Pacific Journal of Mathematics

NEW EXAMPLES OF W_r-MINIMAL HYPERSURFACES IN A SPHERE

GUOXIN WEI

Volume 236 No. 2 June 2008

PACIFIC JOURNAL OF MATHEMATICS
Vol. 236, No. 2, 2008

NEW EXAMPLES OF W_r-MINIMAL HYPERSURFACES IN A SPHERE

GUOXIN WEI

A hypersurface $x : M^n \to S^{n+1}(1)$ is called a W_r-minimal hypersurface if it is a critical hypersurface of the Generalized Willmore functional. In this paper, we give some new nontrivial examples of W_r-minimal hypersurfaces of $S^{n+1}(1)$.

1. Introduction

Let M be an n-dimensional compact hypersurface of the $(n+1)$-dimensional unit sphere $S^{n+1}(1)$. If h_{ij} denotes the components of the second fundamental form of M, then we can choose a proper basis for TM such that $h_{ij} = \lambda_i \delta_{ij}$, where the λ_i are the principal curvatures of M. Then the r-th mean curvature σ_r of M is defined by

$$C_n^r \sigma_r = \sum_{1 \le i_1 < \cdots < i_r \le n} \lambda_{i_1} \cdots \lambda_{i_r} \quad \text{for } r = 1, \ldots, n, \tag{1-1}$$

where $C_n^r = n!/(r!(n-r)!)$ is a binomial coefficient. For convenience, we define $\sigma_0 = 1$. When $\sigma_k = 0$, a hypersurface M is said to be *k-minimal*.

The *Generalized Willmore functional* is the functional

$$W_r(M) = \begin{cases} \int_M Q_r^{n/r} dv & \text{if } r \text{ is odd and } 3 \le r < n, \\ \int_M Q_r^{n/r} dv & \text{if } r \text{ is even and } 2 \le r < n \text{ and } Q_r \ge 0, \\ \int_M Q_n dv & \text{if } 2 \le r = n, \end{cases}$$

where

$$Q_r = \sum_{k=0}^{r} (-1)^{k+1} C_r^k \sigma_1^{r-k} \sigma_k. \tag{1-2}$$

When $r = 2$, we know that $Q_2 = \sigma_1^2 - \sigma_2$ is a nonnegative function on M, and the functional

$$W_2(M) = \int_M (\sigma_1^2 - \sigma_2)^{n/2} dv$$

MSC2000: primary 53A30; secondary 53C20.
Keywords: W_r-minimal hypersurface, Willmore hypersurface, rotational hypersurface.

is called the *Willmore functional* (see [Chen 1974; Wang 1998; Li 2001]). B. Y. Chen [1974] and C. P. Wang [1998] proved that the Willmore functional is an invariant under conformal transformations of S^{n+1}. Its critical points are called *Willmore hypersurfaces*. In particular, if $n = 2$, the critical points are called Willmore surfaces. There has been important progress on Willmore hypersurfaces in recent years. For example, R. Bryant proved a duality theorem for Willmore surfaces. H. Li [2001] proved an integral inequality of Simons type for Willmore hypersurfaces.

For general $2 \le r \le n$, it was shown in [Guo 2007] that the functional $W_r(M)$ is also an invariant under conformal transformations of S^{n+1}. A hypersurface $x : M^n \to S^{n+1}(1)$ is called a *W_r-minimal hypersurface* if it is a critical hypersurface of the Generalized Willmore functional W_r. Guo also proved that M is a W_r-minimal hypersurface if and only if M satisfies

$$
\begin{aligned}
\text{(1-3)}\quad & \triangle\big(Q_r^{(n-r)/r}(Q_{r-1}+\sigma_1^{r-1})\big)\\
&\quad + (C_{n-1}^{r-1})^{-1}\sum_{k=2}^{r}(-1)^{k+1}C_{n-k}^{r-k}T_{(k-1)ij}\big(Q_r^{(n-r)/r}\sigma_1^{r-k}\big)_{,ij}\\
&\quad + Q_r^{(n-r)/r}(n^2\sigma_1^2-n(n-1)\sigma_2+n)(Q_{r-1}+\sigma_1^{r-1}) - n\sigma_1 Q_r^{(n)/r}\\
&+ Q_r^{(n-r)/r}(C_{n-1}^{r-1})^{-1}\sum_{k=2}^{r}(-1)^{k+1}C_{n-k}^{r-k}C_n^k\sigma_1^{r-k}(n\sigma_1\sigma_k-(n-k)\sigma_{k+1}+k\sigma_{k-1}) = 0,
\end{aligned}
$$

where $\triangle$ is the Laplacian, $(\,\cdot\,)_{,ij}$ is the covariant derivative relative to the induced metric, and the $T_{(k)ij}$ are the components of the *k-th Newton transformation* $T_{(k)}$; see [Cao and Li 2007; Reilly 1973].

Equation (1-3) is such a complicated equation to deal with that people know few examples of W_r-minimal hypersurfaces in $S^{n+1}(1)$; few examples are known even for W_2-minimal hypersurfaces (that is, Willmore hypersurfaces). H. Li and L. Vrancken [2003] got some new examples of Willmore surfaces in a sphere. In this paper, we obtain numerous nontrivial examples of W_r-minimal hypersurfaces. In fact, we show two theorems:

Theorem 1.1. *For $n \ge 3$, let M be an n-dimensional compact $(n-1)$-minimal rotational hypersurface in $S^{n+1}(1)$. Then M is a W_r-minimal hypersurface.*

Theorem 1.2. *For $n \ge 3$ and $1 \le j \le n-2$, there are no compact j-minimal rotational and W_r-minimal hypersurfaces of S^{n+1} other than round geodesic spheres.*

Remark. From [Palmas 1999] and [Wei 2007], we know that there exist many compact immersed k-minimal rotational hypersurfaces in a unit sphere $S^{n+1}(1)$ for $1 \le k \le n-1$.

Remark. It is easy to verify that only the hypersurface in Theorem 1.1 conformally equivalent to the hypersurface $S^1(\sqrt{(n-1)/n}) \times S^{n-1}(\sqrt{1/n})$ is that hypersurface itself.

Remark. When $r = 2$, both theorems reduce to theorems due to G. Wei [2008].

2. Preliminaries

In this section, let us introduce *rotational hypersurfaces* in a sphere. Let M be a rotational hypersurface of S^{n+1}, that is, one left invariant by the orthogonal group $O(n)$ considered as a subgroup of isometries of $S^{n+1}(1)$. Let us parametrize the profile curve α in $S^2(1)$ by $y_1 = y_1(s) \geq 0$, $y_{n+1} = y_{n+1}(s)$, and $y_{n+2} = y_{n+2}(s)$. We take $\varphi(t_1, \dots, t_{n-1}) = (\varphi_1, \dots, \varphi_n)$ to be an orthogonal parametrization of the unit sphere $S^{n-1}(1)$. It follows that the rotational hypersurface (see [do Carmo and Dajczer 1983; Leite 1990]) $x : M^n \hookrightarrow S^{n+1}(1) \subset R^{n+2}$ defined by

$$(s, t_1, \dots, t_{n-1}) \mapsto (y_1(s)\varphi_1, \dots, y_1(s)\varphi_n, y_{n+1}(s), y_{n+2}(s)),$$
$$\varphi_i = \varphi_i(t_1, \dots, t_{n-1}), \quad \text{and} \quad \varphi_1^2 + \dots + \varphi_n^2 = 1$$

is a parametrization of a rotational hypersurface generated by a curve $y_1 = y_1(s)$, $y_{n+1} = y_{n+1}(s)$, and $y_{n+2} = y_{n+2}(s)$. Since the curve $\{y_1(s), y_{n+1}(s), y_{n+2}(s)\}$ belongs to $S^2(1)$ and the parameter s can be chosen as its arc length, we have

$$y_1^2(s) + y_{n+1}^2(s) + y_{n+2}^2(s) = 1 \quad \text{and} \quad \dot{y}_1^2(s) + \dot{y}_{n+1}^2(s) + \dot{y}_{n+2}^2(s) = 1, \tag{2-1}$$

where the dot denotes the derivative with respect to s. From (2-1) we can obtain $y_{n+1}(s)$ and $y_{n+2}(s)$ as functions of $y_1(s)$. In fact, we can write

$$\begin{aligned} y_1(s) &= \cos r(s), \\ y_{n+1}(s) &= \sin r(s) \cos\theta(s), \\ y_{n+2}(s) &= \sin r(s) \sin\theta(s). \end{aligned} \tag{2-2}$$

We can deduce from (2-1) and (2-2) that

$$\dot{r}^2 + \dot{\theta}^2 \sin^2 r = 1. \tag{2-3}$$

It follows from (2-3) that $\dot{r}^2 \leq 1$. Combining these with $\dot{r}^2 = \dot{y}_1^2/(1-y_1^2)$, we have $\dot{y}_1^2 + y_1^2 \leq 1$.

Writing $f(s) = y_1(s)$ gives a theorem:

Lemma 2.1 [do Carmo and Dajczer 1983]. *Let M^n be a rotational hypersurface of $S^{n+1}(1)$. Then the principal curvatures λ_i of M^n are*

$$\lambda_i = \lambda = -\frac{\sqrt{1 - f^2 - \dot{f}^2}}{f} \tag{2-4}$$

for $i = 1, \dots, n-1$ *and*

$$\lambda_n = \mu = \frac{\ddot{f} + f}{\sqrt{1 - f^2 - \dot{f}^2}}. \tag{2-5}$$

If M is a k-minimal rotational hypersurface in $S^{n+1}(1)$ with $k < n$, then we can deduce that

$$0 = C_n^k \sigma_k = C_{n-1}^{k-1} \lambda^{k-1} \mu + C_{n-1}^k \lambda^k.$$

That is,

$$\lambda^{k-1}\{(n-k)\lambda + k\mu\} = 0. \tag{2-6}$$

Putting (2-4) and (2-5) into (2-6) gives another theorem:

Lemma 2.2 [Palmas 1999]. *The rotational hypersurface M^n in $S^{n+1}(1)$ is k-minimal with $k < n$ if and only if f satisfies the differential equation*

$$(n-k)(1 - f^2 - \dot{f}^2)^{k/2} - k(1 - f^2 - \dot{f}^2)^{(k-2)/2}(\ddot{f} + f) f = 0. \tag{2-7}$$

Equation (2-7) is equivalent to its first order integral

$$f^{n-k}(1 - f^2 - \dot{f}^2)^{k/2} = K, \tag{2-8}$$

where K is a constant.

For a constant solution $f = f_0$ in (2-7), one has that

$$f_0^2 = \frac{n-k}{n} \quad \text{and} \quad K_0 = \left(\frac{k}{n}\right)^{k/2} \left(\frac{n-k}{n}\right)^{(n-k)/2}.$$

Moreover, the constant solutions of Equation (2-7) correspond to the Riemannian product $S^1(\sqrt{k/n}) \times S^{n-1}(\sqrt{(n-k)/n})$.

Equation (2-8) tells us that a local solution f of (2-7) paired with its first derivative is a subset, denoted by $(f, \dot{f})$, of a level curve of the function G_k defined by

$$G_k(u, v) = u^{n-k}(1 - u^2 - v^2)^{k/2}, \tag{2-9}$$

with $u > 0$ and $u^2 + v^2 \le 1$.

Let us map the open half plane $\{(u, v) \mid u > 0\}$ by the level curve $G_k = K$. See Figure 1.

Each curve is a smooth union of two graphs

$$v = \pm\sqrt{1 - u^2 - \left(\frac{K}{u^{n-k}}\right)^{2/k}},$$

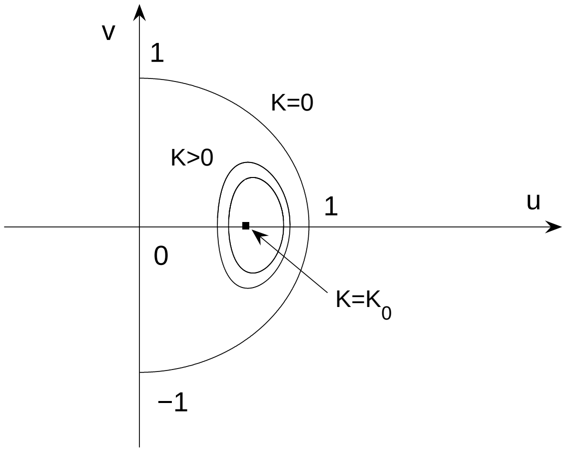

Figure 1. Level curves for $K \geq 0$.

except for the level K_0 given by (2-9). The level curve $G_k = K_0$ consists of the unique critical point of G_k, which is on the horizontal axis, as can be seen from

$$\nabla G_k(u, v) = u^{n-k-1}(1-u^2-v^2)^{(k-2)/2}((n-k)(1-v^2)-nu^2, -kuv).$$

For $K = 0$, the level curve $u^2 + v^2 = 1$ is a semicircle. For $K \neq 0$, we can get easily that the level curve is closed in the open half plane (in fact, in the semicircular region; see Figure 1).

We consider the foliation of the open half plane by level curves $G_k = K$. Since G_m has a maximum at K_0, we know $K \in [0, K_0]$. Clearly any curve at an intermediate level K is compact and the associated solution $r(s)$ attains a unique minimum $r_1 > 0$.

Now we have to consider two cases.

Case 1: $K = 0$. This gives us a totally geodesic n-sphere. In fact, from $K = 0$ and Equation (2-8), we get $f^2 + \dot{f}^2 = 1$. Integration of $f^2 + \dot{f}^2 = 1$ with $f(0) = 0$, we obtain $f = \sin s$ and $\theta =$ constant, so the profile curve is a great circle which generates a totally geodesic n-sphere.

Case 2: $K \in (0, K_0]$. In this case, we have

$$f^2 + \dot{f}^2 < 1 \quad \text{and } 0 < f < 1. \tag{2-10}$$

We then claim that M has two distinct principal curvatures, that is, $\lambda \neq \mu$. In fact, if $\lambda = \mu$, then we see from (2-4), (2-5), and (2-10) that

$$-(\ddot{f} + f)f = 1 - f^2 - \dot{f}^2. \tag{2-11}$$

Then from Equation (2-7) and (2-10), we obtain that

$$(n-k)(1-f^2-\dot{f}^2)-k(\ddot{f}+f)f=0. \tag{2-12}$$

By (2-11) and (2-12), we have $n(1-f^2-\dot{f}^2)=0$. This contradicts (2-10) and hence proves our claim.

3. The rotational k-minimal hypersurfaces in Case 2

In this section, we will recall some basic formulas for submanifolds of a sphere; see [Cheng 2001; Li 1996]. Let M be an n-dimensional compact k-minimal rotational hypersurface in $S^{n+1}(1)$. For any $p\in M$, we choose a local orthonormal frame $e_1,\dots,e_n,e_{n+1}$ in $S^{n+1}(1)$ around p such that $e_1,\dots,e_n$ are tangent to M. Take the corresponding dual coframe $\omega_1,\dots,\omega_n,\omega_{n+1}$. We fix the following conventions for the ranges of indices:

$$1\le A,B,C\le n+1;\quad 1\le i,j,k\le n;\quad 1\le a,b,c\le n-1.$$

The structure equations of $S^{n+1}(1)$ are

$$d\omega_A=\textstyle\sum_B\omega_{AB}\wedge\omega_B\quad\text{with } \omega_{AB}=-\omega_{BA},$$
$$d\omega_{AB}=\textstyle\sum_C\omega_{AC}\wedge\omega_{CB}-\omega_A\wedge\omega_B.$$

Restricted to M, we have $\omega_{n+1}=0$; thus

$$0=d\omega_{n+1}=\textstyle\sum_i\omega_{n+1i}\wedge\omega_i.$$

From Cartan's lemma, we obtain

$$\omega_{in+1}=\textstyle\sum_j h_{ij}\omega_j=\lambda_i\omega_i, \tag{3-1}$$

where $h_{ij}=h_{ji}=\lambda_i\delta_{ij},\ \lambda_1=\dots=\lambda_{n-1}=\lambda$, and $\lambda_n=\mu$.

Then the structure equation of M is

$$d\omega_i=\textstyle\sum_j\omega_{ij}\wedge\omega_j\quad\text{with } \omega_{ij}=-\omega_{ji},$$
$$d\omega_{ij}=\textstyle\sum_k\omega_{ik}\wedge\omega_{kj}-\frac{1}{2}R_{ijkl}\omega_k\wedge\omega_l,$$

where R_{ijkl} is the curvature tensor of the induced metric on M.

The Gauss equation is

$$R_{ijkl}=(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})+(h_{ik}h_{jl}-h_{il}h_{jk}),$$
$$n(n-1)r=n(n-1)+n^2H^2-S,$$

where r is the normalized scalar curvature, $H=\frac{1}{n}\sum_i h_{ii}$ is the mean curvature, and $S=\sum_{i,j}h_{ij}^2$ is the norm square of the second fundamental form of M.

The Codazzi equations are $h_{ijk} = h_{ikj}$, where the covariant derivative of h_{ij} is defined by

$$\sum_k h_{ijk}\omega_k = dh_{ij} + \sum_k h_{kj}\omega_{ki} + \sum_k h_{ik}\omega_{kj}. \tag{3-2}$$

The second covariant derivative of h_{ij} is defined by

$$\sum_l h_{ijkl}\omega_l = dh_{ijk} + \sum_l h_{ljk}\omega_{li} + \sum_l h_{ilk}\omega_{lj} + \sum_l h_{ijl}\omega_{lk}. \tag{3-3}$$

By exterior differentiation of (3-2), we have the Ricci identities

$$h_{ijkl} - h_{ijlk} = \sum_m h_{mj}R_{mikl} + \sum_m h_{im}R_{mjkl}.$$

In Case 2, we know from Section 2 that M has two distinct principal curvatures, that is, $\lambda \neq \mu$.

From (2-4) and (2-10), we can obtain that

$$\lambda \neq 0. \tag{3-4}$$

We see from (2-6) and (3-4) that

$$(n-k)\lambda + k\mu = 0. \tag{3-5}$$

Lemma 3.1 [Ôtsuki 1970, p. 150]. *Let M be an n-dimensional compact hypersurface in a unit sphere $S^{n+1}(1)$ such that the multiplicities of principal curvatures are all constant. Then the distribution of the space of principal vectors corresponding to each principal curvature is completely integrable. In particular, if the multiplicity of a principal curvature is greater than* 1, *then this principal curvature is constant on each integral submanifold of the corresponding distribution of the space of principal vectors.*

By Lemma 3.1 and (3-5), we have

$$\lambda_{,1} = \cdots = \lambda_{,n-1} = 0 \quad \text{and} \quad \mu_{,1} = \cdots = \mu_{,n-1} = 0. \tag{3-6}$$

By means of (3-2), we obtain

$$h_{ijk}\omega_k = \delta_{ij}d\lambda_j + (\lambda_i - \lambda_j)\omega_{ij}. \tag{3-7}$$

Summarizing the arguments above, we obtain

$$\begin{aligned} h_{ijk} &= 0 \quad \text{if } i \neq j \text{ and } \lambda_i = \lambda_j, \\ h_{aab} &= 0, \quad h_{aan} = \lambda_{,n}, \\ h_{nna} &= 0, \quad h_{nnn} = \mu_{,n}. \end{aligned} \tag{3-8}$$

By using methods similar to those in [Ôtsuki 1970], we can prove this:

Proposition 3.1. *Let M be an n-dimensional k-minimal hypersurface in $S^{n+1}(1)$ with $n \geq 3$ and $k < n$ and with two distinct principal curvatures λ and μ whose multiplicities are $n-1$ and 1, respectively. Then M is a locus of the moving $(n-1)$-dimensional submanifold $M_1^{n-1}(s)$ along which the principal curvature λ is constant. $M_1^{n-1}(s)$ is locally isometric to an $(n-1)$-sphere $S^{n-1}(c(s)) = E^n(s) \cap S^{n+1}(1)$ of constant curvature; λ satisfies the second order ordinary differential equation*

$$\frac{d^2\lambda}{ds^2} = \frac{n+k}{n\lambda}\Big(\frac{d\lambda}{ds}\Big)^2 - \frac{n(n-k)\lambda^3}{k^2} + \frac{n\lambda}{k}, \tag{3-9}$$

where $E^n(s)$ is an n-dimensional linear subspace in the Euclidean space R^{n+2} which is parallel to a fixed E^n.

4. Proofs of the theorems

Proof of Theorem 1.1. Let M be an n-dimensional compact $(n-1)$-minimal rotational hypersurface in $S^{n+1}(1)$ with $n \geq 3$. From Section 2, we know that we have to consider two cases.

Case 1: M is a totally geodesic n-sphere, that is, $h_{ij} = 0$, $\sigma_1 = \cdots = \sigma_n = 0$ and $Q_1 = \cdots = Q_n = 0$; it follows that (1-3) holds. Hence, we can easily get that M is a W_k-minimal hypersurface.

Case 2: M has two distinct principal curvatures $\lambda_1 = \cdots = \lambda_{n-1} = \lambda$ and $\lambda_n = \mu$; moreover $\lambda \neq 0$ and $\mu = -\lambda/(n-1)$.

From (3-1) and (3-3), we have

$$\lambda_{,ij}\omega_j = d\lambda_{,i} + \lambda_{,j}\omega_{ji}. \tag{4-1}$$

By using (3-7), (3-8),we obtain $\omega_{an} = \lambda_{,n}\omega_a/(\lambda-\mu)$.

Therefore, we have $d\omega_n = \sum_i \omega_{ni} \wedge \omega_i = 0$, which shows that we may put $\omega_n = ds$.

Then we have

$$\omega_{an} = \frac{(n-1)\lambda_{,n}}{n\lambda}\omega_a = (\log\lambda^{(n-1)/n})'\omega_a,$$

where the prime denotes the derivative with respect to s.

Letting $i = a$ in (4-1), we see from (3-6) and (3-8) that

$$\begin{aligned}\lambda_{,aj}\omega_j &= d\lambda_{,a} + \lambda_{,j}\omega_{ja} = \lambda_{,n}\omega_{na}\\ &= \lambda_{,n}\frac{\lambda_{,n}}{\mu-\lambda}\omega_a = -\frac{(n-1)}{n\lambda}(\lambda_{,n})^2\omega_a.\end{aligned}$$

It follows that

$$
(4\text{-}2)\qquad \begin{aligned} &\lambda_{,aa} = -\frac{(n-1)}{n\lambda}(\lambda_{,n})^2 && \text{for } 1 \le a \le n-1,\\ &\lambda_{,al} = 0 && \text{if } a \ne l,\ \ 1 \le a \le n-1, \text{ and } 1 \le l \le n. \end{aligned}
$$

Letting $i = n$ in (4-1), we know from (3-6) and (3-9) that

$$
\begin{aligned} \lambda_{,nj}\omega_j &= d\lambda_{,n} + \lambda_{,j}\omega_{jn} = d\lambda_{,n}\\ &= \Big(\frac{2n-1}{n\lambda}(\lambda_{,n})^2 - \frac{n\lambda^3}{(n-1)^2} + \frac{n\lambda}{n-1}\Big)\omega_n, \end{aligned}
$$

it follows that

$$
(4\text{-}3)\qquad \lambda_{,na} = \begin{cases} 0 & \text{if } 1 \le a \le n-1.\\ \dfrac{2n-1}{n\lambda}(\lambda_{,n})^2 - \dfrac{n\lambda^3}{(n-1)^2} + \dfrac{n\lambda}{n-1} & \text{if } a = n. \end{cases}
$$

In this case, we see from (3-4), (3-5), (1-1), and (1-2) that $\sigma_1 = \lambda(n-2)/(n-1)$,

$$
(4\text{-}4)\qquad \begin{aligned} \sigma_k &= \frac{1}{C_n^k}\sum_{1\le i_1<\dots<i_k\le n} \lambda_{i_1}\dots\lambda_{i_k}\\ &= \frac{1}{C_n^k}\{C_{n-1}^{k-1}\lambda^{k-1}\mu + C_{n-1}^k\lambda^k\} = \frac{n-1-k}{n-1}\lambda^k, \end{aligned}
$$

and

$$
(4\text{-}5)\qquad \begin{aligned} Q_r &= \sum_{k=0}^{r}(-1)^{k+1}C_r^k\sigma_1^{r-k}\sigma_k\\ &= \sum_{k=0}^{r}(-1)^{k+1}C_r^k\Big(\frac{n-2}{n-1}\Big)^{r-k}\frac{n-1-k}{n-1}\lambda^r\\ &= (-1)^r(r-1)\Big(\frac{1}{n-1}\Big)^r\lambda^r \ne 0. \end{aligned}
$$

It is sufficient to prove

$$
(4\text{-}6)\qquad \begin{aligned} &\triangle(Q_r^{(n-r)/r}(Q_{r-1}+\sigma_1^{r-1}))\\ &\qquad + (C_{n-1}^{r-1})^{-1}\sum_{k=2}^{r}(-1)^{k+1}C_{n-k}^{r-k}T_{(k-1)ij}(Q_r^{(n-r)/r}\sigma_1^{r-k})_{,ij} = 0 \end{aligned}
$$

and

$$
(4\text{-}7)\qquad \begin{aligned} &(n^2\sigma_1^2 - n(n-1)\sigma_2 + n)(Q_{r-1}+\sigma_1^{r-1}) - n\sigma_1 Q_r\\ &\quad + (C_{n-1}^{r-1})^{-1}\sum_{k=2}^{r}(-1)^{k+1}C_{n-k}^{r-k}C_n^k\sigma_1^{r-k}(n\sigma_1\sigma_k - (n-k)\sigma_{k+1} + k\sigma_{k-1}) = 0. \end{aligned}
$$

By a direct calculation, we see from (4-4) and (4-5) that

$$
\begin{aligned}
(C_{n-1}^{r-1})^{-1}&\sum_{k=2}^{r}(-1)^{k+1}C_{n-k}^{r-k}C_n^k\sigma_1^{r-k}(n\sigma_1\sigma_k-(n-k)\sigma_{k+1})\\
&=\sum_{k=2}^{r}(-1)^{k+1}\frac{n^2(r-1)(r-2)\cdots(r-k+1)}{k!}\Big(\frac{n-2}{n-1}\Big)^{r-k+1}\frac{n-1-k}{n-1}\lambda^{r+1}\\
&\quad+\sum_{k=2}^{r}(-1)^{k}\frac{n(n-2-k)(n-k)}{n-1}\frac{(r-1)(r-2)\cdots(r-k+1)}{k!}\Big(\frac{n-2}{n-1}\Big)^{r-k}\lambda^{r+1}\\
&=\frac{n(-n^2+3n-3)}{(n-1)^2}\Big((-1)^r\Big(\frac{1}{n-1}\Big)^{r-1}+\Big(\frac{n-2}{n-1}\Big)^{r-1}\Big)\lambda^{r+1}\\
&\qquad\qquad+(-1)^r\frac{n(r-1)}{n-1}\Big(\frac{1}{n-1}\Big)^{r-2}\lambda^{r+1},\\
(C_{n-1}^{r-1})^{-1}&\sum_{k=2}^{r}(-1)^{k+1}C_{n-k}^{r-k}C_n^k\sigma_1^{r-k}k\sigma_{k-1}\\
&=\sum_{k=2}^{r}(-1)^{k+1}\frac{n(r-1)(r-2)\cdots(r-k+1)}{(k-1)!}\Big(\frac{n-2}{n-1}\Big)^{r-k}\frac{n-k}{n-1}\lambda^{r-1}\\
&=(-1)^r n(r-2)\Big(\frac{1}{n-1}\Big)^{r-1}\lambda^{r-1}-\frac{n(n-r)}{n-1}\lambda^{r-1},
\end{aligned}
$$

and

$$
\begin{aligned}
(n^2\sigma_1^2&-n(n-1)\sigma_2+n)(Q_{r-1}+\sigma_1^{r-1})-n\sigma_1Q_r\\
&=\Big(n^2\Big(\frac{n-2}{n-1}\Big)^2\lambda^2-n(n-1)\frac{n-3}{n-1}\lambda^2\Big)\\
&\qquad\times\Big((-1)^{r-1}(r-2)\Big(\frac{1}{n-1}\Big)^{r-1}\lambda^{r-1}+\Big(\frac{n-2}{n-1}\Big)^{r-1}\lambda^{r-1}\Big)\\
&\quad+n\Big((-1)^{r-1}(r-2)\Big(\frac{1}{n-1}\Big)^{r-1}\lambda^{r-1}+\Big(\frac{n-2}{n-1}\Big)^{r-1}\lambda^{r-1}\Big)\\
&\qquad\qquad+(-1)^{r+1}n\frac{n-2}{n-1}(r-1)\Big(\frac{1}{n-1}\Big)^{r}\lambda^{r+1}\\
&=\frac{n(n^2-3n+3)}{(n-1)^2}\Big((-1)^{r-1}(r-2)\Big(\frac{1}{n-1}\Big)^{r-1}+\Big(\frac{n-2}{n-1}\Big)^{r-1}\Big)\lambda^{r+1}\\
&\quad+(-1)^{r+1}n(r-1)\frac{n-2}{n-1}\Big(\frac{1}{n-1}\Big)^{r}\lambda^{r+1}+(-1)^{r-1}n(r-2)\Big(\frac{1}{n-1}\Big)^{r-1}\lambda^{r-1}\\
&\qquad\qquad+n\Big(\frac{n-2}{n-1}\Big)^{r-1}\lambda^{r-1}.
\end{aligned}
$$

Using (4-4), (4-5), and the last three equations, we obtain

$$(n^2\sigma_1^2 - n(n-1)\sigma_2 + n)(Q_{r-1} + \sigma_1^{r-1}) - n\sigma_1 Q_r + (C_{n-1}^{r-1})^{-1}\sum_{k=2}^{r}(-1)^{k+1}C_{n-k}^{r-k}C_n^k\sigma_1^{r-k}(n\sigma_1\sigma_k - (n-k)\sigma_{k+1} + k\sigma_{k-1}) = 0.$$

Note that

$$\triangle(\lambda^{n-1}) = (n-1)(n-2)\lambda^{n-3}\sum_{i=1}^{n}(\lambda_{,i})^2 + (n-1)\lambda^{n-2}\triangle\lambda = (n-1)(n-2)\lambda^{n-3}(\lambda_{,n})^2 + (n-1)\lambda^{n-2}\triangle\lambda \tag{4-8}$$

and

$$\begin{aligned}(\lambda^{n-k})_{,aa} &= (n-k)(n-k-1)\lambda^{n-k-2}(\lambda_{,a})^2 + (n-k)\lambda^{n-k-1}\lambda_{,aa}\\ &= (n-k)\lambda^{n-k-1}\lambda_{,aa}\\ &= -\frac{(n-k)(n-1)}{n\lambda}\lambda^{n-k-1}(\lambda_{,n})^2,\\ (\lambda^{n-k})_{,nn} &= (n-k)(n-k-1)\lambda^{n-k-2}(\lambda_{,n})^2 + (n-k)\lambda^{n-k-1}\lambda_{,nn}.\end{aligned} \tag{4-9}$$

Next we will prove that Equation (4-6) holds.

We recall the *k-th Newton transformation* defined by

$$T_{(k)} = s_k I - s_{k-1}A + \cdots + (-1)^{k-1}s_1 A^{k-1} + (-1)^k A^k \quad \text{for } k = 0, 1, \ldots, n,$$

where $A = (h_{ij})$ and $s_k = C_n^k\sigma_k$. Then we know that the matrix of $T_{(k)}$ is (also see [Cao and Li 2007; Reilly 1973])

$$T_{(r)ij} = \frac{1}{r!}\delta_{j_1\ldots j_r j}^{i_1\ldots i_r i}h_{i_1 j_1}\cdots h_{i_r j_r}, \tag{4-10}$$

where $\delta_{j_1\ldots j_r j}^{i_1\ldots i_r i}$ is the generalized Kronecker symbol. If its i's and j's are integers between 1 and n, then $\delta_{j_1\ldots j_r j}^{i_1\ldots i_r i}$ is $+1$ or -1 if the i's are distinct and the j's are an even or odd permutation, respectively, of the i's. It is zero in all other cases.

Since $h_{i_k j_k} = \lambda_{i_k}\delta_{i_k j_k}$ and from the definition of $\delta_{j_1\ldots j_r j}^{i_1\ldots i_r i}$, we know that

$$T_{(r)ij} = 0 \quad \text{if } i \neq j.$$

From $\lambda + (n-1)\mu = 0$ and (4-10), we obtain

$$\begin{aligned}T_{(r)11} &= \cdots = T_{(r)n-1n-1}\\ &= \frac{1}{r!}\delta_{i_1\ldots i_r 1}^{i_1\ldots i_r 1}h_{i_1 i_1}\cdots h_{i_r i_r} = C_{n-2}^{r-1}\lambda^{r-1}\mu + C_{n-2}^r\lambda^r\\ &= C_{n-2}^r\lambda^r - \frac{1}{n-1}C_{n-2}^{r-1}\lambda^r.\end{aligned} \tag{4-11}$$

and $T_{(r)nn} = (1/r!)\delta^{i_1\dots i_r n}_{i_1\dots i_r n} h_{i_1 i_1}\cdots h_{i_r i_r} = C^r_{n-1}\lambda^r$.

On one hand, we can deduce from (4-8) that

$$
\begin{aligned}
\text{(4-12)}\quad & \triangle(Q_r^{(n-r)/r}(Q_{r-1}+\sigma_1^{r-1})) \\
&= \triangle\bigg(\Big((-1)^r(r-1)\Big(\frac{1}{n-1}\Big)^r\Big)^{(n-r)/r} \\
&\qquad\qquad \times\Big((-1)^{r-1}(r-2)\Big(\frac{1}{n-1}\Big)^{r-1}+\Big(\frac{n-2}{n-1}\Big)^{r-1}\Big)\lambda^{n-1}\bigg) \\
&= (-1)^{n-r}\Big((r-1)\Big(\frac{1}{n-1}\Big)^r\Big)^{(n-r)/r} \\
&\qquad\qquad \times\Big((-1)^{r-1}(r-2)\Big(\frac{1}{n-1}\Big)^{r-1}+\Big(\frac{n-2}{n-1}\Big)^{r-1}\Big)\triangle\lambda^{n-1} \\
&= (-1)^{n-r}\Big((r-1)\Big(\frac{1}{n-1}\Big)^r\Big)^{(n-r)/r} \\
&\qquad\qquad \times\Big((-1)^{r-1}(r-2)\Big(\frac{1}{n-1}\Big)^{r-1}+\Big(\frac{n-2}{n-1}\Big)^{r-1}\Big) \\
&\qquad\qquad \times((n-1)(n-2)\lambda^{n-3}(\lambda_{,n})^2+(n-1)\lambda^{n-2}((n-1)\lambda_{,11}+\lambda_{,nn})).
\end{aligned}
$$

Using (4-11), one can easily check that

$$
\begin{aligned}
\text{(4-13)}\quad & (C^{r-1}_{n-1})^{-1}\sum_{k=2}^{r}(-1)^{k+1}C^{r-k}_{n-k}T_{(k-1)ij}(Q_r^{(n-r)/r}\sigma_1^{r-k})_{,ij} \\
&= (C^{r-1}_{n-1})^{-1}\sum_{k=2}^{r}(-1)^{k+1}C^{r-k}_{n-k}T_{(k-1)ii}(Q_r^{(n-r)/r}\sigma_1^{r-k})_{,ii} \\
&= (C^{r-1}_{n-1})^{-1}\sum_{k=2}^{r}(-1)^{k+1}C^{r-k}_{n-k}\Big((n-1)T_{(k-1)11}(Q_r^{(n-r)/r}\sigma_1^{r-k})_{,11} \\
&\qquad\qquad + T_{(k-1)nn}(Q_r^{(n-r)/r}\sigma_1^{r-k})_{,nn}\Big) \\
&= (-1)^{n-r}(C^{r-1}_{n-1})^{-1}\Big((r-1)\Big(\frac{1}{n-1}\Big)^r\Big)^{(n-r)/r} \\
&\quad \times\Big(\sum_{k=2}^{r}(-1)^{k+1}C^{r-k}_{n-k}((n-1)C^{k-1}_{n-2}-C^{k-2}_{n-2})\Big(\frac{n-2}{n-1}\Big)^{r-k}\Big)\lambda^{k-1}(\lambda^{n-k})_{,11} \\
&\quad + (-1)^{n-r}(C^{r-1}_{n-1})^{-1}\Big((r-1)\Big(\frac{1}{n-1}\Big)^r\Big)^{(n-r)/r} \\
&\qquad\qquad \times\Big(\sum_{k=2}^{r}(-1)^{k+1}C^{r-k}_{n-k}C^{k-1}_{n-1}\Big(\frac{n-2}{n-1}\Big)^{r-k}\lambda^{k-1}(\lambda^{n-k})_{,nn}\Big).
\end{aligned}
$$

Using (4-12) and (4-13), we get that Equation (4-6) is equivalent to

$$\Big((-1)^{r-1}(r-2)\Big(\frac{1}{n-1}\Big)^{r-1}+\Big(\frac{n-2}{n-1}\Big)^{r-1}\Big) \\ \times\big((n-1)(n-2)\lambda^{n-3}(\lambda_{,n})^2+(n-1)\lambda^{n-2}((n-1)\lambda_{,11}+\lambda_{,nn})\big) \\ +(C_{n-1}^{r-1})^{-1}\Big(\sum_{k=2}^{r}(-1)^{k+1}C_{n-k}^{r-k}((n-1)C_{n-2}^{k-1}-C_{n-2}^{k-2})\Big(\frac{n-2}{n-1}\Big)^{r-k}\Big)\lambda^{k-1}(\lambda^{n-k})_{,11} \\ +(C_{n-1}^{r-1})^{-1}\Big(\sum_{k=2}^{r}(-1)^{k+1}C_{n-k}^{r-k}C_{n-1}^{k-1}\Big(\frac{n-2}{n-1}\Big)^{r-k}\lambda^{k-1}(\lambda^{n-k})_{,nn}\Big)=0,$$

that is,

$$\Big((-1)^{r-1}(r-2)\Big(\frac{1}{n-1}\Big)^{r-1}+\Big(\frac{n-2}{n-1}\Big)^{r-1}\Big) \\ \times\big((n-1)(n-2)\lambda^{n-3}(\lambda_{,n})^2+(n-1)\lambda^{n-2}[(n-1)\lambda_{,11}+\lambda_{,nn}]\big) \\ +(C_{n-1}^{r-1})^{-1}\sum_{k=2}^{r}(-1)^{k+1}C_{n-k}^{r-k}((n-1)C_{n-2}^{k-1}-C_{n-2}^{k-2})\Big(\frac{n-2}{n-1}\Big)^{r-k}(n-k)\lambda^{n-2}\lambda_{,11} \\ +(C_{n-1}^{r-1})^{-1}\sum_{k=2}^{r}(-1)^{k+1}C_{n-k}^{r-k}C_{n-1}^{k-1}\Big(\frac{n-2}{n-1}\Big)^{r-k}\lambda^{k-1} \\ \times\big((n-k)(n-k-1)\lambda^{n-k-2}(\lambda_{,n})^2+(n-k)\lambda^{n-k-1}\lambda_{,nn}\big)=0. \tag{4-14}$$

From (4-2), (4-3), (4-8), and (4-9), we observe all the terms of Equation (4-14) have factors of either $\lambda_{,nn}$ or $(\lambda_{,n})^2$. So, if we can show that the coefficients of these terms are 0, we will conclude Equation (4-6) holds. The coefficient of $\lambda^{n-2}\lambda_{,nn}$ on the left side of Equation (4-14) is

$$=\Big((-1)^{r-1}(r-2)\Big(\frac{1}{n-1}\Big)^{r-1}+\Big(\frac{n-2}{n-1}\Big)^{r-1}\Big)(n-1) \\ +(C_{n-1}^{r-1})^{-1}\sum_{k=2}^{r}(-1)^{k+1}C_{n-k}^{r-k}C_{n-1}^{k-1}\Big(\frac{n-2}{n-1}\Big)^{r-k}(n-k)$$

$$=\Big((-1)^{r-1}(r-2)\Big(\frac{1}{n-1}\Big)^{r-1}+\Big(\frac{n-2}{n-1}\Big)^{r-1}\Big)(n-1) \\ +\sum_{k=2}^{r}(-1)^{k+1}C_{r-1}^{k-1}\Big(\frac{n-2}{n-1}\Big)^{r-k}((n-1)+(-k+1))$$

$$=\Big((-1)^{r-1}(r-2)\Big(\frac{1}{n-1}\Big)^{r-1}+\Big(\frac{n-2}{n-1}\Big)^{r-1}\Big)(n-1) \\ +\sum_{k=1}^{r-1}(-1)^{k}C_{r-1}^{k}\Big(\frac{n-2}{n-1}\Big)^{r-k-1}(n-1)+\sum_{k=0}^{r-2}(-1)^{k}(r-1)C_{r-2}^{k}\Big(\frac{n-2}{n-1}\Big)^{r-k-2},$$

which equals zero. The coefficient of $\lambda^{n-3}(\lambda_{,n})^2$ on the left side of (4-14) is

$$
\begin{aligned}
&= \Big((-1)^{r-1}(r-2)\Big(\frac{1}{n-1}\Big)^{r-1} + \Big(\frac{n-2}{n-1}\Big)^{r-1}\Big) \times \Big((n-1)(n-2) - \frac{(n-1)^3}{n}\Big) \\
&\quad + (C_{n-1}^{r-1})^{-1}\sum_{k=2}^{r}(-1)^{k+1}C_{n-k}^{r-k}((n-1)C_{n-2}^{k-1} - C_{n-2}^{k-2})\Big(\frac{n-2}{n-1}\Big)^{r-k}(n-k)(-1)\frac{n-1}{n} \\
&\qquad\quad + (C_{n-1}^{r-1})^{-1}\sum_{k=2}^{r}(-1)^{k+1}C_{n-k}^{r-k}C_{n-1}^{k-1}\Big(\frac{n-2}{n-1}\Big)^{r-k}(n-k)(n-k-1) \\
&= \Big((-1)^{r-1}(r-2)\Big(\frac{1}{n-1}\Big)^{r-1} + \Big(\frac{n-2}{n-1}\Big)^{r-1}\Big)\frac{(1-n)}{n} \\
&\qquad + \sum_{k=2}^{r}(-1)^{k}C_{r-1}^{k-1}\Big(\frac{n-2}{n-1}\Big)^{r-k}\frac{k(n-k)}{n} \\
&\qquad\quad + \sum_{k=2}^{r}(-1)^{k+1}(r-1)C_{r-2}^{k-2}\Big(\frac{n-2}{n-1}\Big)^{r-k} \times \frac{(n-k)}{n} \\
&= \Big((-1)^{r-1}(r-2)\Big(\frac{1}{n-1}\Big)^{r-1} + \Big(\frac{n-2}{n-1}\Big)^{r-1}\Big)\frac{(1-n)}{n} \\
&\qquad + \frac{n-1}{n}\Big((-1)^{r}\Big(\frac{1}{n-1}\Big)^{r-1} + \Big(\frac{n-2}{n-1}\Big)^{r-1} - (r-1)\Big(\frac{n-2}{n-1}\Big)^{r-2}\Big) \\
&\qquad\qquad + \frac{2(n-2)(r-1)}{n}\Big(\frac{n-2}{n-1}\Big)^{r-2} \\
&\qquad + \frac{(r-1)(n-3)}{n}\Big((-1)^{r}\Big(\frac{1}{n-1}\Big)^{r-2} - \Big(\frac{n-2}{n-1}\Big)^{r-2}\Big) \\
&\qquad\qquad + (-1)^{r-1}\frac{(r-1)(r-2)}{n}\Big(\frac{1}{n-1}\Big)^{r-3} \\
&\qquad + (-1)^{r-1}\frac{(n-2)(r-1)}{n}\Big(\frac{1}{n-1}\Big)^{r-2} + (-1)^{r}\frac{(r-1)(r-2)}{n}\Big(\frac{1}{n-1}\Big)^{r-3},
\end{aligned}
$$

which equals zero. In summary, Equation (4-6) and (4-7) are valid, which completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. Since $\triangle$ and T_{k-1} are the self-adjoint operators (see also [Cao and Li 2007; Reilly 1973]), we obtain from (1-3) that the necessary condition for M to be W_r-minimal is that

$$
\begin{aligned}
&\int_M \big(Q_r^{(n-r)/r}(n^2\sigma_1^2 - n(n-1)\sigma_2 + n)(Q_{r-1} + \sigma_1^{r-1}) - n\sigma_1 Q_r^{n/r}\big)dv \\
&\quad + \int_M \big(Q_r^{(n-r)/r}(C_{n-1}^{r-1})^{-1} \\
&\qquad\qquad \times \sum_{k=2}^{r}(-1)^{k+1}C_{n-k}^{r-k}C_n^k\sigma_1^{r-k}(n\sigma_1\sigma_k - (n-k)\sigma_{k+1} + k\sigma_{k-1})\big)dv = 0.
\end{aligned}
$$

If M is j-minimal rotational hypersurface for $n \geq 3$ and $1 \leq j \leq n-2$ and is not totally geodesic, then we have

$$(n-j)\lambda + j\mu = 0, \quad \sigma_k = \frac{j-k}{j}\lambda^k, \quad Q_r = (-1)^r(r-1)\Big(\frac{1}{j}\Big)^r\lambda^r, \quad \text{and } \lambda \neq 0.$$

A straightforward calculation shows that

$$Q_r^{(n-r)/r} n(Q_{r-1} + \sigma_1^{r-1}) + Q_r^{(n-r)/r}(C_{n-1}^{r-1})^{-1}\sum_{k=2}^{r}(-1)^{k+1}C_{n-k}^{r-k}C_n^k\sigma_1^{r-k}k\sigma_{k-1} = 0$$

and

$$\begin{aligned}Q_r^{(n-r)/r}&(n^2\sigma_1^2 - n(n-1)\sigma_2)(Q_{r-1}+\sigma_1^{r-1}) - n\sigma_1 Q_r^{n/r}\\ &+ Q_r^{(n-r)/r}(C_{n-1}^{r-1})^{-1}\sum_{k=2}^{r}(-1)^{k+1}C_{n-k}^{r-k}C_n^k\sigma_1^{r-k}(n\sigma_1\sigma_k - (n-k)\sigma_{k+1})\\ &= (-1)^r\frac{n(r-1)}{j^{r+1}}Q_r^{(n-r)/r}(j-n+1)\lambda^{r+1}\\ &= (-1)^n n(r-1)^{n/r} j^{-n-1}(j-n+1)\lambda^{n+1},\end{aligned}$$

so

$$\begin{aligned}&\int_M \big(Q_r^{(n-r)/r}(n^2\sigma_1^2 - n(n-1)\sigma_2 + n)(Q_{r-1}+\sigma_1^{r-1}) - n\sigma_1 Q_r^{n/r}\big)dv\\ &\quad+ \int_M \big(Q_r^{(n-r)/r}(C_{n-1}^{r-1})^{-1}\\ &\qquad\times\sum_{k=2}^{r}(-1)^{k+1}C_{n-k}^{r-k}C_n^k\sigma_1^{r-k}(n\sigma_1\sigma_k - (n-k)\sigma_{k+1} + k\sigma_{k-1})\big)dv\\ &\quad= (-1)^n n(r-1)^{n/r} j^{-n-1}(j-n+1)\int_M \lambda^{n+1}dv,\end{aligned}$$

which does not equal zero.

Hence, for $n \geq 3$ and $1 \leq j \leq n-2$, there are no compact j-minimal rotational and W_r-minimal hypersurfaces of S^{n+1} other than round geodesic spheres.

This completes the proof of Theorem 1.2. □

Acknowledgments

This work was supported by the Japan Society for Promotion of Science. The author would like to express his thanks to Professor Q. M. Cheng for his hospitality.

References

[Cao and Li 2007] L. Cao and H. Li, "r-minimal submanifolds in space forms", *Ann. Global Anal. Geom.* **32**:4 (2007), 311–341. MR 2346221 [do Carmo and Dajczer 1983] M. do Carmo and M. Dajczer, "Rotation hypersurfaces in spaces of constant curvature", *Trans. Amer. Math. Soc.* **277** (1983), 685–709. MR 85b:53055 Zbl 0518.53059

[Chen 1974] B.-Y. Chen, "Some conformal invariants of submanifolds and their applications", *Boll. Un. Mat. Ital.* (4) **10** (1974), 380–385. MR 51 #6663 Zbl 0321.53042

[Cheng 2001] Q.-M. Cheng, "Hypersurfaces in a unit sphere $S^{n+1}(1)$ with constant scalar curvature", *J. London Math. Soc.* (2) **64**:3 (2001), 755–768. MR 2002k:53116 Zbl 1023.53044

[Guo 2007] Z. Guo, "Generalized Willmore functionals and related variational problems", *Differential Geom. Appl.* **25**:5 (2007), 543–551. MR 2351429 Zbl 05229639

[Leite 1990] M. L. Leite, "Rotational hypersurfaces of space forms with constant scalar curvature", *Manuscripta Math.* **67**:3 (1990), 285–304. MR 91d:53085 Zbl 0695.53040

[Li 1996] H. Li, "Hypersurfaces with constant scalar curvature in space forms", *Math. Ann.* **305**:4 (1996), 665–672. MR 97i:53073 Zbl 0864.53040

[Li 2001] H. Li, "Willmore hypersurfaces in a sphere", *Asian Journal Math.* **5**:2 (2001), 365–377. MR 2003e:53077 Zbl 1025.53031

[Li and Vrancken 2003] H. Li and L. Vrancken, "New examples of Willmore surfaces in S^n", *Ann. Global Anal. Geom.* **23**:3 (2003), 205–225. MR 2004b:53105 Zbl 1033.53049

[Ôtsuki 1970] T. Ôtsuki, "Minimal hypersurfaces in a Riemannian manifold of constant curvature.", *Amer. J. Math.* **92** (1970), 145–173. MR 41 #9157

[Palmas 1999] O. Palmas, "Complete rotation hypersurfaces with H_k constant in space forms", *Bol. Soc. Brasil. Mat.* (*N.S.*) **30**:2 (1999), 139–161. MR 2000f:53078 Zbl 1058.53044

[Reilly 1973] R. C. Reilly, "Variational properties of functions of the mean curvatures for hypersurfaces in space forms", *J. Differential Geometry* **8** (1973), 465–477. MR 49 #6102 Zbl 0277.53030

[Wang 1998] C. Wang, "Moebius geometry of submanifolds in S^n", *Manuscripta Math.* **96**:4 (1998), 517–534. MR 2000a:53019 Zbl 0912.53012

[Wei 2007] G. Wei, "Complete hypersurfaces with $H_k = 0$ in a unit sphere", *Differential Geom. Appl.* **25**:5 (2007), 500–505. MR 2351426

[Wei 2008] G. Wei, "New examples of Willmore hypersurfaces in a sphere", *Houston J. Math.* (2008). To appear.

Received November 8, 2007.

GUOXIN WEI
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE AND ENGINEERING
SAGA UNIVERSITY
840-8502, SAGA
JAPAN

wei@ms.saga-u.ac.jp