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WITH POTENTIAL**

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The aim of this work is to prove partial regularity of the p -harmonic maps with potential. The main difficulty caused by the potential is in finding the equation satisfied by the scaling function, which breaks down the blow-up processing. We directly estimate the Morrey energy to avoid the difficulties caused by blowing up.

1. Introduction

Suppose (M, g) and (N, h) are two smooth compact Riemannian manifolds. Let $u : M \rightarrow N$ be a map, and let $H(u, x)$ be a function on $N \times M$. We consider the functional

$$(1-1) \quad E(u) = \int_M (e(u) - H(u, x)),$$

where $e(u) = (1/p)(\text{trace } u^*h)^{p/2}$ is the p -energy density of u . Here and below (usually), we leave implicit the integration measure. By Nash's embedding theorem, we can think of N as a submanifold that is isomorphically embedded in some Euclidean space \mathbb{R}^k . Since N is a submanifold of \mathbb{R}^k , we may consider the tangent space $T_y N$ at point $y \in N$ as a subspace of \mathbb{R}^k . Let $P(y) : \mathbb{R}^k \rightarrow T_y N$ be the orthogonal projection. A critical point of the functional $E(u)$ is called a p -harmonic map with potential (for $p = 2$, see [Fardoun and Ratto 1997]). The Euler–Lagrange equation of $E(u)$ is

$$(1-2) \quad -\text{div}(|\nabla u|^{p-2} \nabla u) = |\nabla u|^{p-2} A(u)(\nabla u, \nabla u) + P(u)(\nabla^N H(u, x)),$$

where $\nabla^N H(u, x)$ is the gradient of H on N and $A(u)(\cdot, \cdot)$ is the second fundamental form on N . For u belonging to $W^{1,p}(M; N)$, as defined in (1-4) below, we

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say u is a weakly p -harmonic map if, for any $\phi \in C_0^\infty(M; \mathbb{R}^k)$,

$$\int_M |\nabla u|^{p-2} \nabla u \nabla \phi = \int_M (|\nabla u|^{p-2} A(u)(\nabla u, \nabla u) + P(u)(\nabla^N H(u, x))) \phi.$$

The Euler–Lagrange equation can be seen as requiring vanishing variation of the target manifold for energy (1-1). On the other hand, we can consider the vanishing variation of the domain. For simplicity, we assume M is a subset Ω in \mathbb{R}^m and set $u_\tau(x) = u(x + \tau\varphi)$, where $\varphi \in C_0^\infty(\Omega, \mathbb{R}^m)$ and τ is small enough. Then computing $\frac{d}{d\tau} E(u(x + \tau\varphi))|_{\tau=0} = 0$ leads to

$$(1-3) \quad \partial_\alpha |\nabla u|^p - p \partial_\beta (|\nabla u|^{p-2} \partial_\beta u \cdot \partial_\alpha u) + p \partial_\alpha H(u, x) = 0$$

in the distributional sense.

The weakly p -harmonic maps with potential H in $W^{1,p}(\Omega, N)$ that satisfy (1-3) are called stationary p -harmonic maps with the potential. Here

$$(1-4) \quad W^{1,p}(\Omega, N) = \{u \in W^{1,p}(\Omega, \mathbb{R}^k) \mid u(x) \in N \text{ for almost all } x \in \Omega\}.$$

Stationary harmonic maps were first introduced in [Schoen 1984]. Rivière’s example [1995] shows that a weakly harmonic map may be discontinuous everywhere.

The partial regularity for stationary harmonic maps was proved by Evans [1991] in the case N is a sphere and by Bethuel [1993] in the general case. S. A. Chang, L. Wang, and P. C. Yang gave an elementary argument in their interesting paper [Chang et al. 1999]. In two-dimensional case, Hélein [1991a] proved that weak harmonic maps are smooth, and J. Qing [1993] proved the boundary regularity. For p -harmonic maps, there are some interesting results by T. Toro and C. Wang [1995] and C. Wang [1998]. There are some papers dealing with the existence of harmonic maps and their heat flows with potential [Fardoun and Ratto 1997; Fardoun et al. 2000; Fardoun and Regbaoui 2002, 2003], but there seem to be very few results about the regularity of such maps and their heat flows. In [Chu and Liu 2006], we proved the partial regularity of p -harmonic maps with potential under strongly restrictive conditions on the potential, when $1 < p \leq 2$. The main difficulty caused by the potential H is in finding the equation satisfied by the scaling function, which breaks down the blow-up processing.

In this note, we shall consider the regularity of stationary p -harmonic maps into the sphere \mathbb{S}^{n-1} . We prove the partial regularity for general potential H when $1 < p < \dim M$. Our method is to estimate directly the Morrey energy.

We prove the following theorem in this paper.

Theorem 1.1. *Let M be a compact Riemannian manifold with $\dim M \geq 3$. Let N be the sphere \mathbb{S}^{n-1} , and let $u : M \rightarrow \mathbb{S}^{n-1}$ be a stationary p -harmonic map with*

smooth potential $H(u, x)$, where $1 < p < \dim M$. There exists a closed subset Σ of M such that $\mathcal{H}^{\dim M - p}(\Sigma) = 0$ and $u \in C^{1,\alpha}(M \setminus \Sigma, N)$.

This theorem also holds when $p = \dim M$, in which case the singular set is empty. For simplicity, we shall also assume throughout that M is \mathbb{R}^m with the standard metric.

2. Decay lemma

First we shall prove a monotonicity formula for stationary p -harmonic maps.

Proposition 2.1. *There exists a constant C^* such that*

$$C^* r_1^{p+1} + \frac{1}{r_1^{m-p}} \int_{B_{r_1}(x)} |\nabla u|^p \leq C^* r_2^{p+1} + \frac{e}{r_2^{m-p}} \int_{B_{r_2}(x)} |\nabla u|^p$$

for any $x \in \mathbb{R}^m$ and $0 < r_1 < r_2 \leq 1$.

Proof. Let $B_r(x) = B_r(0) = B_r$, and set $\zeta = \phi(|y|)y$, where

$$\phi(s) = \begin{cases} 1 & \text{if } s \leq r, \\ 1 + (r - s)/h & \text{if } r < s < r + h, \\ 0 & \text{if } s > r + h. \end{cases}$$

Then $\partial_j \zeta^j = \phi' y^j y^j / |y| + \phi \delta_{ij}$, and stationary condition (1-3) becomes

$$\begin{aligned} \int (|\nabla u|^p (\phi' |y| + \phi m) - p |\nabla u|^{p-2} \partial_i u \partial_j u (\phi' y^i y^j / |y| + \phi \delta_{ij})) \\ + \int p (\partial_{u^i} H(u, x) \partial_j u^i + \partial_{x_j} H(u, x)) y^j \phi = 0. \end{aligned}$$

Letting $h \rightarrow 0^+$, we have $\int_{\Omega} |\nabla u|^p \phi' |y| \rightarrow -r \int_{\partial B(r)} |\nabla u|^p$ and

$$\int_{\Omega} |\nabla u|^{p-2} \partial_i u \partial_j u \phi' y^i y^j |y|^{-1} \rightarrow -r^{-1} \int_{\partial B(r)} |\nabla u|^{p-2} |ru_r|^2,$$

where $ru_r = y \nabla_y u$. Thus we obtain

$$\begin{aligned} (m - p) \int_{B_r} |\nabla u|^p - r \int_{\partial B_r} |\nabla u|^p + pr^{-1} \int_{\partial B_r} |\nabla u|^{p-2} |ru_r|^2 \\ + p \int_{B_r} (\partial_{u^i} H(u, x) \partial_j u^i + \partial_{x_j} H(u, x)) y^j = 0, \end{aligned}$$

that is,

$$\begin{aligned} \frac{d}{dr} \left(\frac{1}{r^{m-p}} \int_{B_r} |\nabla u|^p \right) &= \frac{1}{r^{m-p+1}} \left((p-m) \int_{B_r} |\nabla u|^p + r \int_{\partial B_r} |\nabla u|^p \right) \\ &= \frac{1}{r^{m-p+1}} \left(pr^{-1} \int_{\partial B_r} |\nabla u|^{p-2} |r u_r|^2 + p \int_{B_r} (\partial_{u^i} H(u) \partial_j u^i + \partial_{x_j} H(u, x)) y^j \right) \\ &\geq \frac{1}{r^{m-p+1}} \left(-cr \int_{B_r} (|\nabla u| + 1) \right) \geq \frac{1}{r^{m-p+1}} \left(-Cr^{m+1} - r \int_{B_r} |\nabla u|^p \right). \end{aligned}$$

Set $z(r) = (1/r^{m-p}) \int_{B_r} |\nabla u|^p$. Then for any $0 < r \leq 1$,

$$\frac{d}{dr} (\exp(r)z(r)) \geq -Cr^p \exp(r) \geq -C^*(p+1)r^p,$$

that is, $\frac{d}{dr} (\exp(r)z(r) + C^*r^{p+1}) \geq 0$, which proves the proposition. \square

For $B_r(x) \subset \mathbb{R}^m$, we define the renormalized energy

$$U(x, r) = r^p + \frac{1}{r^{m-p}} \int_{B_r(x)} |\nabla u|^p.$$

In order to prove [Theorem 1.1](#), we need the following lemmas.

Lemma 2.1. *Suppose u is a stationary p -harmonic map into the sphere \mathbb{S}^{n-1} . Then*

$$U(x_0, r/2) \leq C_0 \left(U(x_0, r)^{1/p} U(x_0, r) + r^{-m} \int_{B_r(x_0)} |u - u_r|^p \right),$$

where C_0 is a universal constant.

Proof. When $N = \mathbb{S}^{n-1}$, [Equation \(1-2\)](#) becomes

$$(2-1) \quad -\operatorname{div}(|\nabla u|^{p-2} \nabla u) = |\nabla u|^p u + P(u)(\nabla^N H(u, x)).$$

Let $\phi \in C_0^\infty(B_r(x_0))$ be a standard cutoff function, with $\phi = 1$ on $B_{r/2}(x_0)$ and $|\nabla \phi^{p-1}| \leq c/r$. By multiplying [Equation \(2-1\)](#) by $\phi^p(u - u_r)$, where $u_r = u_{x_0, r} = \int_{B_r(x_0)} u$, and then integrating, we obtain

$$\begin{aligned} (2-2) \quad \int |\nabla u|^p \phi^p &= \int |\nabla u|^p u \cdot \phi^p(u - u_r) + \int P(u)(\nabla^N H(u, x)) \cdot \phi^p(u - u_r) \\ &\quad - \int |\nabla u|^{p-2} \nabla u \nabla \phi^p(u - u_r) \\ &\leq Cr^m + \int |\nabla u|^p u \cdot \phi^p(u - u_r) \\ &\quad + \frac{1}{4} \int |\nabla u|^p \phi^p + \frac{C}{r^p} \int_{B_r(x_0)} |u - u_r|^p, \end{aligned}$$

where we used $|P(u)\nabla^N H| \leq C$. We estimate the term $\int |\nabla u|^{p-2} \nabla u \cdot \phi^P(u - u_r)$ as follows. Taking wedge product with respect to u in (2-1), we have

$$-\nabla \cdot (|\nabla u|^{p-2} \nabla u \wedge u) = P(u)\nabla^N (H(u, x)) \wedge u.$$

In particular,

$$\begin{aligned} -\nabla \cdot (\phi^{p-1} |\nabla u|^{p-2} \nabla u \wedge u) &= -\nabla \phi^{p-1} |\nabla u|^{p-2} \nabla u \wedge u \\ &\quad + \phi^{p-1} P(u)\nabla^N (H(u, x)) \wedge u. \end{aligned}$$

For $p' = p/(p-1)$, we have

$$(2-3) \quad \int |\nabla \cdot (\phi^{p-1} |\nabla u|^{p-2} \nabla u \wedge u)|^{p'} \leq Cr^m + \frac{C}{r^{p'}} \int_{B_r(x_0)} |\nabla u|^p.$$

On the other hand, by Hélein's trick [1991a; 1991b],

$$|\nabla u|^p u = |\nabla u|^{p-2} \sum_{\alpha, j} (u^i \partial_\alpha u^j - u^j \partial_\alpha u^i) \frac{\partial u^j}{\partial x_\alpha} = |\nabla u|^{p-2} (\nabla u \wedge u) \nabla u,$$

and we have

$$\begin{aligned} \int |\nabla u|^p u \cdot \phi^P(u - u_r) &= \int |\nabla u|^{p-2} (\nabla u \wedge u) \nabla u \cdot \phi^P(u - u_r) \\ &= \int \phi^{p-1} |\nabla u|^{p-2} (\nabla u \wedge u) \nabla (\phi(u - u_r)) \cdot (u - u_r) \\ &\quad - \int \phi^{p-1} |\nabla u|^{p-2} (\nabla u \wedge u) (\nabla \phi)(u - u_r) \cdot (u - u_r) \\ &=: I_1 + I_2. \end{aligned}$$

Noting that $|u| = 1$, we have by Hölder's inequality that

$$(2-4) \quad |I_2| \leq \frac{1}{4} \int |\nabla u|^p \phi^P + Cr^{-p} \int_{B_r(x_0)} |u - u_r|^p.$$

In order to estimate I_1 , we recall that the Hardy space $\mathcal{H}^1(\mathbb{R}^m)$ is the set of functions $g \in L^1$ such that the maximal function

$$g^*(x) = \sup_{r>0} \left| \frac{1}{r^m} \int g(y) \psi((x-y)/r) dy \right|$$

is also in L^1 . Here ψ represents any smooth function with support in the unit ball and with $\int \psi = 1$. A norm on \mathcal{H}^1 is $\|g\|_{\mathcal{H}^1} = \|g\|_{L^1} + |g^*|_{L^1}$.

A fundamental theorem of Fefferman asserts that the dual space of \mathcal{H}^1 is BMO (see [Strömberg and Torchinsky 1989]), and

$$\left| \int gf \right| \leq c \|g\|_{\mathcal{H}^1} \|f\|_{BMO}.$$

We have the following lemmas.

Lemma 2.2. $V(x) := \phi^{p-1} |\nabla u|^{p-2} (\nabla u \wedge u) \nabla(\phi(u - u_r))$ belongs to $\mathcal{H}^1(\mathbb{R}^m)$.
Also

$$\|V\|_{\mathcal{H}^1(\mathbb{R}^m)} \leq C \left(r^{m+1} + \int_{B_r(x_0)} |\nabla u|^p + r^{-p} \int_{B_r(x_0)} |u - u_r|^p \right),$$

where C is an absolute constant.

Proof. Set $\psi_\rho(\cdot) = (1/\rho^m)\psi(\cdot/\rho)$ and

$$(\psi_\rho * V)(x) = \frac{1}{\rho^m} \int V(y) \psi((x-y)/\rho) dy.$$

Making use of integration by parts, we have

$$\begin{aligned} (\psi_\rho * V)(x) &= \rho^{-m} \int \phi^{p-1} |\nabla u|^{p-2} (\nabla u \wedge u) \nabla(\phi(u - u_r)) \psi\left(\frac{x-y}{\rho}\right) dy \\ &= \rho^{-m} \int \nabla(\phi^{p-1} |\nabla u|^{p-2} (\nabla u \wedge u) (\phi(u - u_r) - c)) \psi\left(\frac{x-y}{\rho}\right) dy \\ &\quad - \rho^{-m} \int \nabla(\phi^{p-1} |\nabla u|^{p-2} (\nabla u \wedge u)) (\phi(u - u_r) - c) \psi\left(\frac{x-y}{\rho}\right) dy \\ &= \rho^{-(m+1)} \int \phi^{p-1} |\nabla u|^{p-2} (\nabla u \wedge u) (\phi(u - u_r) - c) \nabla \psi\left(\frac{x-y}{\rho}\right) dy \\ &\quad - \rho^{-m} \int \nabla(\phi^{p-1} |\nabla u|^{p-2} (\nabla u \wedge u)) (\phi(u - u_r) - c) \psi\left(\frac{x-y}{\rho}\right) dy \\ &=: I_{11} + I_{12}, \end{aligned}$$

where $c = (\phi(u - u_r))_\rho$. Using Hölder's inequality and the Sobolev–Poincaré inequality, we obtain

$$\begin{aligned} |I_{11}| &\leq \frac{C}{\rho} \left(\int_{B_\rho(x)} |\phi^{p-1} |\nabla u|^{p-2} (\nabla u \wedge u)|^{q'} \right)^{1/q'} \left(\int_{B_\rho(x)} |\phi(u - u_r) - c|^q \right)^{1/q} \\ &\leq C \left(\int_{B_\rho(x)} |\phi \nabla u|^{(p-1)q'} \right)^{1/q'} \left(\int_{B_\rho(x)} |\nabla(\phi(u - u_r))|^{q^*} \right)^{1/q^*} \\ &\leq CM(|\phi \nabla u|^{(p-1)q'})^{1/q'}(x) M(|\nabla(\phi(u - u_r))|^{q^*})^{1/q^*}(x), \end{aligned}$$

where $1 < q' < p'$, $1/q' + 1/q = 1$, $q^* = mq/(m+q)$ and $p < q < pm/(m-p)$.

$$\begin{aligned} |I_{12}| &\leq C \left(\int_{B_\rho(x)} |\nabla(\phi^{p-1} |\nabla u|^{p-2} (\nabla u \wedge u))|^{q'} \right)^{1/q'} \left(\int_{B_\rho(x)} |\phi(u - u_r) - c|^q \right)^{1/q} \\ &\leq C \left(\int_{B_\rho(x)} |\nabla(\phi^{p-1} |\nabla u|^{p-2} (\nabla u \wedge u))|^{q'} \right)^{1/q'} \left(\int_{B_\rho(x)} |\nabla(\phi(u - u_r))|^{q^*} \right)^{1/q^*} \rho \\ &\leq CM(|r \nabla(\phi^{p-1} |\nabla u|^{p-2} (\nabla u \wedge u))|^{q'})^{1/q'}(x) M(|\nabla(\phi(u - u_r))|^{q^*})^{1/q^*}(x), \end{aligned}$$

where $M(f)(x)$ is the maximum function of f . Thus by (2-3),

$$\begin{aligned}
& \int \sup_{\rho} |(\psi_{\rho} * V)(x)| \\
& \leq C \int M(|\phi \nabla u|^{(p-1)q'})^{1/q'}(x) M(|\nabla(\phi(u - u_r))|^{q^*})^{1/q^*}(x) \\
& \quad + C \int M(|r \nabla(\phi^{p-1} |\nabla u|^{p-2} (\nabla u \wedge u))|^{q'})^{1/q'}(x) M(|\nabla(\phi(u - u_r))|^{q^*})^{1/q^*}(x) \\
& \leq C \int M(|\phi \nabla u|^{(p-1)q'})^{p'/q'}(x) + C \int M(|\nabla(\phi(u - u_r))|^{q^*})^{p'/q^*}(x) \\
& \quad + C \int M(|r \nabla(\phi^{p-1} |\nabla u|^{p-2} (\nabla u \wedge u))|^{q'})^{p'/q'}(x) \\
& \quad + C \int M(|\nabla(\phi(u - u_r))|^{q^*})^{p'/q^*}(x) \\
& \leq C \left(\int |\nabla u|^p \phi^p + |r \nabla(\phi^{p-1} |\nabla u|^{p-2} (\nabla u \wedge u))|^{p'} + |\nabla(\phi(u - u_r))|^p \right) \\
& \leq C \left(r^{m+1} + \int_{B_r(x_0)} |\nabla u|^p + r^{-p} \int_{B_r(x_0)} |u - u_r|^p \right) \\
& \leq Cr^m + C \int_{B_r(x_0)} |\nabla u|^p. \quad \square
\end{aligned}$$

From the definition of I_1 , we may assume $u - u_r = 0$ on $\mathbb{R}^m \setminus B_r(x_0)$. Hence for $x \in B_r(x_0)$ and $t \leq r$, the monotonicity inequality implies

$$\begin{aligned}
\int_{B_t(x)} |u - u_r - (u - u_r)_{x,t}| &= \int_{B_t(x)} |u - u_{x,t}| \\
&\leq C \left(t^{p-m} \int_{B_t(x)} |\nabla u|^p \right)^{1/p} \leq CU(x_0, r)^{1/p},
\end{aligned}$$

that is,

$$(2-5) \quad \|u - u_r\|_{BMO(\mathbb{R}^m)} \leq CU(x_0, r)^{1/p}.$$

From Lemma 2.2 and (2-5) we have

$$(2-6) \quad |I_1| \leq C \left(r^m + \int_{B_r(x_0)} |\nabla u|^p \right) U(x_0, r)^{1/p}.$$

Combining (2-2) and (2-4) with (2-6), we obtain

$$\begin{aligned}
\int_{B_{r/2}(x_0)} |\nabla u|^p &\leq CU(x_0, r)^{1/p} \left(r^m + \int_{B_r(x_0)} |\nabla u|^p \right) \\
&\quad + C \left(r^m + r^{-p} \int_{B_r(x_0)} |u - u_r|^p \right). \quad \square
\end{aligned}$$

Lemma 2.3. *There is a universal constant C_1 such that for each $\tau \in (0, 1/2]$, there exists a constant $\epsilon_1 > 0$ such that, for $U(x_0, r) \leq \epsilon_1^p$,*

$$\frac{1}{(\tau r)^m} \int_{B_{\tau r}(x_0)} |u - u_{\tau r}|^p \leq C_1 \tau^p U(x_0, r).$$

Proof. The proof is by contradiction. For fixed positive τ and $C_1 > 0$, if the lemma were false, then there would exist positive x_k and r_k such that $U(x_k, r_k) = \lambda_k^p \rightarrow 0$. However

$$(2-7) \quad \frac{1}{(\tau r_k)^m} \int_{B_{\tau r_k}(x_k)} |u - u_{\tau r_k}|^p > C_1 \tau^p \lambda_k^p.$$

Transform variables into the unit ball $B_1(0)$, and set $v_k(x) \equiv (u_k(x) - (u_k)_{0,1})/\lambda_k$, where $u_k = u(x_k + r_k x)$ and $(u_k)_{0,1} = \int_{B_1(0)} u_k$. We have

$$\int_{B_1(0)} |\nabla v_k|^p \leq 1 \quad \text{and} \quad \int_{B_1(0)} v_k = 0.$$

Therefore $\{v_k\}$ is bounded in $W^{1,p}(B_1(0))$. Suppose

$$(2-8) \quad \nabla v_k \rightharpoonup \nabla v \text{ in } L^p(B_1(0)) \text{ as } v_k \rightarrow v \text{ almost everywhere in } B_1(0).$$

It is easy to show that v_k satisfies the equation

$$-\operatorname{div}(|\nabla v_k|^{p-2} \nabla v_k) = \lambda_k |\nabla v_k|^p u_k + (r_k^p / \lambda_k^{p-1}) P(u_k) \nabla^N H(u_k, x_k + r_k x)$$

in $B_1(0)$. From the definition of $U(x_k, r_k)$, we know that $r_k^p \leq \lambda_k^p$. This yields $r_k^p / \lambda_k^{p-1} \leq \lambda_k \rightarrow 0$. Hence v satisfies $-\operatorname{div}(|\nabla v|^{p-2} \nabla v) = 0$ in $B_1(0)$. Then, for $\tau \in (0, 1/2]$, the regularity of p -Laplace equations gives

$$(2-9) \quad \frac{1}{\tau^m} \int_{B_\tau(0)} |v - (v)_{0,\tau}|^p \leq C_2 \tau^p,$$

where C_2 is a constant that is independent of k . Now scaling (2-7) to v_k , letting $k \rightarrow \infty$ in (2-7), and using (2-8), we obtain $(1/\tau^m) \int_{B_\tau(0)} |v - (v)_{0,\tau}|^p \geq C_1 \tau^p$, which contradicts (2-9) if we choose $C_1 > C_2$. \square

Now we use the lemmas above to prove the following decay result.

Proposition 2.2. *For any $\gamma \in (0, 1)$, there exist a constant $\theta \in (0, 1/2]$ and an $\epsilon_0 > 0$ such that $U(x_0, \theta r) \leq \theta^{p\gamma} U(x_0, r)$ for $U(x_0, r) < \epsilon_0^p$.*

Proof. Let $\gamma \in (0, 1)$. For every $\theta \in (0, 1/2]$ to be determined later, there exists an integer $k \geq 0$ with $1/2^{k+2} < \theta \leq 1/2^{k+1}$. Write

$$G(x_0, r) \equiv \frac{1}{r^m} \int_{B_r(x_0)} |u - (u)_{x_0,r}|^p.$$

Take $\tau = 1/2$ in [Lemma 2.3](#). If $\varepsilon_0 \leq \varepsilon_1$, then $G(x_0, 2^{-k}r) \leq C_1 2^{-kp} U(x_0, r)$. By this and [Lemma 2.1](#),

$$\begin{aligned} U(x_0, 2^{-(k+1)}r) &\leq C_0(U(x_0, 2^{-k}r)^{1/p} U(x_0, 2^{-k}r) + G(x_0, 2^{-k}r)) \\ &\leq C_0(\varepsilon_0 U(x_0, 2^{-k}r) + G(x_0, 2^{-k}r)) \\ &\leq C_0 \varepsilon_0 U(x_0, 2^{-k}r) + C_0 C_1 2^{-kp} U(x_0, r), \end{aligned}$$

that is,

$$(2-10) \quad U(x_0, 2^{-(k+1)}r) \leq C_0 \varepsilon_0 U(x_0, 2^{-k}r) + C_0 C_1 2^{-kp} U(x_0, r).$$

Making use of (2-10) and taking ε_0 such that $2^p C_0 \varepsilon_0 \leq \min\{1/2, \varepsilon_1\}$, we get

$$\begin{aligned} U(x_0, \theta r) &\leq 2^{m-2} U(x_0, 2^{-(k+1)}r) \\ &\leq 2^{m-2} U(x_0, r) ((C_0 \varepsilon_0)^{k+1} + C_1 C_0 \sum_{i=0}^k (C_0 \varepsilon_0)^{k-i} 2^{-ip}) \\ &\leq 2^{m-2} U(x_0, r) ((C_0 \varepsilon_0)^{k+1} + C_1 C_0 2^{-kp+1}). \end{aligned}$$

Since $(k+1) \log 2 \leq -\log \theta < (k+2) \log 2$, we thus have

$$(C_0 \varepsilon_0)^{k+1} = \exp\{(k+1) \log(C_0 \varepsilon_0)\} \leq \theta^{|\log(C_0 \varepsilon_0)|/\log 2}$$

and

$$C_1 C_0 2^{-kp+1} = 2^{1+2p} C_1 C_0 \exp(-(k+2)p \log 2) \leq 2^{1+2p} C_1 C_0 \theta^p.$$

Then, taking $\theta \in (0, 1/2]$ such that $2^{m-2}(1 + 2^{1+2p} C_1 C_0) \theta^{p(1-\gamma)} \leq 1$, we obtain $U(x_0, \theta r) \leq \theta^{p\gamma} U(x_0, r)$. \square

3. The proof of [Theorem 1.1](#)

For any $\rho < r/4$, we can find an integer $k \geq 0$ such that $\theta^{k+1}r < 2\rho \leq \theta^k r$. For any $x \in B_\rho(x_0)$, we have by the monotonicity formula in [Proposition 2.1](#) and the decay formula in [Proposition 2.2](#) that

$$U(x, \rho) \leq C U(x_0, 2\rho) \leq C \theta^{p-m} U(x_0, \theta^k r) \leq C \theta^{p-m} \theta^{kp\gamma} U(x_0, r).$$

Since $(k+1) \log \theta + \log r < \log 2\rho \leq k \log \theta + \log r$, we obtain

$$C \theta^{p-m} \theta^{kp\gamma} = C \theta^{p-m} \exp(kp\gamma \log \theta) \leq C \theta^{p-m-p\gamma} 2^{p\gamma} (\rho/r)^{p\gamma}.$$

Thus

$$U(x, \rho) \leq C(\theta, \gamma, p) (\rho/r)^{p\gamma} U(x_0, r).$$

Let $\Omega_0 = \{x \in \mathbb{R}^m \mid U(x, r) < \epsilon_0^p\}$. Then Ω_0 is open, and the standard covering argument implies $\mathcal{H}^{m-p}(\Omega \setminus \Omega_0) = 0$. Furthermore if $x \in \Omega_0$, then $U(y, \rho) \leq C(\theta, \gamma, p) \rho^{p\gamma}$ for all y near x and for sufficiently small radii $\rho > 0$. Hence u

belongs to $C^{0,\gamma}$; see [Giaquinta 1983]. Now let $x \in \Omega_0$ and $R > 0$. We consider the Dirichlet boundary problem

$$\begin{aligned} -\operatorname{div}(|\nabla w|^{p-2}\nabla w) &= 0 \quad \text{for } y \in B_{2R}(x), \\ w &= u \quad \text{for } y \in \partial B_{2R}(x). \end{aligned}$$

The theory of regularity of elliptic systems [Hamburger 1992; Tolksdorf 1984] implies

$$M^p \equiv \sup_{B_R(x)} |\nabla w|^p \leq C \int_{B_{2R}(x)} (1 + |\nabla w|^p)$$

and $\operatorname{osc}(\nabla w, B_\rho(x)) \leq C(1+M)(\rho/R)^\mu$, where $\rho \leq R$ and $0 < \mu < 1$. Consequently,

$$(3-1) \quad \int_{B_\rho(x)} |\nabla w - (\nabla w)_{x,\rho}|^p \leq C(1+M)\rho^m \left(\frac{\rho}{R}\right)^{p\mu}$$

and

$$(3-2) \quad \int_{B_\rho(x)} |\nabla u - (\nabla u)_{x,\rho}|^p \leq C(1+M)\rho^m \left(\frac{\rho}{R}\right)^{p\mu} + C \int_{B_\rho(x)} |\nabla u - \nabla w|^p.$$

Next we estimate

$$\int_{B_\rho(x)} |\nabla u - \nabla w|^p.$$

Extending $u - w = 0$ in $\Omega \setminus B_{2R}(x)$, multiplying (2-1) by $u - w$, integrating, and using the Hölder continuity of u , we get

$$\begin{aligned} &\int_{B_{2R}(x)} (|\nabla u|^{p-2}\nabla u - |\nabla w|^{p-2}\nabla w)(\nabla u - \nabla w) \\ &= \int_{B_{2R}(x)} |\nabla u|^p u(u-w) + \int_{B_{2R}(x)} (\nabla_u H(u, x) - (\nabla_u H(u, x) \cdot u)u)(u-w) \\ &\leq CR^\gamma \int_{B_{2R}(x)} (|\nabla u|^p + 1). \end{aligned}$$

For $p \geq 2$, we have

$$(3-3) \quad \int_{B_{2R}(x)} |\nabla u - \nabla w|^p \leq CR^\gamma \int_{B_{2R}(x)} (|\nabla u|^p + 1).$$

For $1 < p < 2$, we have as in [Liu 1997] that

$$\begin{aligned} &\int_{B_{2R}(x)} |\nabla u - \nabla w|^p \\ &\leq \left(\int_{B_{2R}(x)} (|\nabla u| + |\nabla w|)^p \right)^{(2-p)/2} \left(\int_{B_{2R}(x)} (|\nabla u| + |\nabla w|)^{p-2} |\nabla u - \nabla w|^2 \right)^{p/2} \end{aligned}$$

$$\begin{aligned} &\leq \left(\int_{B_{2R}(x)} (|\nabla u| + |\nabla w|)^p \right)^{(2-p)/2} \\ &\quad \times \left(\frac{1}{p-1} \int_{B_{2R}(x)} (|\nabla u|^{p-2} \nabla u - |\nabla w|^{p-2} \nabla w) (\nabla u - \nabla w) \right)^{p/2} \\ &\leq C R^{\gamma p/2} \int_{B_{2R}(x)} (|\nabla u|^p + 1), \end{aligned}$$

where we have used $\int_{B_{2R}(x)} |\nabla w|^p \leq C \int_{B_{2R}(x)} |\nabla u|^p$. Set

$$\Lambda(x, R) = \frac{1}{R^m} \int_{B_{2R}(x)} (|\nabla u|^p + 1).$$

Then from (3-1), (3-2), and (3-3), we get

$$\begin{aligned} \int_{B_\rho(x)} |\nabla u - (\nabla w)_{x,\rho}|^p &\leq 2^{p-1} \int_{B_\rho(x)} (|\nabla u - \nabla w|^p + |\nabla w - (\nabla w)_{x,\rho}|^p) \\ &\leq C(1 + M)\rho^m (\rho/R)^{p\mu} + C\Lambda R^{\gamma p/2+m}, \end{aligned}$$

and thus

$$\int_{B_\rho(x)} |\nabla u - (\nabla u)_{x,\rho}|^p \leq C(\rho^m (\rho/R)^{p\mu} + R^{p\gamma/2+m}).$$

Take $\rho = \frac{1}{2}R^{1+\theta}$ and $\theta = (p\gamma/2)/(m + p\mu) > 0$. Then

$$\int_{B_\rho(x)} |\nabla u - (\nabla u)_{x,\rho}|^p \leq C R^{m+p\gamma/2} \leq C \rho^{m+\theta p\mu/(1+\theta)}.$$

Hence $\nabla u \in C^{0,\alpha}(\Omega_0)$. □

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