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We consider the evolution of the graph of $f : \mathbb{R}^n \to \mathbb{R}^n$ in $\mathbb{R}^n \times \mathbb{R}^n$ by the mean curvature flow. We prove that the flow exists smoothly for all time if the differential of f has a positive lower bound. Moreover, at each time, the flow remains the graph of a map f_t.

1. Introduction

The mean curvature flow deforms the initial surface in the direction of its mean curvature vector. Let $f : \Sigma_1 \to \Sigma_2$ be a smooth map, and denote the graph of f by Σ which is a submanifold of $\Sigma_1 \times \Sigma_2$ by the embedding $F = \mathrm{id} \times f$. Chen, Li, and Tian [2002], Ecker and Huisken [1989], Li and Li [2003], and Wang [2001; 2002] have studied the deformation of f by the mean curvature flow. The key idea of these papers is to consider the quantity $*\omega$, where ω is the volume form on Σ_1 and $*$ is the Hodge operator with respect to the metric induced on Σ by F. In fact $*\omega$ is the Jacobian of the projection from Σ to Σ_1, and $*\omega > 0$ if and only if Σ is a graph over Σ_1, by the implicit function theorem. In [Chen et al. 2002; Li and Li 2003; Wang 2001; 2002], the authors independently obtained the results of long time existence and convergence under the condition $*\omega > 1/\sqrt{2}$. We remark that their results depend on the choice of subspace over which Σ is written as a graph. In this article, we investigate smooth maps of $\mathbb{R}^n$. The advantage of this kind of map is that we can find a good representation of Σ via Lewy transformation. This technique is used by Yuan [2002] to prove a Bernstein theorem for special Lagrangian graphs. The difficult is that this case is noncompact. We don't know the behavior of solutions Σ_t for large time t. The surfaces Σ_t may diverge to ∞.

Our result is as follows.

Theorem 1.1. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth map and Σ be the graph of f. If $Df \geq \varepsilon$ for some $\varepsilon > 0$, then the mean curvature flow of the graph of f remains a graph and exists for all time.*

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2. Evolution equation

First we recall the evolution equation of $*\omega$ along the mean curvature flow. Let $f:\mathbb{R}^n\to\mathbb{R}^n$ be a smooth map. The graph of f is an embedded submanifold Σ in $\mathbb{R}^n\times\mathbb{R}^n$ by $F=\mathrm{id}\times f$. Let ω be the standard volume form on $\mathbb{R}^n$. Around any point $p\in\Sigma$, we choose orthonormal frames $\{e_i\}_{i=1,\dots,n}$ for $T\Sigma$ and $\{e_\alpha\}_{\alpha=n+1,\dots,2n}$ for $N\Sigma$. In the sequel, we use Roman letters $i, j, k, \dots$ for tangent indices; we use Greek letters $\alpha, \beta, \gamma, \dots$ for normal indices. We consider the quantity $*\omega$, where $*$ is Hodge operator with respect to the induced metric on Σ. Then $*\omega=\omega(e_1,\dots,e_n)=\omega(\pi_1(e_1),\dots,\pi_1(e_n))$, where $\pi_1:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ is the projection map from R^{2n} to the tangent space of the domain $\mathbb{R}^n$. In fact $*\omega$ is the Jacobian of the projection from $T\Sigma$ to the domain $\mathbb{R}^n$. By the singular value decomposition (see [Bretscher 1997]), there exist orthonormal bases $\{a_i\}_{i=1,\dots,n}$ for the domain $\mathbb{R}^n$ and $\{a_\alpha\}_{\alpha=n+1,\dots,2n}$ for the target $\mathbb{R}^n$, such that $\lambda_{i\alpha}=\langle df(a_i), a_\alpha\rangle$ is diagonal. For simplicity, we denote $\lambda_{i(n+i)}=\lambda_i$. Now $\{e_i=(1+\lambda_{i\alpha}^2)^{-1/2}(a_i+\sum_\alpha\lambda_{i\alpha}a_\alpha)\}$ forms an orthonormal basis for $T\Sigma$ and $\{e_\alpha=(1+\lambda_{i\alpha}^2)^{-1/2}(a_\alpha-\sum_i\lambda_{i\alpha}a_i)\}$ forms an orthonormal basis for $N\Sigma$. In this setting,

$$*\omega=\frac{1}{\prod_{i=1}^n(1+\lambda_i^2)}.$$

Define the second fundamental form of Σ as $h_{ij}^\alpha=\langle\nabla_{e_i}e_j, e_\alpha\rangle$. Recall that $*\omega$ satisfies the equation

$$\left(\frac{d}{dt}-\Delta\right)*\omega = *\omega\Big(\sum_{i,j,\alpha}(h_{ij}^\alpha)^2-2\sum_{k,i<j}\lambda_i\lambda_j h_{ik}^{n+i}h_{jk}^{n+j}+2\sum_{k,i<j}\lambda_i\lambda_j h_{ik}^{n+j}h_{jk}^{n+i}\Big); \tag{2-1}$$

see [Wang 2002; Chen et al. 2002; Li and Li 2003]. This formula plays the important role in these papers.

3. Long time existence

Using Equation (2-1), we now begin to prove our theorem.

Theorem 3.1. *If $Df\ge\varepsilon$ for some $\varepsilon>0$, then the mean curvature flow exists for all time.*

Proof. Step 1: The key idea is to seek a good representation of Σ via Lewy transformation such that $|\lambda_i|\le 1-\delta$ on the initial surface for some $\delta>0$. This is inspired by Yuan's work [2002]. We rotate the $(x, y)\in\mathbb{R}^n\times\mathbb{R}^n$ coordinate system to $(\bar{x},\bar{y})$ by $\pi/4$, that is, we set $\bar{x}=(x+y)/\sqrt{2}$ and $\bar{y}=(-x+y)/\sqrt{2}$. Then Σ

has a new parametrization

$$\bar{x} = \tfrac{1}{\sqrt{2}}(x + f(x)) \quad \text{and} \quad \bar{y} = \tfrac{1}{\sqrt{2}}(-x + f(x)).$$

Since $Df \geq \varepsilon$, we have

$$\begin{aligned} |\bar{x}^2 - \bar{x}^1|^2 &= \tfrac{1}{2}\big(|x^2 - x^1|^2 + 2(x^2 - x^1)\cdot(f(x^2) - f(x^1)) + |f(x^2) - f(x^1)|^2\big) \\ &\geq \tfrac{1}{2}|x^2 - x^1|^2. \end{aligned}$$

It follows that Σ is still graph over the whole $\bar{x}$ space $\mathbb{R}^n$. That means Σ has the representation $(\bar{x}, f(\bar{x}))$. Any tangent vector to Σ takes the form

$$\tfrac{1}{\sqrt{2}}((I + Df(x))e, (-I + Df(x))e),$$

where $e \in \mathbb{R}^n$. It follows that

$$D\bar{f}(\bar{x}) = (I + Df(x))^{-1}(-I + Df(x)).$$

Noting that $Df \geq \varepsilon$, we have

$$-I + \delta \leq (D\bar{f}) \leq I - \delta \quad \text{for some } \delta > 0.$$

For the sake of convenience, we still denote the eigenvalues of $d\bar{f}$ by λ_i. Now we have already shown that $|\lambda_i| \leq 1 - \delta$ for some δ. By the theorem in [Tsui and Wang 2004] we know that this condition can be preserved along the mean curvature flow. After putting this into Equation (2-1), its right side term in parentheses becomes

$$\begin{aligned} &\sum_{i,j,\alpha}(h_{ij}^{\alpha})^2 - 2\sum_{k,i<j}\lambda_i\lambda_j h_{ik}^{n+i}h_{jk}^{n+j} + 2\sum_{k,i<j}\lambda_i\lambda_j h_{ik}^{n+j}h_{jk}^{n+i} \\ &\qquad = \delta|A|^2 + (1-\delta)\sum_{i,k}(h_{ik}^{n+i})^2 + (1-\delta)\sum_{i<j,k}((h_{jk}^{n+i})^2 + (h_{ik}^{n+j})^2) \\ &\qquad\qquad - 2\sum_{k,i<j}\lambda_i\lambda_j h_{ik}^{n+i}h_{jk}^{n+j} + 2\sum_{k,i<j}\lambda_i\lambda_j h_{ik}^{n+j}h_{jk}^{n+i} \\ &\qquad \geq \delta|A|^2 + (1-\delta)\sum_{i,k}(h_{ik}^{n+i})^2 + (1-\delta)\sum_{i<j,k}((h_{jk}^{n+i})^2 + (h_{ik}^{n+j})^2) \\ &\qquad\qquad - 2(1-\delta)\sum_{i<j,k}|h_{ik}^{n+i}h_{jk}^{n+j}| - 2(1-\delta)\sum_{i<j,k}|h_{ik}^{n+j}h_{jk}^{n+i}| \\ &\qquad \geq \delta|A|^2. \end{aligned}$$

Thus we have

$$\left(\frac{d}{dt} - \Delta\right) * \omega \geq \delta|A|^2. \tag{3-1}$$

According to the maximum principle for parabolic equations, $\min_{\Sigma_t} *\omega$ is nondecreasing in time. So $*\omega$ has a positive lower bound, and this implies that Σ_t remains the graph of a map $f_t : \mathbb{R}^n \to \mathbb{R}^n$ whenever the flow smoothly exists.

Step 2: The remaining proof is routine. Fix any point (X_0, t_0). Then the backward heat kernel $\rho(F, X_0, t, t_0)$ at (X_0, t_0) is given by

$$\rho(F, X_0, t, t_0) = \frac{1}{(4\pi(t_0-t))^{n/2}} \exp\Big(\frac{-|F(x,t)-X_0|^2}{4(t_0-t)}\Big).$$

First we find a weighted monotonicity formula for $\int_{\Sigma_t}(1/*\omega)\rho(F, X_0, t, t_0)\,d\mu_t$ (see [Chen and Li 2004]). Recalling that

$$\frac{d}{dt} d\mu_t = -|H|^2 d\mu_t,$$
$$\Big(\frac{\partial}{\partial t} + \Delta\Big)\rho = -\Big(|H + \frac{(F-X_0)^\perp}{2(t_0-t)}| - |H|^2\Big),$$

and combining these equations and Equation (3-1), we get

$$\begin{aligned}
\frac{d}{dt}\int_{\Sigma_t} \frac{1}{*\omega}\rho(F, X_0, t, t_0) &\\
\le \int_{\Sigma_t} \phi\rho\Delta\frac{1}{*\omega} - \delta\int_{\Sigma_t}\frac{|A|^2}{*\omega^2}\rho - 2\int_{\Sigma_t}\frac{|\nabla *\omega|^2}{*\omega^3}\rho &\\
- \int_{\Sigma_t}\frac{1}{*\omega}\Big(\Delta\rho + \Big(\Big|H + \frac{(F-X_0)^\perp}{2(t_0-t)}\Big|^2 - |H|^2\Big)\rho\Big) &\\
- \int_{\Sigma_t}\frac{1}{*\omega}\rho|H|^2 &\\
\le -\int_{\Sigma_t}\rho\Big(\frac{2}{*\omega^3}|\nabla *\omega|^2 + \frac{1}{*\omega}\Big|H + \frac{(F-X_0)^\perp}{2(t_0-t)}\Big|^2 + \delta\frac{|A|^2}{*\omega^2}\Big) &\\
+ \int_{\Sigma_t}\Big(\rho\Delta\frac{1}{*\omega} - \frac{1}{*\omega}\Delta\rho\Big) &\\
\le -\Big(\int_{\Sigma_t}\frac{1}{*\omega}\rho(F, X_0, t, t_0)\Big|H + \frac{(F-X_0)^\perp}{2(t_0-t)}\Big|^2 d\mu_t &\\
+ \delta\int_{\Sigma_t}\frac{|A|^2}{*\omega^2}\rho(F, X_0, t, t_0)\,d\mu_t &\\
+ 2\int_{\Sigma_t}\frac{|\nabla *\omega|^2}{*\omega^3}\rho(F, X_0, t, t_0)\,d\mu_t\Big). &
\end{aligned} \tag{3-2}$$

From this we see that $\lim_{t\to t_0}\int_{\Sigma_t}(1/*\omega)\rho$ exists.

Let λ_i be positive numbers tending to ∞ as $i \to \infty$ and let F_i be the diverging sequence obtained by translating F by X_0 and then dilating parabolically by λ_i, that is, by taking $(F, t) \to (\lambda_i(F-X_0), \lambda_i^2(t-t_0))$. Denote the new time parameter by s. Then $t = t_0 + s/\lambda_i^2$. Thus,

$$F_i(x, s) = \lambda_i(F(x, t_0 + \lambda_i^{-2} s) - X_0).$$

Let $d\mu_s^i$ denote the induced volume form on Σ_s^i by F_i. Notice that $*\omega$ is invariant under the parabolic dilation. It is clear that

$$\int_{\Sigma_t} \frac{1}{*\omega}\rho(F, X_0, t, t_0)\, d\mu_t = \frac{1}{*\omega}\int_{\Sigma_s^i} \rho(F_i, 0, s, 0)\, d\mu_s^i.$$

Therefore we get

$$\begin{aligned}\frac{d}{ds}\int_{\Sigma_s^i} \frac{1}{*\omega}\rho(F_i, 0, s, 0)\, d\mu_s^i \le -\Big(\int_{\Sigma_s^i} \frac{1}{*\omega}\rho(F_i, 0, s, 0)\Big|H_i + \frac{F_i^{\perp}}{2(t_0 - t)}\Big|^2 d\mu_s^i \\ + \delta\int_{\Sigma_s^i} \frac{|A_i|^2}{*\omega^2}\rho(F_i, 0, s, 0)\, d\mu_s^i \\ + 2\int_{\Sigma_s^i} \frac{|\nabla *\omega|^2}{*\omega^3}\rho(F_i, 0, s, 0) d\mu_s^i\Big).\end{aligned}$$

Note $t_0 + \lambda_i^{-2} s \to t_0$ for any fixed s as $i \to \infty$ and $\lim_{t\to t_0} \int_{\Sigma_t} (1/*\omega)\rho d\mu_t$ exists. By the above monotonicity formula, this implies that, for any fixed s_1 and s_2,

$$\begin{aligned}0 \leftarrow \int_{\Sigma_{s_1}^i} \frac{1}{*\omega}\rho(F_i, 0, s_1, 0)\, d\mu_{s_1}^i - \int_{\Sigma_{s_2}^i} \frac{1}{*\omega}\rho(F_i, 0, s_2, 0)\, d\mu_{s_2}^i \\ \ge \int_{s_1}^{s_2}\int_{\Sigma_s^i} \frac{1}{*\omega}\rho(F_i, 0, s, 0)\Big|H_i + \frac{F_i^{\perp}}{2(t_0 - t)}\Big|^2 d\mu_s^i \\ + \delta\int_{s_1}^{s_2}\int_{\Sigma_s^i} \frac{|A_i|^2}{*\omega^2}\rho(F_i, 0, s, 0)\, d\mu_s^i \\ + 2\int_{s_1}^{s_2}\int_{\Sigma_s^i} \frac{|\nabla *\omega|^2}{*\omega^3}\rho(F_i, 0, s, 0)\, d\mu_s^i.\end{aligned}$$

Since $*\omega$ is bounded below, we have

$$\int_{s_1}^{s_2}\int_{\Sigma_s^i} |A_i|^2\rho(F_i, 0, s, 0) \to 0 \quad \text{as } i \to \infty.$$

This implies that for any compact $K \subset R^{2n}$,

$$\int_{\Sigma_{s_i}^i \cap K} |A_i|^2 \to 0 \quad \text{as } i \to \infty. \tag{3-3}$$

Now we claim that for the graphic mean curvature flow the fact that $*\omega$ has a positive lower bound implies that (X_0, t_0) is a regular point. Without loss of generality, we assume the origin 0 is on the boundary of $\Sigma^i_{s_i}$. Since

$$*\,\omega = \frac{1}{\prod_{i=1}^n (1+\lambda_i^2)} \le \frac{1}{1+\lambda_i^2} = \frac{1}{1+|df_t|^2}$$

and because $*\,\omega$ has a positive lower bound, Σ_t can be written as the graph of a map $f_t : \Sigma_1 \to \Sigma_2$ with uniformly bounded $|df_t|$. We consider the diverging sequence of f in $\mathbb{R}^{2n}$ by λ_i given by

$$\tilde{f}_i(y) = \lambda_i f_{t_0+\lambda_i^{-2} s_i}(y),$$

where $y \in \mathbb{R}^n$. It is clear that $|d\tilde{f}_i|$ is also uniformly bounded and $\lim_{i\to\infty} \tilde{f}_i(0) = 0$. By the Arzela theorem, $\tilde{f}_i \to \tilde{f}_\infty$ in C^α on any compact set. Note that by [Ilmanen 1995, inequality (29)], we have

$$|A_i| \le |\nabla d\tilde{f}_i| \le C(1+|d\tilde{f}_i|^3)|A_i|,$$

where $\nabla d\tilde{f}_i$ is measured with respect to the induced metric on $\Sigma^i_{s_i}$. From Equation (3-3) we know that

$$\int_{\Sigma^i_{s_i}\cap K} |\nabla d\tilde{f}_i|^2 \to 0 \quad \text{as } i \to \infty,$$

which implies that $\tilde{f}_i \to \tilde{f}_\infty$ in $C^\alpha \cap W^{1,2}_{loc}$ and that the second derivative of $\tilde{f}_\infty$ is 0. By [Chen and Li 2004, Main Theorem], we know that $\Sigma^i_{s_i} \to \Sigma^\infty$ and that Σ^∞ is independent of s. Therefore Σ^∞ is the graph of a linear function. Therefore

$$\lim_{i\to\infty} \int \rho(F_i, 0, s_i, 0)\, d\mu^i_{s_i} = \int \rho(F_\infty, 0, -1, 0)\, d\mu^\infty = 1,$$

This implies that

$$\begin{aligned}\lim_{t\to t_0} \int \rho(F, X_0, t, t_0) &= \lim_{i\to\infty} \int \rho(F, X_0, t_0+\lambda_i^{-2}s_i, t_0)\\ &= \lim_{i\to\infty} \int \rho(F_i, 0, s_i, 0)\, d\mu^i_{s_i} = 1. \end{aligned} \tag{3-4}$$

White's regularity theorem [2002] tells us that if $\lim_{t\to t_0} \rho(F, X_0, t, t_0) \le 1$, then (X_0, t_0) is a regular point. Thus (3-4) tells us that (X_0, t_0) is a regular point. We have proved the theorem. □

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