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In the curve complex for a surface, a handlebody set is the set of loops that bound properly embedded disks in a given handlebody bounded by the surface. A boundary set is the set of nonseparating loops in the curve complex that bound two-sided, properly embedded surfaces. For a Heegaard splitting, the distance between the boundary sets of the handlebodies is zero if and only if the ambient manifold contains a nonseparating, two sided incompressible surface. We show that every vertex in the curve complex is within two edges of a point in the boundary set.

1. Introduction

The curve complex $C(\Sigma)$ for a compact, connected, closed, orientable surface Σ is the simplicial complex whose vertices are loops (isotopy classes of essential, simple closed curves) in Σ and whose simplices correspond to sets of pairwise disjoint loops in Σ. Given a handlebody H and a homeomorphism $\phi : \Sigma \to \partial H$, we can define the following subsets of $C(\Sigma)$.

The *handlebody set* $\mathbf{H}$ is the set of loops that bound properly embedded (essential) disks in H. The *genus g boundary set* $\mathbf{H}^g$ is the set of nonseparating loops such that each bounds a properly embedded, two-sided, incompressible, genus-g surface in H. Note that $\mathbf{H}^0$ is a proper subset of $\mathbf{H}$, specifically the set of all the nonseparating loops in $\mathbf{H}$. Define the *boundary set* to be the union

$$\mathbf{H}^\infty = \bigcup_{g \geq 0} \mathbf{H}^g.$$

We will say that a set A of vertices in $C(\Sigma)$ is *k-dense* if every vertex in $C(\Sigma)$ is within k edges of a point in A.

Theorem 1. *If Σ has genus 3 or greater, then $\mathbf{H}^\infty$ is 2-dense in $C(\Sigma)$.*

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The proof presented here does not work for genus two surfaces. However, Schleimer has shown in [2005] that the orbit of a vertex of $C(\Sigma)$ under the action of the Torelli group is 5-dense. This implies that for a genus two handlebody, $\mathbf{H}^\infty$ is n-dense for some $n \leq 5$.

In contrast to $\mathbf{H}^\infty$, a fixed genus boundary set $\mathbf{H}^g$ has a geometric structure much closer to $\mathbf{H}$, which is not k-dense for any k. This is demonstrated by the following two Lemmas, the first of which is a corollary of Conclusion III.15 in [Jaco 1980] and the second of which follows from a Theorem of Scharlemann [2006].

Lemma 2. *If Σ has genus three or greater and $v \in \mathbf{H}$ then*

$$d(v, \mathbf{H}^g) = 1$$

for every $g > 0$. If Σ has genus two then

$$d(v, \mathbf{H}^g) > 1$$

for every g.

Lemma 3. *For $g \geq 1$, the set $\mathbf{H}^g$ is disjoint from $\mathbf{H}$ and contained in a $2g$ neighborhood of $\mathbf{H}$.*

For this paper, every 3-manifold will be compact, connected, closed and orientable. A *Heegaard splitting* for such a 3-manifold M is a triple (Σ, H_1, H_2) where $\Sigma \subset M$ is a compact, connected, closed, orientable surface and $H_1, H_2 \subset M$ are handlebodies such that

$$\partial H_1 = \Sigma = \partial H_2 \quad \text{and} \quad M = H_1 \cup H_2.$$

The inclusion maps from ∂H_1 and ∂H_2 onto Σ determine handlebody sets $\mathbf{H}_1$ and $\mathbf{H}_2$, respectively. The *distance* of the Heegaard splitting, as defined by Hempel in [2001] is the distance $d(\Sigma) = d(\mathbf{H}_1, \mathbf{H}_2)$ between the two handlebody sets.

The inclusion maps also determine boundary sets $\mathbf{H}_1^g$, $\mathbf{H}_2^h$, $\mathbf{H}_1^\infty$, $\mathbf{H}_2^\infty$, allowing us to generalize this distance to the (g, h)-*distance*

$$d^{g,h}(\Sigma) = d(\mathbf{H}_1^g, \mathbf{H}_2^h)$$

and the *boundary distance*

$$d^\infty(\Sigma) = d(\mathbf{H}_1^\infty, \mathbf{H}_2^\infty).$$

The set $\mathbf{H}^\infty$ is precisely the set of vertices representing simple closed curves whose homology class is nontrivial in Σ, but trivial in H. For a Heegaard splitting, it encodes homology information about the ambient manifold. In particular, the boundary distance determines precisely when a manifold has infinite homology (and therefore a nonseparating, incompressible surface).

Lemma 4. *The following are equivalent*:

(1) *the first homology group of M is infinite*;

(2) *M contains a nonseparating*, *two sided*, *closed incompressible surface*;

(3) $d^{\infty}(\Sigma) = 0$ *and*

(4) $d^{0,\infty}(\Sigma) = 0$.

The proof is given in Section 2. The equivalence of (1) and (2) is well known, but we give a very simple, geometric proof via the boundary set. Theorem 1 is proved in Section 3.

For any Heegaard splitting (Σ, H_1, H_2) of a non-Haken 3-manifold, Lemma 4 implies that the boundary set in $C(\Sigma)$ determined by H_2 must be completely disjoint from the boundary set for H_1. Hempel showed that there are handlebody sets that are arbitrarily far apart in the curve complex. The same is not true for boundary sets. In particular, Theorem 1 implies that for any Heegaard splitting (Σ, H_1, H_2) of genus 3 or greater, $d^{\infty}(\Sigma)$ is equal to either 0, 1 or 2. For non-Haken manifolds, we have the following.

Corollary 5. *For any Heegaard splitting* (Σ, H_1, H_2) *of a non-Haken* 3*-manifold* M, $d^{\infty}(\Sigma)$ *is equal to* 1 *or* 2.

2. Nonseparating surfaces

The following Lemma will not be used until Section 3, but the method of proof gives a good introduction to the proof of Lemma 7. Recall that an element α of a $\mathbf{Z}$ module G is called *primitive* if there is no $\beta \in G$ such that $\alpha = k\beta$ for some $k \neq \pm 1$.

Lemma 6. *Let $\ell_1, \ldots, \ell_k$ be pairwise disjoint*, *essential loops in the boundary of a genus-g handlebody H with $g > k$. Then there is a properly embedded*, *nonseparating surface $F \subset H$ such that ∂F is disjoint from each ℓ_j and the homology class defined by ∂F in $H(\Sigma)$ is primitive.*

Proof. Let $D_1, \ldots, D_g$ be a system of disks for H, that is, a collection of properly embedded, essential disks whose complement in H is a single ball. Orient the boundaries of the disks and the loops $\ell_1, \ldots, \ell_k$, then form the matrix $A = (a_{ij})$ such that a_{ij} is the algebraic intersection number of D_i and ℓ_j.

If we replace one of the disks in the system by a disk slide, the matrix for the new system of disks can be constructed from A by adding or subtracting one row from the other. Thus we can perform elementary row operations on A by choosing new systems of disks for H. In particular, we can make A upper triangular.

Because A has more rows than columns, if A is upper triangular then the bottom row consists of all zeros. In other words, the disk D_g has algebraic intersection 0 with each loop ℓ_j.

If D_g intersects the loop ℓ_1, there must be a pair of adjacent intersections in ℓ_1 with opposite orientations. By attaching a band from ∂D_g to itself, along the arc of the loop ℓ_1, we can form a new surface F_1 whose boundary (consisting of two loops) has algebraic intersection zero with each loop ℓ_j, but whose geometric intersection number with the collection of curves $\ell_1, \ldots, \ell_k$ is strictly lower than that of D_g. The surface F_1 is two sided because the intersections have opposite orientations and nonseparating because D_g is nonseparating.

If ∂F_1 intersects ℓ_1, we can form a new surface F_2 by attaching a band, and so on. Continuing in this manner for each ℓ_j, we form a surface F which is properly embedded, two-sided, nonseparating and such that ∂F is disjoint from each ℓ_j.

Attaching a band to the boundary of F_i does not change the homology class of the boundary, so the homology class of ∂F is equal to the class of ∂D_g. Because ∂D_g is represented by a connected loop, its homology class is primitive, as is the homology class of ∂F. □

We will now use the idea of attaching bands to eliminate intersections to prove the implication (1) ⇒ (4) of Lemma 4.

Lemma 7. *If the first homology of M is infinite then $d^{0,\infty}(\Sigma) = 0$.*

Proof. Let $D_1, \ldots, D_g$ be a system of disks for H_1 and $D'_1, \ldots, D'_g$ be a system of disk for H_2. Orient the boundaries of both systems of disks. Let A be the matrix of algebraic intersection numbers of the boundaries. Because the first homology is infinite, the determinant of A must equal zero.

As in the last proof, we can perform row operations on A by taking disk slides of the disks $D_1, \ldots, D_g$. Because the determinant of A is zero, some sequence of disk slides will leave A with all zeros in the bottom row. Thus after a sequence of disk slides, we can assume D_g has algebraic intersection 0 with each D'_j.

By attaching bands to the boundary of D_g as in the proof of Lemma 6, we can form a properly embedded, two sided, nonseparating surface F whose boundary is disjoint from $D'_1, \ldots, D'_g$. Thus each boundary component of F bounds a disk in H_2. The union of F and these disks is a properly embedded, two sided, nonseparating closed surface in M.

Recall that F was constructed from D_g by attaching bands to its boundary. The last band defines a boundary compression for F corresponding to an isotopy pushing this last band into H_2. After this isotopy, the second to last band defines a second isotopy, and so on. The final result is a surface isotopic to F which intersects H_1 in a disk isotopic to D_g.

The intersection of this surface with H_2 is orientable, two-sided and nonseparating because F has these properties. Thus ∂D_g is in both $\mathbf{H}^0_1$ and $\mathbf{H}^\infty_2$ so

$$d^{0,\infty}(\Sigma) = d(\mathbf{H}^0_1, \mathbf{H}^\infty_2) = 0.$$

□

Proof of Lemma 4. Lemma 7 implies that for any Heegaard splitting (Σ, H_1, H_2),

$$d(\mathbf{H}_1^0, \mathbf{H}_2^\infty) = 0$$

so (1) $\Rightarrow$ (4). Because $\mathbf{H}_1^0$ is contained in $\mathbf{H}_1^\infty$, (4) $\Rightarrow$ (3) is immediate.

Let (Σ, H_1, H_2) be a Heegaard splitting for M. If $d^\infty(\Sigma) = 0$ then there is a simple closed curve $\ell \subset \Sigma$ such that ℓ bounds two-sided, nonseparating properly embedded surfaces $F \subset H_1$ and $F' \subset H_2$. The union $F \cup F'$ is a two-sided, nonseparating closed surface embedded in M. Compressing $F \cup F'$ to either side produces at least one new two-sided, nonseparating surface. By compressing repeatedly, we eventually find a closed, nonseparating, two-sided incompressible surface in M. Thus (3) $\Rightarrow$ (2).

The final step, (2) $\Rightarrow$ (1), is a classical result. If M contains a two-sided, nonseparating, closed surface $S \subset M$, let p be a point in S. There is a path

$$\alpha : [0, 1] \to M$$

from p to itself that does not cross S. The homology class of α has infinite order so the first homology of M is infinite. $\square$

3. Density

Proof of Theorem 1. We will prove the following: Let ℓ be a loop in ∂H and assume the genus of H is at least 3. Then there is an essential loop ℓ' disjoint from ℓ and a properly embedded, two-sided, nonseparating surface F such that ∂F is a single, nonseparating loop disjoint from ℓ'.

By Lemma 6, there is a properly embedded surface $F'' \subset H$ such that $\partial F''$ is disjoint from ℓ and defines a primitive element of the homology. Of all the properly embedded surfaces with boundary disjoint from ℓ and homologous to $\partial F''$, let F' be one with minimal number of boundary components. Each component of $\partial F'$ has an orientation induced by F' and thus defines an element of the first homology of Σ.

For each component C of $\Sigma \setminus (\ell \cup \partial F')$, an orientation for a loop in ∂C induces an orientation of C. Assume two components of ∂C come from loops of $\partial F'$ and induce the same orientation of C. Because the induced orientations agree, adding a band between them produces a new orientable surface with fewer boundary components, but homologous boundary. Thus the minimality assumption implies that each component C of $\Sigma \setminus (\ell \cup \partial F')$ has at most one boundary loop coming from $\partial F'$ inducing each possible orientation. Thus it has at most two boundary loops coming from $\partial F'$ and these induce opposite orientations on C.

Assume for contradiction each component of $\Sigma \setminus (\ell \cup \partial F')$ is planar. Each component that is disjoint from ℓ has exactly two boundary components. A planar surface with two boundary loops is an annulus so each component disjoint from ℓ must be an annulus. There are either two components of $\Sigma \setminus (\ell \cup \partial F')$ with one boundary loop each on ℓ, or one component with two boundary loops on ℓ. In the first case, the two components are pairs of pants or annuli, while in the second, the component is a four punctured sphere or a pair of pants. The union of such components and a collection of annuli is a genus-one or genus-two surface. This contradicts the assumption that Σ has genus at least three, so we conclude that some component must be nonplanar.

Let C be a nonplanar component. There is a simple closed curve $\ell' \subset C$ such that ℓ' separates a once-punctured torus from C. In Σ, the loop ℓ' separates a once-punctured torus that contains no components of $\partial F'$. Let F be a surface whose boundary is homologous to F', disjoint from the once-punctured torus bounded by ℓ' and such that the number of boundary components of F' is minimal over all such surfaces.

Once again, each component of $\Sigma \setminus (\ell' \cup \partial F)$ has at most two boundary components on loops in ∂F, with opposite induced orientations. Because ℓ' bounds a surface disjoint from ∂F, each component of $\Sigma \setminus \partial F$ must also have at most two boundary loops on ∂F.

If a component C of $\Sigma \setminus \partial F$ has a single boundary component, this loop is homology trivial in Σ. Attaching a boundary parallel surface to F removes this loop so minimality of ∂F implies that each component has two boundary loops.

If C has two boundary loops (with opposite induced orientations) then these loops determine the same element of the homology of Σ. Because Σ is connected, this implies that any two loops of ∂F (with their induced orientations) determine the same element of the homology. Thus the element of the homology determined by ∂F is of the form $k\beta$ where k is the number of boundary components of F.

By Lemma 6, ∂F determines a primitive element of the homology of Σ, so k must be 1. In other words, the boundary of F is connected and ∂F determines an element of $\mathbf{H}^\infty$. By construction, ∂F is disjoint from a loop ℓ' that is disjoint from ℓ. Thus the vertex $v \in C(\Sigma)$ determined by ℓ is distance at most 2 from $\mathbf{H}^\infty$. □

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