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# LEFT-SYMMETRIC SUPERALGEBRAIC STRUCTURES ON THE SUPER-VIRASORO ALGEBRAS

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# LEFT-SYMMETRIC SUPERALGEBRAIC STRUCTURES ON THE SUPER-VIRASORO ALGEBRAS

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**We classify the compatible left-symmetric superalgebraic structures on the super-Virasoro algebras satisfying certain natural conditions.**

## 1. Introduction

Left-symmetric algebras (also called pre-Lie algebras, quasiassociative algebras, or Vinberg algebras) are a class of natural algebraic systems appearing in many fields in mathematics and mathematical physics. They were first mentioned by A. Cayley [1890] as a kind of rooted tree algebra and later arose again from the study of convex homogeneous cones [Vinberg 1963], affine manifolds and affine structures on Lie groups [Koszul 1961], and deformation of associative algebras [Gerstenhaber 1963]. They play an important role in the study of symplectic and complex structures on Lie groups and Lie algebras [Andrada and Salamon 2005; Chu 1974; Dardié and Medina 1996b; 1996a; Lichnerowicz and Medina 1988], phase spaces of Lie algebras [Bai 2006; Kupershmidt 1994; 1999a], certain integrable systems [Bordemann 1990; Svinolupov and Sokolov 1994], classical and quantum Yang–Baxter equations [Diatta and Medina 2004; Etingof and Soloviev 1999; Golubchik and Sokolov 2000; Kupershmidt 1999b], combinatorics [Ebrahimi-Fard 2002], quantum field theory [Connes and Kreimer 1998], vertex algebras [Bakalov and Kac 2003], and operads [Chapoton and Livernet 2001]. [Burde 2006] gives a survey.

The super version of left-symmetric algebras, the left-symmetric superalgebras, also appeared in many other fields; see for example [Chapoton and Livernet 2001; Gerstenhaber 1963; Vasil'eva and Mikhalëv 1996]. To our knowledge, they were first introduced by Gerstenhaber [1963] to study the Hochschild cohomology of associative algebras.

On the other hand, the Virasoro and super-Virasoro algebras are not only a class of important infinite-dimensional Lie algebras and Lie superalgebras but are also of considerable interest in physics. For example, they are the fundamental algebraic

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structures in conformal and superconformal field theory. As Kupershmidt [1999a] pointed out, a compatible left-symmetric algebra structure on the Virasoro algebra can be regarded as the "nature of the Virasoro algebra"; in fact, on the Virasoro algebra $\mathscr{V}$ given there, such an algebraic structure satisfies

$$cc = x_m c = c x_m = 0 \quad \text{and} \quad x_m x_n = f(m, n) x_{m+n} + \omega(m, n) c, \tag{1-1}$$

where $f(m, n)$ and $\omega(m, n)$ are two complex-valued functions, and $\{x_m, c \mid m \in \mathbb{Z}\}$ is a basis of the Virasoro algebra $\mathscr{V}$ satisfying

$$[c, x_n] = 0 \quad \text{and} \quad [x_m, x_n] = (m - n) x_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}. \tag{1-2}$$

The condition (1-1) is natural since it means that the compatible left-symmetric algebra is still graded and $c$ is also a central extension given by $\omega(m, n)$. Moreover, in [Kong et al. 2007], we proved that any compatible left-symmetric algebraic structure on the Virasoro algebra $\mathscr{V}$ satisfying (1-1) is isomorphic to one of the examples given in [Kupershmidt 1999a].

In this paper, we study the compatible left-symmetric superalgebraic structures on the super-Virasoro algebras. Motivated by the study in the case of the ordinary Virasoro algebra, we classify such left-symmetric superalgebras satisfying some natural conditions like (1-1). The paper is organized as follows. In Section 2, we give some necessary definitions, notations, and basic results on left-symmetric superalgebras and the super-Virasoro algebras. We also give the classification of compatible left-symmetric algebraic structures on the ordinary Virasoro algebra satisfying (1-1). In Section 3, we study the compatible left-symmetric superalgebraic structures on the centerless super-Virasoro algebras satisfying certain natural conditions. In Section 4, we discuss the nontrivial central extensions of the left-symmetric superalgebras obtained in Section 3 whose supercommutator is a super-Virasoro algebra.

Throughout, all algebras are over the complex field $\mathbb{C}$ and the indices $m, n, l \in \mathbb{Z}$ and $r, s, t \in \mathbb{Z} + \theta$ for $\theta = 0$ or $\theta = 1/2$, unless otherwise stated.

## 2. Preliminaries and fundamental results

Let $(A, \cdot)$ be an algebra over a field $\mathbb{F}$. $A$ is said to be a superalgebra if the underlying vector space of $A$ is $\mathbb{Z}_2$-graded, that is, $A = A_{\bar{0}} \oplus A_{\bar{1}}$, and $A_\alpha \cdot A_\beta \subset A_{\alpha+\beta}$, for $\alpha, \beta \in \mathbb{Z}_2$. An element of $A_{\bar{0}}$ is called even and an element of $A_{\bar{1}}$ is called odd.

**Definition 2.1.** A Lie superalgebra is a superalgebra $A = A_{\bar{0}} \oplus A_{\bar{1}}$ with an operation $[\ ,\ ]$ satisfying the conditions

$$[a, b] = -(-1)^{\alpha\beta}[b, a], \tag{2-1}$$

$$[a, [b, c]] = [[a, b], c] + (-1)^{\alpha\beta}[b, [a, c]] \tag{2-2}$$

where $a \in A_\alpha,\ b \in A_\beta,\ c \in A$, and $\alpha, \beta \in \mathbb{Z}_2$.

**Definition 2.2.** A superalgebra $A$ is called a left-symmetric superalgebra if the associator

$$(x, y, z) := (x \cdot y) \cdot z - x \cdot (y \cdot z) \tag{2-3}$$

of $A$ satisfies

$$(x, y, z) = (-1)^{\alpha\beta}(y, x, z) \quad \text{for all } x \in A_\alpha,\ y \in A_\beta,\ z \in A,\ \alpha, \beta \in \mathbb{Z}_2. \tag{2-4}$$

Obviously, if $A = A_{\bar{0}} \oplus A_{\bar{1}}$ is a Lie superalgebra or a left-symmetric superalgebra, then $A_{\bar{0}}$ is an ordinary Lie algebra or a left-symmetric algebra, respectively. Letting $A$ be a left-symmetric superalgebra, it is easy to show that the supercommutator

$$[x, y] = x \cdot y - (-1)^{\alpha\beta} y \cdot x \quad \text{for all } x \in A_\alpha,\ y \in A_\beta,\ \alpha, \beta \in \mathbb{Z}_2 \tag{2-5}$$

defines a Lie superalgebra $\mathcal{G}(A)$, which is called the subadjacent Lie superalgebra of $A$, and $A$ is also called the compatible left-symmetric superalgebraic structure on the Lie superalgebra $\mathcal{G}(A)$.

On the other hand, we recall the definition of the super-Virasoro algebras. There are two super-Virasoro algebras that correspond to $N = 1$ [Ramond 1971] and $N = 2$ [Neveu and Schwarz 1971a; 1971b] superconformal field theory. In fact, let $\theta = 0$ or $1/2$, which corresponds to the Ramond case or the Neveu–Schwarz case, respectively. Let $\mathcal{SV} = \mathcal{SV}_{\bar{0}} \oplus \mathcal{SV}_{\bar{1}}$ denote a super-Virasoro algebra with a basis $\{L_m, G_r, c \mid m \in \mathbb{Z},\ r \in \mathbb{Z} + \theta\}$. The superbrackets are defined as

$$\begin{aligned}
&[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0},\\
&[L_m, G_r] = \Big(\frac{m}{2} - r\Big)G_{m+r},\\
&[G_r, G_s] = 2L_{r+s} + \frac{c}{12}(4r^2 - 1)\delta_{r+s,0},\\
&[\mathcal{SV}_{\bar{0}}, c] = [\mathcal{SV}_{\bar{1}}, c] = 0,
\end{aligned} \tag{2-6}$$

where the even subspace $\mathcal{SV}_{\bar{0}}$ is spanned by $\{L_m, c \mid m \in \mathbb{Z}\}$ and the odd subspace $\mathcal{SV}_{\bar{1}}$ is spanned by $\{G_r \mid r \in \mathbb{Z} + \theta\}$. Obviously, $\mathcal{SV}_{\bar{0}}$ is nothing but an ordinary Virasoro algebra. A class of compatible left-symmetric algebraic structures on the Virasoro algebra satisfying (1-1) were given in [Kupershmidt 1999a]. Such left-symmetric algebras were classified in [Kong et al. 2007].

**Theorem 2.3** [Kong et al. 2007]. *Any compatible left-symmetric algebraic structure on the Virasoro algebra $\mathcal{SV}_{\bar{0}}$ satisfying (1-1) is isomorphic to one of the (mutually nonisomorphic) left-symmetric algebras given by the multiplication*

$$L_m L_n = \frac{-n(1+\epsilon n)}{1+\epsilon(m+n)} L_{m+n} + \frac{c}{24}(m^3 - m + (\epsilon - \epsilon^{-1})m^2)\delta_{m+n,0} \tag{2-7}$$

*for all* $m, n \in \mathbb{Z}$, *where* $c$ *is an annihilator and* $\operatorname{Re}\epsilon > 0$, $\epsilon^{-1} \notin \mathbb{Z}$ *or* $\operatorname{Re}\epsilon = 0$, $\operatorname{Im}\epsilon > 0$.

## 3. Compatible left-symmetric superalgebraic structures on the centerless super-Virasoro algebras

Let $\widetilde{\mathcal{SV}} = \widetilde{\mathcal{SV}}_{\bar{0}} \oplus \widetilde{\mathcal{SV}}_{\bar{1}}$ be a centerless super-Virasoro algebra with a basis $\{L_m, G_r \mid m \in \mathbb{Z},\ r \in \mathbb{Z}+\theta\}$, and let the superbrackets be given by (2-6) with $c = 0$. Theorem 2.3 motivates us to consider the compatible left-symmetric superalgebraic structures on $\widetilde{\mathcal{SV}}$ that also satisfy the "graded" condition, that is, the multiplications of the compatible left-symmetric superalgebraic structures on $\widetilde{\mathcal{SV}}$ that satisfy

$$
\text{(3-1)} \qquad \begin{aligned} L_m \cdot L_n &= f(m,n) L_{m+n}, & L_m \cdot G_r &= g(m,r) G_{m+r}, \\ G_r \cdot L_m &= h(r,m) G_{m+r}, & G_r \cdot G_s &= d(r,s) L_{r+s}, \end{aligned}
$$

where $f$, $g$, $h$, and $d$ are $\mathbb{C}$-valued functions. Then the supercommutators give the super-Virasoro algebra $\widetilde{\mathcal{SV}}$ if and only if $f(m,n)$, $g(m,r)$, $h(r,m)$, and $d(r,s)$ satisfy

$$
\text{(3-2)} \qquad \begin{aligned} & f(m,n) - f(n,m) = m - n, \\ & d(r,s) + d(s,r) = 2, \qquad g(m,r) - h(r,m) = m/2 - r. \end{aligned}
$$

Furthermore, the functions $f(m,n)$, $g(m,r)$, $h(r,m)$, and $d(r,s)$ define a left-symmetric superalgebra with a basis $\{L_m, G_r \mid m \in \mathbb{Z},\ r \in \mathbb{Z}+\theta\}$ if and only if they satisfy the equations

$$
\begin{aligned}
(L_m, L_n, L_l) &= (-1)^{0\cdot 0}(L_n, L_m, L_l), & (L_m, L_n, G_r) &= (-1)^{0\cdot 0}(L_n, L_m, G_r), \\
(L_m, G_r, L_n) &= (-1)^{0\cdot 1}(G_r, L_m, L_n), & (L_m, G_r, G_s) &= (-1)^{0\cdot 1}(G_r, L_m, G_s), \\
(G_r, G_s, L_m) &= (-1)^{1\cdot 1}(G_s, G_r, L_m), & (G_r, G_s, G_t) &= (-1)^{1\cdot 1}(G_s, G_r, G_t).
\end{aligned}
$$

These equations are equivalent to the equations

$$
\text{(3-3)} \qquad \begin{aligned}
(m-n) f(m+n,l) &= f(n,l) f(m,n+l) - f(m,l) f(n,m+l), \\
(m-n) g(m+n,r) &= g(n,r) g(m,n+r) - g(m,r) g(n,m+r), \\
(m/2-r) h(m+r,n) &= h(r,n) g(m,n+r) - f(m,n) h(r,m+n), \\
(m/2-r) d(m+r,s) &= d(r,s) f(m,r+s) - g(m,s) d(r,m+s), \\
2 f(r+s,m) &= h(s,m) d(r,m+s) + h(r,m) d(s,m+r), \\
2 g(r+s,t) &= d(s,t) h(r,s+t) + d(r,t) h(s,r+t).
\end{aligned}
$$

**Proposition 3.1.** *Any compatible left-symmetric superalgebraic structure* $\widetilde{V}$ *on* $\widetilde{\mathcal{SV}}$ *satisfies* (3-1) *if and only if the functions in* (3-1) *satisfy* (3-2) *and* (3-3).

By Theorem 2.3, we only need to consider the case that

$$f(m,n)=\frac{-n(1+\epsilon n)}{1+\epsilon(m+n)}, \tag{3-4}$$

where $\operatorname{Re}\epsilon>0,\ \epsilon^{-1}\notin\mathbb{Z}$ or $\operatorname{Re}\epsilon=0,\ \operatorname{Im}\epsilon>0$.

**Theorem 3.2.** *For a fixed $\epsilon$ satisfying $\operatorname{Re}\epsilon>0,\ \epsilon^{-1}\notin\mathbb{Z}$ or $\operatorname{Re}\epsilon=0,\ \operatorname{Im}\epsilon>0$ and $f(m,n)$ satisfying (3-4), there is exactly one solution satisfying (3-2) and (3-3) given by*

$$\begin{aligned} g(m,r)&=\frac{-(\frac{m}{2}+r)(1+2\epsilon r)}{1+2\epsilon(m+r)},\\ h(r,m)&=\frac{-m(1+\epsilon m)}{1+2\epsilon(m+r)}, \qquad d(r,s)=\frac{1+2\epsilon s}{1+\epsilon(r+s)}, \end{aligned}\tag{3-5}$$

*for $m,n\in\mathbb{Z}$ and $r,s\in\mathbb{Z}+\theta$, which define a compatible left-symmetric superalgebra $\widetilde{V}_\epsilon$ on $\widetilde{\mathcal{SV}}$.*

*Proof.* It is easy to verify that $f(m,n)$ given in (3-4) and $g(m,r)$, $h(r,m)$, and $d(r,s)$ given in (3-5) satisfy (3-2) and (3-3). On the other hand, set

$$\begin{aligned} G(m,r)&=g(m,r)\frac{1+2\epsilon(m+r)}{1+2\epsilon r},\\ H(r,m)&=h(r,m)\frac{1+2\epsilon(m+r)}{1+\epsilon m}, \qquad D(r,s)=d(r,s)\frac{1+\epsilon(r+s)}{1+2\epsilon s}. \end{aligned}$$

Then we only need to prove that

$$G(m,r)=-m/2-r, \qquad H(r,m)=-m, \qquad D(r,s)=1.$$

We rewrite equations (3-2) and (3-3) involving $g(m,r)$, $h(r,m)$, $d(r,s)$ as

$$G(m,r)(1+2\epsilon r)-H(r,m)(1+\epsilon m)=(m/2-r)(1+2\epsilon(m+r)), \tag{3-6}$$

$$D(r,s)(1+2\epsilon s)+D(s,r)(1+2\epsilon r)=2+2\epsilon(r+s), \tag{3-7}$$

and

$$(m-n)G(m+n,r)=G(n,r)G(m,n+r)-G(m,r)G(n,m+r) \tag{3-8}$$

$$(m/2-r)H(m+r,n)=H(r,n)G(m,n+r)+nH(r,m+n), \tag{3-9}$$

$$(m/2-r)D(m+r,s)=-(r+s)D(r,s)-G(m,s)D(r,m+s), \tag{3-10}$$

$$-2m=H(s,m)D(r,m+s)+H(r,m)D(s,m+r), \tag{3-11}$$

$$2G(r+s,t)=D(s,t)H(r,s+t)+D(r,t)H(s,r+t). \tag{3-12}$$

Letting $r=s$ in equation (3-7), we have

$$D(s,s)=1 \quad \text{for all } s\in\mathbb{Z}+\theta. \tag{3-13}$$

In fact, $D(r,s)\neq 0$ for all $r,s\in\mathbb{Z}+\theta$. Otherwise, assume there exist $r_1$ and $s_1$ such that $D(r_1,s_1)=0$. Letting $r=s=r_1$ and $m=s_1-r_1$ in (3-11), we have

$$-(s_1-r_1)=H(r_1,s_1-r_1)D(r_1,s_1)=0.$$

Hence $r_1=s_1$. This contradicts (3-13).

Let $m=0$ and $r=s$ in (3-10) and (3-6), $r=s=-t$ in (3-12), $m=-2s, r=s$ in (3-6) and (3-10), and $m=-2s\neq 0, r=3s$ in (3-10). We know that

$$\begin{aligned} &G(0,s)=-s, && G(2s,-s)=0,\\ &H(s,0)=0, && H(s,-2s)=2s,\\ &D(-s,s)=D(3s,s)=1 \quad \text{for all } s\in\mathbb{Z}+\theta. \end{aligned} \tag{3-14}$$

Letting $m=-2(n+r)$ in (3-9) and $m+r+s=0$ in (3-11), we have

$$\begin{aligned} 0&=(n+2r)H(-2n-r,n)+nH(r,-n-2r),\\ -2m&=H(-m-r,m)+H(r,m). \end{aligned} \tag{3-15}$$

Letting $r=s$ in (3-11), we have $-m=H(r,m)D(r,m+r)$. So

$$H(r,m)=-m/D(r,m+r). \tag{3-16}$$

By (3-13)–(3-16), we have

$$\begin{aligned} &D(-2n-r,-n-r)=D(r,-n-r),\\ &1/D(r,m+r)+1/D(-m-r,-r)=2. \end{aligned} \tag{3-17}$$

Letting $-n-r=s$ and $m+r=s$ in (3-17), we have $D(r,s)=D(2s+r,s)$ and $D(-s,-r)=D(-s,-2s-r)$ for all $r,s\in\mathbb{Z}+\theta$. Thus by induction, we know that

$$D(r,s)=D(2ks+r,s) \quad \text{and} \quad D(-s,-r)=D(-s,-2ks-r) \tag{3-18}$$

for all $k\in\mathbb{Z}$. Therefore, $D(r,r)=D((2k+1)r,r)=D(r,(2k+1)r)=1$ for all $k\in\mathbb{Z}$. Let $r=s$ in (3-12). Then by (3-16) and (3-18), we have

$$G(2s,t)=D(s,t)H(s,s+t)=D(s,t)\frac{-s-t}{D(s,2s+t)}=-s-t.$$

Letting $m=2s$ in (3-6), we have

$$H(t,2s)=-2s \quad \text{for all } s,t\in\mathbb{Z}+\theta. \tag{3-19}$$

*Case* I. $\theta=1/2$. Then $D(\theta,\pm\theta)=D(\pm\theta,\theta)=1$. Hence $D(k+\theta,\pm\theta)=D(\pm\theta,k+\theta)=1$. That is, $D(r,\pm 1/2)=D(\pm 1/2,r)=1$ for all $r\in\mathbb{Z}+\theta$. Assume that for any $|r_1|\leq|s_1|$, we have $D(r_1,s_1)=1$. Then

$$D(r_1,s_1)=D(2ks_1+r_1,s_1)=1 \quad \text{and} \quad D(s_1,r_1)=D(s_1,2ks_1+r_1)=1.$$

For any $r \in \mathbb{Z}+\theta$, there exist $k \in \mathbb{Z}$, $r_1 \in \mathbb{Z}+\theta$, and $|r_1| \le |s_1|$ such that $r = 2ks_1 + r_1$. Therefore $D(r, s_1) = D(s_1, r) = 1$ for all $r \in \mathbb{Z}+\theta$. Hence by induction, we know that $D(r, s) = 1$ for any $r, s \in \mathbb{Z}+\theta$. Therefore $H(r, m) = -m$ and $G(m, r) = -m/2 - r$ for all $m \in \mathbb{Z}$ and $r \in \mathbb{Z}+\theta$.

*Case* II. $\theta = 0$. Let $m = -2t \neq 0$ and $s = r = 2t$ in (3-11). Then by (3-19), we have $2t = H(2t, -2t)D(2t, 0) = 2tD(2t, 0)$. Therefore

$$D(2t, 0) = 1 \quad \text{and} \quad D(0, 2t) = 1 \quad \text{for all } t \in \mathbb{Z}. \tag{3-20}$$

Letting $r = 0$ and $m = s \neq 0$ in (3-10) and (3-11), we have

$$\begin{aligned} m/2 &= -mD(0, m) - G(m, m), \\ -2m &= H(m, m) + H(0, m). \end{aligned} \tag{3-21}$$

So

$$H(m, m) = -2m - H(0, m) = -2m + \frac{m}{D(0, m)} = -2m - \frac{2m^2}{m+2G(m, m)}.$$

By (3-6), we have $H(m, m) = -m$ and $G(m, m) = -3m/2$, or

$$H(m, m) = \frac{-m}{1+\epsilon m} \quad \text{and} \quad G(m, m) = \frac{-3m-4\epsilon m^2}{2+4\epsilon m}. \tag{3-22}$$

In fact, the latter case cannot hold for any $m \neq 0$. Otherwise, assume that there exists a nonzero integer $m_1$ satisfying (3-22) with $m_1$ replacing $m$. Then

$$D(0, m_1) = -\frac{1}{2} - \frac{G(m_1, m_1)}{m_1} = \frac{1+\epsilon m_1}{1+2\epsilon m_1}.$$

Letting $m = -s = m_1$ and $r = 0$ or $-m_1$ in (3-10), we have

$$\begin{aligned} \frac{m_1}{2} &= m_1 D(0, -m_1) - G(m_1, -m_1), \\ \frac{3m_1}{2} D(0, -m_1) &= 2m_1 - G(m_1, -m_1)D(-m_1, 0). \end{aligned}$$

Hence $(3/2)D(0, -m_1) = 2 - D(0, -m_1)D(-m_1, 0) + (1/2)D(-m_1, 0)$. By (3-7), we have $D(0, -m_1) = 1$ and $D(-m_1, 0) = 1$, or

$$D(0, -m_1) = \frac{3 - \epsilon m_1}{1 - 2\epsilon m_1} \quad \text{and} \quad D(-m_1, 0) = -1 - \epsilon m_1.$$

Since $\epsilon \neq 0$ and $\epsilon^{-1} \notin \mathbb{Z}$, we know that $1/D(0, m_1) + 1/D(-m_1, 0) \neq 2$, which contradicts (3-17). Hence $H(m, m) = -m$ and $G(m, m) = -3m/2$ for all $m \in \mathbb{Z}$. By (3-21) and (3-6), we have $H(0, m) = -m$ and $G(m, 0) = -m/2$. Letting $r = 0$ and $m \neq 0$ in (3-8) and (3-9), we have

$$n^2 - m^2 = -nG(m, n) + mG(n, m), \qquad (m/2)H(m, n) = -nG(m, n) - n(m+n).$$

So $H(m,n)+2G(n,m)=-2(m+n)$. By (3-6), (3-13), and (3-16), we know that $H(m,n)=-n$ and $G(n,m)=-n/2-m$ and $D(m,n)=1$ for all $m,n\in\mathbb{Z}$. □

## 4. Compatible left-symmetric superalgebraic structures on the super-Virasoro algebras

We now consider the central extensions of the left-symmetric superalgebras obtained in Section 3 whose supercommutator is a super-Virasoro algebra $\mathcal{SV}$.

Let $\widetilde{A}$ be a left-symmetric superalgebra, and let $\omega:\widetilde{A}\times\widetilde{A}\to\mathbb{C}$ be a bilinear form. It defines a multiplication on the space $A=\widetilde{A}\oplus\mathbb{C}c$ by the rule

$$(x+\lambda c)\cdot(y+\mu c)=x\cdot y+\omega(x,y)c\quad\text{for all } x,y\in A,\ \lambda,\mu\in\mathbb{C}. \tag{4-1}$$

Let

$$B(x,y,z):=\omega(x\cdot y,z)-\omega(x,y\cdot z). \tag{4-2}$$

Then it is easy to show that $A$ is a left-symmetric superalgebra if and only if

$$B(x,y,z)=(-1)^{\alpha\beta}B(y,x,z) \tag{4-3}$$

for all $x\in\widetilde{A}_\alpha,\ \ y\in\widetilde{A}_\beta$, and $z\in\widetilde{A}$, where $\alpha,\beta\in\mathbb{Z}_2$. The algebra $A$ is called a central extension of $\widetilde{A}$. By construction, the bilinear form

$$\Omega(x,y)=\omega(x,y)-(-1)^{\alpha\beta}\omega(y,x) \tag{4-4}$$

for all $x\in\widetilde{A}_\alpha,\ \ y\in\widetilde{A}_\beta$ and $z\in\widetilde{A}$, where $\alpha,\beta\in\mathbb{Z}_2$, defines a central extension of the Lie superalgebra $\mathcal{G}(A)$.

Let the left-symmetric superalgebra $\widetilde{V}_\epsilon$ on a centerless Virasoro algebra $\widetilde{\mathcal{SV}}$ be given through Theorem 3.2. Since a super-Virasoro algebra $\mathcal{SV}$ is a central extension of a centerless super-Virasoro algebra $\widetilde{\mathcal{SV}}$, it is natural to consider the central extension $V_\epsilon=\widetilde{V}_\epsilon\oplus\mathbb{C}c$ of $\widetilde{V}_\epsilon$ such that $V_\epsilon$ is a compatible left-symmetric superalgebraic structure on the super-Virasoro algebra $\mathcal{SV}$ with $c$ being the annihilator of $V_\epsilon$, that is, the products of $V_\epsilon$ are given by

$$\begin{aligned}
L_m\cdot L_n&=f(m,n)L_{m+n}+\omega(L_m,L_n)c,\\
L_m\cdot G_r&=g(m,r)G_{m+r}+\omega(L_m,G_r)c,\\
G_r\cdot L_m&=h(r,m)G_{m+r}+\omega(G_r,L_m)c,\\
G_r\cdot G_s&=d(r,s)L_{r+s}+\omega(G_r,G_s)c,\\
c\cdot c=c\cdot L_m&=L_m\cdot c=c\cdot G_r=G_r\cdot c=0,
\end{aligned} \tag{4-5}$$

where the functions $f(m,n)$, $g(m,r)$, $h(r,m)$, and $d(r,s)$ satisfy (3-4) and (3-5).

For convenience, set

$$\omega(L_m, L_n) = \varphi(m, n), \qquad \omega(L_m, G_r) = \psi(m, r),$$
$$\omega(G_r, L_m) = \rho(r, m), \qquad \omega(G_r, G_s) = \sigma(r, s). \tag{4-6}$$

So the supercommutators of $V_\epsilon$ give a super-Virasoro algebra $\mathcal{SV}$ if and only if $\varphi(m, n)$, $\psi(m, r)$, $\rho(r, m)$, and $\sigma(r, s)$ satisfy

$$\varphi(m, n) - \varphi(n, m) = \tfrac{1}{12}(m^3 - m)\delta_{m+n,0},$$
$$\sigma(r, s) + \sigma(s, r) = \tfrac{1}{12}(4r^2 - 1)\delta_{r+s,0}, \tag{4-7}$$
$$\psi(m, r) - \rho(r, m) = 0.$$

By (4-3), we have

$$B(L_m, L_n, L_l) = (-1)^{0\cdot 0} B(L_n, L_m, L_l),$$
$$B(L_m, L_n, G_r) = (-1)^{0\cdot 0} B(L_n, L_m, G_r),$$
$$B(L_m, G_r, L_n) = (-1)^{0\cdot 1} B(G_r, L_m, L_n),$$
$$B(L_m, G_r, G_s) = (-1)^{0\cdot 1} B(G_r, L_m, G_s),$$
$$B(G_r, G_s, L_m) = (-1)^{1\cdot 1} B(G_s, G_r, L_m),$$
$$B(G_r, G_s, G_t) = (-1)^{1\cdot 1} B(G_s, G_r, G_t).$$

These are equivalent to the equations

$$(m - n)\varphi(m + n, l) = f(n, l)\varphi(m, n + l) - f(m, l)\varphi(n, m + l), \tag{4-8}$$
$$(m - n)\psi(m + n, r) = g(n, r)\psi(m, n + r) - g(m, r)\psi(n, m + r), \tag{4-9}$$
$$(\tfrac{m}{2} - r)\rho(m + r, n) = h(r, n)\psi(m, n + r) - f(m, n)\rho(r, m + n), \tag{4-10}$$
$$(\tfrac{m}{2} - r)\sigma(m + r, s) = d(r, s)\varphi(m, r + s) - g(m, s)\sigma(r, m + s), \tag{4-11}$$
$$2\varphi(r + s, m) = h(s, m)\sigma(r, m + s) + h(r, m)\sigma(s, m + r), \tag{4-12}$$
$$2\psi(r + s, t) = d(s, t)\rho(r, s + t) + d(r, t)\rho(s, r + t). \tag{4-13}$$

**Proposition 4.1.** *Any compatible left-symmetric superalgebraic structure $V$ on $\mathcal{SV}$ satisfies (4-5) if and only if the functions in (4-5) satisfy (3-4), (3-5) and (4-7)–(4-13).*

If a central extension $V_\epsilon$ of $\widetilde{V}_\epsilon$ given by $\omega$ satisfying Equation (4-5) defines a compatible left-symmetric superalgebraic structure on $\mathcal{SV}$, then $\varphi(m, n)$ defines a central extension of $\widetilde{\mathcal{SV}}_{\bar{0}}$. By Theorem 2.3, we know that

$$\varphi(m, n) = \tfrac{1}{24}(m^3 - m + (\epsilon - \epsilon^{-1})m^2)\delta_{m+n,0}. \tag{4-14}$$

**Theorem 4.2.** *For a fixed $\epsilon \in \mathbb{C}$ satisfying $\operatorname{Re}\epsilon > 0$, $\epsilon^{-1} \notin \mathbb{Z}$ or $\operatorname{Re}\epsilon = 0$, $\operatorname{Im}\epsilon > 0$, suppose the functions $f(m, n)$, $g(m, r)$, $h(r, m)$, and $d(r, s)$ satisfy (3-4) and (3-5),* *and suppose $\varphi(m,n)$ satisfies (4-14). Then there is exactly one solution satisfying (4-7)–(4-13). It is given by*

$$\sigma(r,s)=\tfrac{1}{24}(4r^2-1+2(\epsilon-\epsilon^{-1})r)\delta_{r+s,0},\qquad \text{(4-15)}$$
$$\phi(m,r)=\rho(r,m)=0$$

*for $m\in\mathbb{Z}$ and $r,s\in\mathbb{Z}+\theta$, and it defines a compatible left-symmetric superalgebra $V_\epsilon$ on $\mathcal{SV}$.*

*Proof.* It is easy to verify that the $\varphi(m,n)$ given in (4-14) and the $\sigma(r,s)$, $\phi(m,r)$, and $\rho(r,m)$ given in (4-15) satisfy (4-7)–(4-13).

On the other hand, let $m=0$ and $r+s\neq 0$ in (4-11). Then we have

$$-r\sigma(r,s)=d(r,s)\varphi(0,r+s)-g(0,s)\sigma(r,s)=s\sigma(r,s).$$

Hence $\sigma(r,s)=0$ for all $r+s\neq 0$. Letting $r=s$ and $m=-2s$ in Equation (4-12), we have $2\varphi(2s,-2s)=h(s,-2s)\sigma(s,-s)+h(s,-2s)\sigma(s,-s)=4s\sigma(s,-s)$. So $\sigma(s,-s)=(1/24)(4s^2-1+2(\epsilon-\epsilon^{-1})s)$. Thus

$$\sigma(r,s)=\tfrac{1}{24}(4r^2-1+2(\epsilon-\epsilon^{-1})r)\delta_{r+s,0}\quad\text{for all } r,s\in\mathbb{Z}+\theta.$$

Next, we prove that $\psi(m,r)=\rho(r,m)=0$ for all $m\in\mathbb{Z}$ and $r\in\mathbb{Z}+\theta$. There are two cases.

*Case* I. $\theta=1/2$. Letting $m=n=0$ in (4-10), we have

$$-r\rho(r,0)=h(r,0)\psi(0,r)-f(0,0)\rho(r,0)=0.$$

So $\rho(r,0)=0$. By (4-7), we know that $\psi(0,r)=0$. Letting $n=0$ in (4-9), we have $m\psi(m,r)=\psi(m,r)g(0,r)-\psi(0,m+r)g(m,r)=-r\psi(m,r)$. Hence $(m+r)\psi(m,r)=0$. Therefore, we have

$$\psi(m,r)=\rho(r,m)=0\quad\text{for all } m\in\mathbb{Z},\ r\in\mathbb{Z}+1/2.$$

*Case* II. $\theta=0$. Letting $n=0$ and $m=-r\neq 0$ in (4-10), we have $\psi(0,0)=\rho(0,0)=0$. Letting $m=n=0$ and $r\neq 0$ in (4-10) and letting $m=r=0$ and $n\neq 0$ in (4-9), we have

$$\rho(r,0)=\psi(0,r)=0\quad\text{and}\quad\psi(n,0)=\rho(0,n)=0\quad\text{for all } r,n\in\mathbb{Z}, r,n\neq 0.$$

Let $r=0$, $m,n\neq 0$ in (4-9) and (4-10), we have

$$\psi(m,n)\frac{n}{1+2\epsilon n}-\psi(n,m)\frac{m}{1+2\epsilon m}=0,$$
$$\frac{m}{2}\psi(n,m)+\psi(m,n)\frac{n(1+\epsilon n)}{1+2\epsilon n}=0.$$

Since $\epsilon\neq 0$ and $\epsilon^{-1}\notin\mathbb{Z}$, we have $\psi(n,m)=\rho(n,m)=0$ for all $m,n\in\mathbb{Z}$. □

By Theorem 2.3, it is easy to show that the $V_\epsilon$ are mutually nonisomorphic for all $\epsilon \in \mathbb{C}$ satisfying $\mathrm{Re}\,\epsilon > 0$, $\epsilon^{-1} \notin \mathbb{Z}$ or $\mathrm{Re}\,\epsilon = 0$, $\mathrm{Im}\,\epsilon > 0$. Altogether, from Theorem 2.3, Proposition 3.1, Theorem 3.2, Proposition 4.1, and Theorem 4.2, we have the following conclusion.

**Theorem 4.3.** *Any compatible left-symmetric superalgebra on a super-Virasoro algebra satisfying (4-5) is isomorphic to one of the following (mutually nonisomorphic) left-symmetric superalgebras given by the multiplications*

$$\begin{aligned}
L_m \cdot L_n &= -\frac{n(1+\epsilon n)}{1+\epsilon(m+n)} L_{m+n} + \frac{c}{24}(m^3 - m + (\epsilon - \epsilon^{-1})m^2)\delta_{m+n,0},\\
L_m \cdot G_r &= -\frac{(m/2+r)(1+2\epsilon r)}{1+2\epsilon(m+r)} G_{m+r},\\
G_r \cdot L_m &= -\frac{m(1+\epsilon m)}{1+2\epsilon(m+r)} G_{m+r},\\
G_r \cdot G_s &= \frac{1+2\epsilon s}{1+\epsilon(r+s)} L_{r+s} + \frac{c}{24}(4r^2 - 1 + 2(\epsilon - \epsilon^{-1})r)\delta_{r+s,0},
\end{aligned}$$

*where* $m, n \in \mathbb{Z}$, $r, s \in \mathbb{Z}+\theta$, *c is an annihilator, and* $\mathrm{Re}\,\epsilon > 0$, $\epsilon^{-1} \notin \mathbb{Z}$ *or* $\mathrm{Re}\,\epsilon = 0$, $\mathrm{Im}\,\epsilon > 0$.

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## References

[Andrada and Salamon 2005] A. Andrada and S. Salamon, "Complex product structures on Lie algebras", *Forum Math.* **17**:2 (2005), 261–295. MR 2006c:17029 Zbl 05039175

[Bai 2006] C. Bai, "A further study on non-abelian phase spaces: left-symmetric algebraic approach and related geometry", *Rev. Math. Phys.* **18**:5 (2006), 545–564. MR 2007f:17044 Zbl 1110.17008

[Bakalov and Kac 2003] B. Bakalov and V. G. Kac, "Field algebras", *Int. Math. Res. Not.* 3 (2003), 123–159. MR 2003k:17033 Zbl 1032.17045

[Bordemann 1990] M. Bordemann, "Generalized Lax pairs, the modified classical Yang–Baxter equation, and affine geometry of Lie groups", *Comm. Math. Phys.* **135**:1 (1990), 201–216. MR 91k:58049 Zbl 0714.58025

[Burde 2006] D. Burde, "Left-symmetric algebras, or pre-Lie algebras in geometry and physics", *Cent. Eur. J. Math.* **4**:3 (2006), 323–357. MR 2007c:17001 Zbl 05209438

[Cayley 1890] A. Cayley, "On the theory of analytic forms called trees", pp. 242–246 in *Collected Mathematical Papers of Arthur Cayley, Vol. 3*, Cambridge Univ. Press, 1890.

[Chapoton and Livernet 2001] F. Chapoton and M. Livernet, "Pre-Lie algebras and the rooted trees operad", *Internat. Math. Res. Notices* 8 (2001), 395–408. MR 2002e:17003 Zbl 1053.17001

[Chu 1974] B. Y. Chu, "Symplectic homogeneous spaces", *Trans. Amer. Math. Soc.* **197** (1974), 145–159. MR 49 #7388 Zbl 0261.53039

[Connes and Kreimer 1998] A. Connes and D. Kreimer, "Hopf algebras, renormalization and noncommutative geometry", *Comm. Math. Phys.* **199**:1 (1998), 203–242. MR 99h:81137 ZBL 0932.16038

[Dardié and Médina 1996a] J.-M. Dardié and A. Médina, "Algèbres de Lie kählériennes et double extension", *J. Algebra* **185**:3 (1996), 774–795. MR 98h:17034 Zbl 0866.17006

[Dardié and Medina 1996b] J.-M. Dardié and A. Medina, "Double extension symplectique d'un groupe de Lie symplectique", *Adv. Math.* **117**:2 (1996), 208–227. MR 97e:58076 Zbl 0843.58045

[Diatta and Medina 2004] A. Diatta and A. Medina, "Classical Yang–Baxter equation and left invariant affine geometry on Lie groups", *Manuscripta Math.* **114**:4 (2004), 477–486. MR 2005i:53102 Zbl 1078.53082

[Ebrahimi-Fard 2002] K. Ebrahimi-Fard, "Loday-type algebras and the Rota–Baxter relation", *Lett. Math. Phys.* **61**:2 (2002), 139–147. MR 2004b:17003 Zbl 1035.17001

[Etingof and Soloviev 1999] P. Etingof and A. Soloviev, "Quantization of geometric classical $r$-matrices", *Math. Res. Lett.* **6**:2 (1999), 223–228. MR 2000c:17019 Zbl 0999.17025

[Gerstenhaber 1963] M. Gerstenhaber, "The cohomology structure of an associative ring", *Ann. of Math.* (2) **78** (1963), 267–288. MR 28 #5102 Zbl 0131.27302

[Golubchik and Sokolov 2000] I. Z. Golubchik and V. V. Sokolov, "Generalized operator Yang–Baxter equations, integrable ODEs and nonassociative algebras", *J. Nonlinear Math. Phys.* **7**:2 (2000), 184–197. MR 2001b:37082 Zbl 01491567

[Kong et al. 2007] X. Kong, H. Chen, and C. Bai, "Classification of graded left-symmetric algebra structures on the Witt and Virasoro algebras", Preprint, 2007. arXiv 0706.4229v2

[Koszul 1961] J.-L. Koszul, "Domaines bornés homogènes et orbites de groupes de transformations affines", *Bull. Soc. Math. France* **89** (1961), 515–533. MR 26 #3090 Zbl 0144.34002

[Kupershmidt 1994] B. A. Kupershmidt, "Non-abelian phase spaces", *J. Phys. A* **27**:8 (1994), 2801–2809. MR 95g:17031 Zbl 0842.58030

[Kupershmidt 1999a] B. A. Kupershmidt, "On the nature of the Virasoro algebra", *J. Nonlinear Math. Phys.* **6**:2 (1999), 222–245. MR 2000d:17030 Zbl 1015.17027

[Kupershmidt 1999b] B. A. Kupershmidt, "What a classical $r$-matrix really is", *J. Nonlinear Math. Phys.* **6**:4 (1999), 448–488. MR 2001a:17023 Zbl 1015.17015

[Lichnerowicz and Medina 1988] A. Lichnerowicz and A. Medina, "On Lie groups with left-invariant symplectic or Kählerian structures", *Lett. Math. Phys.* **16**:3 (1988), 225–235. MR 90j:53067 Zbl 0665.53046

[Neveu and Schwarz 1971a] A. Neveu and J. H. Schwarz, "Factorizable dual model of poins", *Nucl. Phys. B.* **31** (1971), 86–112.

[Neveu and Schwarz 1971b] A. Neveu and J. H. Schwarz, "Quark model of dual pions", *Phys. Rev. D* **4** (1971), 1109–1111.

[Ramond 1971] P. Ramond, "Dual theory for free fermions", *Phys. Rev. D* **3** (1971), 2415–2418. MR 49 #1971

[Svinolupov and Sokolov 1994] S. I. Svinolupov and V. V. Sokolov, "Vector-matrix generalizations of classical integrable equations", *Teoret. Mat. Fiz.* **100**:2 (1994), 214–218. In Russian; translated in *Theoret. and Math. Phys.* **100**:2 (1995), 959–962. MR 95k:35154 Zbl 0875.35121

[Vasil'eva and Mikhalëv 1996] E. A. Vasil'eva and A. A. Mikhalëv, "Free left-symmetric superalgebras", *Fundam. Prikl. Mat.* **2**:2 (1996), 611–613. MR 1793397

[Vinberg 1963] È. B. Vinberg, "The theory of homogeneous convex cones", *Trudy Moskov. Mat. Obšč.* **12** (1963), 303–358. MR 28 #1637 Zbl 0138.43301

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