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CURVATURE

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We show that proper Dupin hypersurfaces M^n for $n \geq 4$ in $\mathbb{R}^{n+1}$ with n distinct principal curvatures and constant Möbius curvature cannot be parametrized by lines of curvature. For $n = 3$, up to Möbius transformations, there is a unique proper Dupin hypersurface, parametrized by lines of curvature, with three distinct principal curvatures and constant Möbius curvature. Moreover, these hypersurfaces are the only conformally flat proper Dupin hypersurfaces $M^3 \subset \mathbb{R}^4$ with three distinct principal curvatures and constant Möbius curvature.

1. Introduction

Dupin surfaces were first studied by Dupin in 1822. In the last three decades, several aspects of Dupin hypersurfaces were studied by many authors [Cecil and Chern 1989; Cecil et al. 2007; Cecil and Jensen 1998; 2000; 1980; Miyaoka 1984; 1989; Niebergall 1991; 1992; Pinkall 1981; 1985b; 1985a; Pinkall and Thorbergsson 1989; Riveros and Tenenblat 2005; Stolz 1999; Thorbergsson 1983]. A hypersurface is said to be Dupin if each principal curvature is constant along its corresponding surface of curvature. A Dupin submanifold M is said to be proper if the number g of distinct principal curvatures is constant on M.

The simplest Dupin submanifolds are the isoparametric hypersurfaces, that is, those whose principal curvatures are constant. Such a hypersurface in Euclidean space is an open subset of $\mathbb{R}^n$, of the sphere S^n, or of $S^k \times \mathbb{R}^{n-k}$, and hence $g \leq 2$. Cartan classified the isoparametric hypersurfaces of the sphere with $g \leq 3$. Besides isoparametric hypersurfaces, other special Dupin hypersurfaces are those that have constant Möbius curvature or constant Lie curvature.

In this paper, we consider those proper Dupin hypersurfaces of Euclidean space having distinct principal curvatures and constant Möbius curvature.

The property of being a proper Dupin hypersurface is invariant under Lie transformations. Therefore the classification of Dupin hypersurfaces is considered up to these transformations. The Möbius (conformal) transformations form a subgroup of the Lie sphere group, and it preserves Möbius curvatures.

Thorbergsson [1983] proved that if M is a proper, Dupin, compact embedded hypersurface then $g = 1, 2, 3, 4,$ or 6. Pinkall [1985a] showed that there are no restrictions on g if M is not compact. By using basic constructions, namely, tubes, cylinders, and rotational submanifolds, he provided hypersurfaces of the Euclidean space for any g. A proper Dupin hypersurface is said to be reducible if it is Lie equivalent to a proper Dupin hypersurface obtained by those basic constructions.

The local classification of Dupin surfaces in $\mathbb{R}^3$ says that such a surface is either totally umbilic or it is a Dupin cyclide. Pinkall [1985a] proved that if M^n is a Dupin hypersurface of $\mathbb{R}^{n+1}$ and $g = 2$, then M is conformally equivalent to an isoparametric hypersurface in S^{n+1}. Pinkall [1985b] also gave a complete classification, up to Lie equivalence, for Dupin hypersurfaces $M^3 \subset \mathbb{R}^4$ with three distinct principal curvatures. In particular, he showed that if M is irreducible, then it is locally Lie equivalent to an isoparametric hypersurface of S^4.

Niebergall [1991; 1992] and more recently Cecil and Jensen [2000] (see also [Cecil et al. 2007]) studied irreducible, proper, Dupin hypersurfaces $M^4 \subset \mathbb{R}^5$ with four distinct principal curvatures. Pinkall [1981] proved that proper Dupin hypersurfaces M^n with $g \geq 3$ that are Lie equivalent to isoparametric hypersurfaces cannot be parametrized by lines of curvature.

We will show that those proper Dupin hypersurfaces M^n in $\mathbb{R}^{n+1}$ for $n \geq 4$ having n distinct principal curvatures and constant Möbius curvature cannot be parametrized by lines of curvature. For $n = 3$, up to Möbius transformations, there is a unique proper Dupin hypersurfaces in $\mathbb{R}^4$, parametrized by lines of curvature, having three distinct principal curvatures and constant Möbius curvature.

Moreover, we obtain all the conformally flat Dupin hypersurfaces of $\mathbb{R}^4$, with constant Möbius curvature. We recall that for $n \geq 4$, the classical Cartan–Schouten theorem (see [Cartan 1917; Schouten 1921]) proves that $M^n \subset \mathbb{R}^{n+1}$ is conformally flat if and only if at least $n - 1$ of the principal curvatures coincide at each point. See Chen [1973] for characterizations of such hypersurfaces when $n \geq 4$. For compact conformally flat hypersurfaces, do Carmo, Dajczer, and Mercuri [1985] characterized their topological types, while Pinkall [1988] classified them in the conformal class. Cecil and Ryan [1980] classified conformally flat hypersurfaces with $n \geq 4$ and with the condition of being taut and hence complete. All these results used the Cartan–Schouten theorem, which does not hold in dimension 3. The classification of conformally flat hypersurfaces $M^3 \subset \mathbb{R}^4$ remains open. Such a hypersurface is said to be generic if all the principal curvatures are pairwise

distinct everywhere. Generic conformally flat hypersurfaces $M^3 \subset \mathbb{R}^4$ with additional assumptions such as constant mean curvature, constant Gauss–Kronecker curvature, or special conditions on the mean curvature vector or the metric were studied by several authors; see [Defever 2000a; 2000b; Garay 1994; Lafontaine 1988; Suyama 2005]. In this paper, we characterize the three dimensional generic conformally flat Dupin hypersurfaces with constant Möbius curvature.

In Section 3, we prove the following result.

Theorem A. *Let $M^n \subset \mathbb{R}^{n+1}$ for $n \geq 4$ be a proper Dupin hypersurface with n distinct principal curvatures and constant Möbius curvature. Then M cannot be parametrized by lines of curvature.*

Theorem A does not hold when we replace Möbius curvatures by Lie curvature. We show in Remark 3.4 that there are Dupin hypersurfaces in $\mathbb{R}^{n+1}$ for $n \geq 4$, parametrized by lines of curvature with n distinct principal curvatures and constant Lie curvature. These hypersurfaces were obtained in [Corro et al. 1999] by using Ribaucour transformations.

For $n = 3$, one has this result:

Theorem B. *Up to Möbius transformations, there is a unique proper Dupin hypersurface M^3 that is immersed in $\mathbb{R}^4$, parametrized by lines of curvature, and has three distinct principal curvatures and constant Möbius curvature.*

Theorem B follows from the classification of Möbius isoparametric hypersurfaces in S^4 obtained by Hu and Li [2005], since one can show that a Dupin hypersurface with constant Möbius curvature is precisely what is called a Möbius isoparametric hypersurface in [Li et al. 2002] when there are at least three distinct principal curvatures. However, Theorem B also follows directly from the fundamental theorem for hypersurfaces of the Euclidean space without using the theory of Möbius first and second fundamental forms introduced in [Wang 1998]. This theory was used in [Hu and Li 2005; Hu et al. 2007; Li et al. 2002] to study Möbius isoparametric hypersurfaces of the sphere. Such hypersurfaces are not necessarily Lie equivalent to isoparametric hypersurfaces of the sphere.

The surface of Theorem B is a cone over a flat torus contained on a sphere. It can be parametrized by

$$(1) \qquad X(x_1, x_2, x_3) = \frac{1}{\sqrt{c}\, e^{x_2}} \Big(\frac{\sin x_1}{\sqrt{c-1}}, \frac{-\cos x_1}{\sqrt{c-1}}, \sin x_3, -\cos x_3 \Big),$$

where $c > 1$ is a real constant. This is a reducible hypersurface, and it is not Lie equivalent to an isoparametric hypersurface.

We observe that, for Dupin hypersurfaces parametrized by lines of curvature, the condition of having constant Möbius curvature is equivalent to having all higher dimensional Laplace invariants equal to zero; see Lemma 3.2. These invariants

were introduced by Kamran and Tenenblat [1996; 1998], and they were used to study a class of Dupin hypersurfaces of $\mathbb{R}^5$ in [Riveros and Tenenblat 2005].

According to Cartan's theorem [Cartan 1917; Lafontaine 1988] and Hertrich-Jeromin [1994], a conformally flat hypersurface $M^3 \subset \mathbb{R}^4$ with three distinct principal curvatures admits a local coordinate system, called a Guichard net. This is a parametrization by lines of curvature. As a consequence of Theorem B, one gets the following result, which can also be obtained from [Hu and Li 2005].

Corollary C. *Let $M \subset \mathbb{R}^4$ be a conformally flat hypersurface with three principal curvatures that are pairwise distinct everywhere. Then M is a Dupin hypersurface with constant Möbius curvature if and only if M is parametrized by ϕX, where ϕ is a conformal transformation of $\mathbb{R}^4$ and X is the cone given by* (1).

The cone (1) is a conformally flat hypersurface of hyperbolic type in the class given by Lafontaine [1988].

In Section 2, we include basic results on Dupin hypersurfaces parametrized by lines of curvature and having distinct principal curvatures. In Section 3, we prove Theorem A, and we show that this theorem does not hold if one replaces Möbius curvature by Lie curvature.

2. Dupin hypersurfaces with distinct principal curvatures

Let Ω be a open set of $\mathbb{R}^n$ and $x = (x_1, x_2, \ldots, x_n) \in \Omega$. Let $X : \Omega \subset \mathbb{R}^n \to \mathbb{R}^{n+1}$ be a proper Dupin hypersurface parametrized by lines of curvature with distinct principal curvatures $-\lambda_i$ for $1 \leq i \leq n$ and let $N : \Omega \subset \mathbb{R}^n \to \mathbb{R}^{n+1}$ be the unit normal vector field of X. Then

$$\langle X_{,i}, X_{,j} \rangle = \delta_{ij} g_{ii}, \tag{2}$$

$$N_{,i} = \lambda_i X_{,i}, \tag{3}$$

$$\lambda_{i,i} = 0, \tag{4}$$

where $1 \leq i, j \leq n$ and the subscript ", i" denotes the derivative with respect to x_i. Moreover,

$$X_{,ij} - \Gamma_{ij}^i X_{,i} - \Gamma_{ij}^j X_{,j} = 0 \quad \text{for } 1 \leq i \neq j \leq n, \tag{5}$$

where Γ_{ij}^k are the Christoffel symbols. Taking the derivative of (3) with respect to x_j for $i \neq j$, we have $N_{ij} = \lambda_{i,j} X_{,i} + \lambda_i X_{,ij}$. Similarly, $N_{ji} = \lambda_{j,i} X_{,j} + \lambda_j X_{,ji}$. Subtracting these equations and substituting $X_{,ij}$ given by (5), it follows that

$$\Gamma_{ij}^i = \frac{\lambda_{i,j}}{\lambda_j - \lambda_i} \quad \text{for } 1 \leq i \neq j \leq n. \tag{6}$$

The Christoffel symbols in terms of the metric (2) are given by

$$(7) \qquad \Gamma_{ij}^k = 0, \qquad \Gamma_{ii}^i = \frac{g_{ii,i}}{2g_{ii}}, \qquad \Gamma_{ii}^j = -\frac{g_{ii,j}}{2g_{jj}}, \qquad \Gamma_{ij}^i = \frac{g_{ii,j}}{2g_{ii}},$$

where i, j, k are distinct.

For Dupin hypersurfaces with distinct principal curvatures, the *Möbius curvature* is defined by

$$(8) \qquad C^{ijk} = \frac{\lambda_i - \lambda_j}{\lambda_k - \lambda_j} \quad \text{for distinct } i, j, k.$$

Hence, for all distinct i, j, k, we have

$$(9) \qquad C^{ijk} = 1 - C^{ikj}, \qquad C^{jik} = 1 - \frac{1}{C^{ikj}}, \qquad C^{ijk} = \frac{1}{C^{kji}}.$$

When $n \geq 4$, we also have

$$(10) \qquad C^{ikj} = C^{iks} C^{skj} \quad \text{for distinct } i, j, k, s.$$

The *Lie curvature* for M^n is a cross ratio of principal curvatures. It is a product of Möbius curvatures defined in [Miyaoka 1989] by

$$(11) \qquad \Psi_{ijkl} = \frac{(\lambda_i - \lambda_k)(\lambda_j - \lambda_l)}{(\lambda_i - \lambda_l)(\lambda_j - \lambda_k)} \quad \text{for distinct } i, j, k, l.$$

For later use, we will obtain some of the properties of the Christoffel symbols and its derivatives and the Gauss equation for Dupin hypersurfaces parametrized by lines of curvature and having distinct principal curvatures.

It follows from (7) that

$$(12) \qquad g_{ii,j} = 2\Gamma_{ij}^i g_{ii}$$

$$(13) \qquad \Gamma_{ii}^j = -\Gamma_{ij}^i \frac{g_{ii}}{g_{jj}}.$$

for $1 \leq i \neq j \leq n$. From (6) and (4), we obtain

$$(14) \qquad \Gamma_{ij,i}^i = \Gamma_{ij,j}^j = \Gamma_{ij}^i \Gamma_{ij}^j \quad \text{for } 1 \leq i \neq j \leq n.$$

From (3) and (13) we get

$$(15) \qquad X_{,ii} = \Gamma_{ii}^i X_{,i} - \sum_{k \neq i} \Gamma_{ik}^i \frac{g_{ii}}{g_{kk}} X_{,k} - \lambda_i g_{ii} N.$$

Using the expressions (7) and (12)–(14), we obtain for $1 \leq i \neq j \leq n$ that

$$(16) \qquad \Gamma_{ii,j}^j = \frac{g_{ii}}{g_{jj}} \left(-\Gamma_{ij,j}^i + 2\Gamma_{ij}^i \Gamma_{jj}^j - 2(\Gamma_{ij}^i)^2 \right).$$

Proposition 2.1. *Let $X : \Omega \subset \mathbb{R}^n \to \mathbb{R}^{n+1}$ for $n \geq 3$ be a proper Dupin hypersurface parametrized by lines of curvature whose principal curvatures $-\lambda_i$ for $1 \leq i \leq n$ are distinct. Then the Gauss equation for the immersion X is given by*

$$(17) \qquad \lambda_i \lambda_j + \frac{L_{ji}}{g_{ii}} + \frac{L_{ij}}{g_{jj}} + \sum_{k \neq i \neq j} \frac{\Gamma_{ik}^i \Gamma_{jk}^j}{g_{kk}} = 0,$$

where $i \neq j$ and

$$(18) \qquad L_{ji} = \Gamma_{ji,i}^i + \Gamma_{ji}^j (\Gamma_{ji}^j - \Gamma_{ii}^i).$$

Proof. For a hypersurface of $\mathbb{R}^{n+1}$, we have the curvature tensor

$$R_{ijlm} = \sum_\gamma g_{i\gamma} \left(\sum_k (\Gamma_{mj}^k \Gamma_{lk}^\gamma - \Gamma_{jl}^k \Gamma_{km}^\gamma) + \Gamma_{mj,l}^\gamma - \Gamma_{jl,m}^\gamma \right).$$

On the other hand, $R_{ijij} = \lambda_i \lambda_j g_{ii} g_{jj}$. From the last two equalities, using the fact that the immersion X is a Dupin hypersurface parametrized by lines of curvature, we get

$$\lambda_i \lambda_j g_{ii} = \sum_k (\Gamma_{ii}^k \Gamma_{jk}^j - \Gamma_{ij}^k \Gamma_{ik}^j) + \Gamma_{ii,j}^j - \Gamma_{ij,i}^j.$$

Now it follows from (13) and (16) that

$$\lambda_i \lambda_j + \frac{1}{g_{ii}} (\Gamma_{ij,i}^j + \Gamma_{ij}^j (\Gamma_{ij}^j - \Gamma_{ii}^i)) + \frac{1}{g_{jj}} (\Gamma_{ij,j}^i + \Gamma_{ij}^i (\Gamma_{ij}^i - \Gamma_{jj}^j)) + \sum_{k \neq i \neq j} \frac{\Gamma_{ik}^i \Gamma_{jk}^j}{g_{kk}} = 0.$$

Therefore we obtain the Gauss equation given by (17). $\square$

3. Dupin hypersurfaces with constant Möbius curvature

In this section, we prove Theorem A, namely, that proper Dupin hypersurfaces M^n in $\mathbb{R}^{n+1}$ for $n \geq 4$ with n distinct principal curvatures and constant Möbius curvature cannot be parametrized by lines of curvature. We also show that this is not true if we replace constant Möbius curvature by constant Lie curvature; see Remark 3.4. We start proving some lemmas.

Lemma 3.1. *Suppose $M^n \subset \mathbb{R}^{n+1}$ for $n \geq 3$ is a proper Dupin hypersurface that is parametrized by lines of curvature and that has n distinct principal curvatures $-\lambda_i$. The Möbius curvatures are constant if and only if, up to a reordering of the*

indices, the principal curvatures are given by

$$
\begin{aligned}
\lambda_1 &= -(C-1)h_2 + h_3 + \sum_{r\geq 4}\left(1 - \frac{1}{C_r}\right)h_r, \\
\lambda_2 &= \left(1 - \frac{1}{C}\right)h_1 + \frac{1}{C}h_3 + \sum_{r\geq 4}h_r, \\
\lambda_3 &= h_1 + h_2 + \sum_{r\geq 4}\left(1 + \frac{1}{C_r(C-1)}\right)h_r, \\
\lambda_s &= C_s\lambda_1 + (1-C_s)\lambda_2 \quad for\ s\geq 4,
\end{aligned}
$$

(19)

where $C, C_s \in R \setminus \{0, 1\}$ for $s \geq 4$ are constants such that $C_s \neq 1/(1-C)$, $C_r \neq C_s$ for $r \neq s$, and $h_i(x_i)$ for $i = 1, \ldots, n$ are arbitrary differentiable functions.

Proof. We will first show that each λ_i is a sum of functions of separated variables. By hypothesis, all Möbius curvatures C^{ikj} are constant, and from (8), we have

(20) $\qquad \lambda_i + \left(C^{ikj} - 1\right)\lambda_k - C^{ikj}\lambda_j = 0 \quad$ for all distinct i, j, k.

Differentiating this equation with respect to x_k, x_j, we obtain $\lambda_{i,kj} = 0$. Therefore

(21) $\qquad \lambda_i = \sum_{r\neq i} f_{ir}(x_r) \quad$ for all $i = 1, \ldots, n$.

We fix indices $i = 1$, $j = 2$, and $k = 3$, and we let $C = C^{132}$. Then it follows from (20) and (21) that

$$
\sum_{r\neq 1} f_{1r}(x_r) + (C-1)\sum_{s\neq 3} f_{3s}(x_s) - C\sum_{t\neq 2} f_{2t}(x_t) = 0.
$$

Considering this expression as a sum of functions of distinct variables we have

$$
\begin{aligned}
a_1 &= (C-1)f_{31} - Cf_{21}, & a_3 &= f_{13} - Cf_{23}, \\
a_2 &= f_{12} + (C-1)f_{32}, & a_r &= f_{1r} + (C-1)f_{3r} - Cf_{2r} \quad \text{for } r \geq 4.
\end{aligned}
$$

and $\sum_{s=1}^{n} a_s = 0$. Therefore

$$
\begin{aligned}
f_{12} &= a_2 - (C-1)f_{32}, & f_{1r} &= a_r - (C-1)f_{3r} + Cf_{2r} \quad \text{for } r \geq 4, \\
f_{13} &= a_3 + Cf_{23}, & f_{21} &= (1/C)((C-1)f_{31} - a_1).
\end{aligned}
$$

Letting $a_1 = a$ in the expression above, it follows from (21) that

(22)
$$\lambda_1 = -(C-1)f_{32} + Cf_{23} - a + C\sum_{r\geq 4} f_{2r} - (C-1)\sum_{r\geq 4} f_{3r},$$

$$\lambda_2 = \frac{C-1}{C}f_{31} + \frac{1}{C}(Cf_{23} - a) + \sum_{r\geq 4} f_{2r},$$

$$\lambda_3 = f_{31} + f_{32} + \sum_{r\geq 4} f_{3r},$$

$$\lambda_s = C_s\lambda_1 + (1-C_s)\lambda_2, \; s \geq 4,$$

where $C_s = C^{s21} = (\lambda_s - \lambda_2)/(\lambda_1 - \lambda_2)$. Since all principal curvatures are distinct, it follows that $C, C_s \in R \setminus \{0, 1\}$, for all $s \geq 4$, $C_s \neq 1/(1-C)$, and $C_r \neq C_s$ for $r \geq 4$ with $r \neq s$.

Now λ_s does not depend on x_s; hence $(C-1)f_{3s} = (C-1+1/C_s)f_{2s} - b_s$ for all $s \geq 4$, where b_s is a constant. Introducing the functions

$$h_1(x_1) = f_{31}(x_1), \qquad\qquad h_3(x_3) = Cf_{23}(x_3) - a,$$
$$h_2(x_2) = f_{32}(x_2) - \sum_{s\geq 4} b_s/(C-1), \qquad h_r(x_r) = f_{2r}(x_r) \quad \text{for } r \geq 4,$$

it follows from (22) that the principal curvatures are given by (19).

Conversely, if the principal curvatures are given by (19) then one can see that

$$\lambda_1 - \lambda_2 = -\frac{C-1}{C}H, \qquad\qquad \lambda_1 - \lambda_s = \frac{(C_s-1)(C-1)}{C}H,$$

$$\lambda_1 - \lambda_3 = -H, \qquad\qquad \lambda_2 - \lambda_s = \frac{C_s(C-1)}{C}H,$$

$$\lambda_2 - \lambda_3 = -\frac{1}{C}H, \qquad\qquad \lambda_3 - \lambda_s = \frac{1+C_sC-C_s}{C}H,$$

and $\lambda_s - \lambda_t = -((C_s - C_t)(C-1)/C)H$, where

$$H = h_1 + Ch_2 - h_3 + \sum_{r\geq 4} \frac{C}{C_r(C-1)}h_r.$$

We conclude that all Möbius curvatures are constant.   □

The following lemma characterizes Dupin hypersurfaces with constant Möbius curvature in terms of the Christoffel symbols.

Lemma 3.2. *Let $M^n \subset \mathbb{R}^{n+1}$ for $n \geq 3$ be a proper Dupin hypersurface that is parametrized by lines of curvature and that has n distinct principal curvatures. The Möbius curvature is constant if and only if*

(23)
$$\Gamma^i_{ik} = \Gamma^j_{jk} \quad \text{for all distinct } i, j, k.$$

Moreover, the Möbius curvature is distinct from 0 and 1.

Proof. If the Möbius curvature is constant then $\Gamma^j_{jk} - \Gamma^i_{ik} = (\log C^{ikj})_{,k} = 0$. Conversely, if (23) holds, then for all distinct i, j, k, we have $(C^{ijk})_{,j} = 0$, $(C^{ikj})_{,k} = 0$, and $(C^{jik})_{,i} = 0$. It follows from (9) that C^{ikj} does not depend on x_i, x_k, and x_j. If $n = 3$ the theorem is proved. If $n \geq 4$, let s be any index distinct from i, j, k; then it follows from (10) that $C^{ikj} = C^{iks}C^{skj}$. Since C^{iks} and C^{skj} do not depend on x_s, we conclude that C^{ikj} does not depend on x_s for all s distinct from i, j, k. Therefore C^{ikj} is constant. Since all λ_i are distinct, we conclude that $C^{ikj} \neq 0$ and $C^{ikj} \neq 1$. $\qquad\square$

We observe that the previous lemma shows that constant Möbius curvature is equivalent to having all higher dimensional Laplace invariants equal to zero; see [Kamran and Tenenblat 1996, [1998]] for the definition of these invariants.

We will need the following lemma to prove the main result of this section.

Lemma 3.3. *Let $X : \Omega \subset \mathbb{R}^n \to \mathbb{R}^{n+1}$ for $n \geq 3$ be a proper Dupin hypersurface parametrized by lines of curvature such that all the principal curvatures $-\lambda_r$ for $1 \leq r \leq n$, are distinct. If all Möbius curvatures are constant then*

$$(24) \qquad \sqrt{g_{ii}}(\lambda_j - \lambda_i) = F_{ji}(x_i) = C^{jik}F_{ki}(x_i) \quad \text{for distinct } i, j, k,$$

where $F_{ji}(x_i)$ is a nonvanishing differentiable function of x_i. Moreover,

$$(25) \qquad \Gamma^i_{ii} = \Gamma^j_{ji} + F'_{ji}/F_{ji} \quad \text{for } i \neq j,$$

and the Gauss equation for $i \neq j$ is given by

$$(26) \qquad \lambda_i \lambda_j + \frac{\lambda_i - \lambda_j}{F^2_{ji}}\tilde{L}_{ji} + \frac{\lambda_j - \lambda_i}{F^2_{ij}}\tilde{L}_{ij} + \sum_{k \neq i \neq j} \frac{(\lambda_{i,k})^2}{F^2_{ik}} = 0,$$

where

$$(27) \qquad \tilde{L}_{ji} = \lambda_{j,ii} + \frac{(\lambda_{j,i})^2}{\lambda_i - \lambda_j} - \lambda_{j,i}\frac{F'_{ji}}{F_{ji}}.$$

Proof. It follows from the last equality of (7), that

$$\frac{\partial}{\partial x_j} \log\left(\sqrt{g_{ii}}\,|\lambda_j - \lambda_i|\right) = 0 \quad \text{for all } i \neq j.$$

Hence $\sqrt{g_{ii}}(\lambda_j - \lambda_i) = F_{ji}(\hat{x}_j)$, where $F_{ji}(\hat{x}_j)$ is independent of x_j. It follows that

$$\frac{F_{ji}(\hat{x}_j)}{\lambda_j - \lambda_i} = \frac{F_{ki}(\hat{x}_k)}{\lambda_k - \lambda_i} \quad \text{for all distinct } i, j, k.$$

Therefore $F_{ji}(\hat{x}_j) = C^{jik}F_{ki}(\hat{x}_k)$. Since all Möbius curvatures are constant, it follows that F_{ji} depends only on x_i, and (24) holds. From (7) we conclude that (25) holds.

The Gauss equation for Dupin hypersurfaces with distinct principal curvatures is given by (17). Observe that

$$\Gamma^{j}_{ji,i} = \frac{\lambda_{j,ii}}{\lambda_i - \lambda_j} + \frac{(\lambda_{j,i})^2}{(\lambda_i - \lambda_j)^2}.$$

Hence it follows from (24), (25), and (18) that $\tilde{L}_{ji} = L_{ji}(\lambda_i - \lambda_j)$ is given by (27). From Lemma 3.2, we have $\Gamma^{j}_{jk} = \Gamma^{i}_{ik}$ for all distinct i, j, k. Using (6) and the equalities above, we conclude that the Gauss equation is given by (26).   □

We now prove our main result.

Proof of Theorem A. Assume that M^n admits a parametrization $X(x_1, \ldots, x_n)$ by lines of curvature. Then the principal curvatures $-\lambda_i$ satisfy

(28) $(\lambda_i \lambda_j)_{,rs} = 0$ for all distinct i, j, r, s.

In fact, we consider Gauss Equation (26) and take its mixed derivative with respect to x_r and x_s for $r \neq s$ and distinct from i and j. Since from Lemma 3.1, we know that $\lambda_{i,jk} = 0$ for all distinct i, j, k. We conclude that (28) holds.

Case: $n \geq 5$. We will first show a contradiction in this case. Without loss of generality we can consider λ_i as in Lemma 3.1 and given by (19) in terms of n functions of one variable $h_j(x_j)$. We will prove that the product $h'_A h'_B$ equals zero for any pair $1 \leq A \neq B \leq n$ by considering three cases. The first case is $h'_r h'_s = 0$ for any $r \neq s$ with $r, s \geq 4$. The second is $h'_i h'_j = 0$ for any $i \neq j$ with $i, j \leq 3$. The third is $h'_i h'_s = 0$ for any $i \leq 3$ with $s \geq 4$.

Assume that $h'_r h'_s \neq 0$ for some $r \neq s$ with $r, s \geq 4$. By considering $(\lambda_1 \lambda_2)_{,rs} = 0$ and $(\lambda_1 \lambda_3)_{,rs} = 0$, we get $\lambda_{2,s} \lambda_{3,r} - \lambda_{2,r} \lambda_{3,s} = 0$, which is a contradiction since $C, C_s \in R \setminus \{0, 1\}$ for $s \geq 4$, $C_s \neq 1/(1 - C)$, and $C_r \neq C_s$ for $r \neq s$.

Similarly, if $h'_i h'_j \neq 0$ for $i \neq j$ and $i, j \leq 3$, then by considering $(\lambda_k \lambda_s)_{,ij} = 0$ and $(\lambda_k \lambda_r)_{,ij} = 0$ where $k \leq 3$ is such that i, j, k are distinct and $r \neq s \geq 4$, we obtain a contradiction.

Finally, if $h'_i h'_s \neq 0$ for $i \leq 3$ and $s \geq 4$, then by considering $(\lambda_j \lambda_k)_{,is} = 0$ and $(\lambda_r \lambda_k)_{,is} = 0$ where $j, k \leq 3$ are such that i, j, k are distinct and $r \neq s \geq 4$, we get again a contradiction.

Hence if $n \geq 5$, at most one function h_{j_0} is not constant, that is, $h'_i = 0$ for all $i \neq j_0$ with $1 \leq i \leq n$. Considering Gauss Equation (26) for the pairs (i, ℓ) and (i, k) such that i, ℓ, k, j_0 are distinct, we conclude that

$$\lambda_i \lambda_\ell = \lambda_i \lambda_k = -\frac{(\lambda_{i,j_0})^2}{F^2_{ij_0}},$$

which is a contradiction since all principal curvatures are pairwise distinct.

Therefore if $n \geq 5$, then all functions h_j are constant, that is, all principal curvatures are constant. This is a contradiction, since we know that there are no isoparametric hypersurfaces in Euclidean space with n distinct principal curvatures when $n \geq 3$.

Case: $n = 4$. This follows from [Hu et al. 2007] and [Pinkall 1981]. However, for the sake of completeness, we will provide a different, simple and direct proof. For $n = 4$, since λ_i are given by (19) in terms of $h_j(x_j)$, it follows from (28) that

(29)
$$\begin{array}{ll}
[(C+1)C_4 - 1]h_1' h_2' = 0, & [(2C-1)C_4 + 1]h_2' h_3' = 0, \\
[(C-2)C_4 + 2]h_1' h_3' = 0, & [(C-2)C_4 + 2]h_2' h_4' = 0, \\
[(2C-1)C_4 + 1]h_1' h_4' = 0, & [(C+1)C_4 - 1]h_3' h_4' = 0.
\end{array}$$

If $h_1' h_2' \neq 0$, then it follows from the left three equations that $C_4 = 1/(C+1)$, $C + 1 \neq 0$, and $h_3' = h_4' = 0$. Now considering the Gauss equation as in (26) for the pairs of indices $i = 3$, $j = 4$ and $i = 1$, $j = 3$, we obtain the system of equations

(30)
$$\lambda_3 \lambda_4 + \frac{(\lambda_{3,1})^2}{(F_{31}(x_1))^2} + \frac{(\lambda_{3,2})^2}{(F_{32}(x_2))^2} = 0,$$

(31)
$$\lambda_1 \lambda_3 + \frac{\lambda_1 - \lambda_3}{(F_{31}(x_1))^2} \tilde{L}_{31} + \frac{(\lambda_{3,2})^2}{(F_{32}(x_2))^2} = 0.$$

From (19), we have

$$\lambda_1 = -(C-1)h_2 + h_3 - C h_4, \qquad \lambda_3 = h_1 + h_2 + \frac{2C}{C-1} h_4,$$
$$\lambda_2 = \left(1 - \frac{1}{C}\right)h_1 + \frac{1}{C}h_3 + h_4, \qquad \lambda_4 = \frac{C-1}{C+1}\left(h_1 - h_2 - \frac{2}{C-1}h_3\right).$$

Substituting these expressions for λ_3 and λ_4 into Equation (30) and separating variables (recall that h_1 and F_{31} depend on x_1, that h_2 and F_{32} depend on x_2, and that h_3 and h_4 are constant) we get

$$\frac{C-1}{C+1}\left(h_1^2 + \frac{2}{C-1}(h_3 + C h_4)h_1 + \frac{2C}{(C-1)^2}h_3 h_4\right) + \frac{(h_1')^2}{F_{31}^2} = \frac{A}{C+1},$$
$$\frac{C-1}{C+1}\left(-h_2^2 - \frac{2}{C-1}(h_3 - C h_4)h_2 + \frac{2C}{(C-1)^2}h_3 h_4\right) + \frac{(h_2')^2}{F_{32}^2} = -\frac{A}{C+1},$$

where A is a real constant. Therefore we conclude that

$$F_{31}^2 = (C+1)\frac{(h_1')^2}{D_{31}} \quad \text{and} \quad F_{32}^2 = (C+1)\frac{(h_2')^2}{D_{32}},$$

where

$$D_{31} = -(C-1)h_1^2 - 2(h_3 + Ch_4)h_1 + A - 2Ch_3h_4/(C-1),$$
$$D_{32} = (C-1)h_2^2 - 2(h_3 - Ch_4)h_2 - A - 2Ch_3h_4/(C-1),$$

and A is a constant. We substitute these expressions in $\tilde{L}_{31}$, defined by (27) and in (31) and then take the mixed derivative with respect to x_1 and x_2. We conclude that $(1-C)h_1'h_2' = 0$, which is a contradiction.

Similarly, for each fixed pair of functions h_i, h_j with $i \neq j$ such that $h_i'h_j' \neq 0$, we consider in the system (29) those equations that involve $h_i'h_j'$, $h_i'h_k'$ and $h_i'h_s'$ where i, j, k, s are distinct. Then we get the constant C_4 given in terms of C and $h_k' = h_s' = 0$. Now, considering Gauss Equation (26) for the pairs of indices (k, s) and (i, k), we obtain two equations. The first equation will express the functions F_{ki} and F_{kj} in terms of h_i and h_j, and the second equation will give $h_i'h_j' = 0$, which is a contradiction.   □

Remark 3.4. Theorem A does not hold when we replace Möbius curvature by Lie curvature. We observe that there are Dupin hypersurfaces in $\mathbb{R}^{n+1}$ for $n \geq 4$ that are parametrized by lines of curvature and have n distinct principal curvatures and constant Lie curvature. In what follows, we will exhibit such Dupin hypersurfaces in $\mathbb{R}^{n+1}$. These hypersurfaces were obtained in [Corro et al. 1999] by applying Ribaucour transformations to a hyperplane. Consider the hypersurface in the Euclidean space $\mathbb{R}^{n+1}$ parametrized by

$$X(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0) - \frac{2\sum_{j=1}^n f_j}{\sum_{j=1}^n (f_j')^2 + b^2}(f_1', \ldots, f_n', -b).$$

Then X is a parametrized Dupin hypersurface, where $(x_1, \ldots, x_n) \in U \subset \mathbb{R}^n$ and

$$(32) \quad f_j = b_{j2}x_j^2 + b_{j1}x_j + b_{j0} \quad \text{for } b \neq 0, \ b_{j2}, b_{j1}, b_{j0} \in R \text{ and } 1 \leq j \leq n.$$

The principal curvatures of X are given by

$$(33) \qquad\qquad -\lambda_l = \frac{4bb_{l2}}{D_l} \quad \text{for } 1 \leq l \leq n,$$

where

$$(34) \qquad\qquad D_l = \sum_{j \neq l}^n (f_j')^2 - 4b_{l2}\sum_{j \neq l}^n f_j + a_l,$$

$$(35) \qquad\qquad a_l = b^2 + b_{l1}^2 - 4b_{l2}b_{l0}.$$

By introducing the notation

$$S_1 = \sum_{r=1}^{n} (f_r')^2 \quad \text{and} \quad S = \sum_{r=1}^{n} f_r,$$

expression (34) reduces to $D_l = S_1 - (f_l')^2 - 4b_{l2}(S - f_l) + a_l$. Using (32) and (35), we have $-(f_l')^2 + 4b_{l2}f_l + a_l = b^2$. Hence for each l we have

$$D_l = S_1 - 4b_{l2}S + b^2.$$

It follows from (33) that

$$\lambda_i - \lambda_k = \frac{4b(b_{k2} - b_{i2})(S_1 + b^2)}{D_i D_k}.$$

If the coefficients b_{j2} are distinct for all j, the principal curvatures have multiplicity one. The Möbius curvatures

$$C^{ikl} = \frac{\lambda_i - \lambda_k}{\lambda_l - \lambda_k} = \frac{(b_{k2} - b_{i2})Dl}{(b_{k2} - b_{l2})D_i}$$

are rational functions, and the Lie curvatures given by (11) are constant

$$\Psi_{ijkl} = \frac{(b_{i2} - b_{k2})(b_{j2} - b_{l2})}{(b_{i2} - b_{l2})(b_{j2} - b_{k2})}.$$

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