Pacific Journal of Mathematics

A {2, 3}-LOCAL SHADOW OF O'NAN'S SIMPLE GROUP

INNA KORCHAGINA AND RICHARD LYONS

Volume 235 No. 2 April 2008

PACIFIC JOURNAL OF MATHEMATICS
Vol. 235, No. 2, 2008

A {2, 3}-LOCAL SHADOW OF O'NAN'S SIMPLE GROUP

INNA KORCHAGINA AND RICHARD LYONS

This paper is a contribution to the ongoing project of Gorenstein, Lyons, and Solomon to produce a complete unified proof of the classification of finite simple groups. A part of this project deals with classification and characterization of bicharacteristic finite simple groups. This paper contributes to that particular situation.

1. Introduction

In this paper we continue the characterization of various bicharacteristic finite simple groups G in the sense of [Korchagina and Lyons 2006] and the earlier papers [Korchagina and Solomon 2003; Korchagina et al. 2002]. The strategy is part of the Gorenstein, Lyons, and Solomon revision project [GLS 1994], but expanded to the case $e(G) = 3$ to make the GLS project fit with the Aschbacher–Smith quasithin theorem [2004]. We shall give appropriate but concise definitions below, and refer the reader to [Korchagina and Lyons 2006] for a fuller discussion of bicharacteristic groups and the context in which they occur in the GLS project.

We use the following notation: G is a finite simple group, p is an odd prime, $m_p(X)$ is the p-rank of an arbitrary group X, $m_{2,p}(G)$ is the maximum value of $m_p(N)$ over all subgroups $N \le G$ such that $O_2(N) \ne 1$, and $e(G)$ is the maximum value of $m_{2,p}(G)$ as p ranges over all odd primes. Moreover $m_p^I(G)$ is the maximum value of $m_p(C_G(z))$ as z ranges over all involutions of G.

We fix an odd prime p and set

$$\mathcal{H} = \mathcal{H}(G) = \{H \le G \mid H \text{ is a 2-local subgroup of } G \text{ and } m_p(H) = m_{2,p}(G)\}.$$

The groups that we consider in this paper satisfy the conditions

$$\text{(H1)} \qquad m_{2,p}(G) = e(G) = 3 \quad \text{and} \quad m_p^I(G) \le 2.$$

We state our theorem, tie it in with the main theorem of our [2006] paper to obtain a corollary, and then discuss the theorem's technical terminology.

Theorem 1.1. *Suppose that G satisfies the conditions*

MSC2000: 20D05.

Keywords: finite simple group, sporadic group, local analysis, bicharacteristic group.

(1) *G is a finite $\mathcal{K}$-proper simple group*;

(2) *G has restricted even type*; *and*

(3) *for some odd prime p, G satisfies (H1) and has weak p-type.*

Then $p = 3$ and there exists $H \in \mathcal{H}$ such that $F^*(H) = O_2(H)$. Moreover, for any $H \in \mathcal{H}$ and any $B \leq H$ such that $B \cong E_{3^3}$, there is a hyperplane B_0 of B such that $L_{3'}(C_G(B_0)) \cong A_6$.

The conclusion of Theorem 1.1 implies that G satisfies all the hypotheses of our [2006, Theorem 1.2]. That theorem in turn yields that G has the structure asserted in the corollary, or $G \cong Sp_8(2)$ or $F_4(2)$. But these last two groups do not satisfy the assumption $m_3^I(G) \leq 2$. Indeed, in both, the centralizer of a long root involution is a parabolic subgroup P with Levi factor isomorphic to $Sp_6(2)$, and so $m_3^I(G) \geq m_3(Sp_6(2)) = 3$. Thus we have a corollary:

Corollary 1.2. *If G satisfies the assumptions of Theorem 1.1, then $G \cong A_{12}$ or G has the centralizer of involution pattern of F_5.*

The $\mathcal{K}$-proper assumption in Theorem 1.1 means that all proper simple sections of G are among the known simple groups, as is appropriate for the inductive classification [GLS 1994].

The hypothesis that G is of weak p-type [Korchagina and Lyons 2006] means

(1A) For every $x \in G$ of order p such that $m_p(C_G(x)) \geq 3$, and for every component L of $E(C_G(x)/O_{p'}(C_G(x)))$, the component $L \in \mathcal{C}_p$, and $O_{p'}(C_G(x))$ has odd order.

Here $\mathcal{C}_p$ is an explicit set of quasisimple $\mathcal{K}$-groups defined for any odd prime p [GLS 1994, p. 100]. Instead of repeating the definition here, we shall use it in combination with the condition (H1) and the Thompson dihedral lemma, obtaining in Lemma 2.3 below a much shorter list of possible components L in Equation (1A). The term "restricted even type" is defined on [p. 95].[1]

Rather than repeat the definition we state its impact on the situation at hand: for any involution $z \in G$, if we set $C = C_G(z)$, then these conditions hold:

(1B1) $O_{2'}(C) = 1$.

(1B2) $m_p(C) < 3$ and $m_r(C) \leq 3$ for all odd primes $r \neq p$.

(1B3) For any component L of C, we have $L \in \mathcal{C}hev(2)$, or $L \cong L_2(17)$, or $L/Z(L) \cong L_3(3)$, $G_2(3)$ (with $p > 3$), M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, J_2, J_3 (with $p > 3$), J_4, HS, or Ru;

[1]Because of our frequent references to external results, we abbreviate with the convention that, unless otherwise indicated, unnamed, bracketed tags implicitly belong to the most recent full citation.

(1B4) If L is as in (3) and $L/Z(L) \cong L_3(4)$, then $Z(L)$ has exponent 1 or 2; and

(1B5) If $L \cong A_6$, then $m_2(C_G(z)) = 3$.

Indeed (1B2) is an immediate consequence of (H1). The definition of "even type" implies that (1) holds, and that any component L in (1B3) lies in the set $\mathcal{C}_2$ (defined in [GLS 1994, p. 100]). But $m_r(L) \leq 3$ for all odd r, with strict inequality if $r = p$. Using the known ranks of simple $\mathcal{K}$-groups [GLS 1998, Tables 3.3.1, 5.6.1] we get the groups listed in (1B3), and the additional groups $L_2(q)$, q a Fermat or Mersenne prime or 9, as possible isomorphism types for L. Now the definition of *restricted* even type implies that $q \leq 17$ if $L \cong L_2(q)$, so either $q = 17$ or $L \in \mathcal{C}hev(2)$. Furthermore, covering groups of $L_3(4)$ by centers of exponent 4 are by definition excluded from $\mathcal{C}_2$, which proves (1B4), and condition (1B5) is part of the definition of "restricted even type."

It is somewhat arbitrary that the definition of $\mathcal{C}_2$ excludes the covering groups $4L_3(4)$. This is because the sporadic group $O'N$, in which the centralizer of an involution has such a component, in GLS emerges from the analysis of groups of odd type in [GLS 2005]. Nevertheless, our assumptions in Theorem 1.1 inevitably lead toward the situation in which $F^*(C_G(z))$ is a covering group of $L_3(4)$ by Z_4, and this situation is prevented only by the definition of $\mathcal{C}_2$. In Bender's terminology, $O'N$ is a "shadow" group in our setup.

As may be expected, the proof of Theorem 1.1 uses many properties of the groups in $\mathcal{C}_p$ as well as those in (1B3). To justify these we generally refer to [GLS 1998] or our [2006] paper.

We also use the following notation from our [2006] paper. Here X is any subgroup of G, and $a \in G$ and $A \leq G$ are respectively any element of order p and any elementary abelian p-subgroup of G. In the notations $C(a, K)$ and $C(A, K)$, K is any product of p-components of $C_G(a)$ or $C_G(A)$, respectively. Also,

$$\widehat{X} = X/O_{p'}(X),\ \ \widehat{C}_G(X) = \widehat{C_G(X)}\ \text{ and }\ \widehat{N}_G(X) = \widehat{N_G(X)};$$
$$L_a = L_{p'}(C_G(a))\ \text{ and }\ L_A = L_{p'}(C_G(A));$$
$$C(a, K) = C_{C_G(a)}(K/O_{p'}(K)) = C_{C_G(a)}(\widehat{K})\ \text{ and }\ C(A, K) = C_{C_G(A)}(\widehat{K});$$
$$\mathcal{A}^o = \{A \leq G \mid A \text{ is an elementary abelian } p\text{-group and } m_p(C_G(A)) \geq 3\}.$$

Also for any group X on which the group Y acts, $\mathrm{Aut}_Y(X)$ is the natural image of Y in $\mathrm{Aut}(X)$.

Section 2 spells out some properties of $\mathcal{K}$-groups. In the next four sections we prove that $p = 3$ and obtain, in Proposition 6.1, two possible specific structures for elements of $\mathcal{H}$. These are analyzed separately in the final section to complete the proof of the theorem.

2. Preliminary lemmas

Lemma 2.1. *Let p be an odd prime and let K be a quasisimple $\mathcal{K}$-group such that $O_{p'}(K)=1$ and $m_{2,p}(K)=m_p(Z(K))$. Then one of the following holds*:

(a) $m_p(K)=1$;

(b) $p=3$ *and* $K/Z(K)\cong U_3(2^n)$, n *odd*, $n>1$; *or*

(c) $K\cong L_2(p^n)$, $n\geq 1$, $p\neq 3$.

Proof. Let K be a group satisfying the hypotheses of the lemma. First notice that if $K/Z(K)\cong L_2(p^n)$, then (c) holds. Thus we may assume by the way of contradiction that $m_p(K)>1$, p and K are not as in (b), and that $K/Z(K)\ncong L_2(p^n)$ with $n\geq 1$. Let us show that in most cases there exists a “contradicting” triple (L,y,z), by which we mean a triple satisfying the conditions

(2A) $\qquad L\leq K,\quad z\in \mathcal{I}_2(L),\quad y\in \mathcal{I}_p(C_L(z))$ but $y\notin Z(K)$.

If such a triple exists, then $m_{2,p}(K)\geq m_p(C_K(z))>m_p(Z(K))$, contrary to our hypothesis.

If $K\in \mathcal{S}por$, then the possible values of p for the various K’s are listed in [GLS 1998, Table 5.6.1]. We take $L=C_K(z)$ for a 2-central involution $z\in K$, except for the case $K\cong He$ with $p=5$, in which case we take $L=C_K(z)$ with z a non-2-central involution. Then [Table 5.3] shows that L contains an element y of order p, and we have a contradicting triple. If $K\in \mathcal{A}lt$, then $K\cong A_n$ or $3A_7$, and $n\geq 2p$ as $m_p(K)>1$. We can take $L\cong A_4\times A_{n-4}$ in the first case, and otherwise $L\cong A_4\times Z_3$ (with $p=3$). Then we can take z in the first direct factor and y in the second, for a contradicting triple. Thus $K\notin \mathcal{S}por\cup\mathcal{A}lt$.

If $K/Z(K)\in \mathcal{C}hev(p)-\{L_2(p^n)\}$, then by [4.5.1], $K/Z(K)$ contains a subgroup $L/Z(K)\cong SL_2(p^m)$ or $Z_2\times L_2(p^m)$ for some $m\geq 1$. This clearly yields a contradicting triple unless $Z(K)\neq 1$, in which cases $p=3$ and $K/Z(K)\cong G_2(3)$, $U_4(3)$ or $\Omega_7(3)$, by [6.1.4]. But in those cases $K/Z(K)$ has a subsystem subgroup isomorphic to $U_3(3)$, which splits over $Z(K)$; see [6.1.4]. Hence $L\cong U_3(3)$ has a contradicting $SL_2(3)$-subgroup.

Thus $K/Z(K)\in \mathcal{C}hev-\mathcal{C}hev(p)$. By [6.1.4], K is a homomorphic image of the universal version K_u of $K/Z(K)$. Suppose first that p divides $|Z(K_u)|$. Thus $K_u\cong SL_n^\epsilon(q)$ or $E_6^\epsilon(q)$ for $\epsilon=\pm 1$ and $q\equiv\epsilon \pmod p$, with $p=3$ in the E_6^ϵ case. In the latter case, K contains a subgroup isomorphic to $SL_6^\epsilon(q)$. Hence, in both situations K contains a contradicting subgroup $L\cong Z_p\times SL_2(q)$, unless $K_u\cong SL_3^\epsilon(q)$ and $p=3$. Even in that case if q is odd, K contains a contradicting subgroup $L\cong SL_2(3)$; and if q is even, then since (b) fails, $\epsilon=+$, and a Borel subgroup of K is a contradicting subgroup. So we may assume that p does not divide $|Z(K_u)|$. Let $E\leq K$ with $E\cong E_{p^2}$. As p does not divide $|Z(K_u)|$, there is a simple algebraic group $\overline{K}$ and a σ-setup $(\overline{K}, \sigma)$ for K such that $E \leq \overline{T}$ for some maximal torus $\overline{T}$ of $\overline{K}$. (See [4.1.16].) Then for some $y \in E^{\#}$, $C_{\overline{K}}(y)$ has a simple component. We set $L = C_K(y)$, so that L has a Lie component [4.9.3]. In particular, $C_K(y)$ contains an involution z and (L, y, z) is a contradicting triple. The proof is complete. □

Lemma 2.2. *Suppose that p is an odd prime and that X is a $\mathcal{K}$-group such that $K = L_{p'}(X)$, $O_p(X) \neq 1$, $X = K\,O_p(X)$, $O_{p'}(X)$ has odd order, and $m_{2,p}(X) \leq 3$. Suppose that every component of $X/O_{p'}(X)$ lies in $\mathcal{C}_p$, and that $e(X) \leq 3$. Then $m_{2,r}(X/O_{p'}(X)) \leq 3$ for all primes $r > 3$. Moreover, $m_{2,3}(X/O_{p'}(X)) \leq 3$ unless possibly $p = 7$ and $K/[K, O_{7'}(K)]$ is the central product of $3A_7$ with either $3A_7$ or $SL_3(7)$.*

Proof. By induction on $|X|$, we may suppose that no proper subgroup of X covers $X/O_{p'}(X)$, whence $O_{p'p}(X) \leq \Phi(X)$. By [GLS 1996, 3.1.5], $O_{p'p}(X)$ is nilpotent. Thus $X/F(X)$ is the direct product of simple groups, and $F(X) = O_{2'}(X)$.

We assume that $m_{2,r}(X/O_{p'}(X)) > 3$ for some odd prime r. Thus,

(2B) $$m_{2,r}(X/O_{p'}(X)) > 3 \geq e(X) \geq m_{2,r}(X),$$

so X possesses 2- and r-subgroups T and R, respectively, for some odd prime r, with R normalizing $T\,O_{p'}(X)$ and $m_r(R/R \cap O_{p'}(X)) > 3$, but $m_r(R) \leq e(X) \leq 3$. Clearly $r \neq p$. We must prove that $r = 3$, $p = 7$, and $K/O_{p'}(K) \cong A_7 \times A_7$ or $A_7 \times L_3(7)$. Since $O_{p'}(X)$ has odd order, a Frattini argument permits us to take R to normalize T. By minimality of X, we have $X = K\,O_p(X)$, $K = [K, K]$, and, with (2B),

(2C) $$O_{p'}(K) = O_r(K) \neq 1.$$

We factor K into p-components as $K = K_1 \cdots K_n$, set $W_i = [K_i, O_r(K_i)]$ for $i = 1, \ldots, n$, and assume as we may that nonquasisimple K_i's come first; that is, for some $0 < m < n$, $W_i \neq 1$ for $i = 1, \ldots, m$, and $W_i = 1$ for $i = m+1, \ldots, n$. Then for any $i = 1, \ldots, n$ and any involution $z_i \in K_i$, we have $m_p(C_X(z_i)) \leq m_{2,p}(X) \leq 3$. As $O_p(X) \neq 1$ this implies that

(2D) $$m_p(C_{K_i}(z_i)) \leq 2 + m_p(O_{r,p}(K_i)).$$

If $K_i/O_r(K_i) \in \mathcal{C}hev(p)$, then the involution centralizer data in [GLS 1998, 4.5.1] and the rank data in [3.3.1] show that (2D) is satisfied only if $K_i/O_r(K_i) \cong L_2(p^n)$, $L_3^{\pm}(p^m)$ for $m \leq 2$, $PSp_4(p)$, $L_4^{\pm}(p)$, $G_2(p)$, or $3G_2(3)$ (with $p = 3$). In all these cases as $r \neq p$, we have $m_r(K_i/O_r(K_i)) \leq 3$; see [4.10.2].

For any p-component K_i of K, regardless of its isomorphism type, this implies that either $O_r(K_i) = W_i$, or $r = 3$ with

(2E) $$K_i/W_i \cong 3\,Mc,\ 3A_7,\ 3O'N,\ 3\,Fi'_{24},\ \text{or}\ SL^{\epsilon}_3(p^s),$$

and $p = 5, 7, 7, 7$ and $p^s \equiv \epsilon \pmod 3$, respectively. This follows from the definition of $\mathcal{C}_p$, the previous paragraph's restrictions if $K/O_r(K) \in \mathcal{C}hev(p)$, and the known multipliers of simple $\mathcal{K}$-groups [6.1.4].

With these preliminaries established, we next prove the lemma by two cases:

Case: $m = 0$, that is, all the K_i are all quasisimple.

By the previous paragraph, we have $r = 3$, $O_3(K)$ is an elementary abelian 3-subgroup of $Z(K)$, and some component K_1 of K has one of the isomorphism types in (2E). If $K = K_1$, then among the groups in (2E), the condition $m_3(K_1/Z(K_1)) \geq m_{2,3}(K_1/Z(K_1)) > 3$ is satisfied only by $K_1 \cong 3\,Fi'_{24}$, by [4.10.2, 5.6.1, 5.6.2]. But then $m_{2,3}(K_1) > 3$, a contradiction. Therefore $K \neq K_1$. Set $K^1 = K_2 \cdots K_n \neq 1$. As K^1 is of even order and $[K^1, K_1] = 1$, $m_3(K_1) \leq 3$. Thus $K_1 \cong 3A_7, 3O'N$, or $SL^{\epsilon}_3(p^s)$. If $K_1 \cong 3O'N$, then $m_{2,3}(K_1) = 3$ by [5.6.2]. Also $p = 7$, and so no component of K is a Suzuki group by definition of $\mathcal{C}_p$. Thus, $C_K(K_1) - K_1$ contains an element of order 3, so $m_{2,3}(K) \geq m_{2,3}(K_1) + 1 \geq 4$, a contradiction. Hence $K_1 \cong 3A_7$ or $SL^{\epsilon}_3(p^s)$. Thus $m_3(K_1) = m_{2,3}(K_1) = m_3(K_1/Z(K_1)) = m_{2,3}(K_1/Z(K_1)) = 2$. If $O_3(K^1) = 1$, then $K = K_1 \times K^1$, and it follows immediately that $m_{2,3}(K) = m_{2,3}(K/O_3(K))$, contrary to assumption. So $O_3(K^1) \neq 1$, and K^1 likewise has a $3A_7$ or $SL^{\epsilon}_3(p^s)$ component. As $m_{2,3}(K) \leq 3$, the only possibility is that $K^1 = K_2$ and $Z(K_1) = Z(K_2)$. If neither K_1 nor K_2 is isomorphic to $3A_7$, then using the facts that $O_p(X) \neq 1$, $m_{2,p}(SL^{\epsilon}_3(p^s)) \geq s$ and $m_p(SL^{\epsilon}_3(p^s)) \geq 2$, we reach the contradiction $m_{2,p}(X) \geq 4$. Therefore without loss $K_1 \cong 3A_7$ and $p = 7$. If $K_2 \cong SL^{\epsilon}_3(7^s)$, then $m_{2,7}(X) \geq 1 + m_{2,7}(K_1) + m_7(K_2) = 2 + s$, so $s = 1$. We have obtained the two exceptional conclusions of the lemma.

Case: $m > 0$, so that $W_1 \neq 1$ and K_1 is not quasisimple.

Set $k = m_2(K_1 \cdots K_m)$. Now $C_{K_1 \cdots K_m}(O_r(K)) \leq O_{r,p}(K) \leq O_{2'}(K)$ by the definition of m. Thus by the Thompson dihedral lemma, $K_1 \cdots K_m$ contains the direct product of k copies of D_{2r}. On the other hand $m_{2,r}(X) \leq e(X) \leq 3$, and so

(2F) $$k \leq 4,\ \text{whence}\ m \leq 2.$$

If $K = K_1$, then since $m_r(K/O_r(K)) \geq m_{2,r}(K/O_r(K)) > 3$, we have $K/O_r(K) \notin \mathcal{C}hev(p)$ by the paragraph following (2D). If $K/O_r(K) \cong A_s$, then since $r \geq 3$ and $m_r(K/O_r(K)) > 3$, certainly $s \geq 12$; see [5.2.10]. But then $k \geq 6$, a contradiction. Likewise if $K/O_r(K) \notin \mathcal{C}hev(p) \cup \mathcal{A}lt$, then using the definition of $\mathcal{C}_p$ and the values of $k = m_2(K/O_r(K))$ and $m_{2,r}(K/O_r(K))$ given in [3.3.1, 5.6.1], we see that the conditions $k \leq 4$ and $m_{2,r}(K/O_r(K)) \geq 4$ are inconsistent. (Note: if $p = 3$

and $K/O_r(K) \cong U_5(2)$, then $r > 3$ so $m_r(K/O_r(K)) < 4$ by [4.10.2].) Therefore $K \neq K_1$.

Fix a characteristic subgroup R of W_1 of exponent r and class at most 2 such that $C_{\mathrm{Aut}(W_1)}(R)$ is an r-group, and choose R minimal subject to these conditions. By minimality either R is elementary abelian or $Z(R) \leq Z(K_1)$. Since $W_1 = [K_1, W_1] \neq 1$, $C_{K_1}(R) \leq O_{r,p}(K_1)$.

Obviously R is K_2-invariant. Let $V \leq K_2$ be a four-group and set $K_1^o = O^{p'}(C_{K_1}(V))$ and $R_v = C_R(v)$ for each $v \in V^\#$. Then K_1^o acts on each $C_R(v)$. The action is nontrivial for some $v \in V^\#$, for otherwise K_1^o would centralize $R = \langle R_v \mid v \in V\rangle$, contradicting $C_{K_1}(R) \leq O_{r,p}(K_1)$. On the other hand $m_r(R_v) \leq m_{2,r}(X) \leq 3$, which immediately implies that $m_r(R_v/\Phi(R_v)) \leq 3$ or $R_v \cong r^{1+4}$. Correspondingly the perfect group $K_1^o/C_{K_1^o}(R_v)$ embeds in $SL_3(r)$ or $Sp_4(r)$. In any case since K_1^o covers $K_1/O_{p'}(K_1)$, we have $m_2(K_1) \leq 2$. Indeed the $Sp_4(r)$ case is impossible since any four-subgroup of $Sp_4(r)$ contains $Z(Sp_4(r))$, whereas $O_2(K_1/C_{K_1}(R)) = 1$ since $O_{p'}(K_1)$ has odd order. Similarly $K_1/C_{K_1}(R_v)$ does not embed in $SL_2(R)$, and so $m_r(R_v/\Phi(R_v)) = 3$; indeed $m_r(R_v) = 3$ since R_v has exponent r and class at most 2. Now $K_1/O_{2'}(K_1)$ is involved in $SL_3(r)$, and on the other hand lies in $\mathcal{C}_p$. If $m_p(K_1) = 1$, then $K_1/O_{2'}(K_1) \cong L_2(p)$ or A_p (or $L_2(8)$ or ${}^2B_2(2^{\frac{5}{2}})$ with $p = 3$ or 5) by definition of $\mathcal{C}_p$. From the known structure of all subgroups of $SL_3(r)$ [6.5.3], and the facts that $r \neq p$ and $K_1/O_{2'}(K_1)$ is simple, we see that whatever the value of $m_p(K_1)$, the only possibilities for the pair $(p, K_1/O_{p'}(K_1))$ are $(3, L_2(9))$, $(5, A_5)$, $(7, L_2(7))$, and $(7, A_7)$. Moreover in the last case $r = 5$. In particular $m_r(K_1/O_r(K)) = 1$.

Similarly, $m_r(K_i/O_r(K_i)) = 1$ for all $1 \leq i \leq m$. Since $m_r(K/O_r(K)) \geq 4$ but $m \leq 2$, it follows that $m < n$, that is, K_n is quasisimple. Therefore $[K_1, K_n] = 1$. As K_n has even order and $e(X) \leq 3$, this yields $m_r(K_1) \leq 3$. Therefore $m_r(K_1) = m_r(R_v) = 3$.

We argue that $m_{2,r}(E(K)) = m_r(Z(E(K)))$. Otherwise, changing indices if necessary, there exist a 2-subgroup $T \leq K_n$ that is not the identity and an element $x \in N_{E(K)-Z(E(K))}(T)$ of order r. Then $N_K(T)$ contains $K_1\langle x\rangle$ so that $m_r(K_1\langle x\rangle) \leq e(X) \leq 3 = m_r(K_1)$. Because $[K_1, x] = 1$, we reach the contradiction. Thus $x \in Z(E(K))$, and our assertion is proved. If $Z(E(K)) \neq 1$, then some K_i for $i > m$ is as in (2E). But then there still exists an element $x \in K_i - Z(K_i)$ of order 3 centralizing an involution of $E(K)$ (see [5.3]), and so $m_{2,r}(E(K)) > m_r(Z(E(K)))$, a contradiction. Therefore $Z(E(K)) = 1$ and so $m_{2,r}(E(K)) = 0$. Using that each K_i lies in $\mathcal{C}_p$, we can apply Lemma 2.1 and conclude that $m_r(E(K)) \leq 1$.

Finally $X = K_1 \times E(K)$ or $K_1K_2 \times E(K)$, with $K_1/O_r(K_1)$ (and $K_2/O_r(K_2)$) and $E(K)$ each having cyclic Sylow r-subgroups. Therefore $m_r(K/O_r(K)) \leq 3$ and the proof is complete. □

p	$m_p^I(X)$	K
3	0	$L_2(8)$, $L_2(3^n)$ for $n > 1$
3	1	$L_3^{\pm}(3)$, A_9, M_{11}, J_3, ${}^3D_4(2)$, ${}^2F_4(2^{\frac{1}{2}})'$, $Sp_4(8)$, $L_2(8) \times L_2(8)$
3	2	$3G_2(3)$, $3J_3$
5	0	${}^2B_2(2^{\frac{5}{2}})$
5	1	A_{10}, ${}^2F_4(2^{\frac{1}{2}})'$, ${}^2F_4(2^{\frac{5}{2}})$, J_2, HS, He, Ru, Co_2, Co_3, Mc, F_3, $A_5 \times {}^2B_2(2^{\frac{5}{2}})$, ${}^2B_2(2^{\frac{5}{2}}) \times {}^2B_2(2^{\frac{5}{2}})$
7	0	A_7
7	1	He, $O'N$, F_3, $A_7 \times A_7$, $A_7 \times L_2(7)$
11	0	A_{11}
11	1	J_4
$p \geq 5$	0	$L_2(p^n)$ for $n \geq 1$
$p \geq 5$	1	$L_3^{\pm}(p)$, $L_2(p) \times L_2(p)$

Lemma 2.3. *Table 1 is a complete list, for all odd primes p, of all groups the $K = E(K)$ such that all components of K lie in $\mathcal{C}_p$, and $K \cong E(X/O_{p'}(X))$ for some group X such that $e(X) \leq 3$, $O_p(X) \neq 1$, $m_p^I(X) \leq 2$, and $O_{p'}(X)$ has odd order.*

Proof. If $e(X/O_{p'}(X)) > 3$, then $p = 7$, and either $E(X/O_{p'}(X)) \cong A_7 \times A_7$ or $A_7 \times L_3(7)$, by Lemma 2.2. The first case is allowed, and the second is impossible. This is because the centralizer of an involution in the first 7-component has 7-rank $m_7(O_{7'7}(X)) + m_7(L_3(7)) = 3$, contrary to assumption. So we may assume that $e(X/O_{p'}(X)) \leq 3$. All other hypotheses immediately go over to $X/O_{p'}(X)$ and so we may assume that $O_{p'}(X) = 1$. Our hypotheses are then the hypotheses of [Korchagina and Lyons 2006, 4.4], plus the assumption $m_p^I(X) \leq 2$. We therefore filter the list in [4.4] through this extra condition. Let L be a component of $E(X)$. If $Z(L) \neq 1$, then L is isomorphic to one of the groups in [4.4(h)]. Note that as $3U_4(3) \leq 3\,Mc$ and $3U_4(3)$ contains $Z_3 \times (SL_2(3) * SL_2(3))$, so $m_3^I(3U_4(3)) \geq 3$, and so $L \ncong 3U_4(3)$ or $3\,Mc$. Thus $L \cong 3G_2(3)$ or $3J_3$. Furthermore, $m_3(L) \geq 3$ by [GLS 1998, 5.6.1, 6.3.1], and so $L = E(X)$ in these cases, as desired. Hence we may assume that $Z(X) = 1$. Since $O_p(X) \neq 1$, we know $3 > m_p^I(E(X) \times O_p(X)) > m_p^I(E(X))$, and thus $m_p^I(E(X)) \leq 1$. Consequently if $E(X)$ is not simple, then it has two components, each of p-rank 1, and our [4.4] gives us only the direct products as we have listed. Finally, if $E(X)$ is simple, then $m_p(C_{E(X)}(z)) \leq 1$ for every involution $z \in E(X)$. Using information about centralizers of involutions from [GLS 1998] we conclude that $E(X)$ is restricted to be as stated in the lemma, and the stated values of $m_p^I(K)$ are correct. The references are [4.5.1, 3.3.1] for components in $\mathcal{C}hev(p)$, [5.2.2d, 5.2.10b] for components in $\mathcal{A}lt$, and [5.3] for components in $\mathcal{S}por$. For components $L \in \mathcal{C}hev(2)$, lower bounds on $m_p^I(L)$ come from subsystem subgroups [2.6.2] of type $A_1(q) \times B_{n-1}(q)$ in $B_n(q)$, ${}^2A_2(q^2)$ and ${}^2B_2(q) \times {}^2B_2(q)$ in ${}^2F_4(q)$, and $A_1(q) \times A_1(q^3)$ in ${}^3D_4(q)$. Upper bounds come from the Borel–Tits theorem and the p-ranks of parabolic subgroups [3.1.3, 2.6.5, 4.10.2]. □

Proposition 2.4. *Let G be a group satisfying the hypotheses of Theorem 1.1. Take $A \in \mathcal{A}^o(G)$. Then $O_{p'}(C_G(A))$ has odd order. If $E(\widehat{C}_G(A)) \neq 1$, then $E(\widehat{C}_G(A))$ is one of the groups in Table 1. Moreover, if $m_p(A) = 2$, then $m_p^I(E(\widehat{C}_G(A))) = 0$. In particular, $E(\widehat{C}_G(A))$ is quasisimple.*

Proof. Set $X = C_G(A)$. By [Korchagina and Lyons 2006, Proposition 5.4], $O_{p'}(X)$ has odd order. Take any $a \in A^{\#}$; the subnormal closure L of $L_{p'}(C_G(A))$ in $C_G(a)$ is a product of p-components of $C_G(a)$ by $L_{p'}$-balance, and $L_{p'}(X) = L_{p'}(C_L(A))$. Since G has weak p-type, the components of $\widehat{L}$ lie in $\mathcal{C}_p$, and hence so do the components of $L_{p'}(X)O_{p'}(X)/O_{p'}(X) = E(\widehat{X})$. Then Lemma 2.3 gives the isomorphism type of $E(\widehat{X})$. If $m_p(A) = 2$, then since $m_p^I(G) \le 2$ by assumption, $m_p^I(\widehat{X}) = m_p(O_p(\widehat{X}))$. From the table, this condition forces $m_p^I(\widehat{X}) = 0$, and the remaining statement of the proposition. □

Remark 2.5. Here it is appropriate to identify and correct an error in [Korchagina and Lyons 2006]. The statement of [5.1a] is incorrect for $p = 7$, and in [line 5] of the "proof", the reference to [4.4] is inadequate to draw the stated conclusion. Lemma 2.2 above indicates how to correct the statement of [5.1a] and fill the gap in its proof by using a variant of [4.4], as follows. Namely, instead of assuming that $K \lhd E(H)$ for some group H such that $e(H) \le 3$ and $O_p(H) \neq 1$, make the following weaker hypothesis: $K \lhd E(X/O_{p'}(X))$ for some group X such that $e(X) \le 3$ and $O_p(X) \neq 1$. Then weaken the conclusion by adding to [Table 4.4] the groups $K \cong A_7 \times A_7$ and $A_7 \times L_3(7)$, with $p = 7$ in both cases.

With [Table 4.4] so modified, the statement of [5.1a] is then correct. In the proof of [5.1a], Lemma 2.2 above shows that either $p = 7$ with $K/O_{p'}(K)$ being one of these two groups, or $e(X/O_{p'}(X)) \le 3$. In the latter case [4.4], as originally stated and proved, shows that K is one of the groups in the original [Table 4.4].

Thus the effective change is to add two groups to [Table 4.4], weaken the hypothesis of [4.4] as stated above, and use Lemma 2.2 in the proof of [5.1] to reduce to the case $O_{p'}(X) = 1$.

In the remainder of our [2006] paper, no changes are necessary in [4.5, 4.6, and Table 4.5], because no new *quasisimple* groups have been added to [Table 4.4]. Indeed, through [6.6], the proofs as originally given are correct, because changes were only made for $p=7$, and because [Table 4.5] is correct as originally stated. From [6.6] to the end, we are in the clear since $p=3$.

Lemma 2.6. *Let G be a group satisfying the hypotheses of Theorem 1.1. Let $A \in \mathcal{A}^o(G)$ and suppose that L is a component of $\widehat{C}_G(A)$. Suppose that $Aut(L)$ contains a subgroup B of order p acting nontrivially on a 2-subgroup T. Suppose also that $m^I_p(L) = m_p(Z(L))$, $C_{Inn(L)}(B)$ has odd order, and $m_{2,p}(L) = 1 + m_p(Z(L))$. Then $p=3$ and $L \cong L_2(3^n)$ for $n \geq 2$.*

Proof. By Proposition 2.4, L and p are as in Lemma 2.3. For these groups the condition $m^I_p(L) = m_p(Z(L))$ implies $m^I_p(L) = 0$ and $Z(L) = 1$. All the groups L in the table that pass this test satisfy $m_{2,p}(L) = 0$ (contrary to assumption) except for $L \cong L_2(3^n)$; see our [2006, 4.4] for the values of $m_{2,p}(L)$. □

Lemma 2.7. *Suppose that $p=3$ and $X = LB$ is a $\mathcal{K}$-group such that $L = E(X)$ is a quasisimple group in Table 1, $B = \langle b, a_1, a_2\rangle \cong E_{3^3}$ for $b \in Z(X)$ with $[a_i, L] \neq 1$, and $E(C_L(a_i))$ has a component $L_i \cong L_2(3^{n_i})$ with $n_i \geq 2$ for $i = 1, 2$. Suppose also that for either value of i, U_i is a four-subgroup of L_i, and $U_1U_2 = U_1 \times U_2$. Suppose finally that $m^I_3(X) \leq 2$. Then*

(a) *$L \cong A_9$, J_3, or $3J_3$; and*

(b) *$C_{Aut(L)}(U_1U_2)$ is the image in $Aut(L)$ of U_1U_2.*

Proof. Clearly L is a pumpup of L_i for $i = 1, 2$. But the possible pumpups of $L_i \cong L_2(3^{n_i})$ with $n_i \geq 2$ for $i = 1, 2$ in Lemma 2.3 are among those given in our [2006, 4.5]. Other than the desired isomorphism types of L, we must rule out $L \cong L_2(3^{3n_i})$ and $L \cong Sp_4(8)$. The first is impossible since $m_2(L_2(3^m)) = 2$ for any m, while $L \geq U_1U_2 \cong E_{2^4}$. Finally if $L \cong Sp_4(8)$, then a_1 acts as a field automorphism on L, and so $X \geq \langle b, a_1\rangle \times C_L(a_1) \cong E_{3^2} \times \Sigma_6 \geq E_{3^2} \times \Sigma_3 \times \Sigma_3$, contradicting $m^I_3(X) \leq 2$. □

Lemma 2.8. *Let G be a group satisfying the hypotheses of Theorem 1.1. Let H be any 2-local subgroup of G, and let L be a component of $E(H)$. Suppose that $p=3$. Then the isomorphism type of L is as specified in (1B3) and (1B4). Furthermore $m_3(L) = 1$ if and only if $L \cong L_2(17)$, $L_2(2^n)$, or $L^\epsilon_3(2^n)$ for some $\epsilon = \pm 1$ such that $2^n \equiv -\epsilon \pmod 3$.*

Proof. Let $z \in Z(O_2(H))$ be any involution, and let K be the subnormal closure of L in $C_G(z)$. By $L_{2'}$-balance and the fact that G has even type, $K \leq E(C_G(z))$ and L is a component of $C_K(O_2(H))$. By [GLS 1998, 7.1.10], for any involutory automorphism α of K, every component K_1 of $C_K(\alpha)$ lies in $\mathcal{C}_2$, and hence by the same argument given for (1B) above, the isomorphism type of K_1 is as given in (1B3) and (1B4) above. Using this fact repeatedly we obtain the first assertion. The 3-ranks of the groups in (1B) are determined in [3.3.1, 4.10.2, 5.6.1], and this yields the final statement. □

Lemma 2.9. *Let $L \in \mathcal{C}_2$ be as in (1B3). Suppose that F and W are subgroups of $Aut(L)$ such that $F \cong A_4$, F normalizes W, and $W \le O_{2'}(C_{Inn(L)FW}(O_2(F)))$. Suppose also that either $L/Z(L)$ is involved in A_9 or J_3, or that $C_{Aut(L)}(F)$ contains an element b of order* 3. *If $W \neq 1$, then $L/Z(L) \cong M_{12}$, b is nontrivial, and $C_{Inn(L)}(O_2(F)) \cong O_2(F) \times \Sigma_3$.*

Proof. This proof uses results from [GLS 1996; 1998]. If L is embeddable in A_9 or J_3, set $b = 1$. Assume that $W \neq 1$. Note first that from (1B) and [1998, 2.5.12, 5.3], $\mathrm{Out}(L)$ is 2-nilpotent, and indeed is a 2-group unless $L \in \mathcal{C}hev(2)$. In any case $O_2(F)$ induces inner automorphisms on L. Hence if $L \in \mathcal{C}hev(2)$, then $W = 1$ by [1998, 3.1.4], a contradiction. Otherwise $\langle W, F \times \langle b \rangle\rangle$ maps into $O^2(\mathrm{Aut}(L)) = \mathrm{Inn}(L)$. Replacing W and $F \times \langle b\rangle$ by their images, we may assume that L is simple and work within L. Write $O_2(F) = \langle u, v\rangle$ and set $C = C_L(u)$. Then $W \le O_{2'}(C_C(O_2(F))) \le L^*_{2'}(C)$ by $L^*_{2'}$-balance [1996, 5.18], so $L^*_{2'}(C) \neq 1$. But the structure of C may be found in [1998, 4.5.1] or [1998, 5.3] according as $L \in \mathcal{C}hev$ or $L \in \mathcal{S}por$. From these tables, we see that $L^*_{2'}(C) = 1$ unless $L \cong M_{12}$, HS, J_2, or Ru, with $C \cong Z_2 \times \Sigma_5$, $Z_2 \times \mathrm{Aut}(A_6)$ $E_{2^2} \times A_5$, or $E_{2^2} \times {}^2B_2(2^{\frac{3}{2}})$, respectively. In particular, since the order of L does not divide $|A_9|$ or $|J_3|$, our hypothesis yields $b \neq 1$.

Now $1 \neq W \le O_{2'}(C_C(v))$. If $C \cong E_{2^2} \times A_5$, or $E_{2^2} \times {}^2B_2(2^{\frac{3}{2}})$, this would contradict [1998, 3.1.4]. Thus $L \cong M_{12}$ or HS, whence $C_C(v) \cong \Sigma_3$ or D_{10}, respectively [1998, 6.5.1]. Since by assumption $b \in C_C(v)$ has order 3, we have $L \cong M_{12}$ and the proof is complete. □

Lemma 2.10. *Let K be one of the groups in Lemma 2.3 corresponding to $p = 3$.*

(a) *Suppose that $K \lhd K_1 = K\langle c\rangle$ and $c^3 = 1$. If J is a component of $C_K(c)$, $J/Z(J) \cong L_2(3^n)$ for some $n > 1$, $C_K(c)$ has no subgroup isomorphic to Σ_6, and K_1 has no subgroup isomorphic to $A_4 \times A_4$, then $K \cong L_2(3^n)$ or $L_2(3^{3n})$, with c acting on K trivially or as a field automorphism.*

(b) *There is no $I \le K$ such that $I/Z(I) \cong L^{\epsilon}_3(2^m)$ for $2^m \equiv \epsilon \pmod 3$, $2^m \not\equiv \epsilon \pmod 9$, and $m > 1$.*

Proof. The possible pumpups of $L_2(3^n)$ in Lemma 2.3 are among those given in our [2006, Table 4.5]. Using this, we see that if (a) fails, then $K \cong A_9$, J_3, $3J_3$, or $Sp_4(8)$, with $J \cong A_6$. But A_9 contains $A_4 \times A_4$, as do J_3 and $3J_3$, inside a subgroup disjoint from the center and isomorphic to an extension of E_{2^4} by $GL_2(4)$ [GLS 1998, 5.3h]. Thus $K \cong Sp_4(8)$. But then c acts as a field automorphism, centralizing $Sp_4(2)$. This contradiction proves (a).

In (b), since $2^m \not\equiv \epsilon \pmod 9$, we have $m \neq 3$. Just the condition that $|L^{\epsilon}_3(2^m)|$ divides $|K|$ reduces the possibilities in Lemma 2.3 (with $p = 3$) to $K \cong L_2(3^n)$, A_9, and $Sp_4(8)$. As $m \ge 2$, a Sylow 2-subgroup S of I satisfies $|S/\Phi(S)| \ge 2^4$

[2.4.1, 3.3.1], so $L_2(3^n)$, whose Sylow 2-subgroups are dihedral, is impossible. Since $L_3(4)$ has an E_{2^4}-subgroup all of whose involutions are conjugate while $|L_3(4)|_2 = |A_9|_2$, it does not embed in A_9. Neither do $L_3^\epsilon(2^m)$ for $m > 3$, as $m_2(L_3^\epsilon(2^m)) > m_2(A_9)$ [3.3.3]. Finally, if $K \cong Sp_4(8)$, let P be a Sylow 3-subgroup of K. Then $\Omega_1(P) \cong E_{3^2}$ and K has more than one conjugacy class of subgroups of order 3 [4.8.2]. On the other hand, $\Omega_1(S) \le I$, while I has a single such class [4.8.2]. The proof is complete. □

Lemma 2.11. *Let $K = L_2(q)$ for q odd. Let $V \le K$ with $V \cong E_{2^2}$, and let R be a 2-subgroup of Aut(K) such that $\langle R, V\rangle$ is a 2-group.*

(a) *Suppose that $R \cap V = 1$, and that either $[R, V] = 1$ or $R \cong Z_{2^m}$ for $m > 1$. Then any involution of R is a field automorphism on K.*

(b) *If $V \le R$ and $[R, V] = 1$, then $R = VF$, where F is a group of field automorphisms of K.*

(c) *Suppose that $q = 3^n$ and $\Sigma \le K$ with $\Sigma \cong \Sigma_4$. Then $\Sigma \le J \le K$ for some $J \cong A_6$, and $J = \langle C_J(z) \mid z \in \mathcal{I}_2(\Sigma)\rangle$.*

Proof. We have $\mathrm{Aut}(K) = \mathrm{Inndiag}(K)\Phi$ where Φ is a group of field automorphisms of K and $\mathrm{Inndiag}(K) \cong PGL_2(q)$. All four-subgroups of K are $\mathrm{Aut}(K)$-conjugate and self-centralizing in $\mathrm{Inndiag}(K)$ [GLS 1998, 4.5.1]. Since $C_K(\Phi)$ contains $L_2(r)$, where r is the prime of which q is a power, then, by replacing V by a conjugate, we obtain $C_{\mathrm{Aut}(K)}(V) = V \times \Phi$. Thus (b) holds. By [4.9.1], all involutions of $V\Phi - V$ are field automorphisms. Hence in proving (a) we may assume that $R \cong Z_{2^m}$ for $m > 1$. Expand $\langle V, R\rangle$ to $S \in \mathrm{Syl}_2(\mathrm{Aut}(K))$. Again, by conjugation, we may assume that S is Φ-invariant. Set $T = S \cap \mathrm{Inndiag}(K)$. Then T is dihedral and has a cyclic maximal subgroup $T_0 \lhd S$. As $T_0 \cap V \ne 1$, we have $R \cap T_0 = 1$. But $S/T_0 = T/T_0 \times (S \cap \Phi)T_0/T_0$, so the involution of R lies in $(S \cap \Phi)T_0 - T_0$, and again is a field automorphism by [4.9.1]. This proves (a).

Finally in (c), since $|K|_2 \ge 8$, we know n is even. Thus K contains a subgroup $J \cong A_6$. Also $N_K(V) \cong \Sigma_4$ [6.5.1], so Σ is determined up to conjugacy, and we may assume that $\Sigma \le J$. The final statement follows easily; indeed Σ is maximal in J, but $C_J(z) \cong D_8$ for all $z \in \mathcal{I}_2(J)$. □

Lemma 2.12. *Let $K \in \mathcal{C}_2$ be simple. Let $K \le H \le \mathrm{Aut}(K)$ and $z \in \mathcal{I}_2(H)$ with $O_{2'}(C_H(z)) \ne 1$. Then either $K \cong L_2(q)$ for q a Fermat or Mersenne prime or 9 with $C_K(z) \cong D_{q\pm1}$, or $K/O_2(K) \cong L_3(4)$ with $C_{K/O_2(K)}(z) \cong U_3(2)$. In all cases $z \notin \mathrm{Inn}(K)$.*

Proof. This is a direct consequence of [GLS 1998, 7.7.1], which specifies all instances of locally unbalancing quasisimple $\mathcal{K}$-groups. □

Lemma 2.13. *Let $K \in \mathcal{C}hev(2)$ with $m_3(K) = 2$, and let $B \le Inndiag(K)$ with $B \cong E_{3^2}$. Suppose that $C_K(b)$ has cyclic Sylow 2-subgroups for each $b \in B^\#$. Then $K \cong A_6$, $U_3(3)$, ${}^2F_4(2^{\frac{1}{2}})'$, or $L_3^\epsilon(2^n)$ for $\epsilon = \pm 1$ with $2^n \equiv \epsilon \pmod 3$.*

Proof. First of all, notice that A_6, $U_3(3)$, ${}^2F_4(2^{\frac{1}{2}})'$, and $L_3^\epsilon(2^n)$ for $\epsilon = \pm 1$ with $2^n \equiv \epsilon \pmod 3$ satisfy the hypotheses of the lemma.

Now, take K to be a group satisfying the hypotheses of this lemma, but not isomorphic to $A_6 \cong B_2'(2)$, $U_3(3) \cong G_2'(2)$, or ${}^2F_4(2^{\frac{1}{2}})'$. We may suppose that $K = {}^d\mathcal{L}(2^m)$. By [GLS 1998, 4.2.2, 4.7.3A, 4.9.1], for each $b \in B^\#$, $O^{2'}(C_K(b))$ is the central product of groups of the form ${}^{d_b}\mathcal{L}_b(2^{m_b})$, with m dividing m_b. Since Sylow 2-subgroups of $C_K(b)$ are cyclic, there is at most one factor, and $O^{2'}(C_K(b)) \cong A_1(2)$, ${}^2B_2(2^{\frac{1}{2}})$ or 1. In the ${}^2B_2(2^{\frac{1}{2}})$ case, we would have $m = 1/2$, so this case cannot occur by [4.7.3A]. Hence $m = 1$ or $C_K(b)$ has odd order. Let $\overline{K}$ be the algebraic group overlying K; then b acts on K as conjugation by some $\bar{b} \in \overline{K}$ with $\bar{b}^3 \in Z(\overline{K})$, and the connected component $C_{\overline{K}}(\bar{b})^o = \overline{T}\,\overline{L}$ where $\overline{T}$ is a maximal torus and $\overline{L}$ is either trivial or isomorphic to A_1. By [4.7.1, 4.8.2], we have $\mathcal{L} = A_1$, A_2, or B_2. If $m = 1$, then the only simple choice for K is $A_2(2)$, which is impossible as $m_3(K) = 2$. Thus $m > 1$, and $\mathcal{L} = A_1$ or A_2. Since $m_3(K) = 2$, $K \cong L_3^\epsilon(2^n)$ for $2^n \equiv \epsilon \pmod 3$. The proof is complete. □

Lemma 2.14. *Let $K = A_6$, M_{11}, or $L_2(8)$. Let D be an elementary abelian 3-subgroup of K of maximal rank. Then $C_{Aut(K)}(D)$ has odd order.*

Proof. This is immediate from [GLS 1998, 3.1.4] for $K = A_6 \cong L_2(9)$, from [5.3a] for $K = M_{11}$, and from [6.5.1] for $K = L_2(8)$, in the last case using the fact that $|\mathrm{Out}(K)|$ is odd [2.5.12]. □

Lemma 2.15. *Let $J \in \mathcal{C}hev(2)$, and suppose that $u \in \mathcal{I}_2(Aut(J))$ and K is a component of $C_J(u)$. Assume that $m_3(J) \le 2$, $B \le C_{Aut(J)}(u)$ with $B \cong E_{3^2}$, and the image of KB in $Aut(J)$ is isomorphic to A_6, M_{11}, or $Aut(L_2(8)) = P\Gamma L_2(8)$. Then for some $b \in B^\#$, $C_J(b)$ contains A_5 and in particular is not solvable.*

Proof. By [GLS 1998, 4.9.6], $K \in \mathcal{C}hev(2)$, and so $K \ncong M_{11}$. Since $K \le E(C_J(u))$, u is a field, graph-field or graph automorphism of J by [3.1.4]. Indeed by [4.9.1, 4.9.2], either $J \cong L_m^\epsilon(2)$, $m = 4$ or 5, $\epsilon = \pm 1$, with $K \cong A_6$ and u a graph automorphism, or $J \cong Sp_4(4)$ or $J \cong L_2(8^2)$, with u a field automorphism. In the last case since $KB \cong P\Gamma L_2(8)$, we may take as b some element of $B^\#$ which induces a field automorphism on J. If $J \cong Sp_4(4)$ then any $b \in B^\#$ satisfies the desired property by [4.8.2]. If $J \cong L_m^\epsilon(2)$, then $\epsilon = +1$ by the hypothesis $m_3(J) \le 2$. Then any $b \in B$ with a four-dimensional commutator space on the natural J-module has the property that we want. The proof is complete. □

Lemma 2.16. *Let K be quasisimple with $\overline{K} := K/Z \cong L_3(4)$, where $Z = Z(K)$ is a 2-group. Let $S \in Syl_2(K)$ and identify $\overline{S}$ with its image in $Aut(K)$. Let $u \in Aut(K)$*

be a (noninner) involution such that u normalizes S and $C_{\bar{K}}(u) \cong U_3(2)$. Then these conditions hold:

(a) $\overline{Z(S)} = Z(\bar{S})$.

(b) *If $B \le K$ with $B \cong E_{3^2}$, then $C_{Aut(K)}(B) = \bar{B} \times \langle u' \rangle$, where $u \in Aut(K)$ is $Aut(K)$-conjugate to u.*

(c) *All involutions in the coset $u\bar{S}$ are S-conjugate, and all involutions in $\bar{S}$ are K-conjugate.*

(d) *u centralizes some involution $y \in Z(S) - Z(K)$.*

(e) *In (d), if $z \in Z(K)$ is an involution, then no two involutions in $\langle y, z \rangle$ are $Aut(K)$-conjugate.*

(f) *$\bar{S}$ has no normal Z_4-subgroup.*

Proof. Part (a) is a direct consequence of [GLS 1998, 6.4.2b], which also implies that $Z(S) \cong E_{2^4}$. Then as $m_2(C_{\bar{S}}(u)) = 1$, u acts freely on $Z(\bar{S})$, which implies (d). Any conjugacy in (d) would have to occur in $C_K(\bar{y}) = \bar{S}$. As $y \in Z(S)$, (e) holds. In (b), $B \in \mathrm{Syl}_3(K)$ is self-centralizing in K since its preimage in $SL_3(4)$ is absolutely irreducible on the natural module; the assertions of (b) and (c) follow from [GLS 2005, 2.1ae] and the fact that u acts freely on $\bar{S}/Z(\bar{S})$ and $Z(\bar{S})$. Finally $\bar{S} = \bar{E}\bar{V}$ where $E_{2^4} \cong \bar{E} \lhd \bar{S}$, $\bar{V} \cong E_{2^2}$, and $C_{\bar{E}}(\bar{v}) = Z(\bar{S})$ for all $\bar{v} \in \bar{V}^{\#}$ and $C_{\bar{S}}(\bar{t}) = \bar{E}$ for all $\bar{t} \in \bar{E} - Z(\bar{S})$ (see [2.1f]). Hence $|\bar{S} : C_{\bar{S}}(\bar{t})| = 4$ for all $\bar{t} \in \bar{S} - Z(\bar{S})$, which implies (f). □

3. {2, 3}-local subgroups

For the rest of the paper, we fix a group G and a prime p satisfying the hypotheses of Theorem 1.1. We begin with some simple properties of 2- and 3-local subgroups of G.

Lemma 3.1. *Let $A \le G$ with $A \cong E_{p^3}$. Then $m_2(C_G(A)) = 0$. In particular, $L_{p'}(C_G(A)) = 1$.*

Proof. If $m_2(C_G(A)) \ne 0$, there exists $t \in \mathscr{I}_2(C_G(A))$. But this is absurd since $m_p(C_G(t)) \le 2$ by (H1). The odd order theorem completes the proof. □

Lemma 3.2. *Let N be any p-local or 2-local subgroup of G. Then N has at most two p-components, and $O^2(N)$ normalizes every p-component of N.*

Proof. Let $L_1, \dots, L_n$ be the p-components of N, and let $P \in \mathrm{Syl}_p(N)$. As p is odd, by [GLS 1996, 16.11], for each i there is $x_i \in P \cap L_i - O_{p'p}(L_i)$ such that $x_i^p = 1 \ne x_i$. Suppose that $n > 1$. Choose any $x \in O_p(N)O_2(N)$ of prime order. If x has order p, then $\langle x_2, \dots, x_n, x \rangle \cong E_{p^n}$ centralizes an involution of $\widehat{L}_1$. If $x^2 = 1$, then $\langle x_1, \dots, x_n \rangle \cong E_{p^n}$ centralizes x. In either case, $n \le m_p^I(G) \le 2$ by (H1), and the result follows. □

Lemma 3.3. *Let $A \in \mathcal{A}^o$ be such that $A \cong E_{p^2}$, and suppose $C_G(A)$ contains a noncyclic elementary abelian 2-group E. Suppose $L = L_{p'}(C_G(A))$. Then $L \neq 1$, L is a single p-component, and E acts nontrivially on $\widehat{L} = L/O_{p'}(L)$. Moreover, $m_p(Z(\widehat{L})) = m_p^I(\widehat{L}) = 0$ and $m_{2,p}(\widehat{L}) \leq 1$.*

Proof. Let $C = C_G(A)$. By Proposition 2.4, $O_{p'}(C)$ has odd order. Hence, E is isomorphic to its image in $\widehat{C}$. Clearly, $A \cong \widehat{A} \leq Z(\widehat{C})$. If $C_E(O_p(\widehat{C})) = 1$, then by the Thompson dihedral lemma [Korchagina and Lyons 2006, 2.2], $\widehat{C}$ contains $E_{p^2} \times D_{2p} \times D_{2p}$, since E is noncyclic. Thus $m_{2,p}^I(\widehat{C}) \geq 3$, whence $m_{2,p}^I(C) \geq 3$, contradicting (H1). Therefore $C_E(O_p(\widehat{C})) > 1$, so E acts nontrivially on $\widehat{L}$. In particular, $\widehat{L} \neq 1$. By Proposition 2.4, $\widehat{L}$ is a quasisimple group from Table 1 with $m_p^I(\widehat{L}) = 0$. The remaining conclusions of the lemma follow immediately from [Proposition 4.4]. □

Lemma 3.4. *Let $A \leq G$ with $A \cong E_{p^3}$. Then any A-invariant p'-subgroup of G is solvable.*

Proof. Otherwise let X be a minimal nonsolvable A-invariant p'-subgroup of G. By minimality, $X/\operatorname{Sol}(X) = K_1 \times \cdots \times K_n$, where $K_1, \ldots, K_n$ are simple groups permuted transitively by A. If $n > 1$, then there is $a \in A - N_A(K_1)$, and $C_X(a)$ is nonsolvable, contradicting the minimality of X. So $n = 1$. As $[A, K_1] = 1$ contradicts (H1), A acts nontrivially on K_1. By [GLS 1998, 5.2.1, 5.3], $K_1 \in \mathcal{C}hev$ and the image of A in $\operatorname{Aut}(K_1)$ is generated by a field automorphism, whence $C_{K_1}(A)$ has even order. This contradicts (H1) and completes the proof. □

4. 3-components of type $L_2(3^n)$

Since $m_{2,p}(G) = 3$ by hypothesis, $\mathcal{H} \neq \varnothing$.

The following two results establish the first and third conclusions of Theorem 1.1. They will underlie the proof of the second conclusion as well.

Proposition 4.1. $p = 3$.

Proposition 4.2. *Let $H \in \mathcal{H}$. Choose any $B \leq H$ such that $B \cong E_{3^3}$, let V be any minimal B-invariant subgroup of $O_2(H)$, and set $B_0 = C_B(V)$ and $L_0 = L_{3'}(C_G(B_0))$. Then there is a $b_0 \in B$ such that these conditions hold*:

(a) *$V \cong E_{2^2}$ and $VB = V\langle b_0\rangle \times B_0 \cong A_4 \times E_{3^2}$*;

(b) *$V \leq L_0$*;

(c) *$\widehat{L}_0 \cong L_2(3^n)$ for some $n \geq 2$*; and

(d) *$m_2(C_G(B_0)) = 2$, and Sylow 2-subgroups of $O^2(C_G(B_0))$ are dihedral.*

Recall that by convention, $\widehat{L}_0 = L_0/O_{3'}(L_0)$.

Proofs. Choose $H \in \mathcal{H}$ and $B \le H$ with $B \cong E_{p^3}$. Let V be a minimal B-invariant subgroup of $O_2(H)$, and set $B_0 = C_B(V)$. Because of (H1), $C_V(B) = 1$. By [GLS 1996, 11.12], $B_0 \cong E_{p^2}$, and there exists $b_0 \in B$ such that b_0 acts irreducibly on V, V is elementary abelian, and $VB = V\langle b_0\rangle \times B_0$. Since $m_2(C_G(B_0)) \ge m_2(V) \ge 2$, Lemma 3.3 implies that $\widehat{L}_0$ is quasisimple with $m_p^I(\widehat{L}_0) = m_p(Z(\widehat{L}_0)) = 0$ and $m_{2,p}(\widehat{L}_0) \le 1 = 1 + m_p(Z(\widehat{L}_0))$; moreover, the image of V in $\mathrm{Aut}(\widehat{L}_0)$ is nontrivial and normalized by b_0. In addition, $C_{\widehat{L}_0}(B) = C_{\widehat{L}_0}(b_0)$ has odd order because $m_p^I(G) \le 2$. By Lemma 2.6, these conditions imply that $p = 3$ and $\widehat{L}_0 \cong L_2(3^n)$ for $n \ge 2$. It remains to prove (b) and (d) of Proposition 4.2. Let $t \in C_G(B_0)$ be any involution. If t or induces a (possibly trivial) field automorphism on $\widehat{L}_0$, then $m_3(C_G(t)) \ge m_3(C_{L_0}(t)) + m_3(B_0) > 2$, contradiction. Therefore t induces a nontrivial inner-diagonal automorphism on $\widehat{L}_0$. As t was arbitrary, $C(B_0, L_0)$ has odd order and $m_2(C_G(B_0)) \le m_2(PGL_2(3^n)) = 2$. We use the fact that $\mathrm{Out}(L_2(3^n))$ is abelian [GLS 1998, 2.5.12]. For one thing, Sylow 2-subgroups of $O^2(C_G(B_0))$ embed in $O^2(\mathrm{Aut}(PGL_2(3^n)))$ and hence in $L_2(3^n)$, so are dihedral. Hence (d) holds. For another, the image of $V = [V, b_0]$ in $\widehat{C}_G(B_0)$ lies in $\widehat{L}_0$, and so $V \le O^{2'}(L_0C(B_0, L_0)) = L_0$, which proves (b). □

5. 2-subgroups of G normalized by E_{3^3}-subgroups

By Propositions 4.1 and 4.2, Theorem 1.1 will be completely proved once we show that $F^*(H) = O_2(H)$ for some $H \in \mathcal{H}$. We prove this by contradiction in the next three sections, thus making the following assumption.

(H2) For all $H \in \mathcal{H}$, $E(H)O_{2'}(H) \ne 1$.

In this section we make our only use of (H2). We prove the following result, from which strong restrictions on the {2, 3}-local structure of G will be deduced in the next section.

Proposition 5.1. *Let $B \le G$ with $B \cong E_{3^3}$, and suppose that T is a nontrivial B-invariant 2-subgroup of G. Then $T \cong E_{2^2}$.*

We prove the proposition by contradiction in a sequence of lemmas. Assuming that $T \ncong E_{2^2}$, we first prove this:

Lemma 5.2. *There exist B-invariant four-subgroups U_1, U_2 of G and elements $a_1, a_2, b \in B$ generating B such that*

$$N := U_1U_2B = U_1\langle a_1\rangle \times U_2\langle a_2\rangle \times \langle b\rangle \cong A_4 \times A_4 \times Z_3.$$

Proof. By (H1), $C_T(B) = 1$. For each hyperplane $B_0 \le B$ set $T_{B_0} = C_T(B_0)$, so that $T_{B_0} = [T_{B_0}, B] \le O^2(C_G(B_0))$. Thus if $T_{B_0} \ne 1$, then $T_{B_0}B$ contains a copy of A_4 centralizing B_0. Thus Proposition 4.2 applies with $H = N_G(T_{B_0})$, giving that T_{B_0} is dihedral and hence $T_{B_0} \cong E_{2^2}$ by the action of B.

Since $T \neq 1$, there is a hyperplane B_1 of B such that $1 \neq T_{B_1} \leq Z(T)$. Since $T \ncong E_{2^2}$, we have $T > T_{B_1}$ and so there is a hyperplane $B_2 \neq B_1$ of B such that $T_{B_2} \neq 1$. The lemma follows with $U_i = T_{B_i}$ for $i = 1, 2$, and with any choices of $b \in B_1 \cap B_2^{\#}$, we have $a_1 \in B_2 - B_1$ and $a_2 \in B_1 - B_2$. □

We set $H = N_G(U_1U_2)$, $C = C_G(U_1U_2)$, and $L = L_{3'}(C_G(b))$. For $i = 1, 2$, set $J_i = L_{3'}(C_G(\langle b, a_i\rangle))$.

Lemma 5.3. *These conditions hold*:

(a) $\widehat{L} \cong A_9$, J_3, *or* $3J_3$; *and*

(b) $U_1U_2 \in Syl_2(N_G(\langle b\rangle) \cap C)$.

Proof. Let $i = 1$ or 2, and $\{i, j\} = \{1, 2\}$. Apply Proposition 4.2 with $N_G(U_j)$, B, and U_j in the roles of H, B, and V. We conclude that $\widehat{J_i} \cong L_2(3^n)$ for $n \geq 2$, and $U_j \leq J_i$. By $L_{3'}$-balance and Lemma 3.2, the subnormal closure L_i of J_i in $C_G(b)$ is a 3-component of L. Moreover $m_3(\langle b\rangle L_i) \geq m_3(\langle b\rangle J_i) = n + 1 \geq 3$, so $C(b, L_i)$ has odd order by Lemma 3.1. Therefore $L_1 = L = L_2$. In particular J_i is a 3-component of $L_{3'}(C_L(a_i))$ for $i = 1, 2$. Moreover $U_1U_2 \leq \langle J_1, J_2\rangle \leq L$. Now Lemma 2.7a implies (a). In particular $m_3(L) \geq 3$ [GLS 1998, 5.6.1]. We set $C_0 = C_{N_G(\langle b\rangle)}(\widehat{L})$ and conclude that C_0 has odd order, since $m_3^I(G) < 3$.

Suppose that (b) fails, so that $N_G(\langle b\rangle) \cap C_G(U_1U_2)$ has a 2-element $t \notin U_1U_2$. Then t normalizes L. But by Lemma 2.7b, $C_{\mathrm{Aut}(\widehat{L})}(\widehat{U}_1\widehat{U}_2)$ is the image in $\mathrm{Aut}(\widehat{L})$ of $\widehat{U}_1\widehat{U}_2$. As $t \notin U_1U_2$, we have $\langle t\rangle U_1U_2 \cap C_0 \neq 1$. This contradicts that C_0 has odd order, so the proof is complete. □

Lemma 5.4. $E(H) = 1$.

Proof. Clearly $C \cap B = \langle b\rangle$. Suppose first that $m_3(C) > 1$. Let $P \in \mathrm{Syl}_3(H)$, so that $m_3(P) = m_3(B) = 3$. By [GLS 1996, 10.11] there is $A \leq C$ such that $A \cong E_{3^2}$ and $A \lhd P$, and then there exists $1 \neq a \in C_{\langle a_1, a_2\rangle}(A)$. Then $A\langle a\rangle \cong E_{3^3}$ and for some $i = 1, 2$, $U_i = [A\langle a\rangle, U_i]$ and $A = C_{A\langle a\rangle}(U_i)$. Now Proposition 4.2 applies with $N_G(U_i)$, U_i, $A\langle a\rangle$, and A in the roles of H, V, B, and B_0. By part (d) of that proposition, $m_2(C_G(A)) = 2$. But A centralizes $U_1U_2 \cong E_{2^4}$, a contradiction. Therefore $m_3(C) \leq 1$, and equality holds as $b \in C$. By Lemma 5.3b, $N_C(\langle b\rangle)/U_1U_2$ has odd order, so $b \in Z(N_C(\langle b\rangle))$. Thus C has a normal 3-complement by Burnside's normal complement theorem [16.5]. But by Lemma 3.4, $O_{3'}(C)$ is solvable, so C is solvable. Hence $E(H) \leq C^{(\infty)} = 1$, as required. □

Now we use (H2). Set $W = O_{2'}(H) = O_{2'}(C)$. By Lemma 5.4 and (H2), $W \neq 1$. Also set $N_i = N_G(U_i)$, $C_i = C_G(U_i)$, and $\overline{N}_i = N_i/O_{2'}(N_i)$ for $i = 1, 2$. Then $W = O_{2'}(C_{C_i}(U_j))$ where, as before, $\{i, j\} = \{1, 2\}$. Obviously $E(\overline{N}_i) = E(\overline{C}_i)$.

Lemma 5.5. $W \leq O_{2'}(C_i)$ *for* $i = 1$ *and* $i = 2$.

Proof. If false, then as $C = C_{C_i}(U_j)$, the theory of balance [GLS 1996, 20.6] provides a WU_1U_2-invariant 2-component I_i of $L_{2'}(N_i)$ such that $[\overline{W}, \overline{I}_i] \neq 1$, and such that if we let $Y = WI_iU_1U_2$ and $\widetilde{Y} = \mathrm{Aut}_Y(\overline{I}_i)$, then $\widetilde{W} \le O_{2'}(C_{\widetilde{Y}}(\widetilde{U}_1\widetilde{U}_2)) = O_{2'}(C_{\widetilde{Y}}(\widetilde{U}_j))$. By Lemma 3.4, no components of $E(\overline{N}_i)$ are $3'$-groups. Thus $E(\overline{N}_i) = O^{3'}(E(\overline{N}_i))$, and so $O^2(C_i)$ normalizes $\overline{I}_i$ by Lemma 3.2. Let $Y^* = YO^2(C_i)$ and write $\widetilde{Y}^* = \mathrm{Aut}_{Y^*}(\overline{I}_i)$. Thus $\widetilde{U}_j\langle\widetilde{a}_j\rangle \cong A_4$. By Lemma 2.8, $\overline{I}_i$ is one of the groups in (1B3). If b centralizes $\overline{I}_i$, then $\overline{I}_i$ is involved in A_9 or J_3 by Lemmas 5.3 and 3.4. Hence by Lemma 2.9, either $\widetilde{W} = 1$ or $C_{\widetilde{I}_i}(\widetilde{U}_j) = \widetilde{U}_j \times \Sigma$ with $\widetilde{b} \in \widetilde{\Sigma} \cong \Sigma_3$. In the first case $[\overline{W}, \overline{I}_i] = 1$, contradiction. Hence the second case holds, so an involution of $\widetilde{\Sigma}$ has a preimage $t \in C_{I_i}(U_1U_2) \cap N_G(\langle b\rangle) - U_1U_2$ such that t is a 2-element. This contradicts Lemma 5.3b and proves the lemma. □

Now we complete the proof of Proposition 5.1. Write $U_1 = \langle u, v\rangle$ and set $C_u = C_G(u)$. Notice that $U_2\langle a_2\rangle \le C_u$, and as $W \le O_{2'}(C_2)$ and $[W, U_1] = 1$, we have $W \le O_{2'}(C_{C_u}(U_2))$, with W being $U_2\langle a_2\rangle$-invariant. Using Lemma 2.9 as in the previous proof, we conclude that $[W, J] = 1$ for every component J of C_u, unless possibly $J/Z(J) \cong M_{12}$. But by Lemma 5.5, $W \le O_{2'}(C_G(U_1)) = O_{2'}(C_{C_u}(v))$, and M_{12} is locally 1-balanced with respect to the prime 2 by [GLS 1998, 7.7.1]. Hence, in any case $[W, J] = 1$, by [GLS 1996, 20.6]. We have therefore shown that $[W, E(C_u)] = 1$.

However, $[W, O_2(C_u)] = 1$ by $L^*_{2'}$-balance [GLS 1996, 5.18]. As G has (restricted) even type, $O_{2'}(C_u) = 1$. Therefore, $[W, F^*(C_u)] = 1$ which contradicts the F^*-Theorem [3.6]. Thus $W = 1$, which proves the proposition.

6. Structure of subgroups $H \in \mathcal{H}$ and of centralizers of involutions

We choose any $H \in \mathcal{H}$, set $V_H = O_2(H)$ and $C_H = C_G(V_H)$, and choose any involution $z \in V_H$. Set $E_z = E(C_G(z))$. Also let $B \le H$ with $B \cong E_{3^3}$, and set $B_0 = C_B(V_H)$, $L_0 = L_{3'}(C_G(B_0))$, and $L_b = L_{3'}(C_G(b))$ for every $b \in B_0^\#$. Using Proposition 5.1 we can now prove the following result.

Proposition 6.1. $V_H \cong E_{2^2}$, $B_0 \cong E_{3^2}$, *and either* (a) *or* (b) *holds*:

(a) $F^*(H) = V_H \times E(H)$ *with* $E(H) \cong A_6$, M_{11}, *or* $L_2(8)$, *and* $C_H \cong E_{2^2} \times$ *Aut*$(L_2(8))$ *in the last case. Moreover,* $\widehat{L}_0 \cong L_2(3^2)$ *or* $L_2(3^4)$.

(b) $F^*(H) = V_H \times O_3(H)$, $O^2(C_H) = O_3(H) \in Syl_3(E_z)$, $E_z/O_2(E_z) \cong L_3(4)$, $C_{E_z/O_2(E_z)}(V_H) \cong U_3(2)$, *and* $C_HB \cong A_4 \times U$ *where* U *is isomorphic to a subgroup of* $U_3(2)$ *of index at most* 2. *Moreover* $\widehat{L}_0 \cong L_2(3^2)$.

By Proposition 5.1, $V_H \cong E_{2^2}$ and $m_{2,3}(H/V_H) < 3$. By (H1) and the fact that $B/B_0 \le \mathrm{Aut}(V_H)$, we have $B_0 \cong E_{3^2}$. Notice that several choices may be possible for B_0. In particular any E_{3^2}-subgroup $B^* \le C_H$ that is normal in some Sylow

3-subgroup of H is a possible choice for B_0, because $m_3(C_H(B^*)) = 3$ by [GLS 1996, 10.20(ii)].

We proceed in a sequence of lemmas, the first of which describes normalizers of subgroups of B_0. By Proposition 4.2, $\widehat{L}_0 \cong L_2(3^n)$ for $n \geq 2$.

Lemma 6.2. *Let $b \in B_0^\#$ and set $L_b = L_{3'}(C_G(b))$. Then*

(a) *$L_b/O_{3'}(L_b) \cong L_2(3^{n_b})$ with $n_b = n$ or $3n$;*

(b) *if there is $1 \neq S \leq C_G(b)$ such that S is a 2-group, $[S, V_H] = 1$ and $S \cap V_H = 1$, then $S \cong Z_2$, $n = n_b = 2$, and $[S, B_0] \neq 1$;*

(c) *$C_{C_H/V_H}(b)$ has a normal 2-complement and cyclic Sylow 2-subgroups of order dividing n;*

(d) *$Aut_G(B_0)$ does not contain $SL(B_0) \cong SL_2(3)$; and*

(e) *if $Aut_G(B_0)$ contains a Q_8-subgroup, then $\widehat{L}_0 \cong A_6$.*

Proof. Choose any $b_0 \in B - B_0$. Then $V_H B = B_0 \times V_H\langle b_0\rangle \cong E_{3^2} \times A_4$. By Proposition 4.2, $V_H \leq L_0$. Write $B_0 = \langle b, b'\rangle$. By $L_{3'}$-balance [GLS 1996, 5.17] and Lemma 2.3, the pumpup L of L_0 in $C_G(b)$ is a single 3-component, and $L_{3'}(\widehat{C}_L(b'))$ is a covering group of $\widehat{L}_0 \cong L_2(3^n)$. Furthermore, $C_{\widehat{L}}(b') = C_{\widehat{L}}(B_0)$ does not contain an isomorphic copy of Σ_6, for if it did, then $m_3^I(G) \geq m_3(B_0) + m_3^I(\Sigma_6) = 2+1$, contradicting (H1). And by Proposition 5.1, $\widehat{L}\langle b'\rangle$ does not contain any subgroup isomorphic to $A_4 \times A_4$. Therefore by Lemma 2.10, $\widehat{L} \cong L_2(3^n)$ or $L_2(3^{3n})$, with b' inducing a field automorphism on $\widehat{L}$ in the latter case. In particular, $m_3(L\langle b\rangle) \geq 3$, and so by (H1),

(6A) $$C(b, L) \text{ has odd order.}$$

This implies that $L = L_b$, so (a) holds. Also, (6A) implies that any 2-subgroup R of $C_G(b)$ acts faithfully on $\widehat{L}_b$. If S is as in (b), we take $R = S \times V_H$, and Lemma 2.11a implies that any involution $s \in S$ induces a field automorphism on L_b. Thus $C_{\langle b\rangle L_b/O_{3'}(\langle b\rangle L_b)}(s)$ contains $Z_3 \times L_2(3^{n_b/2})$. As $m_3(C_G(s)) \leq m_3^I(G) < 3$, we conclude that $n_b = 2$, and it follows that $S = \langle s\rangle$, and $n = 2$ by (a). Similarly, as $m_3(C_{L_b}(s)) = 1$, the fact that $m_3(C_G(s)) < 3$ implies that $[s, B_0] \neq 1$, and (b) is completely proved. In (c), we take $R \in \mathrm{Syl}_2(C_{C_H}(b))$, so that, by Lemma 2.11b, $R = V_H F$, where F is a group of field automorphisms of $\widehat{L}_b$. In particular R/V_H is cyclic. Hence $C_{C_H/V_H}(b)$ has a normal 2-complement [GLS 1996, 16.7] and (c) holds.

Since $\mathrm{Out}(\widehat{L}_0)$ is abelian, $N := [N_G(B_0), N_G(B_0)]$ induces inner automorphisms on $\widehat{L}_0$. If $\mathrm{Aut}_G(B_0)$ contains Q_8, then the image of N in $\mathrm{Aut}(B_0)$ contains an involution, and so $C(B_0, L_0)$ has even order. As $m_3^I(G) \leq 2$, $\widehat{L}_0 \cong A_6$ in this case, proving (e). Continuing, we have $O^2(\mathrm{Aut}(\widehat{L})) = \mathrm{Inn}(\widehat{L})$ and so $O^2(N_G(B_0))$ maps into $\mathrm{Inn}(\widehat{L}_0)$. Therefore if $\mathrm{Aut}_G(B_0)$ contains $SL_2(B_0)$, then $\mathrm{Aut}_{C(B_0,L_0)}(B_0)$

contains $SL_2(B_0)$, and in particular $C(B_0, L_0)$ contains commuting elements z and y of orders 2 and 3, respectively. But then $C_G(z)$ contains y and covers $\widehat{L}_0$, so $m_3(C_G(z)) \geq 3$, a contradiction. Hence (d) holds, and the proof is complete. □

Lemma 6.3. *Suppose that $O_{2'}(H) \neq 1$. Then Proposition 6.1b holds.*

Proof. Set $W = O_{2'}(H)$, and write $V_H = \langle z, z' \rangle$. Then $W = O_{2'}(C_{C_G(z)}(z'))$. Since G has restricted even type, $O_{2'}(C_G(z)) = 1$, and hence by [GLS 1996, 20.6], W acts faithfully on the product J_1 of all $\langle z' \rangle W$-invariant components of $E_z := E(C_G(z))$ that are locally unbalanced with respect to z'. Let J be a component of J_1. Again as G has restricted even type, $J \in \mathcal{C}_2$ with $O_{2'}(J) = 1$, and so by Lemma 2.12, either $J \cong L_2(q)$ for q a Fermat or Mersenne prime or 9 with $C_J(z') \cong D_{q\pm 1}$, or $J/O_2(J) \cong L_3(4)$ with $C_{J/O_2(J)}(z') \cong U_3(2)$. In all cases z' induces a noninner automorphism on J. Denote by S a V_H-invariant Sylow 2-subgroup of J.

Assume first that $J/O_2(J) \ncong L_3(4)$. Then $S \cap V_H = 1$. Now $B_0 \leq C_H \leq C_G(z)$. If $C_{B_0}(J)$ contains a nonidentity element b, then since $S \leq C_G(b)$ and $S \cap V_H = 1$, Lemma 6.2b is contradicted. Therefore $C_{B_0}(J) = 1$, so $m_3(C_{\mathrm{Aut}(J)}(z')) \geq 2$. Given the possible isomorphism types of J, we have a contradiction. Therefore $J/O_2(J) \cong L_3(4)$.

Then $C_{C_G(z)}(J)$ is a $3'$-group as $m_3^I(G) \leq 2$. Since every component of J_1 has order divisible by 3, we have $J = J_1 = O^{3'}(E_z)$. If $J \neq E_z$, then E_z has a component I such that $I/Z(I) \cong {}^2B_2(2^{\frac{n}{2}})$ for $n \geq 3$. But then $m_2(C_{V_H I}(b)) \geq 5$, contradicting Lemma 6.2b. Therefore $E_z = J$.

Finally $O_{2'}(C_H)E(C_H) \leq L^*_{2'}(C_G(z))$ by $L^*_{2'}$-balance [GLS 1996, 5.18]. But Sylow 2-subgroups of $L_3(4)$ are self-centralizing in $\mathrm{Aut}(L_3(4))$ [GLS 1998, 3.1.4], and so $L^*_{2'}(C_G(z)) = E_z$. Thus $O_{2'}(C_H) = O_{2'}(C_{E_z}(z')) \cong E_{3^2}$. By the remark before Lemma 6.2, we may take $B_0 = O_{2'}(C_H)$. Then $\mathrm{Aut}_{E_z}(B_0) \cong Q_8$, but $\mathrm{Aut}_G(B_0)$ does not contain $SL(B_0)$, by Lemma 6.2d. Therefore $\mathrm{Aut}_G(B_0)$ is a 2-group, whence $B_0 = O^2(C_H)$. By Lemma 6.2e, $\widehat{L}_0 \cong L_2(3^2)$ and the lemma is proved. □

For the rest of the proof of Proposition 6.1, we may assume that $O_{2'}(H) = 1$, whence $F^*(H) = V_H E(H)$ and $E(H) \neq 1$, because $m_3(H) = 3$.

Lemma 6.4. *Either $E(H)$ is quasisimple with $m_3(E(H)) = 2$, or Proposition 6.1a holds with $C_H \cong \mathrm{Aut}(L_2(8))$.*

Proof. Let $H_1, \ldots, H_m$ be the components of $E(H)$. By Lemma 3.4, 3 divides $|H_i|$ for each $i = 1, \ldots, m$. But $m_3(E(H)) \leq m_3(C_H) = m_3(B_0) = 2$. Thus if the first alternative of the lemma fails, either $m = 2$ or $m = 1$ with $E(H) = H_1$ and $m_3(H_1) = 1$. In the former case, take a nontrivial 3-element $b \in H_1$ and $S \in Syl_2(H_2)$. Then $S \leq C_G(b)$, $[S, V_H] = 1$ and $S \cap V_H = 1$. As S is noncyclic, we have a contradiction with Lemma 6.2b. Thus the latter case holds.

Because of Lemma 6.2c, for all $b \in B_0^\#$, $C_{H_1}(b)$ is solvable with cyclic Sylow 2-subgroups. In particular, $C_{B_0}(H_1) = 1$ and any $b \in B_0 - H_1$ induces a noninner automorphism on H_1. In particular 3 divides $|\mathrm{Out}(H_1)|$.

By Lemma 2.8, $H_1 \cong L_2(2^n)$ for $n \geq 2$ or $L_3^\epsilon(2^n)$ for $n \geq 1$ with $\epsilon = \pm 1$ and $2^n \equiv -\epsilon \pmod 3$. Hence $C_{H_1}(b) \cong L_2(2^{n/3})$ or $L_2^\epsilon(2^{n/3})$. As this must have cyclic Sylow 2-subgroups, the only possibility is $H_1 \cong L_2(8)$. Then $H_1\langle b\rangle \cong \mathrm{Aut}(H_1)$, and since $C_{H_1}(b)$ has a Sylow 2-subgroup S of order 2, Lemma 6.2b implies that $\widehat{L}_0 \cong A_6$. Thus Proposition 6.1a holds, as asserted. □

Lemma 6.5. *If $m_3(E(H)) = 2$, then $E(H) \cong A_6$ or M_{11}, and $\widehat{L}_0 \cong A_6$. Moreover $n = 2$ or 4 in the first case, and $n = 2$ in the last case.*

Proof. Set $X = E(H)$, a quasisimple group by Lemma 6.4. By Lemma 2.8, $X \in \mathcal{C}hev(2)$, $X \cong L_3(3)$, or X is isomorphic to one of the sporadic groups listed in (1B3). Most of these will be ruled out using Lemma 6.2c, reducing us to the following cases:

(6B) $$X \cong A_6,\ L_3(3),\ U_3(3),\ {}^2F_4(2^{\frac{1}{2}})',\ \text{ or } M_{11},\ \text{ or } X/O_2(X) \cong L_3^\epsilon(2^n),$$

where $\epsilon = \pm 1$, $2^n \equiv \epsilon \pmod 3$, and $n > 1$.

Indeed all the sporadic cases in (1B3) except $X \cong M_{11}$ violate Lemma 6.2c; see [GLS 1998, Table 5.3]. Suppose then that $X \in \mathcal{C}hev(2)$. If some $b \in B_0^\#$ acts as an element of $\mathrm{Aut}(X) - \mathrm{Inndiag}(X)$, then either b induces a graph automorphism on X, in which case $m_2(C_X(b)) > 1$ by [4.7.3A], or b induces a field or graph-field automorphism, in which case the facts that $C_X(b)$ has cyclic Sylow 2-subgroups and $|X|_3 \neq 1$ imply with [4.9.1] that $X \cong L_2(2^3)$. This is a contradiction as $m_3(L_2(2^3)) = 1$. Hence B_0 induces inner-diagonal automorphisms on X. Then X is as in (6B) by Lemma 2.13. So we are indeed reduced to the cases (6B).

If $X \cong U_3(3)$ or ${}^2F_4(2^{\frac{1}{2}})'$, then $C_X(b)$ contains a Z_4-subgroup, contradicting Lemma 6.2b.

If $X \cong L_3(3)$, then $\mathrm{Out}(X)$ is a 2-group, so $O^2(H) \cong A_4 \times X$. Moreover a maximal parabolic subgroup $Y \leq X$ satisfies $F^*(Y) \cong E_{3^2}$ and $Y/F^*(Y) \cong SL(Y)$. By the remark before Lemma 6.2, we may assume that $B_0 = O_3(Y)$, and then Lemma 6.2d is contradicted.

Suppose that $X \cong L_3^\epsilon(2^m)$. If $2^m \equiv 1 \pmod 9$, then there is $B_1 \leq X$ such that B_1 is normal in a Sylow 3-subgroup of X, and for some $b \in B_1^\#$, $C_X(b)$ contains $L_2(2^m)$; namely we can take B_1 to be the image of a diagonalizable subgroup of $SL_3^\epsilon(2^m)$. By the remark before Lemma 6.2, we may take $B_0 = B_1$. However, as $m > 1$, this contradicts Lemma 6.2c. Therefore $2^m \not\equiv \epsilon \pmod 9$, so $|X|_3 = 9$ and we may take $B_0 \in \mathrm{Syl}_3(X)$, whence $\mathrm{Aut}_X(B_0) \cong Q_8$. But in this case $n \not\equiv 0 \pmod 3$, so $O^{3'}(\mathrm{Aut}(X)) \cong PGL_3^\epsilon(2^m)$. Hence $C_B(X) \neq 1$. Choose $b \in C_B(X)$. If X acted faithfully on $O_{3'3}(C_G(b))/O_{3'}(C_G(b))$, then, by the Thompson dihedral lemma, $C_G(b)$ would contain the direct product of $\langle b\rangle$ and $m_2(X)$ copies of Σ_3, and so $m_3^I(G) \geq m_2(X)$. This is absurd as $m_2(X) > 2$ but $m_3^I(G) < 3$. Thus X centralizes $O_{3'3}(C_G(b))/O_{3'}(C_G(b))$, whence $X \leq L_b := L_{3'}(C_G(b))$. By Lemma 2.10, $\widehat{L}_b \cong Mc$ or $U_4(3)$, and then $m_3^I(C_G(b)) > 2$, a contradiction.

Consequently $X \cong A_6$ or M_{11} as claimed. If $X \cong M_{11}$, then for any $b \in B_0^\#$, $C_X(b)$ contains an involution and so $\widehat{L}_0 \cong A_6$ by Lemma 6.2b. If $X \cong A_6$, then $N_X(B_0)$ contains a subgroup $\langle t\rangle \cong Z_4$ acting faithfully on B_0. Then $\langle t\rangle$ normalizes L_0 and $\langle t\rangle \cap V_H = 1$. Suppose that $\widehat{L}_0 \cong L_2(3^n)$ for $n > 2$. Then $C_G(\widehat{L}_0)$ has odd order as $m_3^I(G) < 3 \leq m_3(L_0)$. Hence by Lemma 2.11a, the involution $t_0 \in \langle t\rangle$ induces a field automorphism on $\widehat{L}_0$. Thus $C_{\widehat{L}_0}(t_0) \cong L_2(3^{n/2})$, so $n/2 \leq m_3^I(G) \leq 2$, whence $n = 2$ or 4. The lemma is proved. □

Lemmas 6.3, 6.4, and 6.5 prove Proposition 6.1.

7. The residual cases

Fix $H \in \mathcal{H}$. First suppose that $E(H) \cong L_2(8)$, M_{11}, or A_6 as in Proposition 6.1a. We keep this assumption in Lemmas 7.1–7.3. Since $\widehat{L}_0 \cong L_2(3^n)$ with n even, $L_0 \cap H = N_{L_0}(V_H)$ contains a subgroup $\Sigma \cong \Sigma_4$, which we fix. Then in H, since $[\Sigma, B_0] = 1$, it is immediate by Lemma 2.14 that $[\Sigma, E(H)] = 1$, since Σ has no nontrivial quotient of odd order.

Lemma 7.1. *For any involution $y \in \Sigma$, we have $E(H) = E(C_G(y))$.*

Proof. First suppose that V is any four-subgroup of Σ such that $E(H) = E(C_G(V))$. We claim that $E(H) = E(C_G(y))$ for all $y \in V^\#$. Fix y and write $V = \langle y, y'\rangle$. Let E_y be the subnormal closure of $E(H)$ in $C_G(y)$. By (1B1) and $L_{2'}$-balance, E_y is a component of $E(C_G(y))$ or the product of two such components interchanged by y', and $E(H)$ is a component of $C_{E_y}(y')$. Since B_0 acts faithfully on $E(H)$, it acts faithfully on each component of E_y. But $m_3(B_0) = 2 = m_3(C_G(y))$, and so E_y is quasisimple. By Lemma 6.2, $C_{E_y}(b)$ is solvable for all $b \in B_0^\#$. Given that $E(H)B_0 \cong \mathrm{Aut}(L_2(8))$, A_6, or M_{11}, and that $E_y \in \mathcal{C}_2$, we conclude by Lemma 2.15 that $E(H) = E_y$, proving our claim.

For any involution $y \in \Sigma$, choose an involution $z \in V_H$ such that $Y := \langle z, y\rangle \cong E_{2^2}$. Then $E(C_G(z)) = E(H)$ by our claim, and it follows immediately using $L_{2'}$-balance again that $E(C_G(Y)) = E(C_G(z)) = E(H)$. Then $E(C_G(y)) = E(H)$ by our claim again, as desired. □

Lemma 7.2. $E(C_G(E(H))) \cong E(H) \cong A_6$.

Proof. Let $L_0^* = \{C_{L_0}(y) \mid \text{is an involution of } \Sigma\}$. By Lemma 7.1, L_0^* normalizes $E(H)$, and of course L_0^* centralizes B_0. Since $\widehat{L}_0 \cong L_2(3^n)$ for $n = 2$ or 4, $O^2(L_0^*)$ covers an A_6-subgroup of $\widehat{L}_0$ containing the image of Σ, by Lemma 2.11. As $[\Sigma, E(H)B_0] = 1$, it follows that $[O^2(L_0^*), E(H)B_0] = 1$.

Let $D \in \mathrm{Syl}_3(L_0^*)$, so that $D \cong E_{3^2}$. If $E(H) \cong L_2(8)$ or M_{11}, then $E(H)B_0$ contains $Z_3 \times \Sigma_3$ and so $m_3^I(G) \geq m_3^I(DE(H)B_0) \geq 3$, contradicting (H1). Therefore $E(H) \cong A_6$. Let W be an A_4-subgroup of $E(H)$. Then $H' := N_G(O_2(W))$ contains $D \times W$, so $H' \in \mathcal{H}$. But $L_0^* \leq C_G(E(H)) \leq H'$ with $L_0^*/O_{3'}(L_0^*) \cong A_6$. Applying Proposition 6.1 and the prior argument in this lemma to H', we deduce that $E(H') \cong A_6$ is the unique nonsolvable composition factor of H'. Therefore $E(H') = E(C_G(H))$, completing the proof. □

Lemma 7.3. *$\widehat{L}_b \cong A_6$ for all $b \in B_0^\#$, and $E(H) = E(C_G(y))$ for all involutions $y \in C_G(E(H))$.*

Proof. Let $T \in \mathrm{Syl}_2(N_{E(H)}(B_0))$, so that $T \cong Z_4$. If $\widehat{L}_0 \cong L_2(3^4)$, then we saw above that T acts faithfully by field automorphisms on $\widehat{L}_0$. However, this is absurd because $E(C_G(E(H))) \cong A_6$ lies in L_0 and centralizes T. Therefore $\widehat{L}_0 \cong A_6$. Likewise for any $b \in B_0^\#$, the involution $t \in T$ inverts b and centralizes $\widehat{L}_0$. If $\widehat{L}_b \cong L_2(3^6)$, then the centralizer of $\widehat{L}_0$ in $\widehat{L}_b$ would be of order 3, so t would centralize $\widehat{L}_b$, contradicting (H1). Therefore $\widehat{L}_b \cong A_6$, proving the first assertion of the lemma.

Let $y \in C_G(E(H))$ be any involution. Then $[y, y'] = 1$ for some involution $y' \in E(C_G(E(H)))$, and we know by Lemma 7.2 that $E(H) = E(C_G(y'))$. Hence we may assume that $y \neq y'$, and then as in Lemma 7.2, with the help of Lemma 2.15 we may again argue that $E(H) = E(C_G(\langle y, y'\rangle)) = E(C_G(y))$. □

Lemma 7.4. *Proposition 6.1b holds.*

Proof. Assume false and continue the above analysis. By Lemma 7.3, $E(H)$ is terminal in G and $E(H) \cong A_6$. Thus by the Aschbacher–Gilman–Solomon component theorem [GLS 1999, Theorem PU_4^*], $E(H)$ is standard in G. Hence by definition of restricted even type (1B2), $m_2(C_G(E(H))) = 1$. But $E(C_G(E(H))) \cong A_6$ by Lemma 7.3, a contradiction. □

Now fix $z \in V_H^\#$ and set $C = C_G(z)$, $Q = O_2(C)$, and $K = E(C)$. By Proposition 6.1b and (1B4), $O_2(K)$ is elementary abelian and $K/O_2(K) \cong L_3(4)$. Also $\widehat{L}_0 \cong A_6$.

Expand V_H to $R \in \mathrm{Syl}_2(C_G(B_0))$.

Lemma 7.5. *These conditions hold*:

(a) $\Omega_1(Q \cap L_0) = \langle z\rangle = \Omega_1(Z(Q))$;

(b) *z is 2-central in G; and*

(c) *for any involution $y \in K - Z(K)$, no two involutions in $\langle y, z\rangle$ are G-conjugate.*

Proof. By Lemma 6.2, R embeds in $\mathrm{Aut}(\widehat{L}_0)$, and since $m_3^I(G) \leq 2$, $C_{L_0}(u)$ is a $3'$-group for all $u \in R^\#$. Therefore the image of RL_0 in $\mathrm{Aut}(\widehat{L}_0)$ is isomorphic to $\widehat{L}_0$, M_{10}, or $PGL_2(9)$, and so R is dihedral or semidihedral of order at most 16. In any case $N_R(V_H) \in \mathrm{Syl}_2(L_0)$. As L_0 has only one class of involutions, we could have chosen R so that $z \in Z(R)$. In particular, we may expand R to $S \in \mathrm{Syl}_2(C)$.

By Lemma 2.16b, $C_{\mathrm{Aut}(K)}(B_0) \cong B_0 \times Z_2$, and so $R \leq V_H C_R(K)$. Set $R_0 = C_R(K) \cap N_R(V_H)$, so that $|R_0| = 4$. If $R_0 \cong E_{2^2}$, then as $R_0 \leq L_0$, R_0 lies in an A_4-subgroup of L_0. Hence $N_G(R_0) \in \mathcal{H}$, and so Proposition 6.1 applies to $N_G(R_0)$ in place of H. This yields a contradiction since $C_G(R_0)$ involves $K/O_2(K) \cong L_3(4)$. Therefore $R_0 \cong Z_4$.

Since R is of maximal class, Q is cyclic or of maximal class, and there is $Q_0 \lhd S$ such that $Z_4 \cong Q_0 \leq Q$. Thus (a) holds. Moreover

(7A) for any $Q_1 \lhd S$ such that $Q_1 \lhd Z_4$ for $z \in Q_1$.

Otherwise, in $\overline{C} = C/C_C(K) \leq \mathrm{Aut}(\overline{K})$, we would have $\overline{Q}_1 \lhd \overline{S}$ with $\overline{Q}_1 \cong Z_4$, contradicting Lemma 2.16f. Now (7A) implies that $\langle z \rangle$ char S, and so $S \in \mathrm{Syl}_2(G)$, proving (b).

Let $y \in K - Z(K)$ be an involution. By Lemma 2.16c, y has a K-conjugate in $Z(S)$, and no two involutions of $\langle y, z \rangle$ are $N_G(K)$-conjugate (we use Lemma 2.16e if $Z(K) \neq 1$). By Burnside's lemma [GLS 1996, 16.2], G-fusion in $\langle y, z \rangle$ is controlled in $N_G(S)$, and hence in C, since $\langle z \rangle$ char S. As $C \leq N_G(K)$, (c) follows, completing the proof of the lemma. □

Lemma 7.6. *If $U \cong E_{2^2}$ and $U \leq C_H \cap V_H K$, then $U = V_H$ or $|U \cap z^G| \leq 1$.*

Proof. By Proposition 6.1b, $C_H \cap V_H K = V_H \times B_0 P$ where $P \leq C_K(V_H)$, and $P \cong Z_4$ or Q_8. Let $\langle u \rangle = \Omega_1(P)$. Then $\langle u \rangle = \Phi(V_H \times P)$ so by a Frattini argument, $C_H(u)$ contains a 3-element t acting nontrivially on V_H. Since $u \in K - Z(K)$, no two involutions in $\langle u, z \rangle$ are G-conjugate, by Lemma 7.5c. Conjugating this statement by t, we find that $z^G \cap V_H P = V_H$. This implies the lemma. □

Lemma 7.7. *Proposition 6.1b does not hold.*

Proof. Suppose that it does hold and continue the preceding argument. Write $V_H = \langle z, z' \rangle$, so that $C_{K/O_2(K)}(z') \cong U_3(2)$. By Lemma 2.16d, there is an involution $u \in C_K(z') - Z(K)$. By Lemma 2.16c, z' and $z'u$ are K-conjugate, modulo $Z(K)$. As $Z(K) \leq \langle z \rangle$ it follows from Lemma 7.6 that $V_H = \langle z', z'u \rangle$ or $V_H = \langle z', zz'u \rangle$. In either case $u \in V_H$, which is absurd as $V_H \cap K = \langle z \rangle \leq Z(K)$. □

As Lemmas 7.4 and 7.7 are in conflict, Theorem 1.1 is proved.

References

[Aschbacher and Smith 2004] M. Aschbacher and S. D. Smith, *The classification of quasithin groups. I, II*, Mathematical Surveys and Monographs **111–112**, American Mathematical Society, Providence, RI, 2004. MR 2005m:20038a Zbl 1065.20023

[GLS 1994] D. Gorenstein, R. Lyons, and R. Solomon, *The classification of the finite simple groups*, Mathematical Surveys and Monographs **40**, American Mathematical Society, Providence, RI, 1994. MR 95m:20014 Zbl 0816.20016

[GLS 1996] D. Gorenstein, R. Lyons, and R. Solomon, *The classification of the finite simple groups. Number 2. Part I. Chapter G: General group theory*, Mathematical Surveys and Monographs **40**, American Mathematical Society, Providence, RI, 1996. MR 96h:20032 Zbl 0836.20011

[GLS 1998] D. Gorenstein, R. Lyons, and R. Solomon, *The classification of the finite simple groups. Number 3. Part I. Chapter A: Almost simple K-groups*, Mathematical Surveys and Monographs **40**, American Mathematical Society, Providence, RI, 1998. MR 98j:20011 Zbl 0890.20012

[GLS 1999] D. Gorenstein, R. Lyons, and R. Solomon, *The classification of the finite simple groups. Number 4. Part II. Chapters 1–4: Uniqueness theorems*, Mathematical Surveys and Monographs **40**, American Mathematical Society, Providence, RI, 1999. MR 2000c:20028 Zbl 0922.20021

[GLS 2005] D. Gorenstein, R. Lyons, and R. Solomon, *The classification of the finite simple groups. Number 6. Part IV: The special odd case*, Mathematical Surveys and Monographs **40**, American Mathematical Society, Providence, RI, 2005. MR 2005m:20039 Zbl 1069.20011

[Korchagina and Lyons 2006] I. Korchagina and R. Lyons, "A {2, 3}-local characterization of the groups A_{12}, $\mathrm{Sp}_8(2)$, $F_4(2)$ and F_5", *J. Algebra* **300**:2 (2006), 590–646. MR 2007c:20032 Zbl 1106.20008

[Korchagina and Solomon 2003] I. Korchagina and R. Solomon, "Toward a characterization of Conway's group Co_3", *Bull. London Math. Soc.* **35** (2003), 793–804. MR 2004e:20024 Zbl 1043.20012

[Korchagina et al. 2002] I. Korchagina, R. Lyons, and R. Solomon, "Toward a characterization of the sporadic groups of Suzuki and Thompson", *J. Algebra* **257**:2 (2002), 414–451. MR 2004b:20027 Zbl 1028.20011

Received May 5, 2007. Revised October 15, 2007.

INNA KORCHAGINA
MATHEMATICS INSTITUTE
ZEEMAN BUILDING
UNIVERSITY OF WARWICK
COVENTRY CV4 7AL
UNITED KINGDOM

I.Korchagina@warwick.ac.uk

RICHARD LYONS
DEPARTMENT OF MATHEMATICS
RUTGERS UNIVERSITY
110 FRELINGHUYSEN ROAD
PISCATAWAY, NJ 08854
UNITED STATES

lyons@math.rutgers.edu