SMOOTHNESS OF ALGEBRAIC SUPERVARIETIES
AND SUPERGROUPS

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We discuss the notion of smoothness in complex algebraic supergeometry and prove that all affine complex algebraic supergroups are smooth. We then prove the stabilizer theorem in the algebraic context, providing some useful applications.

1. Introduction

The category of differentiable supermanifolds was introduced and discussed in several works — among which [Batchelor 1984; 1985; Berezin 1987; Kostant 1977; Leites 1980; Manin 1988] — from different point of views, especially in connection with important physical applications stemming from string theory and ultimately related to the problem of classification of elementary particles.

In this paper we are interested in algebraic supergeometry and its relation with its differential counterpart. In his foundational work on supermanifolds, Manin [1988] defined the notion of a superscheme and discussed some important examples.

Along the same lines we want to understand the concept of smoothness in complex algebraic supergeometry. Given the algebraic nature of the problems in the theory of supermanifolds, we believe that a deep analysis of the superalgebraic category can shed light also on the differential one. Moreover this is the correct category to work with when one wants to discuss quantum deformations of the geometric objects.

In ordinary algebraic geometry smoothness is a local notion, strongly linked to the dimension of the local ring of the variety at the point. Unfortunately, due to the presence of the odd nilpotents, it is not easy to generalize the idea of dimension of a ring to the supergeometric context. To overcome this problem, we define smoothness as a property of the completion of the local superring of the supervariety at a given point: a point is *smooth* if the local superring is isomorphic to a power series superring. We are then able to show that any supervariety admits a unique supermanifold structure in a neighborhood of a smooth point, as in the classical

case, through the application of the implicit function theorem, after reduction to local complete intersection.

Using Cartier's Theorem adapted to supergeometry we can then prove that all algebraic supergroups are smooth; in other words, *all affine algebraic supergroups are also Lie supergroups*. We apply this to the stabilizer supergroup functor of an action of an affine supergroup on an affine supervariety. After showing that the stabilizer is representable, hence a supergroup, we show that classical supergroups are smooth.[1] This fact is generally known; it is treated in a different context in [Gruson 1994] and [Varadarajan 2004, p. 289], for example. However, we provide an independent proof using algebraic techniques, which we believe can be of help to other questions on differentiable supermanifold and may yield other examples of algebraic Lie supergroups.

This paper is organized as follows:

In Section 2, we review basic facts of algebraic and differential supergeometry, such as the definition of supermanifolds, supervarieties and their functor of points.

In Section 3 we define smooth points of a supervariety. We then prove the super version of the classical result that a smooth point of a complex algebraic variety admits a supermanifold structure in a suitable neighborhood.

In Section 4 we prove that all (closed) points of complex algebraic groups are smooth.

In Section 5 we prove the Stabilizer Theorem, which states that the stabilizer functor for the action of an affine algebraic supergroup on an affine supervariety is representable by a supergroup, so it is a smooth variety, hence a supermanifold.

As an application, in Section 6, we show that the classical supergroup functors as described in [Deligne and Morgan 1999, p. 70] are representable, and consequently they are Lie supergroups.

2. Basic definitions of supergeometry

In this section we recall some basic definitions and facts in supergeometry. For more details see [Varadarajan 2004; Caston and Fioresi 2007; Deligne and Morgan 1999; Manin 1988].

Let k be the ground field. A *superalgebra* A is a $\mathbb{Z}_2$-graded algebra, $A = A_0 \oplus A_1$. We denote by $p(x)$ the parity of an homogeneous element $x \in A$. We say that A is *commutative* if

$$xy = (-1)^{p(x)p(y)} yx.$$

Definition 2.1. A *superspace* $S = (|S|, \mathcal{O}_S)$ is a topological space $|S|$ endowed with a sheaf of superalgebras $\mathcal{O}_S$ such that the stalk $\mathcal{O}_{S,x}$ is a local superalgebra

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[1]For a list of classical supergroups see [Deligne and Morgan 1999, p. 70], for instance.

for all $x \in |S|$. A *morphism* $\phi : S \to T$ of superspaces is given by $\phi = (|\phi|, \phi^*)$, where $\phi : |S| \to |T|$ is a map of topological spaces and $\phi^* : \mathbb{O}_T \to \phi^* \mathbb{O}_S$ is a sheaf morphism such that $\phi_x^*(m_{|\phi|(x)}) = m_x$ where $m_{|\phi|(x)}$ and m_x are the maximal ideals in the stalks $\mathbb{O}_{T,|\phi|(x)}$ and $\mathbb{O}_{S,x}$ respectively.

The most important examples of superspaces are given by supermanifolds and superschemes.

Definition 2.2. Consider the superspace $\mathbb{C}^{p|q} = (\mathbb{C}^p, \mathcal{H}_{\mathbb{C}^{p|q}})$, where

$$\mathcal{H}_{\mathbb{C}^{p|q}}|_U = \mathcal{H}_{\mathbb{C}^p}|_U \otimes \mathbb{C}[\xi_1, \ldots, \xi_q], \qquad U \text{ open in } \mathbb{C}^p$$

where $\mathbb{C}[\xi_1, \ldots, \xi_q]$ is the exterior algebra generated by $\xi_1, \ldots, \xi_q$ and $\mathcal{H}_{\mathbb{C}^p}$ denotes the sheaf of holomorphic functions on $\mathbb{C}^p$.

A *complex supermanifold* of dimension $p|q$ is a superspace $M = (|M|, \mathcal{H}_M)$ which is locally isomorphic to $\mathbb{C}^{p|q}$; that is, for all $x \in |M|$ there exist open sets $V_x \subset |M|$, $U \subset \mathbb{C}^p$ such that

$$\mathbb{O}_M|_{V_x} \cong \mathcal{H}_{\mathbb{C}^{p|q}}|_U$$

Definition 2.3. A *superscheme* S is a superspace $(|S|, \mathbb{O}_S)$ such that $(|S|, \mathbb{O}_{S,0})$ is a quasicoherent sheaf of $\mathbb{O}_{S,1}$-modules. A *morphism* of supermanifolds or of superschemes is a morphisms of the corresponding superspaces.

Superschemes can be characterized by a local model:

Definition 2.4 (Spec A). Let A be a superalgebra and let $\mathbb{O}_{A_0}$ be the structural sheaf of the ordinary scheme $\underline{\text{Spec}}(A_0) = (\text{spec } A_0, \mathbb{O}_{A_0})$ (spec A_0 denotes the prime spectrum of the commutative ring A_0). The stalk of the sheaf at the prime $p \in$ spec A_0 is the localization of A_0 at p. As for any superalgebra, A is a module over A_0. We have indeed a sheaf $\mathbb{O}_A$ of $\mathbb{O}_{A_0}$-modules over spec A_0 with stalk A_p, the localization of the A_0-module A over the prime $p \in$ spec A_0:

$$A_p = \{f/g \mid f \in A, \ g \in A_0 - p\}.$$

A_p contains a unique two-sided maximal ideal generated by the maximal ideal in the local ring $(A_p)_0$ and the generators of $(A_p)_1$ as A_0-module.

$\mathbb{O}_A$ is a sheaf of superalgebras and $(\text{spec } A_0, \mathbb{O}_A)$ is a superscheme that we denote with $\underline{\text{Spec}} A$.

The next proposition shows that $\underline{\text{Spec}} A$ is the local model for superschemes.

Proposition 2.5. *A superspace S is a superscheme if and only if it is locally isomorphic to $\underline{\text{Spec}} A$ for some superalgebra A; that is, for all $x \in |S|$, there exists $U_x \subset |S|$ open such that $(U_x, \mathbb{O}_S|_{U_x}) \cong \underline{\text{Spec}} A$. (Clearly A depends on U_x.)*

Proof. See [Caston and Fioresi 2007, §3]. □

Definition 2.6. We say that a superscheme X is *affine* if it is isomorphic to $\underline{\mathrm{Spec}}\, A$ for some algebra A and we call $k[X] := A$ the *coordinate ring* of the affine superscheme X. Let I^{odd} be the ideal generated by the odd nilpotents of A. If $k[X]/I^{\mathrm{odd}}$ is the coordinate ring of an ordinary affine algebraic variety (called the *reduced variety* associated to X) and $(|X|, \mathcal{O}_{X,0})$ is a coherent sheaf of $\mathcal{O}_{X,1}$-modules, we say that X is an *affine algebraic variety*.

Remark 2.7. There is an equivalence of categories between superalgebras and affine superschemes. This equivalence is treated in detail in [Caston and Fioresi 2007, §3].

Definition 2.8. Let X be a supervariety. The *functor of points* of X is given by

$$h_X : (\text{salg}) \rightarrow (\text{sets}), \qquad h_X(A) = \mathrm{Hom}(\underline{\mathrm{Spec}}\, A, X),$$

where (salg) is the category of commutative superalgebras. If X is an affine supervariety $h_X(A) = \mathrm{Hom}(k[X], A)$. If h_X is group-valued we say that X is an *affine supergroup*. This is equivalent to the fact that $k[X]$ is a Hopf superalgebra. This is also the same as giving a multiplication $m : X \times X \rightarrow X$ and an inverse $i : X \rightarrow X$ satisfying the usual commutative diagrams.

More generally, we say that $G : (\text{salg}) \rightarrow (\text{sets})$ is a *supergroup functor* if it is group-valued. Clearly, a representable supergroup functor is an affine supergroup.

3. Smoothness of complex algebraic supervarieties

Let $k = \mathbb{C}$.

Let $X = (|X|, \mathcal{O}_X)$ be a supervariety and let $P \in |X|$ be a *closed point*; that is, P corresponds to a maximal ideal. Let m_P be the maximal ideal in $\mathcal{O}_{X,P}$.

Definition 3.1. We say that P is *smooth* if

$$\widehat{\mathcal{O}_{X,P}} \cong \mathbb{C}[[x_1, \ldots, x_r, \xi_1, \ldots, \xi_s]], \qquad \widehat{\mathcal{O}_{X,P}} = \varprojlim \mathcal{O}_{X,P}/m_P^n$$

where the x_i and ξ_j are respectively even and odd variables. In this case we say that the *dimension* of the supervariety X at P is $r|s$. Notice that the dimension is well defined, that is if $\mathbb{C}[[x_1, \ldots, x_r, \xi_1, \ldots, \xi_s]] \cong \mathbb{C}[[x_1, \ldots, x_m, \xi_1, \ldots, \xi_n]]$ then $r = m$, $n = s$.

Smoothness of a point of a supervariety cannot be checked at the classical level as the next examples show.

Example 3.2. Consider the supervariety X with coordinate ring

$$\mathbb{C}[X] = \mathbb{C}[x, y, \xi, \eta]/(\xi\eta).$$

Its reduced variety is the affine plane, where all the closed points are smooth in the classical sense. It is immediate to check that this supervariety has no smooth points according to Definition 3.1.

Example 3.3. Take the supervariety with coordinate ring $\mathbb{C}[x, y, \xi, \eta]/(\xi x + \eta y)$. Again its reduced variety is the affine plane. One can check that all (closed) points are smooth except the origin.

Since the notion of smoothness is local we can assume that X is an affine supervariety, with coordinate ring $\mathbb{C}[X] = \mathbb{C}[x_1, \ldots, x_m, \xi_1, \ldots, \xi_n]/I$, where $I = (f_1, \ldots, f_p, \phi_1, \ldots, \phi_q)$. In this case $\mathcal{O}_{X,P}$ is the localization of $\mathbb{C}[X]$ at the point P (see Definition 2.4).

Definition 3.4. As in the classical setting, we define the *jacobian* of $f_1, \ldots, f_p$, $\phi_1, \ldots, \phi_q \in \mathbb{C}[x_1, \ldots, x_m, \xi_1, \ldots, \xi_n]$ at a point P as

$$\mathrm{Jac}(f, \phi)(P) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(P) & \cdots & \frac{\partial f_1}{\partial x_m}(P) & \frac{\partial f_1}{\partial \xi_1}(P) & \cdots & \frac{\partial f_1}{\partial \xi_n}(P) \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial \phi_q}{\partial x_1}(P) & \cdots & \frac{\partial \phi_q}{\partial x_m}(P) & \frac{\partial \phi_q}{\partial \xi_1}(P) & \cdots & \frac{\partial \phi_q}{\partial \xi_n}(P) \end{pmatrix}$$

(for the definition of $\partial f/\partial x$ see for example [Varadarajan 2004]). The *rank* of the jacobian is given by $a|b$ where a and b are the ranks of the $m \times p$, $n \times q$ diagonal blocks.

Lemma 3.5. *Let the notation be as above. Let $P \in |X|$ be a closed point, i.e., a maximal ideal m_P in $\mathbb{C}[X]$. Then*

$$\mathrm{rk}(\mathrm{Jac}(f, \phi)(P)) = m|n - \dim(m_P/m_P^2).$$

Proof. The proof is the same as in ordinary case (see for example [Hartshorne 1977, p. 32]). We sketch it. We have a natural identification

$$F : M_P/M_P^2 \cong \mathbb{C}^{m|n}, \quad f \mapsto df_P := (\frac{\partial f}{\partial x_1}(P), \ldots, \frac{\partial f}{\partial \xi_n}(P)),$$

where M_P denotes the maximal ideal corresponding to the point P in $\mathbb{C}[x_1, \ldots, x_m, \xi_1, \ldots, \xi_n]$. Viewing the rows of $\mathrm{Jac}(f, \phi)$ as vectors in $\mathbb{C}^{m|n}$ the above identification tells us immediately that

$$\mathrm{rk}(\mathrm{Jac}(f, \phi)(P)) = \dim(I + M_P^2)/M_P^2$$

where $I = (f_1, \ldots, f_p, \phi_1, \ldots, \phi_q)$. Since localizations commute with quotients, we have

$$m_P/m_P^2 \cong (M_P/I)/((M_P^2 + I)/I) = M_P/(M_P^2 + I).$$

Hence

$$\mathrm{rk}(\mathrm{Jac}(f, \phi)(P)) = \dim(I + M_P^2)/M_P^2 = \dim M_P/M_P^2 - \dim M_P/(M_P^2 + I). \quad \square$$

If A is a local superring with maximal ideal m, let $\mathrm{Gr}(A) = \bigoplus m^i/m^{i+1}$.

Proposition 3.6. *Let P be a smooth point of an affine supervariety X with dimension $r|s$ in P.*

(1) *m_P/m_P^2 has dimension $r|s$.*

(2) *$\mathrm{Gr}(\mathcal{O}_{X,P}) = \mathbb{C}[x_1, \ldots, x_r, \xi_1, \ldots, \xi_s]$, where $\mathrm{Gr}(\mathcal{O}_{X,P}) = \oplus_i m_P^i/m_P^{i+1}$.*

(3) *$\mathrm{rk}(\mathrm{Jac}(f, \phi)(P)) = m|n - r|s$.*

Proof. Statements (1) and (2) follow from Lemma A.5 in the Appendix, and (3) is a consequence of Lemma 3.5. $\square$

Remark 3.7. The proof of this result resembles the one for the commutative setting. One difference that may cause confusion is this: When we localize $\mathbb{C}[X]$ to obtain $\mathcal{O}_{X,P}$ we are using a maximal ideal of the even part $\mathbb{C}[X]_0$, that is, the ideal generated by $x_i - a_i$ and $\xi_i \xi_j$, for all i and $j < i$, where the $a_i \in \mathbb{C}$ are the coordinates of the closed point P. On the other hand, when we are completing the local superalgebra $\mathcal{O}_{X,P}$ we are taking the inverse limit of the system $\mathcal{O}_{X,P}/m_P^n$, where m_P is the maximal ideal of this superalgebra, hence it is a graded object and it will necessarily contain all the odd generators.

Remark 3.8. If P is smooth and X has dimension $r|s$ at P, m_P/m_P^2 is generated by $r|s$ elements. Hence, by Theorem A.6, m_P is generated by $r|s$ elements.

Remark 3.9. The affine supervariety X is embedded in $\mathbb{C}^{m|n}$ via the chosen explicit presentation of its coordinate ring $\mathbb{C}[X]$. Hence we can give to the set of closed points of X a complex topology inherited from this embedding. However this topology is independent from the embedding; this is a classical fact, still valid in this setting since it is a topological question. We want to show that the closed points of the supervariety X equipped with this complex topology, admit a unique *supermanifold structure* in a suitable complex neighbourhood U of the smooth point P. In other words, we want to show that

$$(*)\qquad\qquad (\mathcal{H}_{\mathbb{C}^{m|n}}/\mathcal{K})|_U \cong (\mathcal{H}_{\mathbb{C}^m}/\mathcal{K}^{\mathrm{red}})|_U \otimes \mathbb{C}[\xi_1, \ldots, \xi_s],$$

where $\mathcal{K}$ is the ideal sheaf whose global sections are generated in $\mathcal{H}_{\mathbb{C}^{m|n}}$ by the ideal I of the supervariety X. The whole question in the supergeometry setting is to show the existence of a local splitting $(*)$. To settle this problem our strategy is to use the implicit functions theorem, which is still valid in this setting. We recall the statement from [Leites 1980, p. 52]:

Theorem 3.10. *Let M be a complex supermanifold, $P \in |U|$, where $U \subset M$, is isomorphic to an open in $\mathbb{C}^{r|s}$. Let K be the ideal in $\mathcal{H}_M(U)$ generated by $g_1, \ldots, g_p, \gamma_1, \ldots, \gamma_q$ vanishing [2] at P and with linearly independent differentials at P. Then there exists a unique subsupermanifold*

$$N = (|N|, \mathcal{H}_N), \qquad \mathcal{H}_N = \mathcal{H}_M|_U/\mathcal{K}$$

where $\mathcal{K}$ is the sheaf of ideals with global sections K and $|N|$ is the topological space whose existence is granted by the classical result.

Remark 3.11. The key for the proof of this result is the fact that any set of functions $g_1, \ldots, g_p, \gamma_1, \ldots, \gamma_q$ with linearly independent differentials at P can be completed to obtain a set of local coordinates in a neighborhood of P. More details on this can be found in [Varadarajan 2004, p. 148].

This theorem allows us, in a special case, to obtain immediately the result we are after.

Corollary 3.12. *Let $P \in |X|$ be a smooth point, and let X have dimension $r|s$ at P. Let the ideal I of the supervariety X be of the form $(f_1, \ldots, f_{m-r}, \phi_1, \ldots, \phi_{n-s})$ (in this case we say that X is a complete intersection). Then in a neighborhood of P, X admits a complex supermanifold structure (in the sense of Observation 3.9).*

Proof. This is a direct application of the Theorem 3.10. X is defined in $\mathbb{C}^{m|n}$ by the polynomials $f_1, \ldots, f_{m-r}, \phi_1, \ldots, \phi_{n-s}$ with $\mathrm{rk}(\mathrm{Jac}(f_i, \phi_j)(P)) = m|n - r|s$. Consider the ideal K generated in $\mathcal{H}(\mathbb{C}^{m|n})$ by the f_i and ϕ_j. Then there exists a unique subsupermanifold N of $\mathbb{C}^{m|n}$ such that $\mathcal{H}_N = (\mathcal{H}_{\mathbb{C}^{m|n}}/\mathcal{K})|_U$ for a suitable neighborhood U of P, $\mathcal{K}$ is the ideal sheaf whose global sections are K. $\qquad\square$

In general the ideal I of the supervariety X is given by $(f_1, \ldots, f_p, \phi_1, \ldots, \phi_q)$ where $p|q > m|n - r|s$. We want to show that, as it happens for the classical setting, X is locally a complete intersection, so that we can conclude our discussion with the same reasoning as in Corollary 3.12. Let $P \in X$ be a smooth point and assume $f_1, \ldots, f_{m-r}, \phi_1, \ldots, \phi_{n-s}$ are such that

$$\mathrm{rk}(\mathrm{Jac}(f_1, \ldots, f_{m-r}, \phi_1, \ldots, \phi_{n-s})(P)) = m|n - r|s.$$

Let X' be the variety corresponding to the ring

$$\mathbb{C}[X'] = \mathbb{C}[x_1, \ldots, x_m, \xi_1, \ldots, \xi_n]/(f_1, \ldots, f_{m-r}, \phi_1, \ldots, \phi_{n-s})$$

and let $\mathbb{O}_{X', P}$ denote its local ring at the closed point P. We are going to show the following:

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[2] We say that $f \in \mathcal{H}_M(U)$ *vanishes at P* if f is annihilated by the morphism $\mathcal{H}_M(U) \to \mathcal{H}_{M,P} \to \mathcal{H}_{M,P}/m_{h,P} \cong \mathbb{C}$, where $m_{h,P}$ is the maximal ideal in $\mathcal{H}_{M,P}$.

1. P is a smooth point of X', and X' has the same dimension as X, that is, $\mathbb{O}_{X',P} = \mathbb{C}[[x_1, \ldots, x_r, \xi_1, \ldots, \xi_s]]$. This implies that X' is a complete intersection.

2. X and X' are locally isomorphic: $\mathbb{O}_{X,P} \cong \mathbb{O}_{X',P}$. Since this holds for every point in a neighbourhood of P, we have $\mathbb{O}_X(U) \cong \mathbb{O}_{X'}(U)$. Hence we can apply Theorem 3.10 to X' to conclude that X admits a supermanifold structure near P.

Lemma 3.13. *There is a commutative diagram*

$$
\begin{array}{ccc}
\mathbb{O}_{X',P} & \longrightarrow\!\!\!\rightarrow & \mathbb{O}_{X,P} \\
\downarrow & & \downarrow \\
\widehat{\mathbb{O}_{X',P}} & \longrightarrow\!\!\!\rightarrow & \widehat{\mathbb{O}_{X,P}}
\end{array}
$$

where the orizontal arrows are surjections, while the vertical ones injections.

Proof. Observe that since there is a surjection $\mathbb{C}[X'] \to \mathbb{C}[X]$ we also have a surjective morphism

$$\mathbb{O}_{X',P} \to \mathbb{O}_{X,P}$$

mapping the maximal ideal onto the maximal ideal (this is a property of localizations). This will give raise to a surjective system

$$\mathbb{O}_{X',P}/{m_P'}^n \to \mathbb{O}_{X,P}/m_P^n.$$

where m_P and m_P' denote the maximal ideals in $\mathbb{O}_{X,P}$ and $\mathbb{O}_{X',P}$. Hence $\widehat{\mathbb{O}_{X',P}} \to \widehat{\mathbb{O}_{X,P}}$ is a surjective map. The vertical arrows are injections since $\cap m_P^i = \cap m_P'^i = (0)$. This happens since this is true in the ordinary case and since the odd variables disappear for large i. $\qquad\square$

Remark 3.14. From Lemma 3.5 we see directly that $\dim(m_P/m_P^2) = \dim(m_P'/m_P'^2)$ and, since the point P is smooth,

$$\dim(m_P/m_P^2) = \dim(m_P'/m_P'^2) = r|s.$$

Hence, by Theorem A.6, both m_P and m_P' are generated by $r|s$ elements.

Lemma 3.15.
$$\widehat{\mathbb{O}_{X',P}} \cong \mathbb{C}[[x_1, \ldots, x_r, \xi_1, \ldots, \xi_s]]/I$$

for a suitable ideal I.

Proof. By the Theorem A.2, there exists a unique map

$$\mathbb{C}[[x_1, \ldots, x_r, \xi_1, \ldots, \xi_s]] \to \widehat{\mathbb{O}_{X',P}}$$

sending the x_i and ξ_j into $r|s$ generators of the maximal ideal m_P'. So the map is surjective and we obtain our result. $\qquad\square$

Proposition 3.16. $\widehat{\mathbb{O}_{X',P}} \cong \widehat{\mathbb{O}_{X,P}} = \mathbb{C}[[x_1, \ldots, x_r, \xi_1, \ldots, \xi_s]]$.

Proof. By Lemma 3.13,

$$\widehat{\mathcal{O}_{X',P}}/J \cong \widehat{\mathcal{O}_{X,P}} = \mathbb{C}[[x_1, \ldots, x_r, \xi_1, \ldots, \xi_s]]$$

By the Theorem A.4 in the Appendix, we get the result.   □

We have proved the local isomorphism in the completions, now we turn our attention to the local rings.

Lemma 3.17. $\mathcal{O}_{X',P} \cong \mathcal{O}_{X,P}.$

Proof. By Proposition 3.16 and Lemma 3.13.   □

This concludes the proof of the following:

Theorem 3.18. *Let X be a complex algebraic supervariety, P a smooth point of X. Then, there exist a neighbourhood of P where we can give to X a unique structure of a complex supermanifold.*

Proof. Assume without loss of generality that X is affine and has dimension $r|s$ at P. Let

$$\mathbb{C}[X] = \mathbb{C}[x_1, \ldots, x_m, \xi_1, \ldots, \xi_n]/(f_1, \ldots, f_p, \phi_1, \ldots, \phi_q)$$

be the coordinate ring of X. Let X' be the algebraic supervariety defined by the coordinate ring

$$\mathbb{C}[X'] = \mathbb{C}[x_1, \ldots, x_m, \xi_1, \ldots, \xi_n]/(f_1, \ldots, f_{m-r}, \phi_1, \ldots, \phi_{n-s})$$

where $\mathrm{rk}(\mathrm{Jac}(f_1, \ldots, f_{m-r}, \phi_1, \ldots, \phi_{n-s})(P)) = m|n - r|s$. Then by Corollary 3.12 the result holds for X' and by Lemma 3.17 X and X' are locally isomorphic.
          □

The next lemma will be crucial in the discussion of smoothness of algebraic supergroups in Section 4.

Lemma 3.19. *Let m_P be generated by $r|s$ elements. Then P is smooth if*

$$\mathrm{Gr}(\mathcal{O}_{X,P}) = \mathbb{C}[x_1, \ldots, x_r, \xi_1, \ldots, \xi_s].$$

Proof. By Theorem A.2, there exists a surjective map

$$\mathbb{C}[[x_1, \ldots, x_r, \xi_1, \ldots, \xi_s]] \to \widehat{\mathcal{O}_{X,P}}$$

sending $x_1, \ldots, x_r, \xi_1, \ldots, \xi_s$ into the generators of the maximal ideal of $\widehat{\mathcal{O}_{X,P}}$. Hence $\widehat{\mathcal{O}_{X,P}} = \mathbb{C}[[x_1, \ldots, x_r, \xi_1, \ldots, \xi_s]]/J$ for some ideal J. Since $\mathrm{Gr}(\mathcal{O}_{X,P}) = \mathrm{Gr}(\widehat{\mathcal{O}_{X,P}})$ by Lemma A.5 the result follows by Lemma A.3.   □

4. Smoothness of supergroups

In this section we want to show that affine algebraic supergroups are smooth, that is all closed points are smooth. In other words we show that the set of closed points of an affine supergroup has a supermanifold structure in the sense of Observation 3.9, hence it is a Lie supergroup. We will do this by using an argument appearing in the classical Cartier's theorem which states that Hopf algebras over a field of characteristic zero are reduced.

It is enough to prove that the identity is a smooth point, since, because of the multiplication law, all closed points have the same local structure.

Let G be an affine algebraic supergroup, $\mathbb{C}[G]$ its Hopf superalgebra with comultiplication Δ, counit ϵ and antipode S. Let $m_1 = \ker \epsilon$ be the maximal ideal of the identity element and let

$$m_1/m_1^2 = \operatorname{span}_{\mathbb{C}}\{t_1, \ldots, t_{r+s}\}$$

where $t_1, \ldots, t_r$ are even and $t_{r+1}, \ldots, t_{r+s}$ are odd.

By an abuse of notation, let $t_1, \ldots, t_{r+s}$, denote also the image of these elements modulo m_1^N.

Lemma 4.1. *The monomials* $t_1^{n_1} \ldots t_{r+s}^{n_{r+s}}$, *with* $\sum_{i=1}^{r+s} n_i = N$ *form a basis for the superspace* m_1^N/m_1^{N+1}. *(Clearly* $n_i = 0, 1$ *if* i *is the index of an odd element,* $i = r+1, \ldots, s$*).*

Proof. The proof is the same as in the classical case, we include it here for completeness; for more details see [Waterhouse 1979, p. 86]. Let $t_1^*, \ldots, t_{r+s}^*$ be the dual basis of $t_1, \ldots, t_{r+s}$. Define the map

$$d_l : \mathbb{C}[G] = \mathbb{C} \oplus m_1 \to m_1/m_1^2 \to \mathbb{C}$$

as $d_l = t_l^* \cdot p, l = 1, \ldots, r+s$, where $p : \mathbb{C}[G] \to m_1/m_1^2$ is the natural projection.

Each d_l gives rise to a derivation $D_l : \mathbb{C}[G] \to \mathbb{C}[G]$ in the following way:

$$D_l(a) := \sum a^{(1)} d_l(a^{(2)}), \qquad \text{where} \quad \Delta(a) = \sum a^{(1)} \otimes a^{(2)}.$$

Observe that

$$\epsilon(D_l(a)) = \sum \epsilon(a^{(1)}) d_l(a^{(2)}) = d_l \sum \epsilon(a^{(1)}) a^{(2)} = d_l(a).$$

Hence $D_i(t_j) \equiv \delta_{ij}$ modulo m_1 (since $\ker \epsilon = m_1$). Let $P(T_1, \ldots, T_{r+s})$ be a homogeneous polynomial of degree n over $\mathbb{C}$; then

$$D_i(P)(t_1, \ldots, t_{r+s}) = \sum D_i(t_j) \frac{\partial P}{\partial T_j}(t_1, \ldots, t_{r+s}).$$

Since $\partial P/\partial T_j(t_1, \ldots, t_{r+s}) \in m_1^{n-1}$, we have $D_i(P) \equiv \partial P/\partial t_j$ modulo m_1^n. Since the congruence $x \equiv y$ modulo m_1^n implies $D_i(x) \equiv D_i(y)$ modulo m_1^{n-1}, we get

$$D_{r+s}^{n_{r+s}} \ldots D_1^{n_1} t_1^{n_1} \ldots t_{r+s}^{n_{r+s}} = n_1! \ldots n_{r+s}! \qquad \mod m_1$$

while on all other monomials the composition of the D_i will give zero. Hence, given a relation P in m_1^{n+1}, one can single out the coefficient of any monomial by applying the correct sequence of D_i's. $\qquad \square$

Corollary 4.2. *The identity $1 \in |G|$ is a smooth point.*

Proof. By Lemma 4.1 and Lemma A.5, the graded associated ring to $\mathbb{O}_{G,1}$ is

$$\mathrm{Gr}(\mathbb{O}_{1,G}) = \mathbb{C}[t_1, \ldots, t_r, \theta_1, \ldots, \theta_s].$$

This implies by Lemma 3.19 that the identity point is smooth. $\qquad \square$

Corollary 4.3. *If G is an affine supergroup, then G is smooth, that is all its closed points are smooth.*

Proof. Let h_G denote the functor of points of G and $\mu : h_G \times h_G \to h_G$ the natural transformation corresponding to the group law. Let $g \in |G|$ be a closed point. g can be identified with an element of $h_G(\mathbb{C}) \subset h_G(A)$. Hence we can define a natural transformation

$$l_g : h_G \to h_G, \qquad l_{g,A}(x) = m_A(g, x), \forall x \in h_G(A)$$

This natural transformation corresponds to an isomorphism of G into itself, hence $\mathbb{O}_{G,1} \cong \mathbb{O}_{G,g}$, so g is smooth. (For more details on the correspondence between natural transformations between functor of points and morphisms of the supervarieties see [Caston and Fioresi 2007] Chapter 3). $\qquad \square$

5. The Stabilizer Theorem

In this section we use the same letter X to denote both a supervariety X and its functor of points h_X. Let G be an affine algebraic supergroup acting on an affine supervariety X, in other words we have a morphism

$$\rho : G \times X \to X, \qquad (g, x) \mapsto g \cdot x, \qquad \forall g \in G(A), x \in X(A)$$

satisfying the usual properties, viewed in the category of supervarieties. Let u be a topological point of X, that is $u \in |X|$ or equivalently $u \in X(\mathbb{C}) = \mathrm{Hom}(\mathbb{C}[X], \mathbb{C})$. Let m_u be the maximal ideal corresponding to u. Notice that u can be viewed naturally as an A-point u_A for all superalgebras A since $\mathbb{C} \subset A$. So we have a morphism

$$\tau : G \to X, \qquad g \mapsto g \cdot u_A$$

or, equivalently,

$$\tilde{\tau} : \mathbb{C}[X] \to \mathbb{C}[G].$$

Definition 5.1. We define the *stabilizer supergroup functor* of a point $u \in |X|$ with respect to the action ρ as the group-valued functor $\mathrm{Stab}_u : (\mathrm{salg}) \to (\mathrm{sets})$ given by

$$\mathrm{Stab}_u(A) = \{g \in G(A) \quad | \quad \tau_A(g) = g \cdot u_A = u_A\},$$

where $\tau_A : G(A) \to X(A)$ or, equivalently,

$$\mathrm{Stab}_u(A) = \{g \in G(A) = \mathrm{Hom}(\mathbb{C}[G], A) \quad | \quad g \cdot \tilde{\tau} = u_A\}$$

We now prove that this functor is representable by an affine supergroup; in the next section we describe some important applications of this result.

Theorem 5.2. *Let G be an affine supergroup acting on an affine supervariety X and let u be a topological point of X. Then* Stab_u *is an affine supergroup.*

Proof. The stabilizer can be described in an equivalent way as

$$\mathrm{Stab}_u(A) = \{g \in G(A) \mid (g \cdot \tau)|_{m_u} = 0\}$$

where $m_u \subset \mathbb{C}[X]$ is the ideal of u. Let I be the ideal in $\mathbb{C}[G]$ generated by $\tilde{\tau}(x)$ for all $x \in m_u$. One can immediately check that $g \in G(A) = \mathrm{Hom}(\mathbb{C}[G], A)$ is in $\mathrm{Stab}_u(A)$ if and only if g factors via $\mathbb{C}[G]/I$, that is, $g : \mathbb{C}[G] \to \mathbb{C}[G]/I \to A$. Thus $\mathrm{Stab}_u(A) = \mathrm{Hom}(\mathbb{C}[G]/I, A)$. $\square$

6. The classical series of Lie supergroups

In [Kac 1977] Kac proved a classification theorem for simple Lie superalgebras. The description of the supergroup functors, corresponding to the classical super-series of Lie superalgebras introduced by Kac, appeared in [Deligne and Morgan 1999, p. 70]; however no representation statement was proved there.

In this section we describe the supergroup functors corresponding to the classical superseries and show they are representable (that is, they are algebraic super-groups), and hence are Lie supergroups by the results of Section 4. For the series $A(m, n)$, $B(m, n)$, $C(n)$ and $D(m, n)$ this result was proved in [Varadarajan 2004, p. 289] with differential techniques.

One should also prove that the Lie superalgebras[3] of these Lie supergroups co-incide with the classical series mentioned above; however this goes beyond the scope of this paper.

The A(n) series. Define $\mathrm{GL}_{m|n}(A)$ as the set of all invertible morphisms $g : A^{m|n} \to A^{m|n}$. This is equivalent to asking that the *Berezinian* or *superdeterminant*

$$\mathrm{Ber}(g) = \mathrm{Ber}\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \det(p - qs^{-1}r) \det s^{-1}$$

---

[3]For the definition of Lie superalgebra of an algebraic supergroup, see [Caston and Fioresi 2007, Chapter 5].

(see [Berezin 1987]) be invertible in A, where p and s are $m \times m$, $n \times n$ matrices of even elements in A, while q and r are $m \times n$, $n \times m$ matrices of odd elements in A. A necessary and sufficient condition for $g \in \mathrm{GL}_{m|n}(A)$ to be invertible is that p and s are invertible. The group-valued functor

$$\mathrm{GL}_{m|n} : (\mathrm{salg}) \to (\mathrm{sets}), \quad A \to \mathrm{GL}_{m|n}(A)$$

is an affine supergroup called the *general linear supergroup* and is represented by the algebra $\mathbb{C}[\mathrm{GL}_{m|n}] := \mathbb{C}[x_{ij}, y_{\alpha\beta}, \xi_{i\beta}, \gamma_{\alpha j}, z, w]/\big((w \det x - 1, z \det y - 1\big)$, where $i, j = 1, \ldots m$ and $\alpha, \beta = 1, \ldots n$.

Consider the morphism

$$\rho : \mathrm{GL}_{m|n} \times \mathbb{C}^{1|0} \to \mathbb{C}^{1|0}, \quad (g, c) \to \mathrm{Ber}(g)c.$$

The stabilizer of the point $1 \in \mathbb{C}^{1|0}$ coincides with $\mathrm{SL}_{m|n}(A)$, the special linear supergroup, defined as the matrices in $\mathrm{GL}_{m|n}(A)$ with Berezinian 1. By Theorem 5.2, $\mathrm{SL}_{m|n}$ is representable, and by Section 4 it is a complex supermanifold. Moreover one can check that $A(m, n) = \mathrm{Lie}(\mathrm{SL}_{m|n})$.

The $B(m, n)$, $C(n)$ and $D(m, n)$ series. Consider the morphism

$$\rho : \mathrm{GL}_{m|2n} \times \mathcal{B} \to \mathcal{B} \qquad (g, \psi(\,\cdot\,,\cdot\,)) \to \psi(g\,\cdot\,, g\,\cdot\,),$$

where $\mathcal{B}$ is the vector superspace of all the symmetric bilinear forms on $\mathbb{C}^{m|2n}$. The stabilizer of the point Φ the standard bilinear form on $\mathbb{C}^{m|2n}$ is the supergroup functor $\mathrm{Osp}_{m|2n}$. Again this is an algebraic supergroup by Theorem 5.2 and it is also a complex supermanifold. One can check that $B(m, n) = \mathrm{Lie}(\mathrm{Osp}_{2m+1|2n})$, $C(n) = \mathrm{Lie}(\mathrm{Osp}_{2|2n-2})$ and $D(m, n) = \mathrm{Lie}(\mathrm{Osp}_{2m|2n})$.

The $P(n)$ series. Define the algebraic supergroup $\pi \mathrm{Sp}_{n|n}$ as we did for $\mathrm{Osp}_{m|n}$, by taking antisymmetric bilinear forms instead of symmetric ones. Consider the action

$$\pi \mathrm{Sp}_{n|n} \times \mathbb{C}^{1|0} \to \mathbb{C}^{1|0}, \quad (g, c) \mapsto \mathrm{Ber}(g)c.$$

By Theorem 5.2, Stab_1 is an affine algebraic supergroup, hence a Lie supergroup. It is corresponding to the $P(n)$ series.

The $Q(n)$ series. Let $D = \mathbb{C}[\eta]/(\eta^2 + 1)$. This is a noncommutative superalgebra. Define the supergroup functor $\mathrm{GL}_n(D) : (\mathrm{salg}) \to (\mathrm{sets})$, with $\mathrm{GL}_n(D)(A)$ the group of automorphisms of the left supermodule $A \otimes D$. In [Deligne and Morgan 1999] is proven the existence of a morphism called the *odd determinant*

$$\mathrm{odet} : \mathrm{GL}_n(D) \to \mathbb{C}^{0|1}.$$

Reasoning as before, define

$$\mathrm{GL}_n(D) \times \mathbb{C}^{0|1} \to \mathbb{C}^{0|1}, \qquad g, c \to \mathrm{odet}(g)c.$$

Then $G = \mathrm{Stab}_1$ is an affine algebraic supergroup and for $n \geq 2$ we define $Qg(n)$ as the quotient of G and the diagonal subgroup $\mathrm{GL}_{1|0}$. This is an algebraic and Lie supergroup and its Lie superalgebra is $Q(n)$.

Appendix: Commutative superalgebra

Let k be the ground field. We assume all superalgebras to be commutative. As before, we denote with latin letters the even elements and with greek letters the odd elements of a superalgebra.

Theorem A.1. *Let A be a finitely generated superalgebra. There exists a unique superalgebra morphism*

$$\phi : k[x_1, \ldots, x_m, \xi_1, \ldots, \xi_n] \to A,$$

(where $k[x_1, \ldots, x_m, \xi_1, \ldots, \xi_n]$ denotes the polynomial superalgebra with even indeterminates x_i and odd indeterminates ξ_j) sending the x_i and the ξ_j to chosen elements in A of the correct parity.

This comes from the universality of the construction of the polynomial superalgebra as it is done for example in [Deligne and Morgan 1999, p. 49].

Theorem A.2. *Let A be a finitely generated superalgebra and let $\hat{A} = \lim_{\longleftarrow} A/n^i$, be its completion with respect an ideal n. Let $\hat{n}$ be the ideal in $\hat{A}$ corresponding to n. There exists a unique superalgebra morphism $\phi : k[[x_1, \ldots, x_m, \xi_1, \ldots, \xi_n]] \to \hat{A}$ sending the x_i and the ξ_j to chosen elements in $\hat{n}$ of the correct parity.*

Proof. This is the same as [Eisenbud 1995, p. 200, Theorem 7.16]. Let's briefly recall it. By Theorem A.1, there is a unique map $k[x_1, \ldots, x_m, \xi_1, \ldots, \xi_n] \to \hat{A}/\hat{n}^i$ sending the x_i and the ξ_j to chosen elements in n. Clearly this maps factors as

$$k[x_1, \ldots, x_m, \xi_1, \ldots, \xi_n] \to$$
$$k[x_1, \ldots, x_m, \xi_1, \ldots, \xi_n]/(x_1, \ldots, x_m, \xi_1, \ldots, \xi_n)^i \to \hat{A}/\hat{n}^i.$$

One can check that

$$\frac{k[x_1, \ldots, x_m, \xi_1, \ldots, \xi_n]}{(x_1, \ldots, x_m, \xi_1, \ldots, \xi_n)^i} \simeq \frac{k[[x_1, \ldots, x_m, \xi_1, \ldots, \xi_n]]}{(x_1, \ldots, x_m, \xi_1, \ldots, \xi_n)^i}$$

hence by the universal property of the inverse limit we have obtained the required map and its uniqueness. $\square$

Lemma A.3. $I = (0)$ *if*

$$\mathrm{Gr}(k[[x_1, \ldots, x_r, \xi_1, \ldots, \xi_s]]/I) \cong \mathrm{Gr}(k[[x_1, \ldots, x_r, \xi_1, \ldots, \xi_s]]).$$

Proof. Let m be the maximal ideal in $k[[x_1, \ldots, x_r, \xi_1, \ldots, \xi_s]]$. There exist i such that $I \subset m^i$ but $I \not\subset m^{i+1}$ otherwise we are done since $I \subset \cap m^i = (0)$. Then

$$(m^i/I)/(m^{i+1} + I/I) = m^i/(m^{i+1} + I) \neq m^i/m^{i+1},$$

which gives a contradiction.   □

Theorem A.4. $I = (0)$ *if*

$$k[[x_1, \ldots, x_r, \xi_1, \ldots, \xi_s]]/I \cong k[[x_1, \ldots, x_r, \xi_1, \ldots, \xi_s]].$$

Proof. This is a consequence of Lemma A.3.   □

Lemma A.5. *Let A be a commutative superalgebra and m a maximal ideal. Let A_m be the localization of A into the even part m_0 of the maximal ideal m and $\widehat{A}_m$ the completion of A_m with respect to the maximal ideal $\tilde{m}$ in A_m. Then*

$$m^i/m^{i+1} \cong \tilde{m}^i/\tilde{m}^{i+1} \cong \widehat{m}^i/\widehat{m}^{i+1}.$$

Proof. This is the same as in the commutative case, because localization and completion commute with quotients.   □

Theorem A.6 (Super-Nakayama's Lemma). *Let A be a local commutative superring with maximal (homogeneous) ideal $\mathfrak{m}$. Let E be a finitely generated module for the ungraded ring A.*

   (i) *If $\mathfrak{m}E = E$, then $E = 0$; more generally, if H is a submodule of E such that $E = \mathfrak{m}E + H$, then $E = H$.*

  (ii) *Let $(v_i)_{1 \leq i \leq p}$ be a basis for the k-vector space $E/\mathfrak{m}E$ where $k = A/\mathfrak{m}$. Let $e_i \in E$ be above v_i. Then the e_i generate E. If E is a supermodule for the superring A, and v_i are homogeneous elements of the vector superspace $E/\mathfrak{m}E$, we can choose the e_i to be homogeneous too (and hence of the same parity as the v_i).*

 (iii) *Suppose E is projective, that is, there is a A-module F such that $E \oplus F = A^N$ where A^N is the free module for the ungraded ring A of rank N. Then E (and hence F) is free, and the e_i above form a basis for E.*

Proof. See [Caston and Fioresi 2007, Appendix].   □

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