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We introduce the notion of twisted generalized complex submanifolds and describe an equivalent characterization in terms of Poisson–Dirac submanifolds. Our characterization recovers a result of Vaisman (2007). An equivalent characterization is also given in terms of spinors. As a consequence, we show that the fixed locus of an involution preserving a twisted generalized complex structure is a twisted generalized complex submanifold. We also prove that a twisted generalized complex manifold has a natural Poisson structure. We also discuss generalized Kähler submanifolds.

1. Introduction

Throughout this paper M will denote a smooth manifold. Generalized complex structures were originally defined by Hitchin [2003], and further studied by Gualtieri [2003]. Examples of generalized complex structures include symplectic and complex manifolds. In order to define generalized complex structures we will recall some structures on $TM \oplus T^*M$. The *Courant bracket* was defined in [Courant 1990] as

$$(1\text{-}1) \qquad [\![X + \xi, Y + \eta]\!] = [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \tfrac{1}{2} d(\eta(X) - \xi(Y))$$

for all $X, Y \in \mathfrak{X}(M)$ and $\xi, \eta \in \Omega(M)$. There also exist smoothly varying nondegenerate symmetric bilinear forms on each fibre of $TM \oplus T^*M$. These are defined as

$$\langle X + \xi, Y + \eta \rangle = \tfrac{1}{2}(\xi(Y) + \eta(X))$$

for all $X, Y \in T_m M$, $\xi, \eta \in T_m^* M$ and $m \in M$.

A *generalized complex structure* is a smooth map $J : TM \oplus T^*M \to TM \oplus T^*M$ such that $J^2 = -\mathrm{Id}$, $JJ^* = \mathrm{Id}$, and the $+i$-eigenbundle of J is involutive with respect to the Courant bracket.

The primary objects of study in this paper are twisted manifolds. A manifold M endowed with a closed 3-form Ω will be called *twisted*. A twisted manifold has

another well known bracket on $\mathfrak{X}(M) \oplus \Omega(M)$: the *twisted Courant bracket*. This bracket was defined in [Ševera and Weinstein 2001] as

$$(1\text{-}2) \quad [\![X + \xi, Y + \eta]\!]_\Omega = [X, Y] + \mathscr{L}_X \eta - \mathscr{L}_Y \xi - \tfrac{1}{2} d(\eta(X) - \xi(Y)) + i_Y i_X \Omega.$$

A *twisted generalized complex structure* is a smooth map $J : TM \oplus T^*M \to TM \oplus T^*M$ such that $J^2 = -\mathrm{Id}$, $JJ^* = \mathrm{Id}$ and the $+i$-eigenbundle of J is involutive with respect to (1-2), rather than (1-1). The triple (M, Ω, J) will be called a *twisted generalized complex manifold*. Indeed, generalized complex manifolds can be defined for arbitrary Courant algebroids. See [Barton 2007] for details.

The aim of this work is to characterize when a submanifold of a twisted generalized complex manifold is also a twisted generalized complex manifold. In the untwisted case, several notions of generalized complex submanifolds have been recently introduced. The notion defined here is similar to the one in [Ben-Bassat and Boyarchenko 2004] and [Vaisman 2007]. A different notion of generalized complex submanifolds appears in [Gualtieri 2003].

Definition 1.1. A *twisted immersion*, from one twisted manifold (N, Υ) to another (M, Ω), is defined as a smooth immersion $h : N \to M$ with $\Upsilon = h^* \Omega$. A *twisted generalized complex immersion* from (N, Υ, J') to (M, Ω, J) is a twisted immersion $h : (N, \Upsilon) \to (M, \Omega)$ such that the pullback of the $+i$-eigenbundle of J is the $+i$-eigenbundle of J'. In this case, N is called a *twisted generalized complex submanifold* of M.

By splitting vectors and covectors, a twisted generalized complex structure can be written as

$$(1\text{-}3) \qquad\qquad J = \begin{pmatrix} \phi & \pi^\sharp \\ \sigma_\flat & -\phi^* \end{pmatrix}.$$

Here ϕ is an endomorphism of TM, $\pi^\sharp : T^*M \to TM$ is the bundle map induced by a bivector field π, and $\sigma_\flat : TM \to T^*M$ is the bundle map induced by a 2-form σ. The fact that $J^2 = -\mathrm{Id}$ also leads to the following formulas:

$$\phi^2 + \pi^\sharp \sigma_\flat = -\mathrm{Id}, \quad \phi \pi^\sharp = \pi^\sharp \phi^*, \quad \text{and} \quad \phi^* \sigma_\flat = \sigma_\flat \phi.$$

These facts, and others, were first noted in [Crainic 2004]. These results were also described using Poisson quasi-Nijenhuis manifolds in [Stiénon and Xu 2007]. The conditions for a submanifold to be twisted generalized complex will be expressed in terms of this splitting. Using the theory of Lie bialgebroids we also show that π from (1-3) is a Poisson bivector field, which, for the untwisted case, is a standard result [Abouzaid and Boyarchenko 2006; Crainic 2004].

Our work was inspired by [Stiénon and Xu 2008], where the reduction of generalized complex structures is studied. The main result was also independently obtained by Vaisman [2007].

This paper is organized as follows. In Section 2, we recall some of the basic facts of Dirac structures. In particular we describe the pull back. In Section 3, we prove that a twisted generalized complex manifold carries a natural Poisson structure. In Section 4, we define the induced generalized complex structure, and characterize when it has the required properties. In Section 5, we prove the main theorem of this paper, and provide examples. Twisted generalized complex involutions are also introduced in this section. In Section 6, we determine when a submanifold of a holomorphic Poisson manifold is itself endowed with an induced holomorphic Poisson structure. Section 7 is a restatement of our main result in terms of spinors. The last section discusses generalized Kähler submanifolds.

2. Dirac structures

The aim of this section is to recall Dirac structures, and their pull backs. Before considering bundles, we will consider a vector space V. In this case a *Dirac structure* is nothing more than a maximal isotropic subspace of $V \oplus V^*$. Let q_1 denote the projection of $V \oplus V^*$ onto V, and q_2 the projection onto V^*.

If L is a Dirac structure then there exists a natural skew-symmetric bilinear form Λ on L defined by

$$\Lambda(X + \xi, Y + \eta) = \xi(Y) = -\eta(X) \text{ for all } X + \xi, Y + \eta \in L.$$

It is easy to see that

$$\Lambda(X + \xi_1, Y + \eta_1) = \Lambda(X + \xi_2, Y + \eta_2) \text{ for all } X + \xi_{1,2}, Y + \eta_{1,2} \in L,$$

and

$$\Lambda(X_1 + \xi, Y_1 + \eta) = \Lambda(X_2 + \xi, Y_2 + \eta) \text{ for all } X_{1,2} + \xi, Y_{1,2} + \eta \in L.$$

Hence, there exists a 2-form ε on $q_1(L)$ defined by

$$\varepsilon(X, Y) = \Lambda(X + \xi, Y + \eta) \text{ for all } X + \xi, Y + \eta \in L,$$

and a 2-form θ on $q_2(L)$ defined by

$$\theta(\xi, \eta) = -\Lambda(X + \xi, Y + \eta) \text{ for all } X + \xi, Y + \eta \in L.$$

If $X \in q_1(L)$ then there exists some $\xi \in V^*$ with $X + \xi \in L$; furthermore $\varepsilon(X, Y) = \xi(Y)$ for all $Y \in q_1(L)$. Thus $i_X \varepsilon = \xi|_{q_1(L)}$, and

$$X + \xi \in L \iff X \in q_1(L) \text{ and } i_X \varepsilon = \xi|_{q_1(L)}.$$

Thus knowing the Dirac structure L is exactly the same as knowing the subspace $q_1(L)$ and the 2-form ε. Similarly, L is equivalent to the pair $(q_2(L), \theta)$. Thus any subspace $R \subseteq V$ endowed with a 2-form ε on R defines a Dirac structure $L(R, \varepsilon)$:

$$L(R, \varepsilon) = \left\{ X + \xi \in R \oplus V^* : i_X \varepsilon = \xi|_R \right\},$$

and any subspace $S \subseteq V^*$ endowed with a 2-form π on S defines a Dirac structure $L(S, \theta)$:

$$L(S, \theta) = \{ X + \xi \in V \oplus S : \theta(\xi, \eta) = -\eta(X) \text{ for all } \eta \in S \}.$$

Details of these constructions can be found in [Courant 1990]. Let W be another vector space and $\varphi : V \to W$ a linear map. The map φ can be used to both pull Dirac structures back from W to V, and push Dirac structures forward from V to W. Let (R, ε) be a Dirac structure on W, with $R \subseteq W$ and $\varepsilon \in \Gamma(\bigwedge^2 R^*)$. A Dirac structure on V is defined by $(\varphi^{-1}R, \varphi^* \varepsilon)$. This Dirac structure is called the *pull back* of (R, ε) under φ. Similarly if (S, θ) is a Dirac structure on V, with $S \subseteq V^*$ and θ defined on S, then $((\varphi^*)^{-1}S, \varphi_* \theta)$ defines a Dirac structure on W. This Dirac structure is called the *push forward* of (S, θ) under φ. These two maps of Dirac structures are denoted by $\mathcal{F}_\varphi$ and $\mathcal{B}_\varphi$. It is very easy to see that for a Dirac structure L on W

$$B_\varphi(L) = \left\{ X + \varphi^* \xi : X \in V, \xi \in W^* \text{ such that } \varphi X + \xi \in L' \right\},$$

and for a Dirac structure L' on V

$$\mathcal{F}_\varphi(L') = \left\{ \varphi X + \xi : X \in V, \xi \in W^* \text{ such that } X + \varphi^* \xi \in L \right\}.$$

Dirac structures can also be defined for a twisted manifold (M, Ω). A *Dirac structure* is a smooth subbundle $L \subseteq TM \oplus T^*M$ for which each fibre is a Dirac structure of the corresponding fibre of $TM \oplus T^*M$, and whose space of sections is closed under the twisted Courant bracket (1-2). The restriction of the twisted Courant bracket to a Dirac structure is a Lie bracket; thus a Dirac structure is naturally a Lie algebroid.

The definitions of push forward and pull back can be reformulated for Dirac structures on manifolds. We will only consider the pull back of a Dirac structure, but more on the push forward can be found in [Bursztyn and Radko 2003] and [Stiénon and Xu 2008]. We note that the pull back of a Dirac structure is automatically a maximal isotropic, but it need not be smooth or involutive.

The last lemma of this section will be used to characterize when the pullback bundle is involutive. Let (M, Ω) and (N, Υ) be two twisted manifolds with an immersion $\varphi : N \to M$. Two sections

$$\sigma_N = X + \xi \in \mathfrak{X}(N) \oplus \Omega(N) \quad \text{and} \quad \sigma_M = Y + \eta \in \mathfrak{X}(M) \oplus \Omega(M)$$

are said to be φ-*related*, denoted by $\sigma_N \overset{\varphi}{\rightsquigarrow} \sigma_M$, if $Y = \varphi_* X$ and $\xi = \varphi^* \eta$. The following lemma is an extension of [Stiénon and Xu 2008, Lemma 2.2]. This lemma is also true for complex sections, which is when it will be applied.

Lemma 2.1. *Assume that $\sigma_N^i \in \Gamma(TN \oplus T^*N)$ and $\sigma_M^i \in \Gamma(TM \oplus T^*M)$ satisfy $\sigma_N^i \overset{\varphi}{\rightsquigarrow} \sigma_M^i$, for $i = 1, 2$. Then, if φ is a twisted immersion,*

$$\llbracket \sigma_N^1, \sigma_N^2 \rrbracket_\Upsilon \overset{\varphi}{\rightsquigarrow} \llbracket \sigma_M^1, \sigma_M^2 \rrbracket_\Omega.$$

Proof. Write $\sigma_N^i = X^i + \xi^i$ and $\sigma_M^i = Y^i + \eta^i$, where $X^i + \xi^i \in \mathfrak{X}(N) \oplus \Omega(N)$ and $Y^i + \eta^i \in \mathfrak{X}(M) \oplus \Omega(M)$, for $i = 1, 2$. Since $\sigma_N^i \overset{\varphi}{\rightsquigarrow} \sigma_M^i$, for $i = 1, 2$, then $\varphi_* X^i = Y^i$ and $\varphi^* \eta^i = \xi^i$. By definition

$$\llbracket \sigma_N^1, \sigma_N^2 \rrbracket_\Upsilon = \left[X^1, X^2 \right] + \mathcal{L}_{X^1} \xi^2 - \mathcal{L}_{X^2} \xi^1 + \tfrac{1}{2} d(\xi^1(X^2) - \xi^2(X^1)) + i_{X^2} i_{X^1} \Upsilon,$$

and

$$\llbracket \sigma_M^1, \sigma_M^2 \rrbracket_\Omega = \left[Y^1, Y^2 \right] + \mathcal{L}_{Y^1} \eta^2 - \mathcal{L}_{Y^2} \eta^1 + \tfrac{1}{2} d(\eta^1(Y^2) - \eta^2(Y^1)) + i_{Y^2} i_{Y^1} \Omega.$$

Now

$$\varphi_* \left[X^1, X^2 \right] = \left[\varphi_* X^1, \varphi_* X^2 \right] = \left[Y^1, Y^2 \right],$$

and

$$\begin{aligned}
\varphi^*(\mathcal{L}_{Y^1} \eta^2) &= \varphi^*(i_{Y^1} d\eta^2 + d i_{Y^1} \eta^2) = \varphi^* i_{\varphi_* X^1} d\eta^2 + \varphi^* d(\eta^2(Y^1)) \\
&= i_{X^1} \varphi^* d\eta^2 + d(\xi^2(X^1)) = i_{X^1} d\xi^2 + d i_{X^1} \xi^2 = \mathcal{L}_{X^1} \xi^2.
\end{aligned}$$

Similarly,

$$\varphi^*(\mathcal{L}_{Y^2} \eta^1) = \mathcal{L}_{X^2} \xi^1.$$

The second last term becomes

$$\varphi^* d(\eta^1(Y^2) - \eta^2(Y^1)) = d(\xi^1(X^2) - \xi^2(X^1)),$$

since

$$\varphi^*(\eta^1(Y^2)) = \varphi^*(\eta^1(\varphi_* X^2)) = (\varphi^* \eta^1)(X^2) = \xi^1(X^2).$$

Finally, because φ is a twisted immersion, the following holds.

$$\varphi^* i_{Y^2} i_{Y^1} \Upsilon = \varphi^* i_{\varphi_* X^2} i_{\varphi_* X^1} \Upsilon = i_{X^2} i_{X^1} \varphi^* \Upsilon = i_{X^2} i_{X^1} \Omega. \qquad \square$$

3. The Poisson bivector field associated to a generalized complex structure

For the usual Courant bracket it is well known that the existence of a generalized complex structure leads to a Poisson bivector [Abouzaid and Boyarchenko 2006; Crainic 2004; Lindström et al. 2005; Hu 2005]. In this chapter we will obtain the same result for arbitrary Courant algebroids and also give a new way of expressing the Poisson bivector.

A *Courant algebroid* [Liu et al. 1997] is a triple consisting of a vector bundle $E \to M$ equipped with a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$, a skew-symmetric bracket $[\![\cdot, \cdot]\!]$ on $\Gamma(E)$, and a smooth bundle map $E \xrightarrow{\rho} TM$ called the *anchor*. These induce a natural differential operator $\mathcal{D} : C^\infty(M) \to \Gamma(E)$ defined by

$$(3\text{-}1) \qquad\qquad \langle \mathcal{D}f, a \rangle = \tfrac{1}{2}\rho(a)f,$$

for all $f \in C^\infty(M)$ and $a \in \Gamma(E)$. These structures must obey the following formulas for all $a, b, c \in \Gamma(E)$ and $f, g \in C^\infty(M)$:

$$(3\text{-}2) \qquad\qquad \rho([\![a, b]\!]) = [\rho(a), \rho(b)],$$

$$(3\text{-}3) \quad [\![\, [\![a, b]\!], c\,]\!] + [\![\, [\![b, c]\!], a\,]\!] + [\![\, [\![c, a]\!], b\,]\!]$$
$$= \tfrac{1}{3}\mathcal{D}\big(\langle [\![a, b]\!], c \rangle + \langle [\![b, c]\!], a \rangle + \langle [\![c, a]\!], b \rangle\big),$$

$$(3\text{-}4) \qquad\qquad [\![a, fb]\!] = f\,[\![a, b]\!] + (\rho(a)f)b - \langle a, b \rangle\,\mathcal{D}f,$$

$$(3\text{-}5) \qquad\qquad \rho \circ \mathcal{D} = 0, \text{ that is, } \langle \mathcal{D}f, \mathcal{D}g \rangle = 0,$$

$$(3\text{-}6) \qquad \rho(a)\langle b, c \rangle = \langle [\![a, b]\!] + \mathcal{D}\langle a, b \rangle, c \rangle + \langle b, [\![a, c]\!] + \mathcal{D}\langle a, c \rangle \rangle.$$

The relation

$$(3\text{-}7) \qquad\qquad [\![\mathcal{D}f, a]\!] + \mathcal{D}\langle \mathcal{D}f, a \rangle = 0$$

is a consequence of these conditions [Roytenberg 1999].

A smooth subbundle L of a Courant algebroid is called a *Dirac subbundle* if it is a maximal isotropic, with respect to $\langle \cdot, \cdot \rangle$, and its space of sections $\Gamma(L)$ is closed under $[\![\cdot, \cdot]\!]$. While not all Courant algebroids are Lie algebroids (since the Jacobi identity is not satisfied), their Dirac subbundles are Lie algebroids.

Example 3.1 ([Courant 1990]). Given a smooth manifold M, the bundle $TM \to M$ carries a natural Courant algebroid structure, where the anchor is the identity map and the pairing and bracket are given, respectively, by

$$\langle X + \xi, Y + \eta \rangle = \tfrac{1}{2}(\xi(Y) + \eta(X)),$$
$$[\![X + \xi, Y + \eta]\!] = [X, Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi + \tfrac{1}{2}d(\xi(Y) - \eta(X)),$$

for all $X, Y \in \mathfrak{X}^\bullet(M)$, and for all $\xi, \eta \in \Omega^1(M)$.

Let E be a Courant algebroid on a smooth manifold M. And let

$$
\begin{array}{ccc}
E & \xrightarrow{\ J\ } & E \\
\downarrow & & \downarrow \\
M & \xrightarrow[\text{Id}]{} & M
\end{array}
$$

be a vector bundle map such that $J^2 = -\mathrm{Id}$. Then the complexification $E_{\mathbb{C}} :=$ $E \otimes \mathbb{C}$ — with the extended $\mathbb{C}$-linear Courant algebroid structure — decomposes as the direct sum $L \oplus \bar{L}$ of the eigenbundles of J. Here L is associated to the eigenvalue $+i$ and its complex conjugate $\bar{L}$ to $-i$. The bundle map J is called a *generalized complex structure* if J is orthogonal with respect to $\langle \cdot, \cdot \rangle$ — this forces L and $\bar{L}$ to be isotropic — and the spaces of sections $\Gamma(L)$ and $\Gamma(\bar{L})$ are closed under the Courant bracket, or equivalently, J is "integrable":

$$[\![Jx, Jy]\!] - [\![x, y]\!] - J([\![Jx, y]\!] + [\![x, Jy]\!]) = 0, \qquad \text{for all } x, y \in \Gamma(E).$$

Since the pairing is nondegenerate, the map

$$\Xi : E \xrightarrow{\text{iso}} E^*, \ e \mapsto \langle e, \cdot \rangle$$

is an isomorphism of vector bundles. One has $\Xi^* = \Xi$ (modulo the canonical isomorphism $(E^*)^* = E$) and $\Xi \circ J + J^* \circ \Xi = 0$.

Proposition 3.2. *The bracket*

(3-8) $$\{f, g\} = 2 \langle J\mathcal{D}f, \mathcal{D}g \rangle, \qquad f, g \in C^\infty(M)$$

is a Poisson structure on the manifold M.

Gualtieri [2007] proved a formula similar to (but slightly less general than) (3-8).

Proof. It is easy to see that this bracket is a skew-symmetric derivation of $C^\infty(M)$. It remains to check that the Jacobi identity is satisfied. Since J is integrable, we have

$$[\![J\mathcal{D}f, J\mathcal{D}g]\!] - [\![\mathcal{D}f, \mathcal{D}g]\!] - J([\![J\mathcal{D}f, \mathcal{D}g]\!] + [\![\mathcal{D}f, J\mathcal{D}g]\!]) = 0,$$

for all $f, g \in C^\infty(M)$. Pairing with $\mathcal{D}h$, we obtain
(3-9)
$$\langle [\![J\mathcal{D}f, J\mathcal{D}g]\!], \mathcal{D}h \rangle - \langle [\![\mathcal{D}f, \mathcal{D}g]\!], \mathcal{D}h \rangle - \langle J([\![J\mathcal{D}f, \mathcal{D}g]\!] + [\![\mathcal{D}f, J\mathcal{D}g]\!]), \mathcal{D}h \rangle = 0.$$

We compute the first term of (3-9):

$$\langle [\![J\mathcal{D}f, J\mathcal{D}g]\!], \mathcal{D}h \rangle$$
$$= \langle [\![J\mathcal{D}f, J\mathcal{D}g]\!] + \mathcal{D} \langle J\mathcal{D}f, J\mathcal{D}g \rangle, \mathcal{D}h \rangle \qquad \text{by (3-5)}$$
$$= \rho(J\mathcal{D}f) \langle J\mathcal{D}g, \mathcal{D}h \rangle - \langle J\mathcal{D}g, [\![J\mathcal{D}f, \mathcal{D}h]\!] + \mathcal{D} \langle J\mathcal{D}f, \mathcal{D}h \rangle \rangle \qquad \text{by (3-6)}$$
$$= 2 \langle \mathcal{D} \langle J\mathcal{D}g, \mathcal{D}h \rangle, J\mathcal{D}f \rangle - \langle J\mathcal{D}g, \mathcal{D} \langle \mathcal{D}h, J\mathcal{D}f \rangle + \mathcal{D} \langle J\mathcal{D}f, \mathcal{D}h \rangle \rangle \qquad \text{by (3-1), (3-7)}$$
$$= 2 \{f, \{g, h\}\} - 2 \{g, \{f, h\}\} \qquad \text{by (3-8)}$$
$$= 2 \{f, \{g, h\}\} + 2 \{g, \{h, f\}\}.$$

By (3-1), (3-2) and (3-5), the second term of (3-9) vanishes. Finally, the third term of (3-9) gives

$$-\big\langle J(\llbracket J\mathscr{D}f, \mathscr{D}g\rrbracket + \llbracket \mathscr{D}f, J\mathscr{D}g\rrbracket), \mathscr{D}h\big\rangle$$
$$= -\langle J(\mathscr{D}\langle \mathscr{D}g, J\mathscr{D}f\rangle - \mathscr{D}\langle \mathscr{D}f, J\mathscr{D}g\rangle), \mathscr{D}h\rangle \qquad \text{by (3-7)}$$
$$= -\{\{f, g\}, h\} + \{\{g, f\}, h\} \qquad\qquad\qquad \text{by (3-8)}$$
$$= 2\{h, \{f, g\}\} \qquad\qquad\qquad\qquad\qquad \square$$

Proposition 3.3. *Let Π denote the bivector field on M associated to the Poisson bracket* (3-8). *We have*

$$\Pi^\sharp = \tfrac{1}{2}\rho \circ J \circ \Xi^{-1} \circ \rho^*,$$

where $\Pi^\sharp : T^*M \to TM$ is the vector bundle map equivalent to $\Pi \in \Gamma(\wedge^2 T^*M)$ via $\Pi(\alpha, \beta) = \beta(\Pi^\sharp \alpha)$, for all $\alpha, \beta \in \Omega^1(M)$. The Hamiltonian vector field associated to the function $f \in C^\infty(M)$ is

$$X_f = \Pi^\sharp df = \rho \circ J \circ \mathscr{D}f.$$

And the characteristic distribution is

$$\Pi^\sharp(T^*M) = \rho J(\ker \rho)^\perp,$$

where $(\ker \rho)^\perp$ refers to the subbundle of E orthogonal to $\ker \rho$ with respect to $\langle \cdot, \cdot \rangle$.

Proof. One has

$$X_f(g) = \{f, g\} = 2\langle J\mathscr{D}f, \mathscr{D}g\rangle = \rho(J\mathscr{D}f)(g)$$

and

$$\Pi^\sharp(df) = X_f = \rho J\mathscr{D}f = \tfrac{1}{2}\rho J\Xi^{-1}\rho^*(df)$$

since (3-1) can be reinterpreted as $\langle \mathscr{D}f, \cdot \rangle = \tfrac{1}{2}\Xi^{-1}\rho^* df$. $\qquad\qquad \square$

Proposition 3.4. *If $E = TM$ is the standard Courant algebroid of Example 3.1 and the matrix representation of J relative to the above direct sum decomposition is shown in* (1-3), *then $\Pi = \pi$.*

Proof. Here $\mathscr{D}$ coincides with the de Rham differential d. Thus

$$\{f, g\}_\Pi = 2\langle J\mathscr{D}f, \mathscr{D}g\rangle = 2\langle \pi^\sharp df - \phi^* df, dg\rangle = dg(\pi^\sharp df) = \{f, g\}_\pi. \qquad \square$$

Recall that the complexification of E decomposes as the direct sum $E_\mathbb{C} = L_+ \oplus L_-$, where $L_\pm$ are Dirac structures (with anchor maps $\rho_\pm$). Thus (L_+, L_-) is a complex Lie bialgebroid [Mackenzie and Xu 1994], where $L_\pm^*$ is identified with $L_\mp$ via Ξ. As shown in [Mackenzie and Xu 1994, Proposition 3.6], to any complex Lie bialgebroid is associated a complex bivector field ϖ on M given by

$$i\varpi^\sharp = \rho_- \circ \Xi^{-1} \circ \rho_+^* = -\rho_+ \circ \Xi^{-1} \circ \rho_-^*.$$

Lemma 3.5. *The Poisson bivector ϖ coming from the Lie bialgebroid structure (L_+, L_-) is real and coincides with Π.*

Proof. It suffices to observe that the following two compositions are both equal to $\varpi^\sharp$:

$$T_{\mathbb{C}}M \xleftarrow{\rho_-} L_- \xleftarrow{-i\cdot} L_- \xleftarrow{\Xi^{-1}} L_+^* \xleftarrow{\rho_+^*} T_{\mathbb{C}}^*M$$

$$T_{\mathbb{C}}M \xleftarrow{\rho_+} L_+ \xleftarrow{+i\cdot} L_+ \xleftarrow{\Xi^{-1}} L_-^* \xleftarrow{\rho_-^*} T_{\mathbb{C}}^*M$$

and that their sum

$$T_{\mathbb{C}}M \xleftarrow{\rho} E_{\mathbb{C}} \xleftarrow{J} E_{\mathbb{C}} \xleftarrow{\Xi^{-1}} E_{\mathbb{C}}^* \xleftarrow{\rho^*} T_{\mathbb{C}}^*M$$

is equal to $2\Pi^\sharp$. $\qquad\qquad\square$

Proposition 3.6. $\rho(L_+) \cap \rho(L_-) = \Delta \otimes \mathbb{C}$ *with* $\Delta = \rho(J \ker \rho)$

Proof. If $v \in \rho(L_+) \cap \rho(L_-)$, then $\overline{v} \in \rho(L_-) \cap \rho(L_+)$. Hence there exists a subbundle Δ of TM such that $\rho(L_+) \cap \rho(L_-) = \Delta \otimes \mathbb{C}$. For all $k \in \ker \rho$, one has $\rho(L_+) \ni \rho(\frac{1+iJ}{2}k) = \frac{i}{2}\rho(Jk)$. Therefore, $\rho(J \ker \rho) \subset \rho(L_+)$. Since J is real, $\rho(J \ker \rho) \subset \rho(L_+) \cap \rho(E) = \Delta$. It remains to prove the converse inclusion: $\Delta \subset \rho(J \ker \rho)$. Since $\Delta = \rho(L_+) \cap \rho(E)$, given $\delta \in \Delta$, there exists $l_+ \in L_+$ such that $\rho(l_+) = \delta = \rho(\overline{l_+})$. Thus $\delta = \rho(\frac{l_+ + \overline{l_+}}{2}) = \rho(J(\frac{l_+ - \overline{l_+}}{2i}))$ with $\frac{l_+ - \overline{l_+}}{2i} \in \ker \rho$. $\square$

Remark 3.7. If $E = TM$ is the standard Courant algebroid of Example 3.1, then $(\ker \rho)^\perp = T^*M = \ker \rho$. Therefore, in this particular case, $\Pi^\sharp(T_{\mathbb{C}}^*M) = \rho(L_+) \cap \rho(L_-)$, recovering Gualtieri's result Gualtieri [2003].

It would be interesting to explore when the symplectic foliation $\Pi^\sharp(T_{\mathbb{C}}^*M)$ coincides with $\rho(L_+) \cap \rho(L_-)$ for arbitrary Courant algebroids.

4. The induced generalized complex structure

Consider two twisted manifolds (M, Ω) and (N, Υ) with an immersion $h : N \to M$. Also, assume that there is a generalized complex structure J on M with eigenbundles L_+ and L_-. The goal of this section is to characterize when the pull backs of L_+ and L_- give a generalized complex structure on N. The pull backs of L_+ and L_- will be called the *induced bundles*, and are given by

$$L'_\pm = \mathcal{B}_h(L_\pm) = \left\{ X + h^*\xi : X \in T_{\mathbb{C}}N, \xi \in T_{\mathbb{C}}^*M \text{ such that } h_*X + \xi \in L_\pm \right\}.$$

By definition, both L'_+ and L'_- are maximal isotropics, but they need not be smooth or involutive subbundles. The bundles may also have nontrivial intersection. The rest of this section is devoted to characterizing when the induced bundles have the desired properties. The first of these properties to be addressed will be the intersection property.

Because L'_+ and L'_- are both maximal isotropics, it suffices to check that they span $T_{\mathbb{C}}N \oplus T^*_{\mathbb{C}}N$. Consider the subbundle $B = TN \oplus T^*M|_N$ of $TM \oplus T^*M$. Its orthogonal, $B^\perp = TN^o$, is the kernel of the natural projection $s : B \to TN \oplus T^*N$, which maps $X + \xi \mapsto X + h^*\xi$. It is not hard to see that $s((B \cap JB) \otimes \mathbb{C}) = L'_+ + L'_-$. Thus the decomposition, $T_{\mathbb{C}}N \oplus T^*_{\mathbb{C}}N = L'_+ \oplus L'_-$, holds if and only if $B = B \cap JB + B^\perp$. The preceding can be summarized in the following proposition.

Proposition 4.1. *The following assertions are equivalent.*

(1) *The subbundle L'_+ is the $+i$-eigenbundle of a—not necessarily smooth— automorphism J' of $TN \oplus T^*N$ such that $J'^2 = -\mathrm{Id}$ and $J'J'^* = \mathrm{Id}$.*

(2) $B = B \cap JB + B^\perp$.   (3) $JB \subseteq B + JB^\perp$.   (4) $JB^\perp \cap B \subseteq B^\perp$.

Conditions (3) and (4) follow from elementary calculations. In the sequel we will assume that the assertions of Proposition 4.1 are satisfied. Consider the restriction of J and s to the J-invariant subspace $B \cap JB$; the latter map will be denoted by s'. The kernel of s' is $B^\perp \cap JB$. Under J, this kernel is mapped to $JB^\perp \cap B$. This must be in $JB \cap B$ and also, by Proposition 4.1, in $B^\perp$, however $B^\perp \subseteq B$. So the image of the kernel of s' is in $B^\perp \cap JB \cap B = B^\perp \cap JB$. Thus the kernel of s' is J-invariant and $J|_{B \cap JB}$ induces an automorphism of $TN \oplus T^*N$:

(4-1)
$$
\begin{array}{ccc}
B \cap JB & \xrightarrow{\ \ J\ \ } & B \cap JB \\
{\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s} \\
TN \oplus T^*N & \xrightarrow[\ \ J'\ \ ]{} & TN \oplus T^*N.
\end{array}
$$

The induced automorphism is nothing but J' from Proposition 4.1. Indeed, the complexification of the above commutative diagram gives

$$
\begin{array}{ccc}
(L_+ \cap B_{\mathbb{C}}) \oplus (L_- \cap B_{\mathbb{C}}) & \xrightarrow{(+i)\mathrm{Id}\oplus(-i)\mathrm{Id}} & (L_+ \cap B_{\mathbb{C}}) \oplus (L_- \cap B_{\mathbb{C}}) \\
\downarrow & & \downarrow \\
L'_+ \oplus L'_- & \xrightarrow[(+i)\mathrm{Id}\oplus(-i)\mathrm{Id}]{} & L'_+ \oplus L'_-.
\end{array}
$$

The next lemma relates condition (4) of Proposition 4.1 to the splitting of J:

$$
\begin{pmatrix} \phi & \pi^\sharp \\ \sigma_\flat & -\phi^* \end{pmatrix}.
$$

Lemma 4.2. *The following assertions are equivalent.*

(1) $JB^\perp \cap B \subseteq B^\perp$.

(2) $TN \cap \pi^\sharp(TN^o) = 0$ *and* $\phi(TN) \subseteq TN + \pi^\sharp(TN^o)$.

Proof. The inclusion $J(TN^o) \cap (TN \oplus T^*M|_N) \subseteq TN^o$ is true if, and only if,

$$\left\{ \begin{array}{l} \xi \in TN^o \\ J\xi \in TN \oplus T^*M|_N \end{array} \right\} \implies J\xi \in TN^o$$

if, and only if,

$$\left\{ \begin{array}{l} \xi \in TN^o \\ \pi^\sharp \xi \in TN \end{array} \right\} \implies \left\{ \begin{array}{l} \pi^\sharp \xi = 0 \\ \phi^* \xi \in TN^o \end{array} \right\}$$

if, and only if,

$$\xi \in TN^o \cap (\pi^\sharp)^{-1}(TN) \implies \left\{ \begin{array}{l} \pi^\sharp \xi = 0 \\ \xi \in (\phi(TN))^o \end{array} \right\}$$

if, and only if,

$$\pi^\sharp(TN^o) \cap TN = 0 \qquad \text{and} \qquad TN^o \cap (\pi^\sharp)^{-1}(TN) \subseteq (\phi(TN))^o.$$

Since π is skew-symmetric $(\pi^\sharp)^{-1}(TN) = (\pi^\sharp(TN^o))^o$, and

$$TN^o \cap (\pi^\sharp(TN^o))^o \subseteq (\phi(TN))^o$$
$$TN + \pi^\sharp(TN^o) \supseteq \phi(TN). \qquad \square$$

According to this lemma, the sum $TN + \pi^\sharp(TN^o)$ must be direct. In the sequel pr will denote the projection $TN \oplus \pi^\sharp(TN^o) \to TN$. If π is degenerate then neither the bundle $TN \oplus \pi^\sharp(TN^o)$, nor the map pr is necessarily smooth.

For any $\xi \in T^*N$ we claim that if $\eta, \eta' \in B \cap JB$ such that $\xi = h^*\eta = h^*\eta'$ then $\pi^\sharp \eta = \pi^\sharp \eta'$. Because η and η' are preimages of ξ they differ by some element of TN^o, and as $B \cap JB$ is J-stable both $\pi^\sharp \eta$ and $\pi^\sharp \eta'$ are in TN. However $TN \cap \pi^\sharp(TN^o) = \{0\}$, and the difference of the two preimages is zero. Thus the assignment $\xi \mapsto \pi^\sharp \eta$ defines a skew-symmetric vector bundle map from T^*N to TN. Its associated bivector field on N will be denoted by π'.

The following technical lemmas will be used to show when J' is smooth.

Lemma 4.3. *Let $X \in TN$ and $\xi \in TN^o$. If $\phi X + \pi^\sharp \xi \in TN$ then $\phi X + \pi^\sharp \xi = (\mathrm{pr} \circ \phi)X$.*

Proof. For any $X \in TN$ the second assertion of Lemma 4.2 gives $\phi X = Y + \pi^\sharp \eta$, where $Y \in TN$ and $\eta \in TN^o$. By definition $Y = (\mathrm{pr} \circ \phi)X$, and $\phi X + \pi^\sharp \xi = Y + \pi^\sharp(\eta + \xi)$. Both Y and $\phi X + \pi^\sharp \xi$ are elements of TN; thus $\pi^\sharp(\eta + \xi)$ is also an element of TN. But $TN \cap \pi^\sharp(TN^o) = \{0\}$, and $\eta + \xi \in TN^o$. Thus $\pi^\sharp(\eta + \xi) = 0$. $\square$

Lemma 4.4. *Let $p_1 : TN \oplus T^*N \to TN$ and $p_2 : TN \oplus T^*N \to T^*N$ be the projections. If $X \in TN$, then*

$$(4\text{-}2) \qquad (p_1 J')X = (\mathrm{pr} \circ \phi)X = \phi X + \pi^\sharp \zeta$$

and

(4-3) $(p_2 J')X = h^*(\sigma_\flat X - \phi^*\zeta),$

where ζ is some element of TN^o such that $X + \zeta \in B \cap JB$.
   If $\xi \in T^*N$, then

(4-4) $(p_1 J')\xi = \pi'^\sharp \xi = \pi^\sharp \eta$

and

(4-5) $(p_2 J')\xi = -(h^*\phi^*)\eta,$

where η is some element of $T^*M|_N \cap B \cap JB$ such that $h^*\eta = \xi$.

Proof. Consider $X \in TN$. Since s is surjective there exists some $\zeta \in TN^o$ such that $X + \zeta \in B \cap JB$ and $s(X + \zeta) = X$. Now $J(X+\zeta) = (\phi X + \pi^\sharp \zeta) + (\sigma_\flat X - \phi^*\zeta) \in B$. Therefore $\phi X + \pi^\sharp \zeta \in TN$ and, by Lemma 4.3, $\phi X + \pi^\sharp \zeta = (\text{pr} \circ \phi)X$. Both (4-2) and (4-3) follow from (4-1).

Now take $\xi \in T^*N$. Again, since s is surjective there exists some $\eta \in T^*M$ such that $\eta \in B \cap JB$ and $s(\eta) = \xi$. Now $J(\xi) = \pi^\sharp \eta - \phi^*\eta = \pi'^\sharp \xi - \phi^*\eta$, which is in B. Both (4-4) and (4-5) follow from (4-1). $\qquad\square$

For the remainder of this section, if L is a smooth vector bundle then $\Gamma(L)$ will denote the space of all — not necessarily smooth — sections of L, and $\Gamma^\infty(L)$ the subspace of smooth sections.

Lemma 4.5. *Let $\xi \in \Gamma(TN^o)$. If $\pi^\sharp \xi \in \Gamma^\infty(TM|_N)$, then $(h^*\phi^*)\xi \in \Gamma^\infty(T^*N)$.*

Proof. As noted previously, if $X \in \Gamma^\infty(TN)$ then $\phi X = Y + \pi^\sharp \eta$, where $Y \in \Gamma(TN)$ and $\eta \in \Gamma^\infty(TN^o)$. Now $(\phi^*\xi)(X) = \xi(\phi X) = \xi(Y) + \xi(\pi^\sharp \eta) = \xi(\pi^\sharp \eta) = -\eta(\pi^\sharp \xi)$. This function and its restriction to TN are smooth because $\pi^\sharp \xi$ is. $\qquad\square$

Lemma 4.6. *Assume $\text{pr} \circ \phi$ is a smooth map and $\eta \in \Gamma(T^*M|_N)$. If $h^*\eta \in \Gamma^\infty(TN)$ and $\pi^\sharp \eta \in \Gamma^\infty(TN)$ then $(h^*\phi^*)\eta \in \Gamma^\infty(T^*N)$.*

Proof. Once again, if $Y \in \Gamma^\infty(TN)$ then

$$\phi Y = (\text{pr} \circ \phi)Y + \pi^\sharp \zeta$$

for some $\zeta \in \Gamma^\infty(TN^o)$. Now

$$((h^*\phi^*)\eta)(Y) = (h^*\eta)(\phi Y) = (h^*\eta)((\text{pr} \circ \phi)Y) + (h^*\eta)(\pi^\sharp \zeta)$$
$$= (h^*\eta)((\text{pr} \circ \phi)Y) - \zeta(\pi^\sharp(h^*\eta)).$$

Thus $((h^*\phi^*)\eta)(Y)$ is a smooth function, and the lemma follows. $\qquad\square$

We are now ready to give the conditions J' must satisfy in order to be smooth.

Proposition 4.7. *The vector bundle automorphism J' of $TN \oplus T^*N$ is smooth if,
and only if, $\mathrm{pr} \circ \phi : TN \to TN$ is smooth and π' is a smooth bivector field on N.*

Proof. First assume that J' is smooth. Thus $(p_1 J')X \in \Gamma^\infty(TN)$ for all $X \in
\Gamma^\infty(TN)$. It follows from (4-2) that $(\mathrm{pr} \circ \phi)$ must be smooth. Also $(p_1 J')\xi \in
\Gamma^\infty(T^*N)$ for all $\xi \in \Gamma^\infty(T^*N)$, and (4-4) shows that $\pi'^\sharp$ is smooth.

Now for the other implication. For every $X \in \Gamma^\infty(T^*N)$ there is some $\zeta \in
\Gamma(TN^o)$ such that (4-2) and (4-3) are satisfied. As J is smooth both $\sigma_\flat$ and ϕ are
smooth. The smoothness of $\mathrm{pr} \circ \phi$ and (4-2) show that $\pi^\sharp \zeta \in \Gamma^\infty(TM|_N)$. Thus,
according to Lemma 4.5, $(h^*\phi^*)\zeta \in \Gamma^\infty(T^*N)$, and the right hand sides of (4-2)
and (4-3) are smooth. Finally $J'X = (p_1 J')X + (p_2 J')X \in \Gamma^\infty(TN \oplus T^*N)$.

Now take $\xi \in \Gamma^\infty(T^*N)$. There must exist $\eta \in \Gamma(T^*M|_N)$ such that (4-4)
holds, (4-5) holds, and $h^*\eta = \xi$. The smoothness of π' and (4-4) show that
$\pi^\sharp \eta \in \Gamma^\infty(TN)$. Now Lemma 4.6 gives $(h^*\phi^*)\eta \in \Gamma^\infty(T^*N)$, and the right hand
sides of (4-4) and (4-5) are smooth. Finally,

$$J'\xi = (p_1 J')\xi + (p_2 J')\xi \in \Gamma^\infty(TN \oplus T^*N). \qquad \square$$

We finish this section by using Lemma 2.1 to show when J' is integrable.

Proposition 4.8. *If J' is smooth then it is integrable.*

Proof. First, observe that the vector bundles $L_\pm \cap B_\mathbb{C} = (I \mp iJ)B_\mathbb{C}$ are smooth.
Since J' is smooth, its eigenbundles $L'_\pm$ are also smooth. It is not hard to check
that any smooth section of L'_+ is h-related to a smooth section of $L_+ \cap B_\mathbb{C}$.

Hence for any $\sigma'_1, \sigma'_2 \in \Gamma^\infty(L'_+)$ there exists $\sigma_1, \sigma_2 \in \Gamma^\infty(L_+ \cap B_\mathbb{C})$ such that
$\sigma_1 \overset{h}{\rightsquigarrow} \sigma'_1$ and $\sigma_2 \overset{h}{\rightsquigarrow} \sigma'_2$. Since L_+ is integrable $[\![\sigma_1, \sigma_2]\!]_\Omega \in \Gamma^\infty(L_+)$, and it follows
from Lemma 2.1 that $[\![\sigma_1, \sigma_2]\!]_\Omega \overset{h}{\rightsquigarrow} [\![\sigma'_1, \sigma'_2]\!]_\Upsilon$. Thus $[\![\sigma'_1, \sigma'_2]\!]_\Upsilon \in \Gamma^\infty(L'_+)$, and L'_+
is involutive with respect to the Υ-twisted bracket. $\qquad \square$

5. Main theorem

The following definition will be used to characterize when a twisted submanifold
is also generalized complex; see [Crainic and Fernandes 2004] for the motivation
of this definition.

Definition 5.1. Let (M, π) be a Poisson manifold. A smooth submanifold N of
M is a *Poisson–Dirac submanifold* of M if $TN \cap \pi^\sharp(TN)^o = \{0\}$, and the induced
Poisson tensor π' on N is smooth.

The next theorem is the main result of this paper. The untwisted version of this
result was obtained independently, using a different method, by Vaisman [2007].

Theorem 5.2. *Let (M, Ω, J) be a twisted generalized complex manifold with $J =
\left(\begin{smallmatrix} \phi & \pi^\sharp \\ \sigma_\flat & -\phi^* \end{smallmatrix} \right)$. A twisted submanifold N of M inherits a twisted generalized complex*

*structure J', making it a twisted generalized complex submanifold, if and only the
following conditions hold*:

(1) *N is a Poisson–Dirac submanifold of (M, π).*

(2) $\phi(TN) \subseteq TN + \pi^{\sharp}(TN^o).$

(3) $\mathrm{pr} \circ \phi : TN \to TN$ *is smooth.*

The generalized complex structure J' on N is given by

$$J' = \begin{pmatrix} \phi' & (\pi')^{\sharp} \\ \sigma'_{\flat} & -(\phi')^* \end{pmatrix}.$$

Here $\phi' = \mathrm{pr} \circ \phi|_{TN}$, π' is the induced Poisson tensor, and

$$\sigma'_{\flat}(X) = h^*(\sigma_{\flat}X - \phi^*\zeta),$$

where $\zeta \in (TN)^o$ such that $X + \zeta \in B \cap JB$, as in Lemma 4.4.

Proof. This theorem is the construction and confirmation of the properties of J'.
Proposition 4.1 combined with Lemma 4.2 shows that $J'^2 = -\mathrm{Id}$ and $J'^*J' = \mathrm{Id}$.
The smoothness of J' follows from Proposition 4.7, and the integrability of its $+i$-
eigenbundle follows from Proposition 4.8. The form of the generalized complex
structure follows from Lemma 4.4. $\square$

For the following examples let $\Omega = 0$.

Example 5.3. Let (M, j) be a complex manifold, and let N be a smooth subman-
ifold of M. There is a generalized complex structure on M given by $\phi = j$, $\sigma = 0$
and $\pi = 0$. Because the Poisson structure is zero, N is automatically a Poisson–
Dirac submanifold. Condition (2) of Theorem 5.2 becomes $j(TN) \subseteq TN$, which
is exactly the requirement for N to be an immersed complex submanifold of M.
Now $\mathrm{pr} \circ j = j|_{TN}$, which is a smooth map. Thus N is a generalized complex
submanifold if, and only if, it is an immersed complex submanifold.

Example 5.4. Let (M, ω) be a symplectic manifold and N a smooth submanifold
of M. The generalized complex structure on M arising from ω is given by $\phi = 0$,
$\sigma_{\flat} = \omega_{\flat}$ and $\pi^{\sharp} = -\omega_{\flat}^{-1}$. Because $\phi = 0$, conditions (2) and (3) of Theorem 5.2
are automatically satisfied. Now N will be a Poisson–Dirac submanifold of M if,
and only if, N is a symplectic submanifold of M. Thus N is a generalized complex
submanifold of M if, and only if, it is a symplectic submanifold.

The last result of this section is an application of Theorem 5.2 to the stable
locus of a twisted generalized complex involution. This result is similar to one for
Poisson involutions [Fernandes and Vanhaecke 2001; Xu 2003]. Let (M, Ω, J) be

a twisted generalized complex manifold. A *twisted generalized complex involution* is a diffeomorphism $\Psi : M \to M$ such that $\Psi^2 = \mathrm{Id}$ and

$$(5\text{-}1) \qquad\qquad \Psi_*^* \circ J = J \circ \Psi_*^*.$$

Here Ψ_*^* is the map from $TM \oplus T^*M$ to $TM \oplus T^*M$ defined by $\Psi_*^*(X + \xi) = \Psi_* X + \Psi^* \xi$.

Corollary 5.5. *Let (M, Ω, J) be a twisted generalized complex manifold and let Ψ be a twisted generalized complex involution of J. The fixed locus, N, of Ψ is a twisted generalized complex submanifold of M.*

Proof. Let ξ be an arbitrary element of T^*M. (5-1) implies that $(\Psi_* \pi^\sharp \Psi^*)\xi = \pi^\sharp \xi$. Hence $\Psi_* \pi = \pi$, and Ψ_* is a Poisson involution. Because Ψ_* is a Poisson involution, [Xu 2003, Proposition 4.1] implies that N is a Dirac submanifold. Thus N is a Poisson–Dirac submanifold, and condition (1) of Theorem 5.2 is satisfied.

Take $X \in TN$. Equation (5-1) implies that $\Psi_*(\phi X) + \Psi^*(\sigma_\flat X) = \phi X + \sigma_\flat X$. The vector field component of this equality proves that $\phi(TN) \subseteq TN$, and condition (2) of Theorem 5.2 is satisfied. Thus $\mathrm{pr} \circ \phi = \phi|_{TN}$, which is a smooth map. Hence condition (3) of Theorem 5.2 is satisfied. $\qquad\square$

6. Holomorphic Poisson submanifolds

Let (M, j, π) be a Poisson Nijenhuis manifold such that $j : TM \to TM$ is an integrable almost complex structure. Such a structure is equivalent to a holomorphic Poisson structure The holomorphic Poisson tensor is given by $\Pi = \pi_j + i\pi$, where $\pi_j^\sharp = \pi^\sharp \circ j^*$.

A generalized complex structure on M [Crainic 2004; Stiénon and Xu 2007] is given by

$$(6\text{-}1) \qquad\qquad J = \begin{pmatrix} j & \pi^\sharp \\ 0 & -j^* \end{pmatrix}.$$

In general, if N is a generalized complex submanifold then the induced generalized complex need not have $\sigma' = 0$.

Recall that $TN \cap \pi^\sharp(TN^o) = \{0\}$ and $\phi(TN) \subseteq TN + \pi^\sharp(TN^o)$. Thus, we can define the composition

$$TN \xrightarrow{\phi} TN \oplus \pi^\sharp(TN^o) \xrightarrow{\mathrm{pr}_2} \pi^\sharp(TN^o).$$

Proposition 6.1. *Consider the generalized complex structure (6-1), and let N be a generalized complex submanifold of M. Now, $\sigma' = 0$ if, and only if,*

$$\phi(TN) \subseteq \pi^\sharp\big((\mathrm{pr}_2 \circ \phi(TN))^o\big).$$

Proof. Take $X \in TN$. Then $\begin{pmatrix} X \\ 0 \end{pmatrix} \in TN \oplus T^*M|_N = B$. Since $B = B \cap JB + B^\perp$ and $B^\perp = TN^o$ there exists a $\zeta \in TN^o$ such that $\begin{pmatrix} X \\ \zeta \end{pmatrix} \in B \cap JB$. But then $J\begin{pmatrix} X \\ \zeta \end{pmatrix} \in B$ too, and so $\phi X + \pi^\sharp \zeta \in TN$.

In other words, given $X \in TN$ there exists $\zeta \in TN^o$ such that $\phi X + \pi^\sharp \zeta \in TN$. Since $\phi(TN) \subseteq TN \oplus \pi^\sharp(TN^o)$, this is equivalent to $\mathrm{pr}_2 \circ \phi(X) = -\pi^\sharp \zeta$. Recall that $\sigma'_\flat(X) = h^*(\sigma_\flat(X) - \phi^*\zeta)$.

Now, assume $\sigma = 0$. Then $\sigma' = 0$ if, and only if $\sigma'_\flat(X) = 0$ for all $X \in TN$. From the previous discussion, this will be true if, and only if $h^*(\phi^*\zeta) = 0$ for all $\zeta \in TN^o$ such that $\pi^\sharp \zeta \in \mathrm{pr}_2 \circ \phi(TN)$, which is equivalent to

$$(6\text{-}2) \qquad TN^o \cap (\pi^\sharp)^{-1}(\mathrm{pr}_2 \circ \phi(TN)) \subseteq (\phi^*)^{-1}(TN^o).$$

Let $A = \mathrm{pr}_2 \circ \phi(TN)$. Now,

$$(\pi^\sharp)^{-1}(A) = ((\pi^\sharp)^*)^{-1}(A) = (\pi^\sharp(A^o))^o,$$

and

$$(\phi^*)^{-1}(TN^o) = (\phi(TN))^o.$$

Hence, (6-2) becomes

$$(TN + \pi^\sharp(A^o))^o \subseteq (\phi(TN))^o,$$

which is equivalent to

$$(6\text{-}3) \qquad\qquad \phi(TN) \subseteq TN + \pi^\sharp(A^o).$$

So a generalized complex submanifold N, of a generalized complex manifold M with generalized complex structure (6-1), will have a generalized complex structure of the same form as (6-1) if and only if $\phi(TN) \subseteq TN + \pi^\sharp(A^o)$. Now, consider the following series of equivalent statements:

$$A \subseteq \pi^\sharp(TN^o),$$
$$A^o \supseteq (\pi^\sharp(TN^o))^o,$$
$$A^o \supseteq (\pi^\sharp)^{-1}(TN),$$
$$\pi^\sharp(A^o) \supseteq \pi^\sharp\big((\pi^\sharp)^{-1}(TN)\big) = TN.$$

Thus (6-3) becomes

$$\phi(TN) \subseteq \pi^\sharp(A^o). \qquad\qquad \square$$

If N is both a complex submanifold of (M, j), and a Poisson–Dirac submanifold of (M, π) then this condition will automatically be satisfied and there will be a generalized complex structure of the form (6-1) on N. Thus N will also be a holomorphic Poisson manifold.

7. Spinors and generalized complex submanifolds

Generalized complex structures may also be realized using Clifford algebras and spinors. The aim of this section is to prove that generalized complex submanifolds can also be realized using spinors. Details for the material in this section can be found in [Gualtieri 2003] and the sections of [Chevalley 1997] cited therein.

Let V be a finite dimensional vector space and let $\mathscr{C}l$ denote the Clifford algebra of $V_\mathbb{C} \oplus V_\mathbb{C}^*$. There is an action of $\mathscr{C}l$ on $\bigwedge^\bullet(V_\mathbb{C}^*)$ defined by

$$(X + \xi) \cdot \mu = i_X \mu + \xi \wedge \mu,$$

for all $X + \xi \in V_\mathbb{C} \oplus V_\mathbb{C}^*$ and $\mu \in \bigwedge^\bullet(V_\mathbb{C}^*)$. The elements of $\bigwedge^\bullet(V_\mathbb{C}^*)$ are called *spinors*. Each spinor μ has a *null space*:

$$L_\mu = \left\{ X + \xi \in V_\mathbb{C} \oplus V_\mathbb{C}^* : (X + \xi) \cdot \mu = 0 \right\}.$$

This subspace is isotropic, and if it is also maximal isotropic then the spinor is called *pure*. Using the notation of Section 2, every pure spinor can be written as

(7-1) $$\mu = c(\det(R^o)) \wedge \exp(\varepsilon),$$

where $c \in \mathbb{C}$ is nonzero, $R \subseteq V_\mathbb{C}$, and $\varepsilon \in \bigwedge^2 R^*$. It is known that pure spinors, up to multiplication by a constant, are in one to one correspondence with maximal isotropics. The maximal isotropics L_μ and $\bar{L}_\mu = L_{\bar{\mu}}$ will have trivial intersection if, and only if,

$$(\mu, \bar{\mu})_{\text{muk}} \neq 0.$$

Here $(\cdot, \cdot)_{\text{muk}}$ is the Mukai pairing.

For each $m \in M$ the previous constructions can be applied to $(T_m M \oplus T_m^* M)^\mathbb{C}$. The bundle formed by these Clifford algebras is called the *Clifford bundle*. In this context spinors are members of $\Omega_\mathbb{C}^\bullet(M)$, and their null spaces are maximal isotropic subbundles of $T_\mathbb{C} M \oplus T_\mathbb{C}^* M$. The following proposition follows from [Alekseev and Xu 2001, Theorem 6.4]. It is also proven in [Gualtieri 2003] for the untwisted case.

Proposition 7.1. *Generalized complex structures are in one to one correspondence with pure spinor line bundles $\mathscr{L} \subseteq \bigwedge^\bullet T_\mathbb{C}^* M$ such that the following hold.*

(1) *If $\mu \in \mathscr{L}$ then $(\mu, \bar{\mu})_{\text{muk}} \neq 0$.*
(2) *For any local nonzero section $\mu \in \Gamma(\mathscr{L}|_U)$ there exists a local section $X + \xi \in \mathfrak{X}_\mathbb{C}(U) \oplus \Omega_\mathbb{C}(U)$ such that $d_\Omega(\mu) = (X + \xi) \cdot \mu$. Here $d_\Omega = d + \Omega$.*

For each point $m \in M$ the spinor line $\mathscr{L}|_m$ is of the form (7-1)

Let $h : N \to M$ be a twisted submanifold with a generalized complex structure defined by a spinor line bundle $\mathscr{L} \subseteq \bigwedge^\bullet T_\mathbb{C}^* M$. This spinor line bundle naturally induces a line bundle in $\Omega_\mathbb{C}^\bullet(N)$ given by $h^* \mathscr{L}$. This induced line bundle could

potentially give a generalized complex structure on N. We will show that the maximal isotropic defined by $h^*\mathcal{L}$ is in fact $\mathcal{B}_h(L_\mathcal{L})$.

Proposition 7.2. *Let (M, J, Ω) be a twisted generalized complex manifold and let θ be the pure spinor line that also gives J. If $h : N \to M$ is a twisted generalized complex submanifold of M, with generalized complex structure J', then the spinor associated to J' is $h^*\theta$.*

Proof. Let L denote the Dirac structure associated to J. The spinor line bundle associated to L is given by

$$\mathcal{L} = \left\{ c(\det(R^o)) \wedge \exp(\varepsilon) : c \in \mathbb{C} \right\}.$$

Now,

$$h^*(c \det(R^o) \wedge \exp(\varepsilon)) = c \det((h^{-1}R)^o) \wedge \exp(h^*\varepsilon).$$

This line bundle is the same as the line bundle associated to $\mathcal{B}_h(L)$. $\square$

With this proposition it is now a simple matter to give the conditions for a twisted generalized complex submanifold in terms of spinors. The involutivity is guaranteed by Lemma 2.1.

Corollary 7.3. *Let M be a twisted generalized complex submanifold, with associated spinor line bundle $\mathcal{L}$. A twisted submanifold $h : N \to M$ is a twisted generalized complex submanifold if, and only if, $h^*\mathcal{L}$ is a pure spinor line bundle and $(h^*\mu, h^*\bar{\mu})_{\mathrm{muk}} \neq 0$ for all $\mu \in \mathcal{L}$.*

8. Generalized Kähler submanifolds

Finally we will consider submanifolds of generalized Kähler structures. A *twisted generalized Kähler structure* on M is a pair of twisted generalized complex structures $J_1, J_2 : TM \oplus T^*M \to TM \oplus T^*M$ such that

(1) J_1 and J_2 commute,
(2) $\langle X + \xi, J_1 J_2 (Y + \eta) \rangle$ is a positive definite metric.

The first proposition of this section gives a condition, in terms of the eigenbundles, for when two complex maps will commute.

Proposition 8.1. *Let W be a real vector space with two maps $\psi_1, \psi_2 : W \to W$ such that $\psi_1^2 = \psi_2^2 = -\mathrm{Id}$. Also, let L_+^k denote the $+i$-eigenbundles of these maps, and L_-^k denote the $-i$-eigenbundles. Using this notation, ψ_1 and ψ_2 commute if, and only if*

$$W_\mathbb{C} = (L_+^1 \cap L_+^2) \oplus (L_+^1 \cap L_-^2) \oplus (L_-^1 \cap L_+^2) \oplus (L_-^1 \cap L_-^2).$$

Proof. First assume the two maps commute. Because of this fact, every $w \in W_{\mathbb{C}}$ can be written as

$$w = \tfrac{1}{4}(\mathrm{Id} - i J_1)(\mathrm{Id} - i J_2)(w) + \tfrac{1}{4}(\mathrm{Id} - i J_1)(\mathrm{Id} + i J_2)(w)$$
$$+ \tfrac{1}{4}(\mathrm{Id} + i J_1)(\mathrm{Id} - i J_2)(w) + \tfrac{1}{4}(\mathrm{Id} + i J_1)(\mathrm{Id} + i J_2)(w)$$
$$:= w_+^+ + w_-^+ + w_+^- + w_-^-.$$

It is clear that $w_\bullet^\pm \in L_\pm^1$, and $w_\pm^\bullet \in L_\pm^2$. Now assume every $w \in W_{\mathbb{C}}$ can be written as $w = w_+^+ + w_-^+ + w_+^- + w_-^-$, where $w_\bullet^\pm \in L_\pm^1$ and $w_\pm^\bullet \in L_\pm^2$. Now $(\psi_2 \circ \psi_1)(w) = \psi_2(i w_+^+ + i w_-^+ - i w_+^- - i w_+^-) = -w_-^+ + w_-^+ + w_+^- - w_-^-$, and $(\psi_1 \circ \psi_2)(w) = -w_+^+ + w_-^+ + w_+^- - w_-^-$. $\square$

Let $J_1, J_2 : TM \oplus T^*M \to TM \oplus T^*M$ be two commuting bundle maps such that $J_1^2 = J_2^2 = -\mathrm{Id}$. Also, let $L_\pm^k$ denote the $+i$-eigenbundles and $-i$-eigenbundles of J_k. We also use the notation of [Section 4](#), in which $B = TN \oplus T^*M|_N$, and $s : B \to TN \oplus T^*N$. The next two lemmas relate the condition above to our conditions.

Lemma 8.2. *The following are equivalent.*

(1) $T_{\mathbb{C}}N \oplus T_{\mathbb{C}}^*N = s((L_+^1 \cap L_+^2) \cap B_{\mathbb{C}}) + s((L_+^1 \cap L_-^2) \cap B_{\mathbb{C}}) + s((L_-^1 \cap L_+^2) \cap B_{\mathbb{C}}) + s((L_-^1 \cap L_-^2) \cap B_{\mathbb{C}})$.
(2) $TN \oplus T^*N = s(B \cap J_1 B \cap J_2 B \cap J_1 J_2 B)$.
(3) $B = B \cap J_1 B \cap J_2 B \cap J_1 J_2 B + B^\perp$.
(4) $J_1 B^\perp \cap B \subseteq B^\perp$, $J_2 B^\perp \cap B \subseteq B^\perp$, *and* $B \cap J_1 J_2 B^\perp \subseteq B^\perp$.

Proof. (1) $\Longrightarrow$ (2): Every $v \in T_{\mathbb{C}}N \oplus T_{\mathbb{C}}^*N$ can be written as $v = s(\tilde{v}_+^+) + s(\tilde{v}_-^+) + s(\tilde{v}_+^-) + s(\tilde{v}_-^-)$, where $\tilde{v}_\pm^+ \in L_+^1 \cap L_\pm^2 \cap B_{\mathbb{C}}$ and $\tilde{v}_\pm^- \in L_-^1 \cap L_\pm^2 \cap B_{\mathbb{C}}$. Now let $\tilde{v} = \tilde{v}_+^+ + \tilde{v}_-^+ + \tilde{v}_+^- + \tilde{v}_-^-$, and so $v = s(\tilde{v})$ and $\tilde{v} \in B_{\mathbb{C}}$. Now $J_1(\tilde{v}) \in B_{\mathbb{C}}$, $J_2(\tilde{v}) \in B_{\mathbb{C}}$, and $J_1 J_2(\tilde{v}) \in B_{\mathbb{C}}$. Thus $\tilde{v} \in J_1(B_{\mathbb{C}})$, $\tilde{v} \in J_2(B_{\mathbb{C}})$, and $\tilde{v} \in J_1 J_2(B_{\mathbb{C}})$. Finally, taking the real parts of each of these gives (2).

(2) $\Longrightarrow$ (1): Every $v \in TN \oplus T^*N$ can be written as $v = s(\tilde{v})$ for some $\tilde{v} \in B \cap J_1 B \cap J_2 B \cap J_1 J_2 B$. Alternately $v = s(\tilde{v})$ for some $\tilde{v} \in B$ such that $J_1(\tilde{v}) \in B$, $J_2(\tilde{v}) \in B$, and $J_1 J_2(\tilde{v}) \in B$. Now we can write

$$\tilde{v} = \tfrac{1}{4}\big((\mathrm{Id} - i J_1) \circ (\mathrm{Id} - i J_2)(\tilde{v}) + (\mathrm{Id} - i J_1) \circ (\mathrm{Id} + i J_2)(\tilde{v})$$
$$+ (\mathrm{Id} + i J_1) \circ (\mathrm{Id} - J_2)(\tilde{v}) + (\mathrm{Id} + i J_1) \circ (\mathrm{Id} + i J_2)(\tilde{v})\big).$$

By definition each of these components is in the intersection of the eigenbundles, and the previous discussion shows that each of these terms is also in $B_{\mathbb{C}}$.

(2) $\Longleftrightarrow$ (3): We know $s(B) = T_{\mathbb{C}}N \oplus T_{\mathbb{C}}^*N$, and $\ker(s) = B^\perp$. Thus these two conditions are equivalent.

(3) $\Longleftrightarrow$ (4): This equivalence is a fairly straightforward calculation:

$$B = B^{\perp} + B \cap J_1 B \cap J_2 B \cap J_1 J_2 B \,,$$

if, and only if

$$B \subseteq B^{\perp} + J_1 B \cap J_2 B \cap J_1 J_2 B \,,$$

if, and only if

$$B \cap (J_1 B^{\perp} + J_2 B^{\perp} + J_1 J_2 B^{\perp}) \subseteq B^{\perp} \,,$$

if, and only if

$$B \cap J_1 B^{\perp} + B \cap J_2 B^{\perp} + B \cap J_1 J_2 B^{\perp} \subseteq B^{\perp} \,,$$

if, and only if

$$B \cap J_1 B^{\perp} \subseteq B^{\perp}, \ B \cap J_2 B^{\perp} \subseteq B^{\perp}, \ \text{and} \ B \cap J_1 J_2 B^{\perp} \subseteq B^{\perp} \,. \qquad \square$$

This last lemma strengthens the conclusions of the first statement in Lemma 8.2.

Lemma 8.3. *If N is a twisted generalized complex submanifold of (M, J_1) and (M, J_2) then the sums in expression* (1), *of the previous proposition, are direct. Also, each of the components in this expression can be rewritten as*

$$s\big((L^1_{\pm} \cap L^2_{\pm}) \cap B_{\mathbb{C}}\big) = F^1_{\pm} \cap F^2_{\pm},$$

where $F^k_{\pm} = \mathcal{B}_i(L^k_{\pm})$.

Proof. The fact that J_1 and J_2 descend to generalized complex structures on N implies that $F^k_+ \cap F^k_- = \{0\}$, and the sums must be direct. Now, by definition $s(L^k_{\pm} \cap B_{\mathbb{C}}) = F^k_{\pm}$ and it is obvious that $s((L^1_{\pm} \cap L^2_{\pm}) \cap B_{\mathbb{C}}) \subseteq F^1_{\pm} \cap F^2_{\pm}$. To see the other inclusion, consider $F^1_+ \cap F^2_+$. This subset will have zero intersection with F^1_- and F^2_-, and so it will not intersect with any of the other components. However, $F^1_+ \cap F^2_+ \subseteq T_{\mathbb{C}} N \oplus T^*_{\mathbb{C}} N$, and the fact that $T_{\mathbb{C}} N \oplus T^*_{\mathbb{C}} N$ is made up of these four components implies that $F^1_+ \cap F^2_+ \subseteq s((L^1_{\pm} \cap L^2_{\pm}) \cap B_{\mathbb{C}})$. $\qquad \square$

We are now ready to prove our last theorem, namely that these conditions are guaranteed to be satisfied by a generalized Kähler structure and so our notion of generalized complex submanifold preserves generalized Kähler structures.

Theorem 8.4. *Let N be a twisted submanifold of a generalized Kähler submanifold (M, J_1, J_2). If N is a twisted generalized complex submanifold of (M, J_1) and (M, J_2), then (N, J'_1, J'_2) is automatically a twisted generalized Kähler manifold.*

Proof. All that we need to show is $J'_1 J'_2 = J'_2 J'_1$, and the metric induced by $G' = J'_1 J'_2$ is positive definite. We start with the commutativity. By Lemma 8.3 and Proposition 8.1, if one of the equivalent conditions in Lemma 8.2 is true then J_1 and J_2 will commute. Consider condition (4) of this lemma. By assumption $J_1 B^{\perp} \cap B \subseteq B^{\perp}$ and $J_2 B^{\perp} \cap B \subseteq B^{\perp}$. All that remains is to show $B \cap J_1 J_2 B^{\perp} \subseteq B^{\perp}$.

Take $v \in B \cap J_1 J_2 B^\perp$, so $v \in B^\perp$ and $J_1 J_2 v \in B$. Thus $\langle v, J_1 J_2 v \rangle = 0$. However, by assumption this metric is positive definite and so $v = 0$. Thus $B \cap J_1 J_2 B^\perp \subseteq \{0\}$, and $B \cap J_1 J_2 B^\perp \subseteq B^\perp$ is always true. It remains to show that $J_1' J_2'$ defines a positive definite metric. Take $v \in TM \oplus T^*M$ and $\tilde{v} \in B \cap J_1 B \cap J_2 B \cap J_1 J_2 B$ such that $s(\tilde{v}) = v$. Because s does not change the inner product

$$\langle v, J_1' J_2'(v) \rangle = \langle s(\tilde{v}), J_1' J_2' s(\tilde{v}) \rangle = \langle s(\tilde{v}), s J_1 J_2(\tilde{v}) \rangle = \langle \tilde{v}, J_1 J_2(\tilde{v}) \rangle,$$

and the positive definiteness of $J_1 J_2$ implies the positive definiteness of $J_1' J_2'$. $\quad\square$

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