Pacific Journal of Mathematics

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Volume 236 No. 2 June 2008

PACIFIC JOURNAL OF MATHEMATICS
Vol. 236, No. 2, 2008

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Let M and N be C^r Banach manifolds with $r \geq 1$. Let P be a submanifold of N and $f : M \to N$ a C^r map. This paper extends the well-known transversality $f \pitchfork P$ mod N to the tangent map $T_x f$ with a sharper singularity by using a new characteristic of the continuity of generalized inverses of linear operators in Banach spaces under small perturbations. We introduce a concept of generalized transversality, written as $f \pitchfork_G P$ mod N. We show that if $f \pitchfork P$ mod N, then $f \pitchfork_G P$ mod N, but the converse is false in general. Then Thom's famous result is expanded into a generalized transversality theorem: if $f \pitchfork_G P$ mod N, then the preimage $S = f^{-1}(P)$ is a submanifold of M with the tangent space $T_x S = (T_x f)^{-1}(T_{f(x)} P)$ for any $x \in S$. As a consequence, when P={y} is a single point set, $f \pitchfork_G P$ mod N if and only if y is a generalized regular value of f. Finally, we give an equivalent geometric description of generalized transversality without the aid of charts.

1. Introduction and preliminaries

E. Zeidler [1988] has pointed out that transversality is certainly one of the most important concepts in modern mathematics, which provides an answer to the question, when is the preimage of a manifold still a manifold? This, the celebrated transversality theorem of Thom, revitalizes the map approach to nonlinear differential equations, as illustrated by a number of examples in [Cafagna 1990]. The result is applied widely to differential topology and dynamic systems in [Abraham et al. 1988; Arnol'd 1988] and [Cafagna 1990].

This paper generalizes transversality and the transversality theorem by using some continuity characteristics of generalized inverses of singular bounded linear operators in Banach spaces under small perturbations. Let M and N be C^r Banach manifolds with $r \geq 1$. Let P be a submanifold of N and $f : M \to N$ a C^r map. We say f is transversal to P and write $f \pitchfork P$ mod N if, for each $x \in f^{-1}(P)$, $R(T_x f) + T_y P = T_y N$ and $(T_x f)^{-1}(T_y P)$ splits $T_x M$, where $y = f(x)$, $T_x f$

MSC2000: 46T05, 47A55, 58C15, 58K99.
Keywords: transversality, perturbation analysis of generalized inverse, Banach manifold, global analysis.
This work is supported by the National Science Foundation of China (grant numbers 10671049 and 10771101) and the Doctoral Program Foundation of the Ministry of Education of China.

denotes the tangent map of f at x, and T_yP and T_yN denote the tangent spaces of P and N at y, respectively. One has the following important result.

Theorem 1.1 [Abraham et al. 1988]. *Suppose $f \pitchfork P$ mod N. Then the preimage $S = f^{-1}(P)$ is a submanifold of M with the tangent space $T_xS = (T_xf)^{-1}(T_yP)$ for any $x \in S$.*

This is Thom's famous result, the transversality theorem, which provides an answer to the question above. The concept of transversality , $f \pitchfork P$ mod N, requires that $R(T_xf) + T_yP = T_yN$ for any $x \in f^{-1}(P)$. However, it is often the case that $R(T_xf) + T_yP \neq T_yN$; see, for instance, the three examples in Section 3. In this case, the singularities of the tangent maps are sharper, and so it is by no means simple to determine what kinds of properties f must have so that the conclusion of the theorem remains true.

Let $B(E, F)$ be the space of all bounded linear operators from a Banach space E to a Banach space F. Recall that an operator $A^+ \in B(F, E)$ is said to be a generalized inverse of $A \in B(E, F)$ if $A^+AA^+ = A^+$ and $AA^+A = A$. It is well known that even in the case of matrices, the generalized inverse $(A + \Delta A)^+$ may not tend to A^+ as $\Delta A \to 0$. Nevertheless, we have the following general theorems.

Theorem 1.2 [Ma 2000b; Ma 2003]. *Let $T_0 \in B(E, F)$ be double splitting. Then for any generalized inverse T_0^+ of T_0 and $T \in B(E, F)$ with $\|(T - T_0)T_0^+\| < 1$ the following conditions are equivalent*:

(i) $R(T) \cap N(T_0^+) = \{0\}$;

(ii) $(I_E - T_0^+T_0)N(T) = N(T_0)$;

(iii) $B = T_0^+(I_F + (T - T_0)T_0^+)^{-1} = (I_E + T_0^+(T - T_0))^{-1}T_0^+$ *is a generalized inverse of T.*

Here $N(\cdot)$ and $R(\cdot)$ denote respectively the null space and the range of operators in the parenthesis, and I_E means the identity on E.

Remark 1.1. Nashed and Chen [1993] first presented the condition (iii) of Theorem 1.2. Thanks to it, some results in [Ma 1999] were improved in [Ma 2000b; Ma 2000a; Ma 2007; Ma 2008]; we will refer to these papers frequently.

Theorem 1.3 [Ma 2000b]. *Let X be a topological space and $T_x : X \to B(E, F)$ an operator-valued map continuous at $x_0 \in X$. Suppose that $T_0 = T_{x_0}$ is double splitting. Then for a generalized inverse T_0^+ of T_0, there exists a neighborhood U_0 at x_0 such that the two conditions*

(i) *T_x has a generalized inverse T_x^+ for all $x \in U_0$) and*

(ii) $\lim_{x \to x_0} T_x^+ = T_0^+$

hold if and only if

$$R(T_x) \cap N(T_0^+) = \{0\} \quad (\textit{near } x_0). \tag{1-1}$$

Hereby one sees that the property (1-1) for T_x is interesting and important. So in [Ma 1999; Ma 2000b], x_0 is called a locally fine point for T_x if T_0 is double splitting and (1-1) is satisfied.

In particular, we have:

Theorem 1.4 [Ma 2000b; Ma 2003]. *If T_0 is a semi-Fredholm operator, then x_0 is a locally fine point of T_x if and only if either*

$$\dim N(T_x) = \dim N(T_0) < \infty$$

or

$$\operatorname{codim} R(T_x) = \operatorname{codim} R(T_0) < \infty \quad (\textit{near } x_0).$$

For a C^1 map from a Banach manifold M into another Banach manifold N, let (U, φ) and (V, ψ) be the charts of M at x and of N at $y = f(x)$, respectively, and let $\bar{f}$ be the representative of f under this pair of charts. By replacing T_x by $\bar{f}'(x_\varphi)$, where $x_\varphi = \varphi(x)$ for $x \in U$ and $\bar{f}'$ stands for the Frechet derivative of $\bar{f}$ at x_φ, Ma [2001] showed that the relative property (1-1) with $T_x = \bar{f}'(x_\varphi)$ is independent of the choice of charts. Then the concept of a generalized regular point (and corresponding generalized regular value) of f is induced from that of a locally fine point of $\bar{f}'(x_\varphi)$, which is equivalent to a subimmersion point of f; that is, $x_0 \in M$ is said to be a generalized regular point of f if $\varphi(x_0)$ is a locally fine point of $\bar{f}'(x_\varphi)$.

This yields the following theorem.

Theorem 1.5 [Ma 2001]. *Let f be a C^1 map from a C^1 Banach manifold M into another Banach manifold N. If $y \in N$ is a generalized regular value of f, which means that the set $f^{-1}(y)$ is empty or consists only of generalized regular points of f, then the preimage $S = f^{-1}(y)$ is a submanifold of M with the tangent space $T_x S = N(T_x f)$ for any $x \in S$.*

To generalize the transversality, Section 2 will supplement some new conditions equivalent to each of the three conditions in Theorem 1.2 and prove that they are all independent of the choice of a generalized inverse T_0^+ of T_0 for T near T_0. Finally, Section 3 will introduce generalized transversality and prove a generalized transversality theorem. As a corollary, we will give an equivalent geometric description of generalized transversality.

2. Perturbation analysis of generalized inverses

The following Theorem 2.1 is a supplement to Theorem 1.2, which is the key to the concept of generalized transversality in this paper.

Theorem 2.1. *Let $T_0 \in B(E, F)$ be double splitting. Then for any generalized inverse T_0^+ of T_0 and $T \in B(E, F)$ with $\|(T - T_0)T_0^+\| < 1$, the following conditions are equivalent*:

(i) $R(T) \cap N(T_0^+) = \{0\}$;

(ii) $(I_E - T_0^+ T_0)N(T) = N(T_0)$;

(iii) $B = T_0^+(I_F + (T - T_0)T_0^+)^{-1} = (I_E + T_0^+(T - T_0))^{-1}T_0^+$ *is a generalized inverse of* T;

(iv) $N(T) \oplus R(T_0^+) = E$;

(v) $N(T) + R(T_0^+) = E$.

Proof. By Theorem 1.2, it suffices to show the equivalence of conditions (ii) and (iv), as well as (ii) and (v). If (ii) holds, $R(B) = R(T_0^+)$ and $N(T) \oplus R(B) = E$, since B is a generalized inverse of T. Then $N(T) \oplus R(T_0^+) = E$. This says that condition (ii) implies both conditions (iv) and (v). Conversely, if either of conditions (iv) and (v) holds, then

$$N(T_0) = (I_E - T_0^+ T_0)E = (I_E - T_0^+ T_0)N(T). \qquad \square$$

Let T_1^+ and T_0^+ be two arbitrary generalized inverses of T_0, and let T_*^+ be the generalized inverse of T_0 corresponding to decomposition

$$E = N(T_0) \oplus R(T_1^+) \quad \text{and} \quad F = R(T_0) \oplus N(T_0^+).$$

Evidently $R(T_*^+) = R(T_1^+)$, and $N(T_*^+) = N(T_0^+)$. Let $\delta = (\|T_*^+\| + \|T_1^+\| + \|T_0^+\|)^{-1}$. Then we have the following significant theorem.

Theorem 2.2. *Let T_1^+ and T_0^+ be two arbitrary generalized inverses of T_0. If $R(T) \cap N(T_0^+) = \{0\}$ for any T with $\|T - T_0\| < \delta$, then $R(T) \cap N(T_1^+) = \{0\}$.*

Proof. By Theorem 2.1, if $R(T) \cap N(T_0^+) = \{0\}$ for any T with $\|T - T_0\| < \delta$, then $R(T) \cap N(T_*^+) = \{0\}$ and $N(T) + R(T_*^+) = E$. So

$$N(T) + R(T_1^+) = N(T) + R(T_*^+) = E.$$

Again by Theorem 2.1, we conclude that $R(T) \cap N(T_1^+) = \{0\}$. $\square$

Remark 2.1. Recalling the definition of a locally fine point x_0 for an operator value map T_x (or a generalized regular point of a C^1 map f), we see that it involves formally a generalized inverse of the operator T_0 (or $f'(x_0)$); however, it is independent of the choice of a generalized inverse by Theorem 2.2. Thus it presents a behavior depending only on T_x (or $f'(x)$) near x_0 in the case of double splitting T_0 (or $f'(x)$).

3. Generalized transversality

Let M and N be C^r Banach manifolds with $r \geq 1$. Let P be a submanifold of N and $f : M \to N$ a C^r map. By the definition of a submanifold, for each point $y \in P$ there exists a chart (V, ψ) of N with $y \in V$ such that the chart space F contains a linear closed subspace F_0 that splits F and such that the image $\psi(V \cap P)$ is an open set in F_0. For abbreviation, in the sequel we will write such a chart (V, ψ) as an $(N.P)$ chart at y.

Before going to the generalized transversality, let us introduce the following two conditions.

(H_1) For each $x \in f^{-1}(P)$, $R(T_x f)+T_y P$ and $(T_x f)^{-1}(T_y P)$ split $T_y N$ and $T_x M$, and $T_y P$ splits $R(T_x f)+T_y P$. Here $y = f(x)$, $T_x f$ denotes the tangent map of f at x, and $T_y(\cdot)$ is the tangent space at y of the manifold in parentheses.

(H_2) For any $x_0 \in f^{-1}(P)$, there exist a neighborhood U_0 at x_0, a pair of charts (U, φ) of M at x_0, and a $(N.P)$ chart at $y_0 = f(x_0)$ such that

$$(\bar{f}'(x_\varphi))^{-1}(F_0) + E_0 = E \quad \text{for all } x \in U \cap U_0,$$

where E is the chart space of (U, φ) and E_0 satisfies $(\bar{f}'(x^0_\varphi))^{-1}(F_0) \oplus E_0 = E$ for $x^0_\varphi = \varphi(x_0)$.

Remark 3.1. In the definition of $f \pitchfork P$ mod N, the assumption that $(T_x f)^{-1}(T_y P)$ splits $T_x M$ and $R(T_x f)+T_y P = T_y N$ implies that (H_1) holds when $f \pitchfork P$ mod N; in the case of $\dim M < \infty$ and $\dim N < \infty$, (H_1) is automatically satisfied.

We will focus our attention on $(T_y \psi)(T_y P) = F_0$ for any $y \in P \cap V$ under a $(N.P)$ chart (V, ψ); this fact follows from that $\psi(P \cap V)$ is open in F_0. Because of the continuity of f, we can assume $f(U) \subset V$ in the sequel.

Next we claim a lemma, which shows that together with (H_1), (H_2) is coordinate independent, that is, geometric.

For each $x_0 \in f^{-1}(P)$, let (U_1, φ_1) at x_0 and (V_1, ψ_1) at $y_0 = f(x_0)$ be other charts with the same properties as (U, φ) and (V, ψ), and let F_1 be the chart space of (V_1, ψ_1) containing a closed subspace F^1_0 such that $\psi_1(P \cap V_1)$ is open in it.

Lemma 3.1. *Suppose that condition (H_1) holds. If condition (H_2) holds under the charts (U, φ) and (V, ψ), then so does (H_2) under two arbitrary charts (U_1, φ_1) and (V_1, ψ_1) as above.*

Proof. Let $\bar{f}$ and $\tilde{f}$ be the representatives of f under the two pairs of charts (U, φ) and (V, ψ) and (U_1, φ_1) and (V_1, ψ_1), respectively. Obviously

$$\tilde{f}(x_{\varphi_1}) = (\psi_1 \circ \psi^{-1}) \circ \bar{f} \circ (\varphi \circ \varphi_1^{-1})(x_{\varphi_1}) \quad \text{for all } x_{\varphi_1} \in \varphi_1(U \cap U_1),$$

and so

$$\widetilde{f}'(x_{\varphi_1}) = (\psi_1 \circ \psi^{-1})'(y_\psi) \cdot \overline{f}' \cdot (\varphi \circ \varphi_1^{-1})'(x_{\varphi_1}) \quad \text{for all } x_{\varphi_1} \in \varphi_1(U \cap U_1), \tag{3-1}$$

where $y_\psi = \psi(y)$.

By the assumptions of the lemma, we can assume that there exist two subspaces F_{01} and F_{02} of the chart space F such that

$$(R(\overline{f}'(x_\varphi^0)) + F_0) \oplus F_{01} = F \quad \text{and} \quad R(\overline{f}'(x_\varphi^0)) + F_0 = F_0 \oplus F_{02},$$

where $(T_{y_0}\psi)(T_{y_0}P) = F_0$ and $(T_{y_0}\psi)(T_{y_0}N) = F$. We can also assume there is a closed subspace E_0 of E such that

$$\overline{f}'(x_\varphi^0)^{-1}(F_0) \oplus E_0 = E.$$

Obviously $F_0^\perp = F_{01} \oplus F_{02}$ is a topological complement of F_0, and

$$F_0^1 = (\psi_1 \circ \psi^{-1})'(y_\psi) F_0 \quad \text{for all } y \in P \cap V_1 \cap V. \tag{3-2}$$

Now, we consider the map $g : \varphi(U) \to B(E, F_0^\perp)$ defined by

$$g(x_\varphi) = P_{F_0, F_0^\perp} \overline{f}'(x_\varphi) \quad \text{for all } x_\varphi \in \varphi(U),$$

where $P_{F_0, F_0^\perp}$ is the projection corresponding to the decomposition $F = F_0 \oplus F_0^\perp$. It is not difficult to verify that x_φ^0 is a locally fine point for $g(x_\varphi)$. Indeed,

$$R(g(x_\varphi^0)) = P_{F_0, F_0^\perp} R(\overline{f}'(x_\varphi^0)) = P_{F_0, F_0^\perp}\{R(\overline{f}'(x_\varphi^0)) + F_0\} = P_{F_0, F_0^\perp}(F_0 \oplus F_{02}) = F_{02}$$

and $u \in N(g(x_\varphi)) \iff \overline{f}'(x_\varphi)u \in F_0 \iff u \in (\overline{f}'(x_\varphi))^{-1}(F_0)$, so that

$$N(g(x_\varphi)) = (\overline{f}'(x_\varphi))^{-1}(F_0) \quad \text{for all } x_\varphi \in \varphi(U). \tag{3-3}$$

Hence by the condition (H$_2$) and Theorem 2.1, we can conclude that x_φ^0 is a locally fine point for $g(x_\varphi)$. To show the condition (H$_2$) holds under the charts (U_1, φ_1) and (V_1, ψ_1), the routine method in global analysis can hardly be applied here; indeed, an ingenious application of both Theorems 2.1 and 1.3 is the essence of our proof below.

Consider the operator-valued map $\widetilde{g} : \varphi_1(U \cap U_1) \to B(E_1, F_1)$ defined by

$$\widetilde{g}(x_{\varphi_1}) = (\psi_1 \circ \psi^{-1})'(y_\psi) \cdot g(x_\varphi) \cdot (\varphi \circ \varphi_1^{-1})'(x_{\varphi_1}) \quad \text{for all } x_{\varphi_1} \in \varphi_1(U \cap U_1),$$

where E_1 is the chart space of (U_1, φ_1). Clearly,

$$R(\widetilde{g}(x_{\varphi_1})) = (\psi_1 \circ \psi^{-1})'(y_\psi) R(g(x_\varphi)).$$

Meanwhile, it is easy to verify by (3-1)–(3-3) that

$$N(\widetilde{g}(x_{\varphi_1})) = (\widetilde{f}'(x_{\varphi_1}))^{-1}(F_0^1). \tag{3-4}$$

In fact,

$$\begin{aligned} u \in \widetilde{g}(x_{\varphi_1}) &\iff (\varphi \circ \varphi_1^{-1})'(x_{\varphi_1})u \in N(g(x_\varphi)) \\ &\iff \bar{f}'(x_\varphi)(\varphi \circ \varphi_1^{-1})'(x_{\varphi_1})u \in F_0 && \text{by (3-3)} \\ &\iff (\psi_1 \circ \psi^{-1})'(y_\psi) \cdot \bar{f}'(x_\varphi) \cdot (\varphi \circ \varphi_1^{-1})'(x_{\varphi_1})u \in F_0^1 && \text{by (3-2)} \\ &\iff u \in (\widetilde{f}'(x_{\varphi_1}))^{-1}(F_0^1) && \text{by (3-1)}. \end{aligned}$$

Moreover, as x_φ^0 is a locally fine point of $g(x_\varphi)$, there exists, by Theorem 1.3, a neighborhood $U_0 \subset U \cap U_1$ at x_0 such that there is a generalized inverse $g^+(x_\varphi)$ of $g(x_\varphi)$ for any $x \in U_0$ and $g^+(x_\varphi) \to g^+(x_\varphi^0)$ as $x_\varphi \to x_\varphi^0$. Hereby we get

$$\widetilde{g}^+(x_{\varphi_1}) = (\varphi_1 \circ \varphi^{-1})'(x_\varphi) \cdot g^+(x_\varphi) \cdot (\psi \circ \psi_1^{-1})'(y_{\psi_1}) \quad \text{for all } x_{\varphi_1} \in \varphi_1(U_0),$$

which is clearly continuous at $x_{\varphi_1}^0$.

Next we claim that $\widetilde{g}^+(x_{\varphi_1})$ is just a generalized inverse of $\widetilde{g}(x_{\varphi_1})$ for each $x \in U_0$. By computing directly,

$$\widetilde{g}(x_{\varphi_1}) \cdot \widetilde{g}^+(x_{\varphi_1}) = (\psi_1 \circ \psi^{-1})'(y_\psi) \cdot g(x_\varphi) \cdot g^+(x_\varphi) \cdot (\psi \circ \psi_1^{-1})'(y_{\psi_1}),$$

which means

$$\begin{aligned} &\widetilde{g}(x_{\varphi_1}) \cdot \widetilde{g}^+(x_{\varphi_1}) \cdot \widetilde{g}(x_{\varphi_1}) \\ &\qquad = (\psi_1 \circ \psi^{-1})'(y_\psi) \cdot g(x_\varphi) \cdot g^+(x_\varphi) \cdot g(x_\varphi) \cdot (\varphi \circ \varphi_1^{-1})'(x_{\varphi_1}) \\ &\qquad = (\psi_1 \circ \psi^{-1})'(y_\psi) \cdot g(x_\varphi) \cdot (\varphi \circ \varphi_1^{-1})'(x_{\varphi_1}) = \widetilde{g}(x_{\varphi_1}) \end{aligned}$$

and

$$\begin{aligned} &\widetilde{g}^+(x_{\varphi_1}) \cdot \widetilde{g}(x_{\varphi_1}) \cdot \widetilde{g}^+(x_{\varphi_1}) \\ &\qquad = (\varphi_1 \circ \varphi^{-1})'(x_\varphi) \cdot g^+(x_\varphi) \cdot g(x_\varphi) \cdot g^+(x_\varphi) \cdot (\psi \circ \psi_1^{-1})'(y_{\psi_1}) \\ &\qquad = (\varphi_1 \circ \varphi^{-1})'(x_\varphi) \cdot g^+(x_\varphi) \cdot (\psi \circ \psi_1^{-1})'(y_{\psi_1}) = \widetilde{g}^+(x_{\varphi_1}). \end{aligned}$$

So $\widetilde{g}^+(x_{\varphi_1})$ is a generalized inverse of $\widetilde{g}(x_{\varphi_1})$ for each $x \in U_0$. On the other hand, by the continuity of $g^+(x_\varphi)$ at x_φ^0, we have

$$\lim_{x \to x_0} \widetilde{g}^+(x_{\varphi_1}) = \widetilde{g}^+(x_{\varphi_1}^0).$$

Thus, by Theorem 1.3 again, we assert that $x_{\varphi_1}^0$ is a locally fine point for $\widetilde{g}(x_{\varphi_1})$, and hence by Theorem 2.1, there are a neighborhood at x_0, still written as $U_0 \subset U \cap U_1$, and a closed subspace $E_0^1 = R(\widetilde{g}^+(x_{\varphi_1}^0))$ of E_1 such that

$$N(\widetilde{g}(x_{\varphi_1})) + E_0^1 = E_1 \quad \text{for all } x \in U_0.$$

Finally by (3-4),

$$(\widetilde{f}'(x_{\varphi_1}))^{-1}(F_0^1) + E_0^1 = E_1 \quad \text{for all } x \in U_0;$$

that is, the condition (H$_2$) holds under the charts (U_1, φ_1) and (V_1, ψ_1). $\square$

We are now ready to introduce generalized transversality.

Definition 3.1. Let $f : M \to N$ be a C^r map and P be a submanifold of N. We say f is generalized transversal to P, and write as $f \pitchfork_G P \bmod N$, if, for each $x_0 \in f^{-1}(P)$, conditions (H$_1$) and (H$_2$) hold for some pair consisting of a chart of M at x_0 and an $(N.P)$ chart at y_0.

Thanks to Lemma 3.1, Definition 3.1 is reasonable.

The next theorem and counterexamples show that the concept of $f \pitchfork_G P \bmod N$ expands that of $f \pitchfork P \bmod N$.

Theorem 3.1. *If* $f \pitchfork P \bmod N$, *then* $f \pitchfork_G P \bmod N$.

Proof. Assume $f \pitchfork P \bmod N$. It has been pointed out that (H$_1$) is satisfied. Hence we only need to examine the condition (H$_2$). Our proof at this time is also based on an application of Theorem 2.1. With the charts (U, φ) of M at x_0 and an $(N.P)$ chart at $y_0 = f(x_0)$ as in Lemma 3.1, consider for each $x_0 \in f^{-1}(P)$ the map $g : \varphi(U) \to B(E, F_0^{\perp})$ defined by

$$g(x_\varphi) = P_{F_0, F_0^{\perp}} \bar{f}'(x_\varphi) \quad \text{for all } x_\varphi \in \varphi(U),$$

where $F_0^{\perp}$ is a topological complement of F_0. Obviously,

$$R(g(x_\varphi^0)) = P_{F_0, F_0^{\perp}}\{R(\bar{f}'(x_\varphi^0)) + F_0\} = P_{F_0, F_0^{\perp}} F = F_0^{\perp},$$

while $N(g(x_\varphi^0)) = (\bar{f}'(x_\varphi^0))^{-1}(F_0)$ by (3-3), which splits E by assumption. Therefore x_φ^0 is a locally fine point for $g(x_\varphi)$. Thus by Theorem 2.1 there is a neighborhood $U_0 \subset U$ at x_0 such that

$$(\bar{f}'(x_\varphi))^{-1}(F_0) + E_0 = E \quad \text{for all } x \in U_0,$$

where $E_0 = R(g(x_\varphi^0)^+)$ and $g(x_\varphi^0)^+$ is a generalized inverse of $g(x_\varphi^0)$. $\square$

The examples below deal with finite-dimensional manifolds, and so for verifying the generalized transversality, one only needs to check the condition (H$_2$).

Example 1. Define $f : \mathbb{R} \to N$ by

$$(u, v, w) = f(x) = (x^2, x, x) \quad \text{for all } x \in \mathbb{R},$$

and let N be the parabolic cylinder in $\mathbb{R}^3$ defined by $u = v^2$. Suppose that P is the parabolic curve in $\mathbb{R}^3$ defined by $u = v^2$ and $w = v$.

Since $f(\mathbb{R}) = P \subsetneq N$, it is clear that f is not transversal to P. However, one can verify that $f \pitchfork_G P \bmod N$.

In fact, by Lemma 3.1, consider the charts $(\mathbb{R}.I)$ and $(N.\psi)$, where ψ is as follows:

$$(u_1, v_1, w_1) = \psi(u, v, w) = (u - v^2, v, w) \quad \text{for all } (u, v, w) \in \mathbb{R}^3.$$

Clearly (N, ψ) is an $(N.P)$ chart at each point of P, and its chart space is the coordinate plane $V_1 O W_1$ containing the subspace

$$F_0 = \{(0, v_1, v_1) : \text{for all } v_1 \in \mathbb{R}\},$$

while $\bar{f}(x) = (\psi \circ f)(x) = (0, x, x)$, $F_0 = R(\bar{f}'(x))$ for any $x \in \mathbb{R}$, and $F_0 = (T_y\psi)(P)$ for any $y \in P$. Then immediately,

$$\bar{f}'(x)^{-1}(F_0) + \{0\} = \mathbb{R} + \{0\} = \mathbb{R} \quad \text{for all } x \in \mathbb{R}.$$

Therefore $f \pitchfork_G P$ mod N.

Example 2. Let $M = \{(x_1, x_2) \in \mathbb{R}^2 : x_1x_2 \neq 0\}$, and define $f : M \to \mathbb{R}^2$ by

$$f(x_1, x_2) = (e^{x_1x_2}, x_1x_2) \quad \text{for all } (x_1, x_2) \in M.$$

Suppose that P consists only of a single point $(e, 1)$ in $\mathbb{R}^2$. Because $T_{(e,1)}P = \{0\}$, we have

$$f'(x) = \begin{pmatrix} x_2e^{x_1x_2} & x_1e^{x_1x_2} \\ x_2 & x_1 \end{pmatrix} \quad \text{for all } (x_1, x_2) \in M. \tag{3-5}$$

Then it is easy to see that for all $x \in M$,

$$f'(x)^{-1}(T_{(e,1)}P) = f'(x)^{-1}(0) = N(f'(x)) , \tag{3-6}$$

and $\dim N(f'(x)) = \operatorname{codim} R(f'(x)) = 1$. Hence

$$R(f'(x)) + T_{f(x)}P = R(f'(x)) \subsetneq \mathbb{R}^2 \quad \text{for all } x \in f^{-1}(P).$$

This means that f is not transversal to P mod $\mathbb{R}^2$. However, $f \pitchfork_G P$ mod N.

In fact, by Theorem 1.4, each $x \in M$ is a locally fine point for $f'(x)$ so that by Theorem 2.1, there exists a neighborhood U_0 at x and $E_0 \subset \mathbb{R}^2$ such that

$$N(f'(z)) + E_0 = f'(z)^{-1}(0) + E_0 = \mathbb{R}^2$$

for any $z \in U_0$, and

$$N(f'(x)) \oplus E_0 = f'(x)^{-1}(0) \oplus E_0 = \mathbb{R}^2,$$

where E_0 is the range of a generalized inverse of $f'(x)$.

Example 3. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by

$$(x, y, z) = f(x_1, x_2) = (x_1, x_1^2, x_2),$$

and let P be the z-axis in $\mathbb{R}^3$. Then, for any $(x_1, x_2) \in \mathbb{R}^2$,

$$f'(x_1, x_2) = \begin{pmatrix} 1 & 0 \\ 2x_1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad f^{-1}(P) = \text{the } x_2\text{-axis in } \mathbb{R}^2,$$

and

$$R(T_{(x_1,x_2)} f) = \{(h_1, 2x_1h_1, h_2) : \text{ for all } (h_1, h_2) \in \mathbb{R}^2\}. \tag{3-7}$$

So for each $(x_1, x_2) \in f^{-1}(P)$, we have $R(T_{(0,x_2)} f) + T_{(0,0,x_2)} P \subsetneq \mathbb{R}^3$, that is, f is not transversal to P mod $\mathbb{R}^3$. However, $f \pitchfork_G P$ mod $\mathbb{R}^3$. Indeed, by (3-7)

$$(T_{(x_1,x_2)} f)^{-1}(T_{(x_1,x_1^2,x_2)} P) = \text{the } x_2\text{-axis in } \mathbb{R}^2,$$

so that $E_0 =$ the x_1-axis fulfills that for any $(x_1, x_2) \in \mathbb{R}^2$,

$$(T_{(x_1,x_2)} f)^{-1}(T_{(x_1,x_1^2,x_2)} P) + E_0 = \mathbb{R}^2.$$

Note that here the natural coordinate system in $\mathbb{R}^3$ is just an $(\mathbb{R}^3.P)$ chart at each $(x, y, z) \in P$. This proves that $f \pitchfork_G P$ mod $\mathbb{R}^3$.

None of the maps f in the examples above are transversal to P, but all are generalized transversal to P.

The following, one of the main theorems in this paper, is like the well-known transversality theorem [Kahn 1980]. It should lead one to recognize the concept of generalized transversality.

Theorem 3.2. *If $f \pitchfork_G P$ mod N, then the preimage $S = f^{-1}(P)$ is a submanifold of M with the tangent space $T_x S = (T_x f)^{-1}(T_{f(x)} P)$ for any $x \in S$.*

Proof. For any $x_0 \in f^{-1}(P)$, let (U, φ) be a chart of M at x_0 with chart space E, and let (V, ψ) be a $(N.P)$ chart at $y_0 = f(x_0)$ with the chart space F, which contains a splitting subspace F_0 such that $\psi(P \cap V)$ is open in it. Because of the continuity of f, we may assume $f(U) \subset V$. Under this pair of charts, let $\bar{f}$ be the representative of f.

Assume that $f \pitchfork_G P$ mod N. Let the subspaces of F, F_{01}, and F_{02} be the same as in the proof of Lemma 3.1. Then we see $F_0^{\perp} = F_{01} \oplus F_{02}$ is a topological complement of F_0. Consider the map $g : \varphi(U) \to F_0^{\perp}$ defined by

$$g(x_\varphi) = P_{F_0, F_0^{\perp}} \bar{f}(x_\varphi) \quad \text{for all } x_\varphi \in \varphi(U),$$

where $P_{F_0, F_0^{\perp}}$ is the projection corresponding to the decomposition $F = F_0 \oplus F_0^{\perp}$. Then $g'(x_\varphi) = P_{F_0, F_0^{\perp}} \bar{f}'(x_\varphi)$ and $g^{-1}(0) = \{x_\varphi \in \varphi(U) : x \in f^{-1}(P \cap V)\}$. (Note that $f(U) \subset V$.)

Replacing g in Lemma 3.1 by g', we assert as in the lemma that

$$R(g'(x_\varphi^0)) = F_{02} \quad \text{and} \quad N(g'(x_\varphi^0)) = (\bar{f}'(x_\varphi^0))^{-1}(F_0),$$

and this means that $g'(x_\varphi^0)$ is double splitting. Moreover, because of the condition (H_2) and by Theorem 2.1, x_φ^0 is a locally fine point for $g'(x_\varphi)$, that is, a generalized regular point of g.

By Theorem 1.2, it is easy to check that a locally fine point is an inner point. Indeed, let $T_x : X \to B(E, F)$ be the same as in Theorem 1.3, and let $x_0 \in X$ be a locally fine point for T_x. Then there exists a neighborhood W_0 at x_0 such that

$$R(T_x) \cap N(T_0^+) = \{0\} \quad \text{for all } x \in W_0.$$

Without loss of generality, one can assume that $W_0 \subset \{x \in X : \|T_x - T_0\| < \|T_0^+\|^{-1}\}$. For any $x_1 \in W_0$, by Theorem 1.2,

$$T_1^+ = (I_E + T_0^+(T_{x_1} - T_0))^{-1} T_0^+$$

is a generalized inverse of T_{x_1} with $N(T_1^+) = N(T_0^+)$. Thus

$$R(T_x) \cap N(T_1^+) = \{0\} \quad \text{for all } x \in W_0 \cap \{x \in X : \|T_x - T_{x_1}\| < \|T_1^+\|^{-1}\};$$

that is, x_1 is also a locally fine point for T_x. This proves that x_0 is an inner point. Therefore, a generalized regular point is an inner point, and hence each point of $\varphi(U)$ may be regarded as a generalized regular point, for otherwise one may shrink U.

Thus, 0 is a generalized regular value of $g(x_\varphi)$, so by Theorem 1.5, $\bar{S}_0 = g^{-1}(0)$ is a submanifold of $\varphi(U)$ with tangent space $T_{x_\varphi}\bar{S}_0 = (T_{x_\varphi}\bar{f})^{-1}(F_0)$ for any $x_\varphi \in \bar{S}_0$. Hence $S_0 = \varphi^{-1}(\bar{S}_0)$ is a submanifold of U, and $T_x S_0 = (T_x f)^{-1}(T_{f(x)}P)$ for any $x \in S_0$. Indeed, let c denote a smooth curve based the point x, and let $[c]$ be the equivalence class of c. Then for any $x \in S_0$,

$$\begin{aligned}
[c] \in T_x S_0 &\iff [\varphi \circ c] \in T_{x_\varphi}\bar{S}_0 = (T_{x_\varphi}\bar{f})^{-1}(F_0) \\
&\iff \bar{f}(\varphi \circ c) = \psi(f \circ c) \subset F_0 \cap V \\
&\iff \frac{d}{dt}\psi(f \circ c)\Big|_{t=0} = v = \frac{d}{dt}\psi(\psi^{-1}(\psi(f(x)) + tv))\Big|_{t=0} \\
&\qquad \text{and } \psi(f \circ c) \subset F_0 \cap \psi(V) \\
&\iff [f \circ c] = [\psi^{-1}(\psi(f(x)) + tv)] \in T_{f(x)}P \\
&\iff [c] \in (T_x f)^{-1}(T_{f(x)}P).
\end{aligned}$$

This proves that $T_x S_0 = (T_x f)^{-1}(T_{f(x)}P)$. The first argument is obvious.

Now we claim that S is a submanifold of M. By the definition of a submanifold in [Zeidler 1988], one merely needs to show that for any $x_0 \in f^{-1}(P)$, there exists an $(M.S)$ chart at x_0. Since $\widetilde{S}_0$ is a submanifold of E, there exists an $(E.\bar{S}_0)$ chart $(\widetilde{U}_0, \widetilde{\varphi}_0)$ at x_φ^0 such that $\widetilde{U}_0 \subset \varphi(U)$. Hereby one gets an $(M.S_0)$ (and also an $(M.S)$) chart $(\varphi^{-1}(\widetilde{U}), \widetilde{\varphi} \circ \varphi)$ at x_0. This proves the theorem. □

Remark 3.2. In view of the counterexamples above, Theorem 3.2 answers in more generality than the well-known transversality theorem the question, when is the preimage of a manifold a manifold?

In what follows, we give a corollary of the generalized transversality theorem.

Corollary 3.1. *If P consists of a single point $y \in N$, then $f \pitchfork_G P$ mod N if and only if y is a generalized regular value of f.*

Proof. If $f \pitchfork_G P$ mod N, then, noting that $T_y P = \{0\}$, we assert that $T_x f$ splits $T_y N$ for each $x \in f^{-1}(y)$, and for each $x_0 \in f^{-1}(y)$ there exists a neighborhood U of x_0 and a chart (U, φ) of M at x_0 such that

$$N(\bar{f}'(x_\varphi)) + E_0 = E \quad \text{for all } x_\varphi \in \varphi(U),$$

where E is the chart space of (U, φ) and $E = E_0 \oplus N(\bar{f}'(x_\varphi^0))$. Thus by Theorems 2.1 and 2.2, x_0 is a generalized regular point of f. This shows that $f^{-1}(y)$ consists only of generalized regular points of f, and therefore y is a generalized regular value of f.

Conversely, if y is a generalized regular value of f, then for $x \in f^{-1}(y)$, $R(T_x f)$ and $(T_x f)^{-1}(0) = N(T_x f)$ split $T_y N$ and $T_x M$, respectively, while by the definition of a generalized regular point of f and Theorem 2.1, it is not difficult to verify that the condition (H_2) is fulfilled; this means that $f \pitchfork_G P$ mod N. □

Remark 3.3. Since a point x that is regular is also generalized regular (that is, any regular value is also a generalized regular value), it is clear that Corollary 3.1 is a generalization of the well-known preimage theorem when P is a set formed by a single point.

In order to give a geometric description of the condition (H_2) without the aid of charts, we will need the following lemma.

Lemma 3.2. *If N is a differentiable manifold and P is a submanifold of N, then for each $y_0 \in P$ there exists a neighborhood V_0 at y_0 such that for any $y \in V_0$, there exists a splitting subspace of $T_y N$, which is identified with $T_y P$ when $y \in P$. We still denote it by $T_y P$ for any $y \in V_0$.*

Proof. We first work on an $(N.P)$ chart (V, ψ) at y_0. For any $u \in F_0$, set

$$\alpha(t) = y_\psi + ut \quad \text{for } t \in (-\varepsilon, \varepsilon),$$

where $y \in V$ and ε is a positive number satisfying that $\alpha(t) \subset F_0 \cap \psi(V)$ for all $t \in (-\varepsilon, \varepsilon)$. Thus we get

$$T_y P = \{[\psi^{-1} \circ \alpha] : \text{for all } u \in F_0\}.$$

For another $(N.P)$ chart (V_1, ψ_1) at y_0, we have by (3-2) that

$$F_0^1 = (\psi_1 \circ \psi^{-1})'(y_\psi) F_0 \quad \text{for all } y \in P \cap V_1 \cap V.$$

Thus the definition of $T_y P$ makes sense, as is required. □

Theorem 3.3. *If M and N are C^r Banach manifolds with $r \geq 1$. Let P be a submanifold of N and $f : M \to N$ a C^r map. Then (H$_2$) is equivalent to the condition that*

(H$_3$) *for any $x_0 \in f^{-1}(P)$, there exists a neighborhood U_0 at x_0 and a subbundle $\bigcup_{x \in U_0} m_x$ of $T_{U_0} M$ (the restriction of TM to U_0) such that m_{x_0} is a topological complement of $(T_{x_0} f)^{-1}(T_{y_0} P)$, and $(T_x f)^{-1}(T_y P) + m_x = T_x M$ for any $x \in U_0$.*

Proof. By the definition of the subbundle, there exists a chart (W, φ) of M at x_0 with $W \subset U$ such that for all $x \in W$,

$$(T_x \varphi)(T_x M) = E \quad \text{and} \quad (T_x \varphi)(m_x) = E_0,$$

where E is the chart space of (W, φ), and E_0 is a split subspace of E. By Lemma 3.2, there exists a neighborhood V_0 at $y_0 = f(x_0)$, and, as mentioned before, we can also assume that $f(W) \subset V_0$ and that there exists a split subspace $T_y P$ of $T_y N$ for any $y \in V_0$.

Now if we assume the condition (H$_3$) holds, then

$$E_0 \oplus (T_{x_0} \varphi)((T_{x_0} f)^{-1}(T_{y_0} P)) = E,$$

where $x_0 \in f^{-1}(P)$, and

$$E_0 + (T_x \varphi)((T_x f)^{-1}(T_y P)) = E,$$

where $y = f(x)$ for all $x \in W$.

Let (V, ψ) be an $(N.P)$ chart at y_0 whose chart space F contains a subspace F_0 such that $\psi(P \cap V)$ is open in F_0. Without loss of generality, one can assume $V = V_0$. Then it is clear that the condition (H$_2$) holds under the charts (W, φ) and (V, ψ). Indeed, let $\bar{f}$ be the representative of f under these charts, and let $T_y P$ be the tangent space given in the proof of Lemma 3.2. Then it follows that $(T_y \psi)(T_y P) = F_0$ for all $y \in V$, so that for all $x \in W$,

$$E = E_0 + (T_x \varphi)((T_x f)^{-1}(T_y P)) = (\bar{f}'(x))^{-1}(F_0) + E_0$$

and
$$E = E_0 \oplus (T_{x_0}\varphi)((T_{x_0} f)^{-1}(T_{y_0} P)) = (\bar{f}'(x_0))^{-1}(F_0) \oplus E_0.$$
This shows the sufficiency part of the theorem.

Now assume (H$_2$) holds for the pair of a chart (U, φ) at x_0 and an $(N.P)$ chart (V, ψ) at y_0. Without loss of generality, we can assume that V fulfills the property for V_0 in Lemma 3.2. We can also assume $U_0 \subset U$ and $f(U_0) \subset V$.

Proceeding as in Lemma 3.2, we set
$$m_x = \{[\varphi^{-1} \circ \beta] : \beta = x_\varphi + ut, \text{ for all } u \in E_0\}$$
for each $x \in U_0$. Obviously $\bigcup_{x \in U_0} m_x$ is a subbundle of $T_{U_0} M$, which fits our requirement. In fact, for any $[c_x] \in T_x M$, let $u \in E$ be such that $c_x = \varphi^{-1}(x_\varphi + ut)$. By condition (H$_2$), there exist $u_0 \in E_0$ and $u_1 \in \bar{f}'(x_\varphi)^{-1}(F_0)$ such that $u = u_0 + u_1$. On the other hand, $\varphi^{-1}(x_\varphi + u_0 t) \in m_x$, and
$$\begin{aligned} u_1 \in \bar{f}'(x_\varphi)^{-1}(F_0) &\iff \bar{f}'(x_\varphi) u_1 \in F_0 \\ &\iff [\bar{f}(x_\varphi + u_1 t)] \in (T\psi)(T_y P) \\ &\iff [f(\varphi^{-1}(x_\varphi + u_1 t))] \in T_y P \\ &\iff [\varphi^{-1}(x_\varphi + u_1 t)] \in (T_x f)^{-1}(T_y P). \end{aligned}$$
Thus by the definition of a linear operator in the tangent space,
$$[\varphi^{-1}(x_\varphi + u_1 t)] + [\varphi^{-1}(x_\varphi + u_0 t)] = [\varphi^{-1}(x_\varphi + ut)].$$
This shows $(T_x f)^{-1}(T_y P) + m_x \supset T_x M$ for any $x \in U_0$. The converse inclusion is obvious. This proves the necessary part of the theorem. □

Thus the generalized transversality has the following equivalent description.

Corollary 3.2. *Let M and N be C^r Banach manifolds with $r \geq 1$. Let P be a submanifold of N and $f : M \to N$ a C^r map. Then f is generalized transversal to P, if and only if the conditions* (H$_1$) *and* (H$_3$) *are satisfied.*

Remark 3.4. When $f \pitchfork P$ mod N, then in a way analogous to Lemma 3.1, we can work on the charts (U, φ) at x_0 and an $(N.P)$ chart at y_0 and find that
$$g(x^0_\varphi) = P_{F_0, F_0^\perp} \bar{f}'(x^0_\varphi) \in B(E, F_0^\perp)$$
is surjective and $N(g(x^0_\varphi)) = \bar{f}'(x^0_\varphi)^{-1}(F_0)$ splits E as
$$E = E_0 \oplus \bar{f}'(x^0_\varphi)^{-1}(F_0).$$
Hence x^0_φ is a locally fine point for $g(x_\varphi)$. Further one can verify the existence of the subbundle $\bigcup_{x \in U_0} m_x$ by the equivalent condition (v) in Theorem 2.1 for a locally fine point x^0_φ; that is, $m_x = \{[\varphi^{-1} \circ \beta] : \beta(t) = x_\varphi + ut \text{ for all } u \in E_0\}$ near x_0.

References

[Abraham et al. 1988] R. Abraham, J. E. Marsden, and T. Ratiu, *Manifolds, tensor analysis, and applications*, 2nd ed., Applied Mathematical Sciences **75**, Springer, New York, 1988. MR 89f:58001 Zbl 0875.58002

[Arnol'd 1988] V. I. Arnol'd, *Geometrical methods in the theory of ordinary differential equations*, 2nd ed., Grundlehren der Mathematischen Wissenschaften **250**, Springer, New York, 1988. MR 89h:58049 Zbl 0648.34002

[Cafagna 1990] V. Cafagna, "Global invertibility and finite solvability", pp. 1–30 in *Nonlinear functional analysis* (Newark, NJ, 1987), edited by P. S. Milojević, Lecture Notes in Pure and Appl. Math. **121**, Dekker, New York, 1990. MR 91a:58028 Zbl 0699.35094

[Kahn 1980] D. W. Kahn, *Introduction to global analysis*, Pure and Applied Mathematics **91**, Academic Press, New York, 1980. MR 81i:58001 Zbl 0443.58001

[Ma 1999] J.-P. Ma, "(1.2) inverses of operators between Banach spaces and local conjugacy theorem", *Chinese Ann. Math. Ser. B* **20**:1 (1999), 57–62. MR 2000c:47117 Zbl 0926.47044

[Ma 2000a] J.-P. Ma, "Local conjugacy theorem, rank theorems in advanced calculus and a generalized principle for constructing Banach manifolds", *Sci. China Ser. A* **43**:12 (2000), 1233–1237. MR 2001j:58007 Zbl 1002.58004

[Ma 2000b] J.-P. Ma, "Rank theorems of operators between Banach spaces", *Sci. China Ser. A* **43**:1 (2000), 1–5. MR 2001e:47008 Zbl 0946.47001

[Ma 2001] J.-P. Ma, "A generalized preimage theorem in global analysis", *Sci. China Ser. A* **44**:3 (2001), 299–303. MR 2002c:46144 Zbl 1029.46126

[Ma 2003] J.-P. Ma, "A rank theorem for operators between Banach spaces", *Chinese Ann. Math. Ser. A* **24**:6 (2003), 669–674. MR 2005h:47003 Zbl 1066.47002

[Ma 2007] J.-P. Ma, "Three classes of smooth Banach submanifolds in $B(E, F)$", *Sci. China Ser. A* **50**:9 (2007), 1233–1239. MR 2370611 Zbl 05221809

[Ma 2008] J.-P. Ma, "Complete rank theorem of advanced calculus and singularities of bounded linear operators", *Front. Math. China* **3**:2 (2008), 1–12.

[Nashed and Chen 1993] M. Z. Nashed and X. Chen, "Convergence of Newton-like methods for singular operator equations using outer inverses", *Numer. Math.* **66**:2 (1993), 235–257. MR 95a:65102 Zbl 0797.65047

[Zeidler 1988] E. Zeidler, *Nonlinear functional analysis and its applications, IV: Applications to mathematical physics*, Springer, New York, 1988. MR 90b:00004 Zbl 0648.47036

Received August 30, 2007.

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