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We derive the variation formula of the $\bar\partial$-energy and of the ∂-energy for a smooth map from a complex Finsler manifold to an Hermitian manifold. Applying the result on a nonlinear elliptic system due to J. Jost and S. T. Yau, we obtain some existence theorems of harmonic maps from strongly Kähler Finsler manifolds to Kähler manifolds. Also, for such maps, we show that the difference between ∂-energy and $\bar\partial$-energy is a homotopy invariant.

1. Introduction

Complex Finsler manifolds are complex manifolds with complex Finsler metrics, which are more general than Hermitian metrics. S. Kobayashi [1975] gave two good reasons for considering complex Finsler structures in a complex manifold. One is that every hyperbolic complex manifold M carries a natural complex Finsler metric in a broad sense. The other is their use as a differential geometric tool for the study of complex vector bundles.

Harmonic maps are important and interesting in both differential geometry and mathematical physics. By using the volume measure induced from the projective sphere bundle, harmonic maps between real Finsler manifolds were investigated in [He and Shen 2005; 2007; Mo 2001; Mo and Yang 2005; Shen and Zhang 2004]. X. Mo [2001] considered the energy functional and the Euler–Lagrange operator of a smooth map from a real Finsler manifold to a Riemannian manifold. Mo and Yang [2005] also gave an existence theorem of harmonic maps from a real Finsler manifold to a Riemannian manifold. In [Shen and Zhang 2004], the second author and Y. Zhang obtained the first and second variation formulas of harmonic maps between two real Finsler manifolds. As an application of the variation formulas, He and Shen [2005; 2007] showed some stability and rigidity results of harmonic maps for real Finsler manifolds. Recently, Nishikawa [2004] studied the harmonic maps from a compact Riemann surface into complex Finsler manifolds by considering the $\bar\partial$-energy.

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In this paper, we shall study harmonic maps from complex Finsler manifolds to Hermitian manifolds — in particular, to Kähler manifolds — by virtue of the $\bar{\partial}$-energy. Of course, the harmonic maps are defined as the critical point of the $\bar{\partial}$-energy (or ∂-energy). By calculating the first variation and applying a result of J. Jost and S. T. Yau [1993] to a nonlinear elliptic system, we shall give some existence theorems on harmonic maps from strongly Kähler Finsler manifolds to Kähler manifolds. Precisely, we prove the following.

Theorem 1.1. *Let (M, G) be a compact strongly Kähler Finsler manifold, and let (N, H) be a compact Kähler manifold with negative sectional curvature. Let $\psi : M \to N$ be continuous, and suppose that ψ is not homotopic to a map onto a closed geodesic of N. Then there exists a harmonic map $\phi : M \to N$ homotopic to ψ.*

Moreover, we also prove a striking theorem, which was found by A. Lichnerowicz [1968/1969] for the case of the smooth maps between Kähler manifolds:

Theorem 1.2. *Let (M, G) be a compact strongly Kähler Finsler manifold and (N, H) be a Kähler manifold. If $\phi : (M, G) \to (N, H)$ is a smooth map, then the difference $K(\phi)$ between ∂-energy and $\bar{\partial}$-energy of ϕ is a smooth homotopy invariant.*

Some technical terms in above will be explained below. The contents of the paper are arranged as follows. Section 2 gives some fundamental definitions and formulas which are necessary for the present paper. In Section 3, we establish the first variation formula of $\bar{\partial}$-energy functional for a smooth map from a complex Finsler manifold to a Hermitian manifold. Section 4 shows some existence theorems of harmonic maps from a compact strongly Kähler Finsler manifold to a compact Kähler manifold. Finally, in Section 5, we derive the first variation formula of ∂-energy functional, obtain the homotopy invariant theorem, and give some of its applications.

2. Preliminaries

Let M be a complex manifold of complex dimension m. Denote the holomorphic tangent bundle of M by $\pi : T^{1,0}M \to M$. For a local complex coordinate system $z = (z^1, \dots, z^m)$ on M, a holomorphic tangent vector v of M is written as

$$v = v^i \partial_i, \qquad \partial_i := \frac{\partial}{\partial z^i}, \qquad \dot{\partial}_j := \frac{\partial}{\partial v^j},$$

and we may take $(z, v) = (z^1, \dots, z^m, v^1, \dots, v^m)$ as a local coordinate system for $T^{1,0}M$. Throughout this paper and unless otherwise stated, we shall fix the index ranges as $1 \le i, j, k, \dots \le m$ and $1 \le \alpha, \beta, \gamma, \dots \le n$.

Let $\widetilde{M} = T^{1,0}M \setminus \{0\}$ denote $T^{1,0}M$ without the zero section. $\{\partial_i, \dot{\partial}_j = \partial/\partial v^j\}$ gives a local holomorphic frame field of the holomorphic tangent bundle $T^{1,0}\widetilde{M}$ of $\widetilde{M}$.

Definition 2.1 [Abate and Patrizio 1994]. A complex Finsler metric on M is an upper continuous function $F : T^{1,0}M \to \mathbb{R}^+$ that satisfies the conditions

(1) $G = F^2(z, v) \in C^\infty(\widetilde{M})$, that is, G is smooth in $\widetilde{M}$;

(2) $G(z, v) \geq 0$, where the equality holds if and only if $v = 0$;

(3) $G(z, \lambda v) = |\lambda|^2 G(z, v)$ for all $(z, v) \in T^{1,0}M$ and $\lambda \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

The pair $(M, G = F^2)$ is called a *complex Finsler manifold*. A complex Finsler metric F is said to be *strongly pseudoconvex* if the complex Hessian

$$(G_{i\bar{j}}) = (\dot{\partial}_i \dot{\partial}_{\bar{j}} G)$$

of G is positively definite on $\widetilde{M}$. In particular, if $G(z, v) = h_{i\bar{j}}(z) v^i \bar{v}^j$ is a Hermitian metric on M, then $G(z, v)$ defines a strongly pseudoconvex Finsler metric on M. All complex Finsler metrics considered in the present paper are always strongly pseudoconvex unless otherwise stated.

Let $\tilde{\pi} : T^{1,0}\widetilde{M} \to \widetilde{M}$ denote the natural projection. The differential $d\pi : T^{\mathbb{C}}\widetilde{M} \to T^{\mathbb{C}}M$ of $\pi : \widetilde{M} \to M$ defines the vertical bundle $\mathcal{V}$ over $\widetilde{M}$ by

$$\mathcal{V} = \ker d\pi \cap T^{1,0}\widetilde{M},$$

which yields a holomorphic vector bundle of rank m over $\widetilde{M}$. A local frame field of $\mathcal{V}$ is given by $\{\dot{\partial}_j\}$. As is described in [Abate and Patrizio 1994], there is another horizontal subbundle $\mathcal{H}$ over $\widetilde{M}$ such that $T^{1,0}\widetilde{M} = \mathcal{V} \oplus \mathcal{H}$ and whose local frame field is $\{\delta_i\}$ given by

$$\delta_i = \partial_i - N_i^j \dot{\partial}_j \quad \text{and} \quad N_j^i = G^{i\bar{l}} G_{\bar{l}, j} := G^{i\bar{l}} \dot{\partial}_{\bar{l}} \partial_j G, \tag{2-1}$$

where $(G^{i\bar{l}}) = (G_{i\bar{l}})^{-1}$. Thus we get a local frame field $\{\delta_i, \dot{\partial}_i\}$ of $T^{1,0}\widetilde{M}$. Let $\{dz^i, \delta v^i\}$ denote the dual frame field of $\{\delta_i, \dot{\partial}_i\}$, where

$$\delta v^i = dv^i + N_j^i dz^j.$$

Associated with the decomposition $T^{1,0}\widetilde{M} = \mathcal{V} \oplus \mathcal{H}$, a Hermitian metric $h_{\widetilde{M}}$ on $\widetilde{M}$ canonically associated with G is defined by requiring $\mathcal{H}$ to be orthogonal to $\mathcal{V}$, so that $h_{\widetilde{M}}$ is given in local coordinates by

$$h_{\widetilde{M}} = G_{i\bar{j}}(z, v) dz^i \otimes d\bar{z}^j + G_{i\bar{j}}(z, v) \delta v^i \otimes \delta \bar{v}^j. \tag{2-2}$$

For a complex Finsler metric on a complex manifold M, there is a unique Chern–Finsler connection ${}^c\nabla$, for which the connection form (ω_j^i) can be written as [Abate and Patrizio 1994]

$$\omega^i_j = G^{\bar{k}i}\partial G_{j\bar{k}} = \Gamma^i_{j,k}dz^k + \gamma^i_{jk}\delta v^k, \tag{2-3}$$

where

$$\Gamma^i_{j,k} = G^{\bar{l}i}\delta_k G_{j\bar{l}} \quad \text{and} \quad \gamma^i_{jk} = G^{\bar{l}i}\dot{\partial}_k G_{j\bar{l}}. \tag{2-4}$$

The differential operator d on functions is decomposed as $d = d_{\mathcal{H}} + d_{\mathcal{V}}$. We also decompose $d_{\mathcal{H}}$ and $d_{\mathcal{V}}$ into (1, 0)-parts and (0, 1)-parts as

$$d_{\mathcal{H}} = \partial_{\mathcal{H}} + \bar{\partial}_{\mathcal{H}} \quad \text{and} \quad d_{\mathcal{V}} = \partial_{\mathcal{V}} + \bar{\partial}_{\mathcal{V}}, \tag{2-5}$$

respectively, where we put $\partial_{\mathcal{H}} f = (\delta f/\delta z^i)dz^i$ and $\partial_{\mathcal{V}} f = (\partial f/\partial v^i)\delta v^i$ for a C^∞ function $f(z, v)$ on TM.

Definition 2.2 [Kobayashi 1975]. A complex Finsler metric $G = F^2$ is said to be *strongly Kähler* if $\Gamma^i_{j,k} = \Gamma^i_{k,j}$.

Equation (2-5) and Definition 2.2 lead to another definition:

Definition 2.3 [Aikou 1991]. Let (M, G) be a complex Finsler manifold. The fundamental form associated with G is $\Phi = \sqrt{-1}G_{i\bar{j}}dz^i \wedge d\bar{z}^j$, which is a real (1, 1) form on $\widetilde{M}$. (M, G) is called a Finsler–Kähler manifold if $d_{\mathcal{H}}\Phi = 0$.

The curvature form of the Chern–Finsler connection is given by $\Omega = (\Omega^i_j) = (\bar{\partial}\omega^i_j)$, which can be written as

$$\Omega^i_j = R^i_{j,k\bar{l}}dz^k \wedge d\bar{z}^l + R^i_{j\bar{l},k}dz^k \wedge d\bar{v}^l + R^i_{jk,\bar{l}}dv^k \wedge d\bar{z}^l + R^i_{jk\bar{l}}dv^k \wedge d\bar{v}^l, \tag{2-6}$$

where

$$R^i_{j,k\bar{l}} = -\delta_{\bar{l}}(\Gamma^i_{j,k}) - \Gamma^i_{jm}\delta_{\bar{l}}(\Gamma^m_{,k}), \qquad R^i_{j\bar{l},k} = -\dot{\partial}_{\bar{l}}(\Gamma^i_{j,k}) - \Gamma^i_{jm}\Gamma^m_{\bar{l},k},$$
$$R^i_{jk,\bar{l}} = -\delta_{\bar{l}}(\Gamma^i_{jk}) = R^i_{kj,\bar{l}}, \qquad R^i_{jk\bar{l}} = -\dot{\partial}_{\bar{l}}(\Gamma^i_{jk}).$$

Setting $R_{i\bar{j},k\bar{l}} = G_{m\bar{j}}R^m_{i,k\bar{l}}$ and so on and using the (1,1)-homogeneity of G, we have

$$\begin{aligned} R_{i\bar{j},k\bar{l}}v^i v^{\bar{j}} &= (-G_{i\bar{j},k\bar{l}} + G^{p\bar{q}}G_{p\bar{j},\bar{l}}G_{\bar{q}i,k})v^i\bar{v}^j \\ &= -G_{,k\bar{l}} + G^{p\bar{q}}G_{p,\bar{l}}G_{\bar{q},k} := R_{k\bar{l}}, \end{aligned} \tag{2-7}$$

and $R_{i\bar{j}k\bar{l}}v^i = R_{i\bar{j}k,\bar{l}}v^i = R_{i\bar{j}\bar{l},k}v^i\bar{v}^j = 0$.

The *projective tangent bundle* $P\widetilde{M}$ of M is defined by $P\widetilde{M} := \widetilde{M}/\mathbb{C}^*$, which has a natural Hermitian metric (2-2) with local homogeneous coordinates (z^i, v^i). The invariant volume form of $P\widetilde{M}$ is (see [Zhong and Zhong 2004])

$$d\mu_{P\widetilde{M}} = \frac{\omega^{m-1}_{\mathcal{V}}}{(m-1)!} \wedge \frac{\omega^m_{\mathcal{H}}}{m!}, \tag{2-8}$$

where $\omega_{\mathcal{V}} = \sqrt{-1}(\ln G)_{i\bar{j}}\delta v^i \wedge \delta\bar{v}^j$ and $\omega_{\mathcal{H}} = \sqrt{-1}G_{i\bar{j}}dz^i \wedge d\bar{z}^j$. The advantage of working on $P\widetilde{M}$ rather than $\widetilde{M}$ is that $P\widetilde{M}$ is compact when M is compact.

Lemma 2.1 [Bland and Kalka 1996]. *Let (M, G) be a complex Finsler manifold. Then*

$$\sqrt{-1}\partial\bar{\partial}\ln G = \sqrt{-1}(\ln G)_{i\bar{j}}\delta v^i \wedge \delta\bar{v}^j + \sqrt{-1}G^{-1}(G_{;k\bar{l}} - G_{\overline{m};k}G^{\overline{m}n}G_{n;\bar{l}})dz^k \wedge d\bar{z}^l = \omega_{\mathcal{V}} + \kappa,$$

where $\kappa := \sqrt{-1}G^{-1}(G_{;k\bar{l}} - G_{\overline{m};k}G^{\overline{m}n}G_{n;\bar{l}})dz^k \wedge d\bar{z}^l$.

Using Lemma 2.1 the volume form of $P\widetilde{M}$ can also be written as

$$d\mu_{P\widetilde{M}} = \frac{(\sqrt{-1}\partial\bar{\partial}\ln G)^{m-1}}{(m-1)!} \wedge \frac{\omega_{\mathcal{H}}^m}{m!}. \tag{2-9}$$

If we denote by $d\sigma$ the pure vertical forms of $(\partial\bar{\partial}G)^{m-1}/(m-1)!$, that is,

$$d\sigma = \frac{(\sqrt{-1}(\ln G)_{i\bar{j}}\delta v^i \wedge \delta\bar{v}^j)^{m-1}}{(m-1)!}, \tag{2-10}$$

then

$$d\mu_{P\widetilde{M}} = d\sigma \wedge \frac{\omega_{\mathcal{H}}^m}{m!} = \det(G_{i\bar{j}})\,d\sigma \wedge dz, \tag{2-11}$$

where $dz = (\sqrt{-1}\sum_{i=1}^m dz^i \wedge d\bar{z}^i)^m$.

3. $\bar{\partial}$-energy and the first variation

Let (M, G) be a complex manifold of dimension m with strongly pseudoconvex Finsler metrics G, and let (N, H) be a Hermitian manifold of complex dimension n. Let $\phi : M \to N$ be a smooth map from M to N. We denote the local holomorphic coordinate systems for M by $\{z^i\}$ and for N by $\{w^\alpha\}$, and express ϕ locally as

$$w^\alpha = \phi^\alpha(z^1, \dots, z^m, \bar{z}^1, \dots, \bar{z}^m) \quad \text{for } 1 \le \alpha, \beta, \dots \le n.$$

As is well known, the differential $d\phi : TM \to TN$ of ϕ extends to a complex linear map between the complexified tangent bundles $T^{\mathbb{C}}M$ and $T^{\mathbb{C}}N$. According to the decompositions

$$T^{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M \quad \text{and} \quad T^{\mathbb{C}}N = T^{1,0}N \oplus T^{0,1}N,$$

we obtain

$$d\phi|_{T^{1,0}M} = \partial\phi + \partial\bar{\phi} \quad \text{and} \quad d\phi|_{T^{0,1}M} = \bar{\partial}\phi + \bar{\partial}\bar{\phi}, \tag{3-1}$$

where

$$\partial\phi : T^{1,0}M \to T^{1,0}N, \quad \bar\partial\phi : T^{0,1}M \to T^{1,0}N, \quad \bar\partial\bar\phi = \overline{\partial\phi}, \quad \partial\bar\phi = \overline{\bar\partial\phi},$$

which are expressed in local coordinates by

$$\partial\phi = \phi_i^\alpha dz^i \otimes \frac{\partial}{\partial w^\alpha} \quad \text{and} \quad \bar\partial\phi = \phi_{\bar i}^\alpha d\bar z^i \otimes \frac{\partial}{\partial w^\alpha}, \tag{3-2}$$

where

$$\phi_i^\alpha = \frac{\partial\phi^\alpha}{\partial z^i} \quad \text{and} \quad \phi_{\bar i}^\alpha = \frac{\partial\phi^\alpha}{\partial \bar z^i}.$$

Obviously, ϕ is holomorphic (respectively antiholomorphic) if and only if $\bar\partial\phi = 0$ (respectively $\partial\phi = 0$).

We set $\widetilde N = T^{1,0}N \setminus \{0\}$ as well as $\widetilde M$. Then G (respectively H) on $\widetilde M$ (respectively $\widetilde N$) can be expressed as

$$G(z, v) = G_{i\bar j}(z, v)v^i\bar v^j \quad \text{and} \quad H(w) = H_{\alpha\bar\beta}(w)d\omega^\alpha d\bar\omega^\beta.$$

Then the $\bar\partial$-energy density of ϕ can be defined naturally by

$$|\bar\partial\phi|^2(z, v) = G^{\bar i j}(z, v)\phi_{\bar i}^\alpha \phi_j^{\bar\beta} H_{\alpha\bar\beta}(\phi(z)). \tag{3-3}$$

By means of the volume measure (2-8) of the projective tangent bundle $P\widetilde M$, we can define the $\bar\partial$-*energy* of ϕ by

$$E_{\bar\partial}(\phi) = \frac{1}{c_M}\int_{P\widetilde M} |\bar\partial\phi|^2 d\mu_{P\widetilde M},$$

where c_M is the standard volume of the $(m-1)$-dimensional complex projective space $\mathbb{C}P^{m-1}$.

We now consider a *smooth variation* of $\phi = \phi_0$ via a family of smooth maps

$$\phi_t : M \to N \quad \text{for } t \in \mathfrak{D} = \{z \in \mathbb{C} \mid |z| < \varepsilon\}.$$

Then the first variation of the $\bar\partial$-energy functional is

$$\frac{\partial}{\partial t}E_{\bar\partial}(\phi_t)\Big|_{t=0} = \frac{1}{c_M}\int_{P\widetilde M}\Big(\frac{\partial}{\partial t}|\bar\partial\phi_t|^2\Big)\Big|_{t=0} d\mu_{P\widetilde M}. \tag{3-4}$$

The variation $\{\phi_t\}$ induces a vector field on the pull-back bundle $\phi_t^{-1}T^{\mathbb{C}}N$:

$$V := d\phi_t\Big(\frac{\partial}{\partial t}\Big) = \partial\phi_t\Big(\frac{\partial}{\partial t}\Big) + \partial\bar\phi_t\Big(\frac{\partial}{\partial t}\Big) = V' + V'', \tag{3-5}$$

with

$$V' = \frac{\partial\phi_t^\alpha}{\partial t}\frac{\partial}{\partial w^\alpha} \quad \text{and} \quad V'' = \frac{\partial\bar\phi_t^\alpha}{\partial t}\frac{\partial}{\partial \bar w^\alpha}.$$

This, valued at $t = 0$, is called the *variation vector field* and is denoted $V_0 := V|_{t=0}$.

On putting [Nishikawa 2004]

$$U = d\phi_t\Big(\frac{\partial}{\partial \bar{t}}\Big) = \partial\bar{\phi}_t\Big(\frac{\partial}{\partial \bar{t}}\Big) + \bar{\partial}\bar{\phi}_t\Big(\frac{\partial}{\partial \bar{t}}\Big) := U' + U'',$$
$$T_i = d\phi_t\Big(\frac{\partial}{\partial z^i}\Big) = \partial\phi_t\Big(\frac{\partial}{\partial z^i}\Big) + \partial\bar{\phi}_t\Big(\frac{\partial}{\partial z^i}\Big) := T_i' + T_i'', \tag{3-6}$$
$$S_i = d\phi_t\Big(\frac{\partial}{\partial \bar{z}^i}\Big) = \bar{\partial}\phi_t\Big(\frac{\partial}{\partial \bar{z}^i}\Big) + \bar{\partial}\bar{\phi}_t\Big(\frac{\partial}{\partial \bar{z}^i}\Big) := S_i' + S_i'',$$

we have from (3-3)

$$\begin{aligned} |\bar{\partial}\phi_t|^2 = G^{\bar{i}j}(z,v)\phi^\alpha_{t\bar{i}}\phi^{\bar{\beta}}_{tj}H_{\alpha\bar{\beta}}(\phi(z)) &= G^{\bar{i}j}\langle \phi^\alpha_{t\bar{i}}\delta_\alpha, \phi^\beta_{t\bar{j}}\delta_\beta\rangle_N \\ &= G^{\bar{i}j}\langle (\phi^\alpha_{t\bar{i}}\partial_\alpha)^{\mathcal{H}}, (\phi^\beta_{t\bar{j}}\partial_\beta)^{\mathcal{H}}\rangle_N \\ &= G^{\bar{i}j}\langle S_i'^{\mathcal{H}}, S_j'^{\mathcal{H}}\rangle_N, \end{aligned} \tag{3-7}$$

where $\langle\cdot,\cdot\rangle_N$ is the Hermitian inner product in the pull-back bundle $\phi_t^{-1}T^{1,0}\widetilde{N}$. By means of (3-5) and (3-6) we have

$$\frac{\partial}{\partial t}|\bar{\partial}\phi_t|^2 = G^{\bar{i}j}\frac{\partial}{\partial t}\langle S_i'^{\mathcal{H}}, S_j'^{\mathcal{H}}\rangle_N.$$

Thus,

$$\begin{aligned} \frac{\partial}{\partial t}\langle S_i'^{\mathcal{H}}, S_j'^{\mathcal{H}}\rangle_N &= V^{\mathcal{H}}\langle S_i'^{\mathcal{H}}, S_j'^{\mathcal{H}}\rangle_N \\ &= \langle \nabla_{V^{\mathcal{H}}}S_i'^{\mathcal{H}}, S_j'^{\mathcal{H}}\rangle_N + \langle S_i'^{\mathcal{H}}, \nabla_{\overline{V^{\mathcal{H}}}}S_j'^{\mathcal{H}}\rangle_N \\ &= \langle \nabla_{S_i'^{\mathcal{H}}}V'^{\mathcal{H}} + [V'^{\mathcal{H}}, S_i'^{\mathcal{H}}] + \theta(V'^{\mathcal{H}}, S_j'^{\mathcal{H}}), S_j'^{\mathcal{H}}\rangle_N \\ &\qquad + \langle \nabla_{V''^{\mathcal{H}}}S_i'^{\mathcal{H}}, S_j'^{\mathcal{H}}\rangle_N + \langle S_i'^{\mathcal{H}}, \nabla_{\overline{V^{\mathcal{H}}}}S_j'^{\mathcal{H}}\rangle_N. \end{aligned} \tag{3-8}$$

On the other hand, it is easy to obtain

$$[V'^{\mathcal{H}}, S_i'^{\mathcal{H}}] = \Big[\frac{\partial\phi^\alpha_t}{\partial t}\delta_\alpha, \frac{\partial\phi^\beta_t}{\partial\bar{z}^i}\delta_\beta\Big] = \Big(\frac{\partial\phi^\alpha_t}{\partial t}\delta_\alpha\Big(\frac{\partial\phi^\beta_t}{\partial\bar{z}^i}\Big) - \frac{\partial\phi^\alpha_t}{\partial\bar{z}^i}\delta_\alpha\Big(\frac{\partial\phi^\beta_t}{\partial t}\Big)\Big)\delta_\beta,$$
$$\nabla_{V''^{\mathcal{H}}}S_i'^{\mathcal{H}} = \frac{\partial\bar{\phi}^\alpha_t}{\partial t}\delta_{\bar{\alpha}}(\partial_{\bar{i}}\phi^\beta_t)\delta_\beta.$$

Combining these two equations yields that

$$[V'^{\mathcal{H}}, S_i'^{\mathcal{H}}] + \nabla_{V''^{\mathcal{H}}}S_i'^{\mathcal{H}} = \nabla_{S_i''^{\mathcal{H}}}V'^{\mathcal{H}}. \tag{3-9}$$

Similarly, we also have

$$[\overline{V''^{\mathcal{H}}}, S_j'^{\mathcal{H}}] + \nabla_{\overline{V'^{\mathcal{H}}}}S_j'^{\mathcal{H}} = \nabla_{S_j''^{\mathcal{H}}}\overline{V''^{\mathcal{H}}}. \tag{3-10}$$

Substituting (3-9) and (3-10) into (3-8), we get

$$\begin{aligned}(3\text{-}11)\quad \frac{\partial}{\partial t}\langle S_i'^{\mathcal{H}}, S_j'^{\mathcal{H}}\rangle_N &= S_i^{\mathcal{H}}\langle V'^{\mathcal{H}}, S_j'^{\mathcal{H}}\rangle_N - \langle V'^{\mathcal{H}}, \nabla_{\overline{S_i^{\mathcal{H}}}} S_j'^{\mathcal{H}}\rangle_N \\ &+ \overline{S_j^{\mathcal{H}}}\langle S_i'^{\mathcal{H}}, \overline{V''^{\mathcal{H}}}\rangle_N - \langle \nabla_{\overline{S_j^{\mathcal{H}}}} S_i'^{\mathcal{H}}, \overline{V''^{\mathcal{H}}}\rangle_N \\ &+ \langle \theta(V'^{\mathcal{H}}, S_i'^{\mathcal{H}}), S_j'^{\mathcal{H}}\rangle_N + \langle S_i'^{\mathcal{H}}, \theta(\overline{V''^{\mathcal{H}}}, S_j'^{\mathcal{H}})\rangle_N.\end{aligned}$$

Substituting (3-11) into (3-4) yields

$$\begin{aligned}(3\text{-}12)\quad \frac{\partial}{\partial t}E_{\bar\partial}(\phi_t) = \frac{1}{c_M}\int_{P\widetilde{M}} G^{\bar i j}\big\{&S_i^{\mathcal{H}}\langle V'^{\mathcal{H}}, S_j'^{\mathcal{H}}\rangle_N + \overline{S_j^{\mathcal{H}}}\langle S_i'^{\mathcal{H}}, \overline{V''^{\mathcal{H}}}\rangle_N \\ &-\langle V'^{\mathcal{H}}, \nabla_{\overline{S_i^{\mathcal{H}}}} S_j'^{\mathcal{H}}\rangle_N - \langle \nabla_{\overline{S_j^{\mathcal{H}}}} S_i'^{\mathcal{H}}, \overline{V''^{\mathcal{H}}}\rangle_N \\ &+ \langle \theta(V'^{\mathcal{H}}, S_i'^{\mathcal{H}}), S_j'^{\mathcal{H}}\rangle_N + \langle S_i'^{\mathcal{H}}, \theta(\overline{V''^{\mathcal{H}}}, S_j'^{\mathcal{H}})\rangle_N\big\}d\mu_{P\widetilde{M}}.\end{aligned}$$

In the following, we denote the canonical connection coefficient for M by ${}^M\Gamma$ and for N by ${}^N\Gamma$.

Let $\widetilde{\Psi} = G^{\bar i j}\langle V'^{\mathcal{H}}, S_j'^{\mathcal{H}}\rangle_N \mathcal{I}_{\delta_{\bar i}}(d\mu_{P\widetilde{M}})$ be an $(m, m-1)$-form, where $\mathcal{I}_{\delta_{\bar i}}$ denotes the inner differential operator with respect to the vector $\delta_{\bar i}$. We have

$$d\widetilde{\Psi} = G^{\bar i j}\{S_i^{\mathcal{H}}\langle V'^{\mathcal{H}}, S_j'^{\mathcal{H}}\rangle_N - \langle V'^{\mathcal{H}}, S_j'^{\mathcal{H}}\rangle_N({}^M\Gamma^{\bar k}_{\bar i,\bar k} - {}^M\Gamma^{\bar k}_{\bar k,\bar i})\}d\mu_{P\widetilde{M}},$$

so that, by Stokes' formula,

$$\int_{P\widetilde{M}} G^{\bar i j} S_i^{\mathcal{H}}\langle V'^{\mathcal{H}}, S_j'^{\mathcal{H}}\rangle_N d\mu_{P\widetilde{M}} = \int_{P\widetilde{M}} G^{\bar i j}\langle V'^{\mathcal{H}}, S_j'^{\mathcal{H}}\rangle_N({}^M\Gamma^{\bar k}_{\bar i,\bar k} - {}^M\Gamma^{\bar k}_{\bar k,\bar i})d\mu_{P\widetilde{M}},$$

if M is compact without boundary. Since

$$\langle V'^{\mathcal{H}}, S_j'^{\mathcal{H}}\rangle_N = V'^{\alpha}\frac{\partial\overline{\phi_t^{\beta}}}{\partial z^j}H_{\alpha\bar\beta},$$

we then obtain

$$\begin{aligned}(3\text{-}13)\quad &\int_{P\widetilde{M}} G^{\bar i j} S_i^{\mathcal{H}}\langle V'^{\mathcal{H}}, S_j'^{\mathcal{H}}\rangle_N d\mu_{P\widetilde{M}} \\ &\qquad = \int_{P\widetilde{M}} G^{\bar i j} V'^{\alpha}\frac{\partial\overline{\phi_t^{\beta}}}{\partial z^j}({}^M\Gamma^{\bar k}_{\bar i,\bar k} - {}^M\Gamma^{\bar k}_{\bar k,\bar i})H_{\alpha\bar\beta}d\mu_{P\widetilde{M}}.\end{aligned}$$

Similarly, we can also get

$$\begin{aligned}(3\text{-}14)\quad &\int_{P\widetilde{M}} G^{\bar i j}\overline{S_j^{\mathcal{H}}}\langle S_i'^{\mathcal{H}}, \overline{V''^{\mathcal{H}}}\rangle_N d\mu_{P\widetilde{M}} \\ &\qquad = \int_{P\widetilde{M}} G^{\bar i j}\frac{\partial\phi_t^{\alpha}}{\partial\bar z^i}V''^{\beta}({}^M\Gamma^{k}_{j,k} - {}^M\Gamma^{k}_{k,j})H_{\alpha\bar\beta}d\mu_{P\widetilde{M}}.\end{aligned}$$

On the other hand, by the definitions of V and S_i, it is easy to see

$$\langle V'^{\mathcal{H}}, \nabla_{\overline{S_i^{\mathcal{H}}}} S_j'^{\mathcal{H}} \rangle_N = V'^{\alpha}\Big(\frac{\partial^2 \overline{\phi_t^{\beta}}}{\partial \bar{z}^i \partial z^j} + \frac{\partial \overline{\phi_t^{\gamma}}}{\partial \bar{z}^i} \frac{\partial \overline{\phi_t^{\sigma}}}{\partial z^j} {}^N\Gamma^{\bar\beta}_{\bar\gamma,\bar\sigma} \Big) H_{\alpha\bar\beta}, \tag{3-15}$$

$$\langle \nabla_{\overline{S_j^{\mathcal{H}}}} S_i'^{\mathcal{H}}, \overline{V''^{\mathcal{H}}} \rangle_N = V''^{\beta}\Big(\frac{\partial^2 \phi_t^{\alpha}}{\partial \bar{z}^i \partial z^j} + \frac{\partial \phi_t^{\gamma}}{\partial \bar{z}^i} \frac{\partial \phi_t^{\sigma}}{\partial z^j} {}^N\Gamma^{\alpha}_{\gamma,\sigma} \Big) H_{\alpha\bar\beta}. \tag{3-16}$$

Recall that [Abate and Patrizio 1994]

$$\langle \theta(V'^{\mathcal{H}}, S_i'^{\mathcal{H}}), S_j'^{\mathcal{H}} \rangle_N = \langle V'^{\mathcal{H}}, \theta^*(S_j'^{\mathcal{H}}, \overline{S_i'^{\mathcal{H}}}) \rangle_N,$$

where $\theta^* = \frac{1}{2} H^{\bar\sigma\alpha} H_{\beta\bar\gamma} ({}^N\Gamma^{\bar\gamma}_{\bar\rho,\bar\sigma} - {}^N\Gamma^{\bar\gamma}_{\bar\sigma,\bar\rho}) dw^{\beta} \wedge d\overline{w^{\rho}} \otimes \delta_{\alpha}$.

We then obtain

$$\begin{aligned} \langle \theta(V'^{\mathcal{H}}, S_i'^{\mathcal{H}}), S_j'^{\mathcal{H}} \rangle_N &= \langle V'^{\mathcal{H}}, \theta^*(S_j'^{\mathcal{H}}, \overline{S_i'^{\mathcal{H}}}) \rangle_N \\ &= \frac{1}{2} \frac{\partial \phi_t^{\alpha}}{\partial t} H^{\bar\beta\delta} H_{\bar\sigma\gamma} \big({}^N\Gamma^{\gamma}_{\rho,\delta} - {}^N\Gamma^{\gamma}_{\delta,\rho} \big) \frac{\partial \phi_t^{\rho}}{\partial \bar{z}^i} \frac{\partial \overline{\phi_t^{\sigma}}}{\partial z^j} H_{\alpha\bar\beta}. \end{aligned} \tag{3-17}$$

Similarly, we can also get

$$\begin{aligned} \langle S_i'^{\mathcal{H}}, \theta(\overline{V''^{\mathcal{H}}}, S_j'^{\mathcal{H}}) \rangle_N &= \langle \theta^*(S_i'^{\mathcal{H}}, \overline{S_j'^{\mathcal{H}}}), \overline{V''^{\mathcal{H}}} \rangle_N \\ &= \frac{1}{2} \frac{\partial \overline{\phi_t^{\beta}}}{\partial t} H^{\bar\delta\alpha} H_{\sigma\bar\gamma} \big({}^N\Gamma^{\bar\gamma}_{\bar\rho,\bar\delta} - {}^N\Gamma^{\bar\gamma}_{\bar\delta,\bar\rho} \big) \frac{\partial \phi_t^{\sigma}}{\partial \bar{z}^i} \frac{\partial \bar\phi_t^{\rho}}{\partial z^j} H_{\alpha\bar\beta}. \end{aligned} \tag{3-18}$$

Inserting (3-13)–(3-18) into (3-12), we have the following result:

Theorem 3.1. *Let (M, G) be a compact complex Finsler manifold and (N, H) be a Hermitian manifold. Let $\phi : M \to N$ be a smooth map from M to N. Then the first variation of $\bar\partial$-energy functional is*

$$\frac{\partial}{\partial t} E_{\bar\partial}(\phi_t)\Big|_{t=0} = -\frac{1}{c_M} \int_{P\widetilde{M}} \big(\overline{V_0^{\beta}} Q^{\alpha} H_{\alpha\bar\beta} + V_0^{\alpha} \overline{Q^{\beta}} H_{\alpha\bar\beta} \big) d\mu_{P\widetilde{M}},$$

where

$$\begin{aligned} Q^{\alpha} = G^{\bar i j} \big(({}^M\Gamma^l_{l;j} - {}^M\Gamma^l_{j;l}) \phi^{\alpha}_{\bar i} + \phi^{\alpha}_{\bar i j} + {}^N\Gamma^{\alpha}_{\sigma,\rho} \phi^{\sigma}_{\bar i} \phi^{\rho}_{j} \\ - \tfrac{1}{2} H^{\bar\delta\alpha} H_{\sigma\bar\gamma} \big({}^N\Gamma^{\bar\gamma}_{\bar\rho,\bar\delta} - {}^N\Gamma^{\bar\gamma}_{\bar\delta,\bar\rho} \big) \phi^{\sigma}_{\bar i} \phi^{\bar\rho}_{j} \big). \end{aligned}$$

By Definition 2.2, we have immediately a corollary:

Corollary 3.1. *Let (M, G) be a compact strongly Kähler Finsler manifold and (N, H) be a Hermitian manifold. Let $\phi : M \to N$ be a smooth map from M to N. Then the first variation of $\bar\partial$-energy functional is*

$$\frac{\partial}{\partial t} E_{\bar\partial}(\phi_t)\Big|_{t=0} = -\frac{1}{c_M} \int_{P\widetilde{M}} \big(\overline{V_0^{\beta}} \Xi^{\alpha} H_{\alpha\bar\beta} + V_0^{\alpha} \overline{\Xi^{\beta}} H_{\alpha\bar\beta} \big) d\mu_{P\widetilde{M}},$$

where

$$\Xi^\alpha = G^{\bar{i}j}\big(\phi^\alpha_{\bar{i}j} + {}^N\Gamma^\alpha_{\sigma\rho}\phi^\sigma_{\bar{i}}\phi^\rho_j - \tfrac{1}{2}H^{\bar{\delta}\alpha}H_{\sigma\bar{\gamma}}\big({}^N\Gamma^{\bar{\gamma}}_{\bar{\rho},\bar{\delta}} - {}^N\Gamma^{\bar{\gamma}}_{\bar{\delta},\bar{\rho}}\big)\phi^\sigma_{\bar{i}}\phi^{\bar{\rho}}_{\bar{j}}\big). \tag{3-19}$$

It is well known that a *harmonic* map is critical point of the first variation of the energy functional. Let

$$\|Q\| := \sup_{V_0\in\mathcal{C}(\phi^{-1}T^{\mathbb{C}}N)}\left\{\frac{\left|\int_{P\widetilde{M}}\overline{V_0^\beta}\,Q^\alpha H_{\alpha\bar{\beta}}\,d\mu_{P\widetilde{M}}\right|}{\|V_0\|}\right\}.$$

Using this, we can make the following definition:

Definition 3.1. ϕ is harmonic if and only if $\|Q\| \equiv 0$; ϕ is said to be strongly harmonic if and only if $Q^\alpha = 0$.

From this definition, we see that a holomorphic (respectively antiholomorphic) map is a (strongly) harmonic map. Also we remark that strong harmonicity implies harmonicity.

4. Existence theorem

The basic problem for harmonic maps can be formulated in the following manner: Let $\phi_0 : M \to N$ be a map between two manifolds M and N. Can ϕ_0 be deformed into a harmonic map $\phi : M \to N$? Mo and Yang [2005] gave an existence theorem of harmonic maps from a real Finsler manifold to a Riemannian manifold. On the other hand, J. Jost and Yau [1993] introduced and studied a nonlinear elliptic system of equations imposed on a map from a Hermitian manifold M into a Riemannian manifold N. In local coordinates, the system is

$$\frac{1}{2}\frac{\partial}{\partial\bar{z}^\beta}\Big(\gamma^{\alpha\bar{\beta}}\frac{\partial f^i}{\partial z^\alpha}\Big) + \frac{1}{2}\frac{\partial}{\partial z^\alpha}\Big(\gamma^{\alpha\bar{\beta}}\frac{\partial f^i}{\partial\bar{z}^\beta}\Big) + \gamma^{\alpha\bar{\beta}}\Gamma^i_{jk}\frac{\partial f^j}{\partial z^\alpha}\frac{\partial f^k}{\partial\bar{z}^\beta} = 0, \tag{4-1}$$

where $\gamma_{\alpha\bar{\beta}}$ is the Hermitian metric of M, Γ^i_{jk} are the Christoffel symbols of N, and $\alpha, \beta, \ldots = 1, \ldots, \dim M$, $i, j, \ldots = 1, \ldots, \dim N$. A disadvantage of this system is that a holomorphic map need not be harmonic unless M is Kähler. So (4-1) is replaced by

$$\gamma^{\alpha\bar{\beta}}\Big(\frac{\partial^2 f^i}{\partial z^\alpha\partial\bar{z}^\beta} + \Gamma^i_{jk}\frac{\partial f^j}{\partial z^\alpha}\frac{\partial f^k}{\partial\bar{z}^\beta}\Big) = 0, \tag{4-2}$$

which is equivalent to (4-1) if M is Kähler. Some existence theorems on the solutions of the system (4-2) were given in [Jost and Yau 1993]. Of course, the act of taking variations for the system has no meaning unless M is Kähler.

In this section we give the existence theorem of harmonic maps from a compact strongly Kähler Finsler manifold to a compact Kähler manifold by means of the result of [Jost and Yau 1993].

First of all, from the formula (2-11) we have a lemma.

Lemma 4.1. *If M is a compact complex Finsler manifold, then for any function $f : P\widetilde{M} \to R$, we have*

$$\int_{P\widetilde{M}} f d\mu_{P\widetilde{M}} = \int_M dz \int_{P_z\widetilde{M}} f \det(G_{i\bar{j}}) d\sigma.$$

Lemma 4.2. *Let G be a complex Finsler metric on M. Put*

$$\gamma^{\bar{i}j}(z) := \frac{\int_{P_z\widetilde{M}} G^{\bar{i}j}(z,v)\det(G_{k\bar{l}}(z,v))d\sigma}{\int_{P_z\widetilde{M}} \det(G_{k\bar{l}}(z,v))d\sigma}.$$

Then $g = \gamma_{i\bar{j}} dz^i \otimes dz^{\bar{j}}$ is a Hermitian metric on M, where $\gamma^{\bar{i}j} = (\gamma_{i\bar{j}})^{-1}$.

Proof. It is easy to see that $(\gamma^{\bar{i}j})$ is a positive definite matrix. Thus we only need to check that g is independent of the local holomorphic coordinate system (U, z^i). Suppose there is another local holomorphic coordinate system $(V, \tilde{z}^i)$, and $U \cap V \neq \varnothing$. Then on $U \cap V$, from $G_{i\bar{j}} dz^i d\bar{z}^j = \widetilde{G}_{i\bar{j}} d\tilde{z}^i d\bar{\tilde{z}}^j$ and $d\tilde{z}^j = (\partial\tilde{z}^j/\partial z^i)dz^i$, we have a series of implications:

$$\widetilde{G}_{i\bar{j}} = G_{k\bar{l}} \frac{\partial\tilde{z}^k}{\partial z^i}\frac{\partial\bar{\tilde{z}}^l}{\partial\bar{z}^j} \quad\Rightarrow\quad \gamma^{\bar{i}j} = \tilde{\gamma}^{\bar{k}l}\frac{\partial\bar{z}^i}{\partial\bar{\tilde{z}}^k}\frac{\partial z^j}{\partial\tilde{z}^l} \quad\Rightarrow\quad \gamma_{i\bar{j}} = \tilde{\gamma}_{k\bar{l}}\frac{\partial z^k}{\partial\tilde{z}^i}\frac{\partial\bar{z}^l}{\partial\bar{\tilde{z}}^j},$$

that is, $\tilde{g}$ is a Hermitian metric. ☐

Let (M, G) be a compact strongly Kähler Finsler manifold and (N, H) be a compact Kähler manifold. Let $\phi : M \to N$ be a smooth map from M to N. By Theorem 3.1, we know that ϕ is a harmonic map if and only if

$$\int_{P\widetilde{M}} G^{\bar{i}j}(\phi^\alpha_{\bar{i}j} + {}^N\Gamma^\alpha_{\sigma,\rho}\phi^\sigma_{\bar{i}}\phi^\rho_j)\overline{V_0^\beta} H_{\alpha\bar{\beta}} d\mu_{P\widetilde{M}} = 0. \tag{4-3}$$

for any variation field V_0. By Lemmas 4.1 and 4.2, Equation (4-3) can be written as

$$\int_M \gamma^{\bar{i}j}(z)(\phi^\alpha_{\bar{i}j} + {}^N\Gamma^\alpha_{\sigma,\rho}\phi^\sigma_{\bar{i}}\phi^\rho_j)\overline{V_0^\beta} H_{\alpha\bar{\beta}}\sigma(z)dz = 0, \tag{4-4}$$

where $\sigma(z) := \int_{P_z\widetilde{M}} \det(G_{k\bar{l}}(z,v))d\sigma$ and $\gamma^{\bar{i}j}$ is a Hermitian metric defined in Lemma 4.2. Since the field V_0 is arbitrary, we see that ϕ is harmonic if and only if

$$\gamma^{\bar{i}j}(z)(\phi^\alpha_{\bar{i}j} + {}^N\Gamma^\alpha_{\sigma,\rho}\phi^\sigma_{\bar{i}}\phi^\rho_j) = 0. \tag{4-5}$$

Let $\phi^\alpha = f^\alpha + \sqrt{-1} f^{n+\alpha}$. Then (4-5) can be reduced as

$$\gamma^{\bar{i}j}(z)\Big(\frac{\partial^2 f^\alpha}{\partial \bar{z}^i \partial z^j} + {}^N\widetilde{\Gamma}^\alpha_{A,B}\frac{\partial f^A}{\partial \bar{z}^i}\frac{\partial f^B}{\partial z^j}\Big) + \sqrt{-1}\gamma^{\bar{i}j}(z)\Big(\frac{\partial^2 f^{n+\alpha}}{\partial \bar{z}^i \partial z^j} + {}^N\widetilde{\Gamma}^{n+\alpha}_{A,B}\frac{\partial f^A}{\partial \bar{z}^i}\frac{\partial f^B}{\partial z^j}\Big) = 0, \tag{4-6}$$

where $1 \le A, B, C, \ldots \le 2n$. Hence, (4-6) is equivalent to

$$\gamma^{\bar{i}j}(z)\Big(\frac{\partial^2 f^C}{\partial \bar{z}^i \partial z^j} + {}^N\widetilde{\Gamma}^C_{A,B}\frac{\partial f^A}{\partial \bar{z}^i}\frac{\partial f^B}{\partial z^j}\Big) = 0. \tag{4-7}$$

Comparing (4-7) with (4-2) and using the existence results of [Jost and Yau 1993], we have immediately the following theorems.

Theorem 4.1. *Suppose (M, G) is a compact strongly Kähler Finsler manifold and (N, H) is a compact Kähler manifold with negative sectional curvature. Suppose $\psi : M \to N$ is continuous and ψ is not homotopic to a map onto a closed geodesic of N. Then there exists a harmonic map $\phi : M \to N$ homotopic to ψ.*

Theorem 4.2. *Let (M, G) be a compact strongly Kähler Finsler manifold, and let (N, H) be a compact Kähler manifold with nonpositive sectional curvature. Let $\psi : M \to N$ be smooth, and suppose $\mathscr{E}(g^*TN) \neq 0$, where $\mathscr{E}$ is the Euler class. Then there exists a harmonic map f homotopic to ψ.*

We can also get the other existence theorems as in [Jost and Yau 1993].

5. ∂-energy and homotopy invariant

Let (M, G) be a complex Finsler manifold of dimension m, and let (N, H) be a Hermitian manifold of complex dimension n. Let $\phi : M \to N$ be a smooth map from M to N. As defined in Section 3, we have the partial energy densities of ϕ as the following squares of complex norms:

$$\begin{aligned} e'(\phi) &= |\partial\phi|^2(z, v) = G^{i\bar{j}}(z, v)\phi^\alpha_i \phi^{\bar{\beta}}_{\bar{j}} H_{\alpha\bar{\beta}}(\phi(z)),\\ e''(\phi) &= |\bar{\partial}\phi|^2(z, v) = G^{\bar{i}j}(z, v)\phi^\alpha_{\bar{i}} \phi^{\bar{\beta}}_{j} H_{\alpha\bar{\beta}}(\phi(z)), \end{aligned} \tag{5-1}$$

where ϕ^α_i (respectively $\phi^\alpha_{\bar{j}}$) is the matrix representation of $\partial\phi$ (respectively $\bar{\partial}\phi$) in the chosen local frame fields. We then have $e(\phi) = e'(\phi) + e''(\phi)$.

By means of the volume measure (2-8) of the projective tangent bundle $P\widetilde{M}$, we can define the ∂-*energy* and $\bar{\partial}$-*energy* of ϕ respectively by

$$E'(\phi) \equiv E_\partial(\phi) = \frac{1}{c_M}\int_{P\widetilde{M}} |\partial\phi|^2 d\mu_{P\widetilde{M}},$$
$$E''(\phi) \equiv E_{\bar{\partial}}(\phi) = \frac{1}{c_M}\int_{P\widetilde{M}} |\bar{\partial}\phi|^2 d\mu_{P\widetilde{M}},$$

where c_M is the standard volume of the $(m-1)$-dimensional complex projective space $\mathbb{C}P^{m-1}$. We also have $E(\phi) = E'(\phi) + E''(\phi)$.

Obviously, ϕ is holomorphic if and only if $E_{\bar\partial} = 0$, and antiholomorphic if and only if $E_\partial = 0$.

Set

$$k(\phi) = e'(\phi) - e''(\phi) \quad \text{and} \quad K(\phi) = E'(\phi) - E''(\phi). \tag{5-2}$$

In Section 3, we have obtained

$$\begin{aligned}\frac{\partial}{\partial t} E_{\bar\partial}(\phi_t) &= \frac{1}{c_M}\int_{P\widetilde{M}} \frac{\partial e''(\phi_t)}{\partial t} d\mu_{P\widetilde{M}} \\ &= \frac{1}{c_M}\int_{P\widetilde{M}} G^{\bar i j}\langle V'^{\mathcal{H}}, ({}^M\Gamma^k_{i,k} - {}^M\Gamma^k_{k,i}) S'^{\mathcal{H}}_j - \nabla_{\overline{S^{\mathcal{H}}_i}} S'^{\mathcal{H}}_j + \theta^*(S'^{\mathcal{H}}_j, \overline{S'^{\mathcal{H}}_i})\rangle_N d\mu_{P\widetilde{M}} \\ &\quad + \frac{1}{c_M}\int_{P\widetilde{M}} G^{\bar i j}\langle ({}^M\Gamma^k_{j,k} - {}^M\Gamma^k_{k,j}) S'^{\mathcal{H}}_i - \nabla_{\overline{S^{\mathcal{H}}_j}} S'^{\mathcal{H}}_i + \theta^*(S'^{\mathcal{H}}_i, \overline{S'^{\mathcal{H}}_j}), U'^{\mathcal{H}}\rangle_N d\mu_{P\widetilde{M}}\end{aligned} \tag{5-3}$$

for a smooth variation $\{\phi_t : M \to N\}$.

Similarly, we have

$$\frac{\partial}{\partial t} E_\partial(\phi_t) = \frac{1}{c_M}\int_{P\widetilde{M}} \frac{\partial e'(\phi_t)}{\partial t} d\mu_{P\widetilde{M}}. \tag{5-4}$$

From (5-1) and (3-6) we have

$$\frac{\partial e'(\phi_t)}{\partial t} = \frac{\partial}{\partial t}|\partial\phi_t|^2 = G^{i\bar j}\frac{\partial}{\partial t}\langle T'^{\mathcal{H}}_i, T'^{\mathcal{H}}_j\rangle_N.$$

Since

$$\begin{aligned}\frac{\partial}{\partial t}\langle T'^{\mathcal{H}}_i, T'^{\mathcal{H}}_j\rangle_N &= V^{\mathcal{H}}\langle T'^{\mathcal{H}}_i, T'^{\mathcal{H}}_j\rangle_N \\ &= \langle \nabla_{V^{\mathcal{H}}} T'^{\mathcal{H}}_i, T'^{\mathcal{H}}_j\rangle_N + \langle T'^{\mathcal{H}}_i, \nabla_{\overline{V^{\mathcal{H}}}} T'^{\mathcal{H}}_j\rangle_N,\end{aligned}$$

we can get

$$\begin{aligned}\frac{\partial}{\partial t} E_\partial(\phi_t) &= \\ \frac{1}{c_M}\int_{P\widetilde{M}} G^{i\bar j}\big(&\langle V'^{\mathcal{H}}, ({}^M\Gamma^{\bar k}_{\bar i,\bar k} - {}^M\Gamma^{\bar k}_{\bar k,\bar i}) T'^{\mathcal{H}}_j - \nabla_{\overline{T^{\mathcal{H}}_i}} T'^{\mathcal{H}}_j + \theta^*(T'^{\mathcal{H}}_j, \overline{T'^{\mathcal{H}}_i})\rangle_N \\ &+ \langle ({}^M\Gamma^{\bar k}_{\bar j,\bar k} - {}^M\Gamma^{\bar k}_{\bar k,\bar j}) T'^{\mathcal{H}}_i - \nabla_{\overline{T^{\mathcal{H}}_j}} T'^{\mathcal{H}}_i + \theta^*(T'^{\mathcal{H}}_i, \overline{T'^{\mathcal{H}}_j}), U'^{\mathcal{H}}\rangle_N\big) d\mu_{P\widetilde{M}}\end{aligned} \tag{5-5}$$

by means of the similar calculation in Section 3.

In the following, suppose that (M, G) is a compact strongly Kähler Finsler manifold and (N, H) is a Kähler manifold. Thus (5-3) and (5-5) can be reduced respectively to

$$\frac{\partial}{\partial t} E_{\bar\partial}(\phi_t) = -\frac{1}{c_M}\int_{P\widetilde{M}} G^{\bar i j}\big(\langle V'^{\mathcal{H}}, \nabla_{\overline{S^{\mathcal{H}}_i}} S'^{\mathcal{H}}_j\rangle_N + \langle \nabla_{\overline{S^{\mathcal{H}}_j}} S'^{\mathcal{H}}_i, U'^{\mathcal{H}}\rangle_N\big) d\mu_{P\widetilde{M}} \tag{5-6}$$

and

$$\frac{\partial}{\partial t}E_{\partial}(\phi_t) = -\frac{1}{c_M}\int_{P\widetilde{M}} G^{i\bar{j}}\big(\langle V'^{\mathcal{H}}, \nabla_{\overline{T_i^{\mathcal{H}}}} T_j'^{\mathcal{H}}\rangle_N + \langle \nabla_{\overline{T_j^{\mathcal{H}}}} T_i'^{\mathcal{H}}, U'^{\mathcal{H}}\rangle_N\big) d\mu_{P\widetilde{M}}. \tag{5-7}$$

In local coordinates, (5-6) and (5-7) can be expressed as

$$\begin{aligned}\frac{\partial}{\partial t}E_{\bar{\partial}}(\phi_t) = -\frac{1}{c_M}\int_{P\widetilde{M}} G^{\bar{i}j}\Big(\frac{\partial \phi_t^\alpha}{\partial t}\Big(\frac{\partial^2 \overline{\phi_t^\beta}}{\partial \bar{z}^i \partial z^j} + \frac{\partial \overline{\phi_t^\gamma}}{\partial \bar{z}^i}\frac{\partial \overline{\phi_t^\sigma}}{\partial z^j}{}^N\Gamma^{\bar{\beta}}_{\bar{\gamma},\bar{\sigma}}\Big)\\ + \frac{\partial \overline{\phi_t^\beta}}{\partial t}\Big(\frac{\partial^2 \phi_t^\alpha}{\partial \bar{z}^i \partial z^j} + \frac{\partial \phi_t^\gamma}{\partial \bar{z}^i}\frac{\partial \phi_t^\sigma}{\partial z^j}{}^N\Gamma^{\alpha}_{\gamma,\sigma}\Big)\Big) H_{\alpha\bar{\beta}}\, d\mu_{P\widetilde{M}}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial t}E_{\partial}(\phi_t) = -\frac{1}{c_M}\int_{P\widetilde{M}} G^{i\bar{j}}\Big(\frac{\partial \phi_t^\alpha}{\partial t}\Big(\frac{\partial^2 \overline{\phi_t^\beta}}{\partial z^i \partial \bar{z}^j} + \frac{\partial \overline{\phi_t^\gamma}}{\partial z^i}\frac{\partial \overline{\phi_t^\sigma}}{\partial \bar{z}^j}{}^N\Gamma^{\bar{\beta}}_{\bar{\gamma},\bar{\sigma}}\Big)\\ + \frac{\partial \overline{\phi_t^\beta}}{\partial t}\Big(\frac{\partial^2 \phi_t^\alpha}{\partial z^i \partial \bar{z}^j} + \frac{\partial \phi_t^\gamma}{\partial z^i}\frac{\partial \phi_t^\sigma}{\partial \bar{z}^j}{}^N\Gamma^{\alpha}_{\gamma,\sigma}\Big)\Big) H_{\alpha\bar{\beta}}\, d\mu_{P\widetilde{M}}.\end{aligned}$$

From the last two equations, it follows that

$$\frac{d}{dt}K(\phi_t) = \frac{\partial}{\partial t}E_{\partial}(\phi_t) - \frac{\partial}{\partial t}E_{\bar{\partial}}(\phi_t) = 0. \tag{5-8}$$

This proves the following theorem.

Theorem 5.1. *Suppose (M, G) is a compact strongly Kähler Finsler manifold and (N, H) is a usual Kähler manifold. Then $K(\phi)$ is a smooth homotopy invariant, that is, it is constant on the connected components of the space $\mathcal{C}(M, N)$ of all smooth maps from M to N.*

Remark. When (M, G) is the usual Kähler manifold, this striking theorem due to Lichnerowicz [1968/1969] was proved in a different way by Eells and Lemaire [1983].

By (5-8), we can get

$$\frac{\partial}{\partial t}E_{\partial}(\phi_t) = \frac{\partial}{\partial t}E_{\bar{\partial}}(\phi_t) = \frac{1}{2}\frac{\partial}{\partial t}E(\phi_t).$$

Thus we have a corollary:

Corollary 5.1. *The E'-, E''-, and E-critical points coincide. Furthermore, in a given homotopy class the E'-, E''-, and E-minima coincide.*

Proof. For the second statement, we note that for ϕ and ϕ_0 in the same homotopy class, $E'(\phi) - E'(\phi_0) = E''(\phi) - E''(\phi_0)$. Thus, if $E'(\phi_0) \le E'(\phi)$ for all ϕ, then $E''(\phi_0) \le E''(\phi)$ for all ϕ.

Similarly, since $E(\phi) = K(\phi) + 2E''(\phi)$, we get $E(\phi) - 2E'(\phi) = E(\phi_0) - 2E''(\phi_0)$ or $E(\phi) - E(\phi_0) = 2E''(\phi) - 2E''(\phi_0)$, so the minima also coincide. □

Corollary 5.2. *If ϕ is a holomorphic or antiholomorphic map from a compact strongly Kähler Finsler manifold to a Kähler manifold, then it is a harmonic map and an absolute minimum of E in its homotopy class.*

Proof. Let $\phi_t : M \times [0, 1] \to N$ be a family of smooth maps satisfying $\phi_0 = \phi$. If ϕ is a holomorphic map, then $E''(\phi) = 0$. So we have

$$E(\phi) = E'(\phi_0) + E''(\phi_0) = E'(\phi_0) - E''(\phi_0) = K(\phi_0) = K(\phi_t) \le E(\phi_t).$$

If ϕ is an antiholomorphic map, the claim follows by a similar proof. □

Theorem 5.1 has another consequence:

Corollary 5.3. *If ϕ_0 and ϕ_1 are homotopy maps from a compact strongly Kähler Finsler manifold to a Kähler manifold such that ϕ_0 is holomorphic and ϕ_1 is antiholomorphic, then ϕ_0 and ϕ_1 are constant. In particular, any homotopically trivial holomorphic (or antiholomorphic) map is constant.*

Remark. Recently, B. Chen and the second author prove that Kähler and strongly Kähler Finsler metrics are in fact equivalent [Chen and Shen 2008]. So, all the strongly Kähler Finsler manifolds in this paper can be changed into Kähler Finsler manifolds.

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