1)i)$ and $t_4 = \exp((\pi/2 + \delta + \gamma_2)i)$.

For the Weierstrass representation, let us define $g : E \to E$ by $g(w) = w$. We wish to find $f \in C^0(E \setminus \{t_1, \ldots, t_L\})$, with at worst integrable singularities at $t_1, \ldots, t_L$, which is analytic in E_0. We define $X \in C^0(E : \mathbb{R}^3) \cap C^2(E_0 : \mathbb{R}^3)$ with

Figure 2. $\gamma_1 \ge \delta, \ \gamma_2 \ge \delta, \ \gamma_1 + \gamma_2 \ge \pi/2 - 2\delta$.
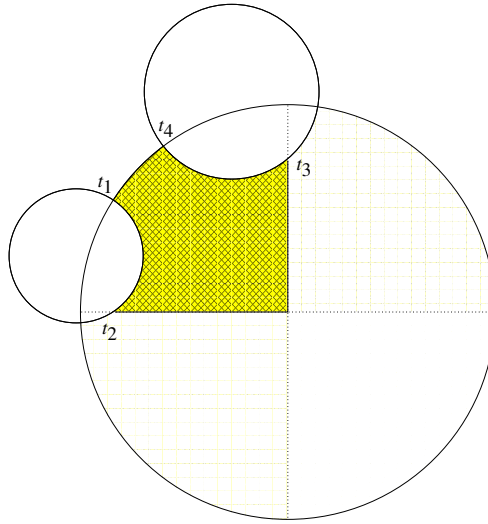
Figure 3. $\gamma_1 \geq \delta$, $\gamma_2 \geq \delta$, $\gamma_1 + \gamma_2 < \pi/2 - 2\delta$.

$$(6) \qquad X(u + iv) = (x(u, v), y(u, v), z(u, v)).$$

and $K(u+iv) = (x(u, v), y(u, v))$ for $u+iv \in E$, such that $X(0) = (0, 0, 0)$ and (a) the analytic functions (f, g) form the Weierstrass representation of X [Osserman 1986]; (b) K is a homeomorphism between σ_1 and AP, σ_2 and PB, σ_3 and OA, and σ_4 and BO (where, for example, OA means the line segment between O and A); and (c)

$$(7) \qquad K \text{ is constant on each component of } \sigma_5.$$

Here we say f has an *integrable singularity* at t_n if and only if $|f(w)| \leq C|w - t_n|^s$ for $-1 < s < 0$ and $C \geq 0$ for w near t_n.

We will now formulate the Riemann–Hilbert problem which we will solve by temporarily assuming the existence of a suitable function f.

The boundary requirements (b) imply

(8a)	$y(u, v) = \cot(\delta)x(u, v) + 1$	for $u + iv \in \sigma_1$,
(8b)	$y(u, v) = \tan(\delta)(x(u, v) - x_0)$	for $u + iv \in \sigma_2$,
(8c)	$x(u, v) = 0$ and $0 \leq y(u, v) \leq 1$	for $u + iv \in \sigma_3$,
(8d)	$y(u, v) = 0$ and $x_0 \leq x(u, v) \leq 0$	for $u + iv \in \sigma_4$,

where $x_0 \in (-\cot(\delta), -\tan(\delta))$ remains to be determined. Decompose $f(u+iv) =$

$f_1(u, v) + i f_2(u, v)$, where f_1 and f_2 are real-valued. Now (a) implies

$$x_w = f(w)(1 - w^2)/2, \qquad y_w = i f(w)(1 + w^2)/2, \qquad z_w = w f(w)$$

for $w \in E$; see for example [Osserman 1986]. Since $d/dw = (\partial/\partial u - i \partial/\partial v)/2$, the equations above yield

$$x_u(u, v) = \text{Re}(f(w)(1 - w^2)), \qquad x_v(u, v) = -\text{Im}(f(w)(1 - w^2)),$$
$$y_u(u, v) = \text{Re}(i f(w)(1 + w^2)), \qquad y_v(u, v) = -\text{Im}(i f(w)(1 + w^2)),$$
$$z_u(u, v) = \text{Re}(2 w f(w)), \qquad z_v(u, v) = -\text{Im}(2 w f(w)),$$

where we use the notation $w = u + iv$.

If we parametrize σ_k by $\sigma_k = \{w_k(t) = u_k(t) + i v_k(t) : t \in I_k\}$, with I_k a closed interval in $\mathbb{R}$ for $k = 1, \ldots, 5$, then by differentiating with respect to t (and denoting d/dt by a prime), we find

(8a) implies $y_u u_1' + y_v v_1' = \cot(\delta)(x_u u_1' + x_v v_1')$;

(8b) implies $y_u u_2' + y_v v_2' = \tan(\delta)(x_u u_2' + x_v v_2')$;

(8c) implies $x_u(u_3(t), 0) = 0$;

(8d) implies $y_v(0, v_4(t)) = 0$; and

(7) implies $x_u u_5' + x_v v_5' = 0$ and $y_u u_5' + y_v v_5' = 0$.

For σ_1, we have $(u_1(t) - u_1)^2 + (v_1(t) - v_1)^2 = r_1^2$, where $r_1 = \tan(\gamma_1)$, which implies

$$\frac{u_1'(t)}{v_1'(t)} = -\frac{v_1(t) - v_1}{u_1(t) - u_1} = -\frac{\text{Im}(w_1(t) - w_1)}{\text{Re}(w_1(t) - w_1)}.$$

For σ_2, we have $(u_2(t) - u_2)^2 + (v_2(t) - v_2)^2 = r_2^2$, where $r_2 = \tan(\gamma_2)$, which implies

$$\frac{u_2'(t)}{v_2'(t)} = -\frac{v_2(t) - v_2}{u_2(t) - u_2} = -\frac{\text{Im}(w_2(t) - w_2)}{\text{Re}(w_2(t) - w_2)}.$$

Recall that $\cot(\delta) = -\text{Re}(w_1)/\text{Im}(w_1) = -u_1/v_1$ and $\tan(\delta) = -\text{Re}(w_2)/\text{Im}(w_2) = -u_2/v_2$.

If we use this information, we obtain

(9) $$\text{Re}\left((a_k(u, v) + i b_k(u, v)) f(u + iv)\right) = 0,$$

when $(u, v) \in \sigma_k$ for $k = 1, \ldots, 5$, where

$$a_1(u, v) + i b_1(u, v) = i e^{i\delta}(w - w_1)\left(e^{-2\delta i} - w^2\right) \quad \text{if } w = u + iv \in \sigma_1,$$
$$a_2(u, v) + i b_2(u, v) = e^{-i\delta}(w - w_2)\left(e^{2\delta i} + w^2\right) \quad \text{if } w = u + iv \in \sigma_2,$$
$$a_3(u, v) + i b_3(u, v) = -1 \qquad\qquad\qquad\qquad \text{if } w = u + iv \in \sigma_3,$$
$$a_4(u, v) + i b_4(u, v) = -1 \qquad\qquad\qquad\qquad \text{if } w = u + iv \in \sigma_4,$$
$$a_5(u, v) + i b_5(u, v) = (u + iv)^2 \qquad\qquad\quad \text{if } w = u + iv \in \sigma_5.$$

We now define $a, b : \partial E \to \mathbb{R}$ by $a(u + iv) = a_k(u, v)$ and $b(u + iv) = b_k(u, v)$ if $u + iv \in \sigma_k$ for $k \in \{1, \dots, 5\}$ and define $G : \partial E \to \mathbb{C}$ by $G(w) = a(w) + ib(w)$. It is possible that $\sigma_5 = \varnothing$, in which case $a_5(u, v) + ib_5(u, v)$ has no impact on G.

Assuming the existence of f, we were led to (9). Now we can formulate the problem of finding X with $g = \mathrm{id}$. We wish to find a function $f \in C^0(E \setminus \{t_1, \dots, t_L\})$ that is analytic in E_0 and satisfies

(10) $$\mathrm{Re}(G(w) f(w)) = 0 \quad \text{for } w \in \partial E \setminus \{t_1, \dots, t_L\}.$$

This is a Hilbert problem with piecewise discontinuous Hölder coefficients G in the notation of [Monakhov 1983, Chapter 1, Section 4]. To use the results in [Monakhov 1983], we need to compute the index of G in an appropriate function class $O(m) = O(t_{n_1}, \dots, t_{n_m})$ for some $m \in \{0, \dots, L\}$. Define $G_1 : \partial E \to \mathbb{C}$ by

$$G_1(w) = -\frac{\overline{G(w)}}{G(w)}.$$

We have $G_1(w) = -1$ for $w \in \sigma_3 \cup \sigma_4$, G_1 is continuous at 0, and $G_1(w) = -(\overline{w}/|w|)^4 = -\overline{w}^4$ for $w \in \sigma_5$. Set $\omega = e^{i\delta}$. Notice also that

$$G_1(w) = |G(w)|^{-2}\overline{\left(\omega(w - w_1)(\overline{\omega}^2 - w^2)\right)}^2 \qquad \text{for } w \in \sigma_1,$$

$$G_1(w) = -|G(w)|^{-2}e^{2\delta i}\overline{\left((w - w_2)(w^2 + \omega^2)\right)}^2 \quad \text{for } w \in \sigma_2.$$

For two corners t_n and t_{n+1} with $n \geq 1$ on ∂E, let $[t_n, t_{n+1}]$ denote the arc of ∂E between t_n and t_{n+1} with t_{L+1} equalling t_1; for example, $[t_1, t_2] = \sigma_1$ and $[t_2, t_3] = \sigma_3 \cup \sigma_4$. For $n \in \{1, \dots, L\}$, let

$$\theta_n = \frac{1}{2\pi}\left(\arg(G_1(t_n - 0)) - \arg(G_1(t_n + 0))\right),$$

where $\arg(G_1(t_n - 0))$ means the limit at t_n of the argument of G_1 along the arc $[t_{n-1}, t_n]$ and $\arg(G_1(t_n + 0))$ means the limit at t_n of the argument of G_1 along the arc $[t_n, t_{n+1}]$. The argument is taken to be continuous along each arc $[t_n, t_{n+1}]$.

To understand the monotonicity behavior of $\arg G_1$ on the different arcs, and since G_1 is a product on σ_1 and σ_2, we consider $w \in \overline{Q_1} \setminus \{-\omega, i\omega\}$ and define

$$\lambda_1(w) = \arg(\overline{\omega}^2 - w^2) \quad \text{and} \quad \lambda_2(w) = \arg(\omega^2 + w^2) = \arg(w^2 - (-\omega^2));$$

note that $\lambda_1(w) \in (-\pi/2 - 2\delta, \pi/2 - 2\delta)$ and $\lambda_2(w) \in (-\pi/2 + 2\delta, \pi/2 + 2\delta)$.

Notice that if $w \in \sigma_3$, then $\lambda_1(w) \in [-\pi/2 - \delta, -2\delta)$ and if $w \in \sigma_4$, then $\lambda_2(w) \in (2\delta, \pi/2 + \delta]$. Let us factor $\overline{\omega}^2 - w^2$ and $\omega^2 + w^2$ and analyze these factors. Define $\alpha_1(w) = \arg(w - \overline{\omega}) \in [3\pi/4 - \delta/2, \pi - \delta/2]$, $\alpha_2(w) = \arg(w + i\omega) \in [\pi/2 + \delta/2, 3\pi/4 + \delta/2]$, $\beta_1(w) = \arg(w + \overline{\omega}) = \arg(w - (-\overline{\omega})) \in (-\pi/2 - \delta, \pi/2 - \delta)$, and $\beta_2(w) = \arg(w - i\omega) \in (\delta - \pi, \delta)$. Then

$$\lambda_1(w) = \alpha_1(w) + \beta_1(w) - \pi \quad \text{and} \quad \lambda_2(w) = \alpha_2(w) + \beta_2(w).$$

Some simple calculations show that if $w_{(1)}(\tau) = w_1 + r_1 e^{i\tau}$ parametrizes the circular arc $C_1 \cap \overline{B_1}$, then

$$\frac{d}{d\tau}\big(\alpha_1(w_{(1)}(\tau))\big) < 0, \quad \frac{d}{d\tau}\big(\beta_1(w_{(1)}(\tau))\big) > 0, \quad \frac{d}{d\tau}\big((\alpha_1 + \beta_1)(w_{(1)}(\tau))\big) > 0,$$

while if $w_{(2)}(\tau) = w_2 + r_2 e^{i\tau}$ parametrizes the circular arc $C_2 \cap \overline{B_1}$, then

$$\frac{d}{d\tau}\big(\alpha_2(w_{(2)}(\tau))\big) < 0, \quad \frac{d}{d\tau}\big(\beta_2(w_{(2)}(\tau))\big) > 0, \quad \frac{d}{d\tau}\big((\alpha_2 + \beta_2)(w_{(2)}(\tau))\big) > 0.$$

Hence $\lambda_1(w)$ is strictly increasing as $w \in \sigma_1$ moves in a clockwise direction with respect to E (that is, counterclockwise on C_1) and $\lambda_2(w)$ is strictly increasing as $w \in \sigma_2$ moves in a clockwise direction with respect to E (that is, counterclockwise on C_2). Similarly, we see that $\lambda_1(u)$ is strictly increasing for $u \in [-1, 0]$ and $\lambda_2(iv)$ is strictly increasing for $v \in [0, 1]$.

5. Proof of Theorem 3.1

Suppose first that $\gamma_1 + \gamma_2 \geq \pi/2 - 2\delta$, so that $L = 3$. Then

$$\begin{aligned}
t_1 &= w_2 + r_2 e^{i\tau_{2A}} \text{ for some } \tau_{2A} \in [\gamma_2 + \delta - \pi, 0), \\
t_3 &= w_2 + r_2 e^{i\tau_{2B}} \text{ for some } \tau_{2B} \in (\delta - \pi/2, 0], \quad \text{with } \tau_{2A} < \tau_{2B}, \\
t_1 &= w_1 + r_1 e^{i\tau_{1A}} \text{ for some } \tau_{1A} \in (-\pi/2, \pi/2 - \delta - \gamma_1], \\
t_2 &= w_1 + r_1 e^{i\tau_{1B}} \text{ for some } \tau_{1B} \in [-\pi/2, -\delta), \quad \text{with } \tau_{1A} > \tau_{1B}.
\end{aligned}$$

We observe that

$$\begin{aligned}
\alpha_2(t_3) &= \arg(t_3 + i\omega) < \pi/2 + \delta, & \beta_2(t_3) &= \arg(t_3 - i\omega) < \delta, \\
\alpha_1(t_2) &= \arg(t_2 - \overline{\omega}) > \pi - \delta, & \beta_1(t_2) &= \arg(t_2 + \overline{\omega}) > -\pi/2 - \delta,
\end{aligned}$$

$\tau_{1A} - \tau_{2A} > 0$ and (4) implies $\tau_{2A} < 0$ and $\tau_{1A} > -\pi/2$. For the sake of definiteness, we set

$$\begin{aligned}
\arg(G_1(t_1 - 0)) &= \pi + 2\delta - 2(\tau_{2A} + \lambda_2(t_1)), \\
\arg(G_1(t_1 + 0)) &= -2(\delta + \tau_{1A} + \lambda_1(t_1)), \\
\arg(G_1(t_2 - 0)) &= -2(\delta + \tau_{1B} + \lambda_1(t_2)), \\
\arg(G_1(t_2 + 0)) &= \arg(G_1(t_3 - 0)) = \pi, \\
\arg(G_1(t_3 + 0)) &= \pi + 2\delta - 2(\tau_{2B} + \lambda_2(t_3)).
\end{aligned}$$

Notice that

$$\begin{aligned}
\theta_1 &= 1/2 + (2\delta + \tau_{1A} - \tau_{2A} + \lambda_1(t_1) - \lambda_2(t_1))/\pi, \\
\theta_2 &= -1/2 - (\delta + \lambda_1(t_2) + \tau_{1B})/\pi \in (-1/2 + 2\delta/\pi, 1/2], \\
\theta_3 &= (\lambda_2(t_3) + \tau_{2B} - \delta)/\pi \in (-1/2 + 2\delta/\pi, 1/2].
\end{aligned}$$

Also

$$\arg(G_1(t_2 - 0)) - \arg(G_1(t_1 + 0)) = 2(\tau_{1A} - \tau_{1B}) + 2(\lambda_1(t_1) - \lambda_1(t_2)) > 0,$$
$$\arg(G_1(t_1 - 0)) - \arg(G_1(t_3 + 0)) = 2(\tau_{2B} - \tau_{2A}) + 2(\lambda_2(t_3) - \lambda_2(t_1)) > 0;$$

hence $\theta_1 + \theta_2 + \theta_3 > 0$. Therefore, $\theta_1 > 1 - 4\delta/\pi > 0$.

Now $\tau_{1A} - \tau_{2A}$ equals the angle formed by the triangle with vertices w_1, t_1, and w_2 at the vertex t_1; notice that $0 < \tau_{1A} - \tau_{2A} \le \pi$. Also, $\beta_1(t_1) - \beta_2(t_1)$ equals the angle at t_1 formed by the line segments $[t_1, -\bar{\omega}]$ and $[t_1, i\omega]$ from t_1 to $-\bar{\omega}$ and $i\omega$, respectively, where the angle begins in the direction $[t_1, i\omega]$ and moves counterclockwise to $[t_1, -\bar{\omega}]$. Notice that $\tau_{1A} - \tau_{2A} < \beta_1(t_1) - \beta_2(t_1)$ and $\pi/4 - \delta < \beta_1(t_1) - \beta_2(t_1) \le 3\pi/4 + \delta$. Further, $\alpha_1(t_1) - \alpha_2(t_1)$ equals the (acute) angle formed by the triangle with vertices $-i\omega$, t_1, and $\bar{\omega}$ at the vertex t_1; notice that $\pi/4 - \delta \le \alpha_1(t_1) - \alpha_2(t_1) < \pi/2 - 2\delta$. In agreement with the conclusion of the previous paragraph, we see that

$$\theta_1 > 1/2 + \big(2\delta + 0 + [(\pi/4 - \delta) + (\pi/4 - \delta) - \pi]\big)/\pi = 0,$$

where the sum in the square brackets is a lower bound for $\lambda_1(t_1) - \lambda_2(t_1)$. In fact, as γ_1 and γ_2 converge to $\pi/2$ from below, θ_2 and θ_3 each converge to $-1/2 + 2\delta/\pi$, and θ_1 converges to $1 - 4\delta/\pi$.

If we set $z_1 = \exp(i(\pi - \delta - \gamma_1))$ and $z_2 = \exp(i(\pi/2 + \delta + \gamma_2))$, so that $z_1 \in C_1 \cap \partial B_1$ and $z_2 \in C_2 \cap \partial B_1$, then the monotonicity of λ_2 on $C_2 \cap B_1$ implies

$$\arg(G_1(t_1 - 0)) \le \pi + 2\delta - 2[\tau_{2A} + \lambda_2(z_2)] = 2\pi - 2\tau_{2A} - 2\delta - 2\gamma_2,$$

and the monotonicity of λ_1 on $C_1 \cap B_1$ implies

$$\arg(G_1(t_1 + 0)) \ge -2\delta - 2[\tau_{1A} + \lambda_1(z_1)] = -\pi - 2\tau_{1A} + 2\delta + 2\gamma_1.$$

Recalling that $\gamma_1 + \gamma_2 \ge \pi/2 - 2\delta$, we obtain

$$0 < \theta_1 \le 1 + (\tau_{1A} - \tau_{2A})/\pi \le 2$$

and $\theta_1 = 2$ if and only if $\gamma_1 + \gamma_2 = \pi/2 - 2\delta$.

Set $\nu_1 = [\theta_1]$, $\nu_2 = 0$, $\nu_3 = 0$, $\alpha_1 = \theta_1 - \nu_1$, $\alpha_2 = \theta_2$, and $\alpha_3 = \theta_3$, where $[\theta_1]$ denotes the greatest integer less than or equal to θ_1. Notice that $\nu_1 + \nu_2 + \nu_3 \in \{0, 1, 2\}$. Let ψ be the conformal map from E onto the unit disk B_1 that maps t_1 to 1, t_2 to i, and t_3 to $-i$. Define $G^* : \partial B_1 \to \mathbb{C}$ by

$$G^*(\zeta) = G(\psi^{-1}(\zeta)) \quad \text{for } \zeta \in \partial B_1.$$

Our Hilbert problem (10) is then equivalent to finding $f^* \in C^0(\overline{B_1} \setminus \{1, \pm i\})$ which is analytic in B_1, has (at worst) integrable singularities at $1, i, -i$, and satisfies

(11) $\text{Re}\big(G^*(\zeta) f^*(\zeta)\big) = 0 \quad \text{for } \zeta \in \partial B_1 \setminus \{1, \pm i\}.$

We now use results from [Monakhov 1983, Chapter 1, Section 4]. Using [2.°(a), p. 49–52] when $\nu_1 = 0$ after setting $\kappa = 0$, [2.°(b), p. 52–53] when $\nu_1 = 1$ after setting $\kappa = 0$, and [2.°(a), p. 49–52] when $\nu_1 = 2$ after setting $\kappa = 1$, we obtain a solution $f = cf_0$ of (10), where f_0 has the form

$$f_0(w) = (\psi(w) - 1)^{\theta_1}(\psi(w) - i)^{\alpha_2}(\psi(w) + i)^{\alpha_3} e^{\Gamma_0(\psi(w))}$$

and $e^{\Gamma_0(\zeta)}$ is a continuous, nonvanishing function on $\overline{B_1}$. Assuming $c \neq 0$, f does not vanish on $E \setminus \{t_1, t_2, t_3\}$. Notice that $\mathrm{Re}(f_0(w)) = 0$ if $w \in (\sigma_3 \cup \sigma_4) \setminus \{t_2, t_3\}$, and so $f_0(0) = ib_0$ for some $b_0 \neq 0$. Now $\mathrm{Im}(f_0(w)) \neq 0$, and so $\mathrm{Im}(b_0 f_0(w)) > 0$ for $w \in (\sigma_3 \cup \sigma_4) \setminus \{t_2, t_3\}$. We set

$$(12) \qquad\qquad -1/c = \int_{t_2}^{0} (1 + u^2) i f_0(u) \, du$$

and observe that $cb_0 > 0$ since $t_2 < 0$. It follows then that $f(0) = ib_0 c$, $\mathrm{Im}(f(w)) > 0$ for $w \in (\sigma_3 \cup \sigma_4) \setminus \{t_2, t_3\}$, and

$$\int_{0}^{t_2} (1 + u^2) i f(u) \, du = 1.$$

Let $X \in C^0(E : \mathbb{R}^3)$ be the minimal surface whose Weierstrass (f, g) representation is given by $g(w) = w$, where f is defined above, and which satisfies $X(0) = (0, 0, 0)$. Recall (6) and notice that $(x(t_2, 0), y(t_2, 0)) = (0, 1)$ and K maps σ_3 onto the line segment joining $(0, 0)$ and $(0, 1)$. Also $x(0, t_3) < 0$ and K maps σ_4 onto the line segment joining $(0, 0)$ and $B = (x(0, t_3), 0)$. With $w = u + iv \in E$, let

$$(13) \qquad \vec{N}(w) = \frac{X_u(w) \times X_v(w)}{|X_u(w) \times X_v(w)|} = \left(\frac{2\,\mathrm{Re}(w)}{|w|^2 + 1}, \frac{2\,\mathrm{Im}(w)}{|w|^2 + 1}, \frac{|w|^2 - 1}{|w|^2 + 1} \right)$$

denote the Gauss map $\vec{N} \in C^0(E : S^2)$ of $X(E)$; here $\vec{N}(w)$ is the downward unit normal to $X(E)$ at $X(w)$.

From the above, setting $h_2(w) = (\psi(w) - 1)^{\alpha_1}(\psi(w) + i)^{\alpha_3} e^{\Gamma_0(\psi(w))}$, and defining $h_3(w)$ similarly, we have

$$(14) \qquad f(w) = (\psi(w) - i)^{\alpha_2} h_2(w) = (\psi(w) + i)^{\alpha_3} h_3(w)$$

with $h_2 \in C^0(E \setminus \{t_1, t_3\})$, $h_3 \in C^0(E \setminus \{t_1, t_2\})$, $h_2(t_2) \neq 0$, and $h_3(t_3) \neq 0$. Since $\psi(w) \in \partial B_1$ with $\arg(\psi(w)) \in [\pi/2, 3\pi/2]$ for $w \in \sigma_3 \cup \sigma_4$, $\psi(t_2) = i$, and $\psi(t_3) = -i$, we see that

$$|\psi(w) - i|^{-\alpha_2} f(w) = \left(\frac{\psi(w) - i}{|\psi(w) - i|} \right)^{\alpha_2} h_2(w) \to e^{-\alpha_2 \pi i} h_2(t_2) \quad \text{as } \sigma_3 \ni w \to t_2,$$

$$|\psi(w) + i|^{-\alpha_3} f(w) = \left(\frac{\psi(w) + i}{|\psi(w) + i|} \right)^{\alpha_3} h_3(w) \to e^{\alpha_3 \pi i} h_3(t_3) \quad \text{as } \sigma_4 \ni w \to t_3.$$

Since $\mathrm{Re}(-f(w)) = 0$ if $w \in (\sigma_3 \cup \sigma_4) \setminus \{t_2, t_3\}$ and so $\mathrm{Re}(|\psi(w) - i|^{-\alpha_2} f(w)) = 0$ and $\mathrm{Re}(|\psi(w) + i|^{-\alpha_3} f(w)) = 0$ for $w \in (\sigma_3 \cup \sigma_4) \setminus \{t_2, t_3\}$, we see that

$$h_2(t_2) = i\lambda_2 e^{\alpha_2 \pi i} \quad \text{and} \quad h_3(t_3) = i\lambda_3 e^{-\alpha_3 \pi i}$$

for some $\lambda_2, \lambda_3 \in \mathbb{R}$. Since $\mathrm{Im}(f) > 0$ on $(\sigma_3 \cup \sigma_4) \setminus \{t_2, t_3\}$, we have $\lambda_2, \lambda_3 > 0$,

$$f(w) = (\psi(w) - i)^{\alpha_2} i\lambda_2 e^{\alpha_2 \pi i} + o(|\psi(w) - i|^{\alpha_2}) \quad \text{for } w \in E \text{ near } t_2,$$

$$f(w) = (\psi(w) + i)^{\alpha_3} i\lambda_3 e^{-\alpha_3 \pi i} + o(|\psi(w) + i|^{\alpha_3}) \quad \text{for } w \in E \text{ near } t_3.$$

Recall that $w_{(1)}(\tau) = w_1 + r_1 e^{\tau i}$ and define

$$u_{(1)}(\tau) = \mathrm{Re}(w_{(1)}(\tau)), \qquad x_1(\tau) = x(u_{(1)}(\tau), v_{(1)}(\tau)),$$

$$v_{(1)}(\tau) = \mathrm{Im}(w_{(1)}(\tau)), \qquad y_1(\tau) = y(u_{(1)}(\tau), v_{(1)}(\tau)).$$

Also recall $w_{(2)}(\tau) = w_2 + r_2 e^{\tau i}$ and define

$$u_{(2)}(\tau) = \mathrm{Re}(w_{(2)}(\tau)), \qquad x_2(\tau) = x(u_{(2)}(\tau), v_{(2)}(\tau)),$$

$$v_{(2)}(\tau) = \mathrm{Im}(w_{(2)}(\tau)), \qquad y_2(\tau) = y(u_{(2)}(\tau), v_{(2)}(\tau)).$$

Recall that the condition $\mathrm{Re}[G(w)f(w)] = 0$

for $w \in \sigma_1 \setminus \{t_1, t_2\}$ implies $y_1'(\tau) = \cot(\delta) x_1'(\tau)$ for $\tau \in (\tau_{1B}, \tau_{1A})$ and

for $w \in \sigma_2 \setminus \{t_1, t_3\}$ implies $y_1'(\tau) = \tan(\delta) x_1'(\tau)$ for $\tau \in (\tau_{1B}, \tau_{1A})$.

Thus $K(w_{(1)}(\tau))$ maps $[\tau_{1B}, \tau_{1A}]$ onto a line segment on the line $y = \cot(\delta)x + 1$ (with $K(w_{(1)}(\tau_{1B})) = K(t_2) = (0, 1)$) and $K(w_{(2)}(\tau))$ maps $[\tau_{2A}, \tau_{2B}]$ onto a line segment on the line $y = \tan(\delta)x + y_0$ for some undetermined y_0.

Do we know that K maps ∂E onto a nonconvex quadrilateral of the type desired? Actually, the answer is no. Figure 4 represents the types of nonconvex quadrilaterals onto which K could map ∂E. We need to show that Figure 4(C) corresponds to the actual image of K. (Different choices of α_1, α_2 and α_3 do yield all of the possibilities illustrated in (A), (B), and (C) of Figure 4.)

Since $\psi(w) \in \partial B_1$ with $\arg(\psi(w)) \in (0, \pi/2)$ when $w \in \sigma_1 \setminus \{t_1, t_2\}$, we see that

$$|\psi(w_{(1)}(\tau)) - i|^{-\alpha_2} f(w_{(1)}(\tau)) = \left(\frac{\psi(w_{(1)}(\tau)) - i}{|\psi(w_{(1)}(\tau)) - i|} \right)^{\alpha_2} h_2(w_{(1)}(\tau)) \to h_2(t_2)$$

as $\tau \to \tau_{1B}^+$. Notice that

$$\frac{d}{d\tau}(x_1(\tau)) = \frac{-r_1 \sin(\tau)}{v_{(1)}(\tau) - v_1} \mathrm{Im}\big((w_{(1)}(\tau) - w_1) f(w_{(1)}(\tau))(1 - w_{(1)}^2(\tau))\big),$$

$$\frac{d}{d\tau}(y_1(\tau)) = \frac{-r_1 \sin(\tau)}{v_{(1)}(\tau) - v_1} \mathrm{Im}\big((w_{(1)}(\tau) - w_1) if(w_{(1)}(\tau))(1 + w_{(1)}^2(\tau))\big).$$
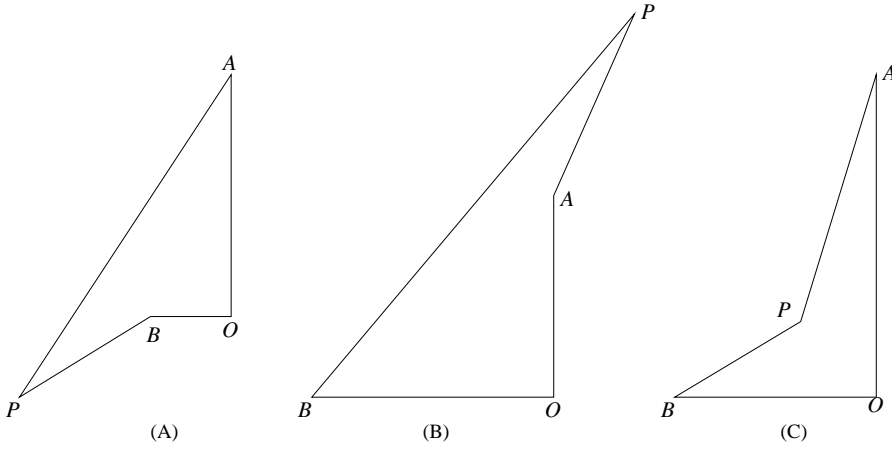
Figure 4. Types of quadrilaterals onto which K could map ∂E.

Recall that $h_2(t_2) = i\lambda_2 e^{\alpha_2 \pi i}$, $t_2 - w_1 = r_1 e^{\tau_{1B} i}$, $\sin(\tau_{1B}) < 0$, $v_1 > 0$, $\text{Im}(t_2) = 0$, and $\alpha_2 \pi + \tau_{1B} + \pi/2 = -(\lambda_1(t_2) + \delta) \in (\delta, \pi/2]$. Hence $\cos(\alpha_2 \pi + \tau_{1B} + \pi/2) \geq 0$ and $\sin(\alpha_2 \pi + \tau_{1B} + \pi/2) > 0$. Now

$$\lim_{\tau \to \tau_{1B}^+} \frac{|\psi(w_1(\tau)) - i|^{-\alpha_2}}{|1 - w_{(1)}^2(\tau)|} \frac{d}{d\tau}(x_1(\tau)) = \frac{-r_1 \sin(\tau_{1B})}{0 - v_1} \text{Im}\big((t_2 - w_1)i\lambda_2 e^{\alpha_2 i}\big)$$

$$= \frac{-r_1 \sin(\tau_{1B})}{-v_1} r_1 \lambda_2 \sin\big(\alpha_2 \pi + \tau_{1B} + \frac{\pi}{2}\big) < 0,$$

$$\lim_{\tau \to \tau_{1B}^+} |\psi(w_1(\tau)) - i|^{-\alpha_2} \frac{d}{d\tau}(y_1(\tau)) = \frac{-r_1 \sin(\tau_{1B})}{0 - v_1}(1 + t_2^2) \text{Im}\big((t_2 - w_1)i^2\lambda_2 e^{\alpha_2 i}\big)$$

$$= \frac{-r_1 \sin(\tau_{1B})}{-v_1} r_1 \lambda_2 (1 + t_2^2) \cos\big(\alpha_2 \pi + \tau_{1B} + \frac{\pi}{2}\big) \leq 0.$$

These inequalities imply that there exists an $\epsilon > 0$ such that $x_1'(\tau) < 0$ and, if $\gamma_1 > \delta$ (so $\tau_{1B} > -\pi/2$), $y_1'(\tau) < 0$ for $\tau \in (\tau_{1B}, \tau_{1B} + \epsilon)$.

We leave it to the reader to verify that if τ_0 is in (τ_{1B}, τ_{1A}) and $x_1'(\tau_0) = 0$, then $f(w_{(1)}(\tau_0)) = 0$; this would use that $|w_{(1)}(\tau_0)| < 1$. Since $f \neq 0$ on $\sigma_1 \setminus \{t_1, t_2\}$, we see that $x_1'(\tau) \neq 0$ and $y_1'(\tau) \neq 0$ if $\tau \in (\tau_{1B}, \tau_{1A})$. Therefore,

$$x_1'(\tau) < 0 \quad \text{and} \quad y_1'(\tau) < 0 \quad \text{for } \tau \in (\tau_{1B}, \tau_{1A}),$$

and K is a homeomorphism from σ_1 onto the line segment between $A = (0, 1)$ $(= K(t_2))$ and $P = (x_P, y_P)$, for some $x_P < 0$ and $y_P = \cot(\delta)x_P + 1$. $K(\sigma_1)$ is a subset of the line $y = \cot(\delta)x + 1$.

Notice that

$$\frac{d}{d\tau}\big(x_2(\tau)\big) = \frac{-r_2\cos(\tau)}{u_{(2)}(\tau) - u_2}\,\text{Im}\big((w_{(2)}(\tau) - w_2)f(w_{(2)}(\tau))(1 - w_{(2)}^2(\tau))\big),$$

$$\frac{d}{d\tau}\big(y_2(\tau)\big) = \frac{-r_2\cos(\tau)}{u_{(2)}(\tau) - u_2}\,\text{Im}\big[(w_{(2)}(\tau) - w_2)if(w_{(2)}(\tau))(1 + w_{(2)}^2(\tau))\big].$$

Arguing as above, we find that

$$\lim_{\tau\to\tau_{2B}^-}|\psi(w_2(\tau)) + i|^{-\alpha_3}x_2'(\tau) = \frac{-r_2\cos(\tau_{2B})}{0 - u_2}r_2\lambda_3(1 - t_3^2)\sin\big(\frac{\pi}{2} + \tau_{2B} - \alpha_2\pi\big) \le 0,$$

$$\lim_{\tau\to\tau_{2B}^-}\frac{|\psi(w_2(\tau)) + i|^{-\alpha_3}}{|1 - w_{(2)}^2(\tau)|}y_1'(\tau) = \frac{-r_2\cos(\tau_{2B})}{0 - u_2}r_2\lambda_3\cos\big(\frac{\pi}{2} + \tau_{2B} - \alpha_2\pi\big) < 0,$$

since $\tau_{2B} + \pi/2 - \alpha_2\pi = \pi/2 - (\lambda_2(t_3) - \delta) \in [0, \pi/2 - \delta)$. Hence $y_2'(\tau) = \tan(\delta)x_2'(\tau) < 0$ for $\tau \in (\tau_{2A}, \tau_{2B})$. Since $y_2(\tau_{2B}) = \text{Im}(K(t_3)) = 0$, $\tau_{2A} < \tau_{2B}$, and $y_2(\tau)$ is a decreasing function of τ, we have $y_2(\tau_{2A}) > 0$. This implies $x_B = \text{Re}(K(t_3)) = x_2(\tau_{2B}) < x_P$. Therefore K maps ∂E homeomorphically onto the boundary of a nonconvex quadrilateral $APBO$ with K mapping σ_1 onto AP, σ_2 onto PB, σ_3 onto BO, and σ_4 onto OA.

Each point $w \in E_0 = E \setminus \partial E$ lies in the open disc B_1, and so the tangent plane to $X(E_0)$ at $X(w)$ is not vertical (for example, (13)). Hence w has an open neighborhood U whose closure is strictly contained in B_1 and whose image $X(U)$ under X is a C^2 function over an open set in the xy-plane. Since the boundary $X(\partial E)$ has a simple projection onto the quadrilateral $APBO$, it follows that $X(E)$ is the graph of a function in $h \in C^2(\Omega(B)) \cap C^0(\overline{\Omega(B)})$ over the xy-plane; see for example [Huff and McCuan 2006, Lemma 1].

If $\gamma_1 + \gamma_2 > \pi/2 - 2\delta$, $\gamma_1 > \delta$, and $\gamma_2 > \delta$, then the stereographic projection of the image of the Gauss map of $X(E)$ is contained in a compact subset of the open unit disk B_1, and hence $\|\nabla h\|$ is uniformly bounded in Ω. From this, it follows that $h \in C^1(\overline{\Omega})$.

Suppose there exists another value $x_1 \in (-\cot(\delta), -\tan(\delta))$ such that if $B_1 = (x_1, 0)$, $\Omega = \Omega(B_1)$, and $\Omega_0 = \Omega_0(B_1)$, then the boundary value problem (1) with $\kappa = \lambda = 0$ and γ as defined in Section 1 has a solution $h \in C^2(\Omega) \cap C^0(\overline{\Omega})$ which satisfies the conditions that $h(0, 0) = 0$ and whose Gauss map $\tilde{G}$ over Ω_0, given in (5), extends to a map $\tilde{G} \in C^0(\overline{\Gamma_0} : \mathbf{S}^2)$ which is injective. From the symmetry of the problem, the comparison principle implies that $h = 0$ on OA and on OB. Suppose $X : E \to \mathbb{R}^3$, $X(w) = (x(w), y(w), z(w))$, is an isothermal parametrization of S_0, the graph of h over Ω_0 which maps t_1 to $(P, h(P))$, t_2 to $(A, h(A))$, and t_3 to $(B, h(B))$. Then it follows from the symmetries in the capillary problem and the symmetries of E that X maps 0 to $(0, 0, 0)$. If we define $\tilde{g} : E \to \mathbb{C}$ by $\tilde{g} = p \circ \tilde{G} \circ X$, where $p(x_1, x_2, x_3) = (x_1 + ix_2)/(1 - x_3)$ is the usual stereographic projection, then

$\tilde{g}$ is an analytic function; see for example [Osserman 1986]. Notice that Section 2 implies $\tilde{g} : \partial E \to \partial E$, $\tilde{g}(t_1) = t_1$, $\tilde{g}(t_2) = t_2$, and $\tilde{g}(t_3) = t_3$. Since $\tilde{G}$ is injective on $\overline{\Gamma}_0$, $\tilde{g}$ is injective, and so $\tilde{g}$ must be a conformal map from E to E that equals the identity; that is, $\tilde{g} = g$. Let us define the analytic function $f : E \to \mathbb{C}$ by $f(w) = x_w(w) - i y_w(w)$. From Section 2, we see that f must be a solution of the Hilbert problem (10) and must therefore be the one obtained earlier here. However, this implies $x_1 = x_B$ ($= \mathrm{Re}(K(t_3))$).

6. Proof of Theorem 3.2

Suppose next that $\gamma_1 + \gamma_2 < \frac{\pi}{2} - 2\delta$, so that $L = 4$. Then

$$t_1 = e^{(\pi-\delta-\gamma_1)i} = w_1 + r_1 e^{(\pi/2-\gamma_1-\delta)i}, \ t_3 = w_2 + r_2 e^{i\tau_{2B}} \text{ for some } \tau_{2B} \in (\delta - \tfrac{\pi}{2}, 0],$$

$$t_4 = e^{(\pi/2+\delta+\gamma_2)i} = w_2 + r_2 e^{(\gamma_2+\delta-\pi)i}, \ t_2 = w_1 + r_1 e^{i\tau_{1B}} \text{ for some } \tau_{1B} \in [-\tfrac{\pi}{2}, -\delta).$$

For the sake of definiteness, we set

$$\arg(G_1(t_1 - 0)) = \pi - 4(\pi - \delta - \gamma_1),$$
$$\arg(G_1(t_1 + 0)) = -2(\delta + (\pi/2 - \gamma_1 - \delta) + (\pi/2 - \gamma_1 - 2\delta)),$$
$$\arg(G_1(t_2 - 0)) = -2(\delta + \tau_{1B} + \lambda_1(t_2)),$$
$$\arg(G_1(t_2 + 0)) = \arg(G_1(t_3 - 0)) = \pi,$$
$$\arg(G_1(t_3 + 0)) = \pi + 2\delta - 2(\tau_{2B} + \lambda_2(t_3)),$$
$$\arg(G_1(t_4 - 0)) = \pi + 2\delta - 2((\gamma_2 + \delta - \pi) + (2\delta + \gamma_2 - \pi/2)),$$
$$\arg(G_1(t_4 + 0)) = \pi - 4(\pi/2 + \delta + \gamma_2).$$

Then

$$\theta_1 = -1/2, \qquad \theta_2 = -1/2 - (\delta + \lambda_1(t_2) + \tau_{1B})/\pi \in (-1/2 + 2\delta/\pi, 1/2],$$
$$\theta_4 = 5/2, \qquad \theta_3 = (\lambda_2(t_3) + \tau_{2B} - \delta)/\pi \in (-1/2 + 2\delta/\pi, 1/2].$$
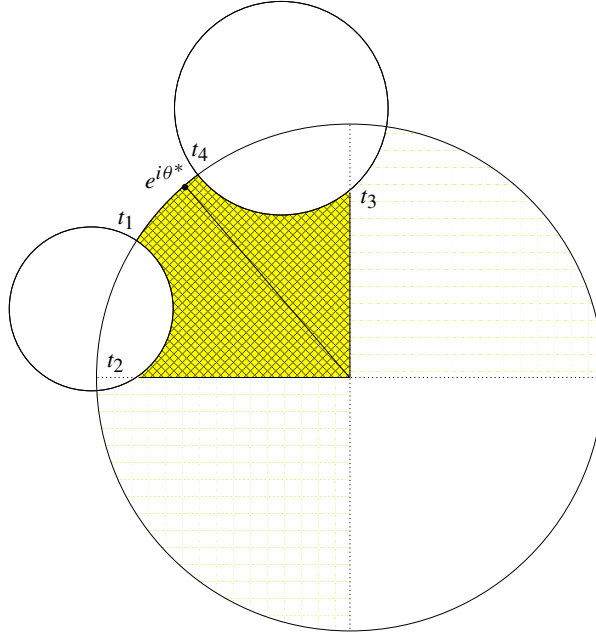
Also $\theta_1 + \theta_2 + \theta_3 + \theta_4 \in (1 + 4\delta/\pi, 3]$.

Let us set $\nu_1 = -1$, $\nu_2 = 0$, $\nu_3 = 0$, $\nu_4 = 2$, $\alpha_1 = \theta_1 + 1$, $\alpha_2 = \theta_2$, $\alpha_3 = \theta_3$, and $\alpha_4 = \theta_4 - 2$. Let ψ be the conformal map from E onto the unit disk B_1 which maps t_1 to 1, t_2 to i, and t_3 to $-i$. Set $\psi(t_4) = \zeta_4$. Define $G^* : \partial B_1 \to \mathbb{C}$ by

$$G^*(\zeta) = G(\psi^{-1}(\zeta)) \quad \text{for } \zeta \in \partial B_1.$$

Our Hilbert problem (10) is then equivalent to finding $f^* \in C^0(\overline{B_1} \setminus \{1, \pm i, \zeta_4\})$ which is analytic in B_1, has (at worst) integrable singularities at $1, i, -i, \zeta_4$, and satisfies

$$(15) \qquad \mathrm{Re}\big(G^*(\zeta)f^*(\zeta)\big) = 0 \quad \text{for } \zeta \in \partial B_1 \setminus \{1, \pm i, \zeta_4.\}.$$

Figure 5. Region E for Theorem 3.2.

Since $v_1 + v_2 + v_3 + v_4 = 1$, this Hilbert problem has a solution of the form

$$f_\mu^*(\zeta) = c(e^{i\mu}\zeta - e^{-i\mu})\Pi(\zeta)e^{\Gamma_0(\zeta)}$$

where $c, \mu \in \mathbb{R}$ and $\Pi(\zeta) = (\zeta - 1)^{\alpha_1}(\zeta - i)^{\alpha_2}(\zeta + i)^{\alpha_3}(\zeta - \zeta_4)^{\alpha_4}$. Recall that $e^{\Gamma_0(\zeta)}$ is nonvanishing and continuous on $\overline{B_1}$.

Now select $\theta^* \in [\pi/2 + \delta + \gamma_2, \pi - \delta - \gamma_1]$, and note that $e^{i\theta^*} \in \sigma_5$; see Figure 5. Set

$$\mu = -\arg(\psi(e^{i\theta^*}))/2 \in [0, -\arg(\zeta_4)/2],$$

so that $e^{-2i\mu} = \psi(e^{i\theta^*})$. With this choice, we see that

$$f_\mu^*(\zeta) = ce^{i\mu}(\zeta - e^{-2\mu i})\Pi(\zeta)e^{\Gamma_0(\zeta)},$$
$$f_\mu(w) = ce^{i\mu}(\psi(w) - e^{-2\mu i})\Pi(\psi(w))e^{\Gamma_0(\psi(w))}.$$

Let $c \in \mathbb{R}$ be given by (12), with the obvious interpretation of $f_0(w)$. If we define $p(\zeta) = 2cG^*(\zeta)e^{i\mu}\Pi(\zeta)e^{\Gamma_0(\zeta)}$, then we see $p(e^{-2\mu i}) \neq 0$, and so

$$(16) \quad \lim_{\theta \uparrow -2\mu} \frac{2G^*(e^{i\theta})f_\mu^*(e^{i\theta})}{|e^{i\theta} - e^{-2i\mu}|} = \lim_{\theta \uparrow -2\mu} p(e^{i\theta})\frac{e^{i\theta} - e^{-2i\mu}}{|e^{i\theta} - e^{-2i\mu}|} = -ie^{-2i\mu}p(e^{-2\mu i})$$

and

$$(17) \quad \lim_{\theta \downarrow -2\mu} \frac{2G^*(e^{i\theta}) f^*_\mu(e^{i\theta})}{|e^{i\theta} - e^{-2i\mu}|} = \lim_{\theta \downarrow -2\mu} p(e^{i\theta}) \frac{e^{i\theta} - e^{-2i\mu}}{|e^{i\theta} - e^{-2i\mu}|} = i e^{-2i\mu} p(e^{-2\mu i}).$$

Using (15), we see that $e^{-2i\mu} p(e^{-2\mu i}) \in \mathbb{R} \setminus \{0\}$.

Let $X_\mu : E \to \mathbb{R}^3$, $X_\mu(u + iv) = (x_\mu(u, v), y_\mu(u, v), z_\mu(u, v))$ be the minimal surface satisfying $X(0) = (0, 0, 0)$ whose Weierstrass (f, g)-representation is given by f_μ and $g(w) = w$. From the arguments in the proof of Theorem 3.1, we see that

(i) there exists $x_B \in (-\cot(\delta), -\tan(\delta))$ and $B = (x_B, 0)$ such that $X_\mu(\partial E)$ projects onto $\partial(\Omega(B))$,

(ii) X_μ is a homeomorphism from E onto $X_\mu(E)$, and

(iii) K_μ maps σ_1 onto AP, σ_2 onto PB, σ_3 onto BO, σ_4 onto OA, and σ_5 to the point P.

Notice, for example, that the argument in the previous section showing $x'_1(\tau) < 0$, $y'_2(\tau) < 0$ and $x'_2(\tau) < 0$, $y'_2(\tau) < 0$ remains valid here (with $x'_1(\tau) < 0$, $y'_1(\tau) < 0$ for $\tau \in (\tau_{1B}, \frac{\pi}{2} - \gamma_1 - \delta)$ and $x'_2(\tau) < 0$, $y'_2(\tau) < 0$ for $\tau \in (\gamma_2 + \delta - \pi, \tau_{2B})$)).

Notice that a classical "tilting argument" (for example [Huff and McCuan 2006, Section 3]), together with the argument in Section 5, shows that $X(E_0)$ is the graph of a function $h_\mu \in C^2(\Omega(B))$. Now h_μ is discontinuous at P and the radial limits Rh_μ of h_μ at P contain a central fan if $\theta^* \in (\pi/2 + \delta + \gamma_2, \pi - \delta - \gamma_1)$ with Rh_μ constant on $[\theta^* - 3\pi/2, \theta^* - \pi/2]$.

Since $K_\mu(w) = P$ for all $w \in \sigma_5$ while $d/d\tau(z_\mu(e^{i\tau})) \neq 0$ for $\tau \in (\arg(\zeta_4), 0)$ with $\tau \neq -2\mu$, we see that h_μ is discontinuous at P if $\theta^* \in (\pi/2 + \delta + \gamma_2, \pi - \delta - \gamma_1)$. Suppose now θ^* is in $(\pi/2 + \delta + \gamma_2, \pi - \delta - \gamma_1)$. Notice that

$$\frac{d}{d\tau}(z_\mu(e^{i\tau})) = -2 \operatorname{Im}(G(e^{i\tau}) f(e^{i\tau}))$$

and, since $e^{-2i\mu} = \psi(e^{i\theta^*})$, (16) and (17) show that this derivative changes sign at $\tau = \theta^*$. This implies $z_\mu(e^{i\tau})$ is increasing on one side of θ^* and decreasing on the other side of θ^* for $\tau \in (\pi/2 + \delta + \gamma_2, \pi - \delta - \gamma_1)$. Set

$$\xi_1 = \delta + \gamma_2 - \pi, \quad \xi_L = \theta^* - 3\pi/2, \quad \xi_R = \theta^* - \pi/2, \quad \xi_2 = \pi/2 - \delta - \gamma_1.$$

From our construction, X_μ has a (boundary) branch point at $e^{i\theta^*}$. From (13), we see that $\vec{N}(e^{i\theta^*}) = (\cos(\theta^*), \sin(\theta^*), 0)$. Let $D = \{u + iv \in \mathbb{C} : u^2 + v^2 < 1, v > 0\}$, and let χ be a conformal map from D to E_0 which maps 0 to $e^{i\theta^*}$. Let $X : D \to \mathbb{R}^3$ be defined by $X = X_\mu \circ \chi$, and let $X(u + iv) = (x(u, v), y(u, v), z(u, v))$ denote the components of X. From [Lancaster and Siegel 1996b, p. 174], we see that

$$x_v(u, 0) = \sin(\theta^*)u + o(|u|) \text{ and } y_v(u, 0) = -\cos(\theta^*)u + o(|u|) \text{ as } u \to 0.$$
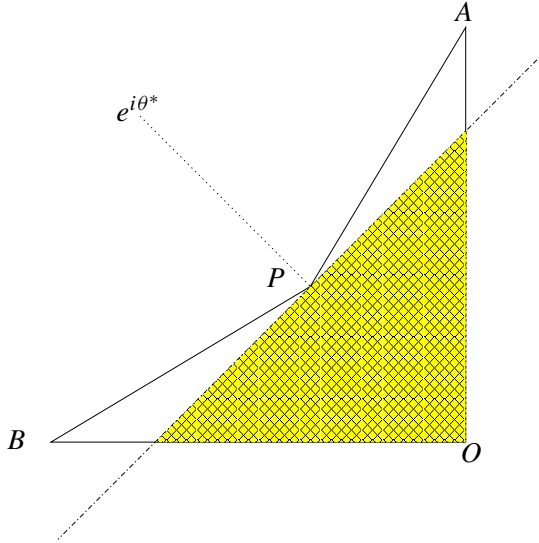
Figure 6. The "central fan region".

Hence

$$\frac{(x_v(u, 0), y_v(u, 0))}{\sqrt{x_v^2(u, 0) + y_v^2(u, 0)}} \to \begin{cases} (\cos(\xi_L), \sin(\xi_L)) & \text{as } u \uparrow 0, \\ (\cos(\xi_R), \sin(\xi_R)) & \text{as } u \downarrow 0. \end{cases}$$

As in Steps 2 and 3 of the proof of [Lancaster and Siegel 1996b, Theorem 1(ii)], this implies that the radial limits of h_μ at P satisfy

$$Rh_\mu(\theta) \text{ is } \begin{cases} \lim_{\tau \to -2\mu^-} z_\mu(\tau) & \text{if } \delta - \pi \leq \theta \leq \xi_1, \\ \text{strictly decreasing} & \text{if } \xi_1 < \theta < \xi_L, \\ \text{constant} & \text{if } \xi_L \leq \theta \leq \xi_R, \\ \text{strictly increasing} & \text{if } \xi_R < \theta < \xi_2, \\ \lim_{\tau \to -2\mu^+} z_\mu(\tau) & \text{if } \xi_2 \leq \theta \leq \frac{\pi}{2} - \delta. \end{cases}$$

The behavior of $Rh_\mu(\cdot)$ on $[\alpha_L, \alpha_R]$ is the definition of a "central fan" of (constant) radial limits (for example [Lancaster and Siegel 1996b]), and the "central fan region" (for example $\{(r\cos(\theta), r\sin(\theta)) \in \Omega : r > 0, \theta \in [\xi_L, \xi_R]\}$) is illustrated in Figure 6. If $\theta^* = \pi/2 + \delta + \gamma_2$, then $\xi_L = \xi_1$, $\xi_L = \delta + \gamma_2$, and $Rh_\mu(\theta)$ is constant on $[\delta - \pi, \xi_R]$ while if $\theta^* = \pi - \delta - \gamma_1$, then $\xi_R = \xi_2$, $\xi_L = -\pi/2 - \delta - \gamma_1$, and $Rh_\mu(\theta)$ is constant on $[\xi_L, \pi/2 - \delta]$. We leave it to the reader to verify the uniqueness of h as in the proof of Theorem 3.1.

7. General results

Theorem 7.1. *Let* $\delta \in (0, \pi/4)$, *and suppose* $\gamma_1, \gamma_2 \in (0, \pi/2)$ *satisfy* (4) *and* $\gamma_1 + \gamma_2 \geq \pi/2 - 2\delta$. *Then there exists a unique* $x_0 \in (-\cot(\delta), -\tan(\delta))$ *such*

that if $B = (x_0, 0)$, $\Omega = \Omega(B)$, and $\Omega_0 = \Omega_0(B)$, then the boundary value problem (1) with $\kappa = \lambda = 0$ and γ as defined in Section 1 has a (unique) solution $h \in C^2(\Omega) \cap C^0(\overline{\Omega} \setminus \{A, B, C, D\})$ which satisfies the conditions that $h(0, 0) = 0$ and whose Gauss map over Ω_0, $\tilde{G} : \Gamma_0 \to \mathbf{S}^2$, given in (5) with $\Gamma_0 = \{(x, y, h(x, y)) : (x, y) \in \Omega_0\}$, extends to a map $\tilde{G} \in C^0(\overline{\Gamma_0} : \mathbf{S}^2)$ which is injective. In addition, if $\gamma_1 + \gamma_2 > \pi/2 - 2\delta$, then $h \in C^1(\overline{\Omega} \setminus \{A, B, C, D\})$. If $\gamma_1 \geq \delta$, then $C^0(\overline{\Omega} \setminus \{B, D\})$, while if $\gamma_2 \geq \delta$, then $C^0(\overline{\Omega} \setminus \{A, C\})$.

Theorem 7.2. *Let $\delta \in (0, \pi/4)$ and $\gamma_1, \gamma_2 \in (0, \pi/2)$ satisfy (4) and*

$$\gamma_1 + \gamma_2 < \pi/2 - 2\delta.$$

For each $\theta_1 \in [\delta + \gamma_2 - \pi, -\pi/2 - \delta - \gamma_1]$, there exists a unique $x_0 = x_0(\theta_1) \in (-\cot(\delta), -\tan(\delta))$ such that if $B = (x_0, 0)$, $\Omega = \Omega(B)$, and $\Omega_0 = \Omega_0(B)$, then the boundary value problem (1) with $\kappa = \lambda = 0$ and γ as defined in Section 1 has a (unique) solution $h \in C^2(\Omega) \cap C^0(\overline{\Omega} \setminus \{A, B, C, D, P, Q, R, S\})$ which satisfies the conditions that $h(0, 0) = 0$ and whose Gauss map $\tilde{G}$ over Ω_0, given in (5), extends to a map $\tilde{G} \in C^0(\overline{\Gamma_0} : \mathbf{S}^2)$ which is injective. This solution is discontinuous at P, Q, R, and S and

$$Rh(\theta; P) = \begin{cases} constant & if \ \theta \in [\delta - \pi, \delta + \gamma_2 - \pi], \\ strictly\ decreasing & if \ \theta \in [\delta + \gamma_2 - \pi, \theta_1], \\ constant & if \ \theta \in [\theta_1, \theta_1 + \pi], \\ strictly\ increasing & if \ \theta \in [\theta_1 + \pi, \pi/2 - \delta - \gamma_1], \\ constant & if \ \theta \in [\pi/2 - \delta - \gamma_1, \pi/2 - \delta]. \end{cases}$$

$Rh(\theta; Q)$, $Rh(\theta; R)$, and $Rh(\theta; S)$ in an appropriately similar manner. If $\gamma_1 \geq \delta$, then $h \in C^0(\overline{\Omega} \setminus \{B, D, P, Q, R, S\})$. If $\gamma_2 \geq \delta$, then $h \in C^0(\overline{\Omega} \setminus \{A, C, P, Q, R, S\})$.

The are eight different possible geometries of E represented by Theorems 7.1 and 7.2. For Theorem 7.1, they are tabulated as follows.

(i)	$\gamma_1 \geq \delta$	$\gamma_2 \geq \delta$	$\gamma_1 + \gamma_2 \geq \pi/2 - 2\delta$	$L = 3$	Figure 2
(ii)	$\gamma_1 \geq \delta$	$\gamma_2 < \delta$	$\gamma_1 + \gamma_2 \geq \pi/2 - 2\delta$	$L = 4$	Figure 7
(iii)	$\gamma_1 < \delta$	$\gamma_2 \geq \delta$	$\gamma_1 + \gamma_2 \geq \pi/2 - 2\delta$	$L = 4$	Figure 7
(iv)	$\gamma_1 < \delta$	$\gamma_2 < \delta$	$\gamma_1 + \gamma_2 \geq \pi/2 - 2\delta$	$L = 5$	Figure 7

For Theorem 7.2, the table is this:

(v)	$\gamma_1 \geq \delta$	$\gamma_2 \geq \delta$	$\gamma_1 + \gamma_2 < \pi/2 - 2\delta$	$L = 4$	Figure 3
(vi)	$\gamma_1 \geq \delta$	$\gamma_2 < \delta$	$\gamma_1 + \gamma_2 < \pi/2 - 2\delta$	$L = 5$	Figure 8
(vii)	$\gamma_1 < \delta$	$\gamma_2 \geq \delta$	$\gamma_1 + \gamma_2 < \pi/2 - 2\delta$	$L = 5$	Figure 8
(viii)	$\gamma_1 < \delta$	$\gamma_2 < \delta$	$\gamma_1 + \gamma_2 < \pi/2 - 2\delta$	$L = 6$	Figure 8

The results of Theorem 3.1 show that Theorem 7.1 is true in case (i) and Theorem 3.2 shows that Theorem 7.2 is correct in case (v). Here we have used the notation of Section 4 to indicate (by L) the number of corners, excluding the origin, which
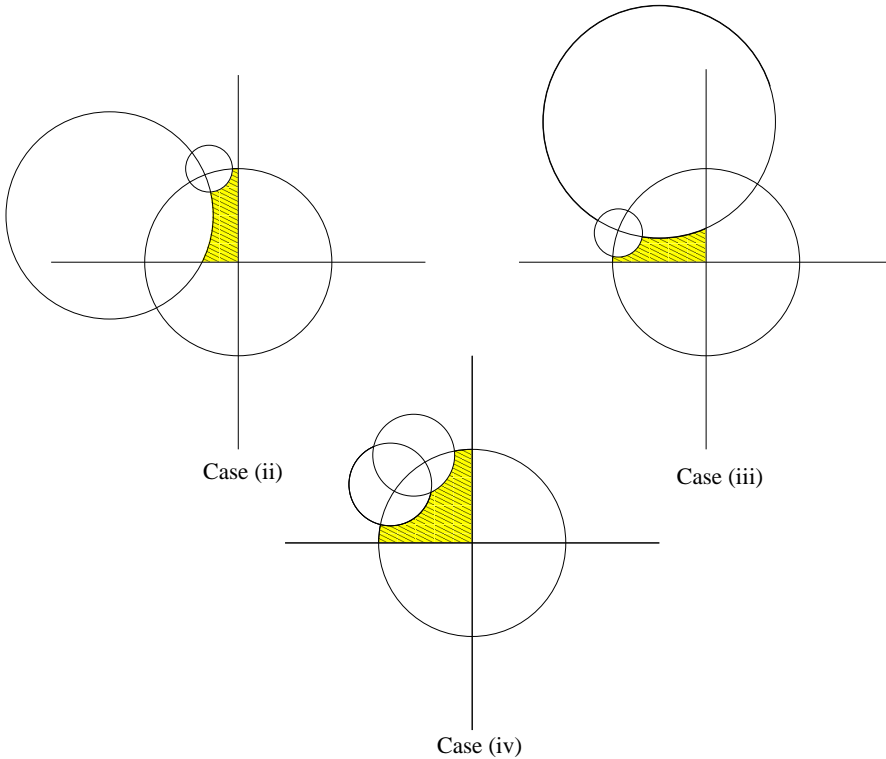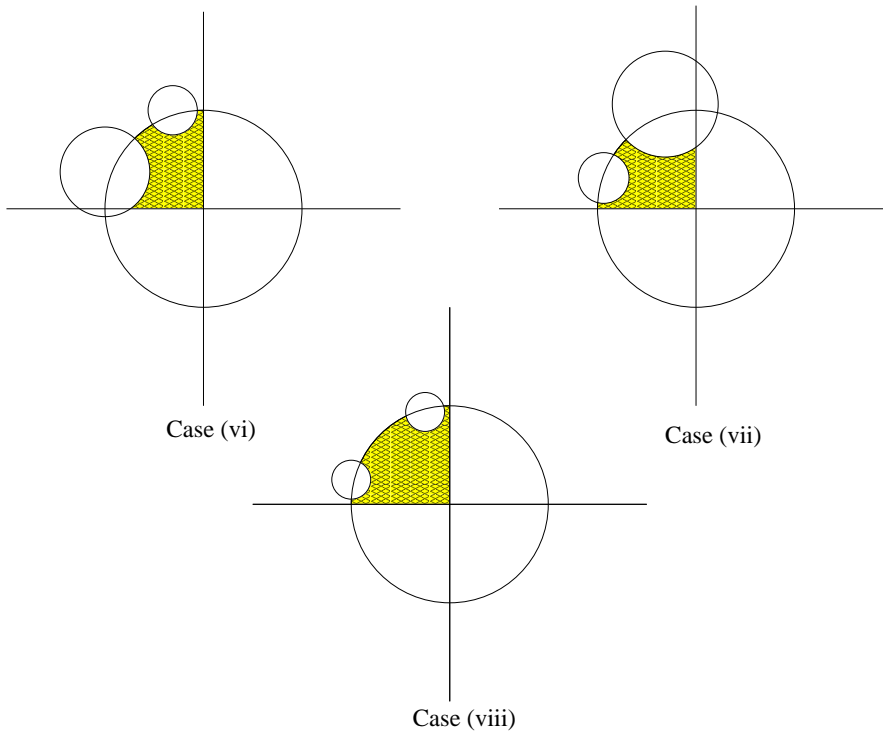
Figure 7. Other Theorem 7.1 cases.

the boundary of E contains. In cases (ii)–(iv), the proof of Theorem 3.1 remains unchanged except for certain details which differ from those in Section 5. The changes for each case are listed below.

(ii) $t_3 = e^{i(\pi/2+\delta-\gamma_2)}$, $t_4 = i$, $\theta_3 = -2\delta/\pi$, $\theta_4 = 0$, and $\alpha_4 = \theta_4$.

(iii) $t_2 = e^{i(\pi-\delta+\gamma_1)}$, $t_4 = -1$, $\theta_2 = 1/2 + 2(\delta - \gamma_1)/\pi$, $\theta_4 = 0$, and $\alpha_4 = \theta_4$.

(iv) $t_2 = e^{i(\pi-\delta+\gamma_1)}$, $t_3 = e^{i(\pi/2+\delta-\gamma_2)}$, $t_4 = i$, $t_5 = -1$, $\theta_2 = 1/2 + 2(\delta - \gamma_1)/\pi$,
   $\theta_3 = -2\delta/\pi$, $\theta_4 = 0$, $\theta_5 = 0$, $\alpha_4 = \theta_4$, and $\alpha_5 = \theta_5$.

In cases (vi)-(viii), the proof of Theorem 3.2 remains unchanged except for certain differences from Section 6:

(vi) $t_3 = e^{i(\pi/2+\delta-\gamma_2)}$, $t_5 = i$, $\theta_3 = -2\delta/\pi$, $\theta_5 = 0$, and $\alpha_5 = \theta_5$.

(vii) $t_2 = e^{i(\pi-\delta+\gamma_1)}$, $t_5 = -1$, $\theta_2 = 1/2 + 2(\delta - \gamma_1)/\pi$, $\theta_5 = 0$, and $\alpha_5 = \theta_5$.

(viii) $t_2 = e^{i(\pi-\delta+\gamma_1)}$, $t_3 = e^{i(\pi/2+\delta-\gamma_2)}$, $t_5 = i$, $t_6 = -1$, $\theta_2 = 1/2 + 2(\delta - \gamma_1)/\pi$,
   $\theta_3 = -2\delta/\pi$, $\theta_4 = 0$, $\theta_5 = 0$, $\alpha_5 = \theta_5$, and $\alpha_6 = \theta_6$.

Case (vi)

Case (vii)

Case (viii)

Figure 8. Other Theorem 7.2 cases.

Remark 7.3. It would be interesting to extend this work to appropriate analytic functions $g : E \to B_1$ which map ∂E homeomorphically onto ∂E but are not injective over E. In this case, although the surface $z = h(x, y)$ would be a strict minimizer of the appropriate capillary functional, the surface $z = h(x, y)$ would have "undulations" or "wobbles" which one might expect to increase surface area.

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