Pacific Journal of Mathematics

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Volume 237 No. 1 September 2008

PACIFIC JOURNAL OF MATHEMATICS
Vol. 237, No. 1, 2008

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We give a classification of the infinitesimal q-Schur algebras that have finite representation type.

1. Introduction

R. Dipper and G. James [1989; 1991] introduced q-analogues of Schur algebras, called q-Schur algebras. The q-Schur algebras are related to Hecke algebras of type A in precisely the same way that classical Schur algebras are related to group algebras of symmetric groups. The q-Schur algebras play an important role in the nondefining representation and cohomology theories of the finite general linear groups. The representations of q-Schur algebras $S_q(n,d)$ are equivalent to the polynomial representations of the quantum linear group $G := G_q(n)$ of a given degree d; see [Parshall and Wang 1991, 11.2], [Donkin 1996, Section 4], and [Cox 1997, 1.6]. Infinitesimal Schur/q-Schur algebras were introduced in [Doty et al. 1996; Cox 1997; 2000] as the dual algebras of the homogeneous components of the infinitesimal thickening (by the torus) of the Frobenius kernel. It turns out that infinitesimal q-Schur algebras control the polynomial representations of G_rT. Here G_rT is the q-analogue of Jantzen subgroups, which can be regarded as infinitesimal thickenings of the Frobenius kernels G_r by the torus T.

It is important to classify the representation type of a finite-dimensional algebra. In the classical case, the representation type of Schur algebras and infinitesimal Schur algebras has been classified; see [Erdmann 1993; Doty et al. 1997; Doty et al. 1999]. In the quantum case, the classification of the representation type of the q-Schur algebras was given in [Erdmann and Nakano 2001]. The representation type of Hecke algebras has also been classified; see [Uno 1992; Erdmann and Nakano 2002; Ariki and Mathas 2004; Ariki 2005]. Here, we will classify the infinitesimal q-Schur algebras of finite representation type.

MSC2000: 20G42, 16G60.

Keywords: representation type, q-Schur algebra, infinitesimal q-Schur algebra, Frobenius kernel.

Supported by the Leverhulme Trust and the National Natural Science Foundation of China (10601037 and 10671142).

Little q-Schur algebras were introduced as homomorphic images of infinitesimal quantum $\mathfrak{gl}_n$ in [Du et al. 2005]; see also [Fu 2007]. The relationship between infinitesimal q-Schur algebras and little q-Schur algebras is similar to that of G_rT and G_r; see [Fu 2005]. We expect that the result of this paper can be used to study the representation type of little q-Schur algebras.

2. Main result

Let k be an algebraically closed field of characteristic $p \geqslant 0$. Let $q \in k$ and $q \neq 0, 1$. Let l be the multiplicative order of q in k^*; that is, either q is a primitive l-th root of unity or $l = \infty$.

The q-Schur algebra $S_q(n, d)$ is a deformation of the classical Schur algebra $S(n, d)$. Let us recall the construction of the q-Schur algebra from the coordinated algebra of the quantum matrix monoid $M_q(n)$ as follows. Following [Dipper and Donkin 1991], let $A_q(n)$ be the k-algebra generated by the n^2 indeterminates c_{ij}, with $1 \leqslant i, j \leqslant n$, subject to the relations

$$\begin{aligned} c_{ij}c_{it} &= c_{it}c_{ij} && \text{for all } i, j, t,\\ c_{ij}c_{st} &= qc_{st}c_{ij} && \text{for } i > s \text{ and } j \leqslant t,\\ c_{ij}c_{st} &= c_{st}c_{ij} + (q-1)c_{sj}c_{it} && \text{for } i > s \text{ and } j > t. \end{aligned}$$

This is a bialgebra, with comultiplication and counit given by

$$\Delta(c_{ij}) = \sum_{t=1}^{n} c_{it} \otimes c_{tj} \quad \text{and} \quad \varepsilon(c_{ij}) = \delta_{ij}.$$

We denote by $M = M_q(n)$ the quantum matrix monoid with the coordinate algebra $k[M] = A_q(n)$. Let $A_q(n, d)$ denote the subspace of elements in $A_q(n)$ of degree d in the c_{ij}. Then the $A_q(n, d)$ are in fact subcoalgebras of $A_q(n)$ for all d. Hence $A_q(n, d)^*$ is an algebra, which is isomorphic to the q-Schur algebra $S_q(n, d)$ by [Dipper and Donkin 1991, 3.2.6].

Theorem 2.1 [Erdmann and Nakano 2001, Theorem 1.4(B)]. *The algebra $S_q(n, d)$ has infinite representation type if and only if q is a primitive l-th root of unity and one of the following holds*:

(1) $n \geqslant 3$ *and* $d \geqslant 2l$;

(2) $n = 2$, $p \neq 0, l \geqslant 3$ *and* $d \geqslant lp$;

(3) $n = 2$, $p \geqslant 3$, $l = 2$ *and d is even with* $d \geqslant 2p$, *or d is odd with* $d \geqslant 2p^2 + 1$.

From now on, we assume $p > 0$ and q is a primitive l-th root of unity. Note that l and p must be coprime.

We recall the definition of infinitesimal q-Schur algebras of [Cox 1997; Cox 2000]. Let J_r be the ideal in $A_q(n)$ generated by ${c_{ij}}^{lp^{r-1}}$ for $1 \leqslant i \neq j \leqslant n$. This is a coideal; thus $A_q(n)/J_r$ is a bialgebra and gives rise to a quantum monoid, which we denote by $M_r D$. Let $A_q(n,d)_r$ be the image of $A_q(n,d)$ under the quotient map $k[M] = A_q(n) \to k[M_r D] = A_q(n)/J_r$. This subspace is also a subcoalgebra of $k[M_r D]$ for all d. The algebra $s_q(n,d)_r = A_q(n,d)_r^*$ is called the infinitesimal q-Schur algebra.

The main result of this paper is as follows.

Theorem 2.2. *Assume k is an algebraically closed field of characteristic $p > 0$ and $q \in k$ is a primitive l-th root of unity. Then the infinitesimal q-Schur algebra $s_q(n,d)_r$ has finite representation type if and only if one of the following holds*:

(1) $n \geqslant 3,\ r \geqslant 2$ *and* $d < 2l$;

(2) $n \geqslant 3,\ r = 1$ *and* $d < l$;

(3) $n = 3,\ l = 3,\ r = 1$ *and* $d = 4, 5$;

(4) $n = 3,\ l = 2,\ r = 1$ *and* $d = 2, 3$;

(5) $n = 2,\ r \geqslant 2$ *and* $d < lp$;

(6) $n = 2,\ l = 2,\ r \geqslant 3$ *and d is odd with* $2p+1 \leqslant d < 2p^2+1$;

(7) $n = 2,\ l = 2,\ r = 2$ *and d is odd with* $d \geqslant 2p+1$;

(8) $n = 2$ *and* $r = 1$.

For convenience, it will be useful to reformulate Theorem 2.2 as follows.

Theorem 2.3. *Assume k is an algebraically closed field of characteristic $p > 0$ and $q \in k$ is a primitive l-th root of unity. Then the infinitesimal q-Schur algebra $s_q(n,d)_r$ has infinite representation type if and only if one of the following holds*:

(1) $n \geqslant 3,\ r \geqslant 2$ *and* $d \geqslant 2l$;

(2) $n \geqslant 4,\ r = 1$ *and* $d \geqslant l$;

(3) $n = 3,\ l \geqslant 4,\ r = 1$ *and* $d \geqslant l$;

(4) $n = 3,\ l = 3,\ r = 1$ *and either* $d = 3$ *or* $d \geqslant 6$;

(5) $n = 3,\ l = 2,\ r = 1$ *and* $d \geqslant 4$;

(6) $n = 2,\ l \geqslant 3,\ r \geqslant 2$ *and* $d \geqslant lp$;

(7) $n = 2,\ l = 2,\ r \geqslant 3$ *and either d is even with $d \geqslant 2p$ or d is odd with* $d \geqslant 2p^2+1$;

(8) $n = 2,\ l = 2,\ r = 2$ *and d is even with* $d \geqslant 2p$.

3. Preliminaries

In this section, we shall prove some general results of infinitesimal q-Schur algebras, which will be used in Sections 4 and 5 to prove our main result.

Let $\overline{G} = GL(n, k)$ be the general linear group with coordinate algebra

$$k[\overline{G}] = k[x_{ij} \text{ for } 1 \leqslant i, j \leqslant n;\ \delta^{-1}], \quad \text{where } \delta = \det(x_{ij})_{n\times n}.$$

There are several different quantum deformations of $\overline{G}$; see [Parshall and Wang 1991; Dipper and Donkin 1991; Du et al. 1991]. We will use the version introduced by Dipper and Donkin: let $\delta_q = \sum_{\pi\in\mathcal{S}_n}(-1)^{\ell(\pi)} c_{1,1\pi}c_{2,2\pi}\cdots c_{n,n\pi}$ be the q-determinant in $A_q(n)$, where $\mathcal{S}_n$ is the symmetric group and $\ell(\pi)$ is the length of π. Since $c_{ij}\delta_q = q^{i-j}\delta_q c_{ij}$ for $1 \leqslant i, j \leqslant n$, we may localize the bialgebra $A_q(n)$ at δ_q. They proved that the localization $A_q(n)_{\delta_q}$ is a Hopf algebra. Let $G = G_q(n)$ be the quantum linear group whose coordinate algebra is $k[G] = A_q(n)_{\delta_q}$.

The torus $T = T_q(n)$ is defined to be the subgroup of G with defining ideal of $k[G]$ generated by all c_{ij} with $i \neq j$. Similarly, we can define $D = D_q(n)$ to be the submonoid of M with defining ideal of $k[M]$ generated by all c_{ij} with $i \neq j$. Following [Du et al. 1991, (3.1)], let F be the quantum Frobenius morphism $F : G \to \overline{G}$ with comorphism $F^\# : k[\overline{G}] \to k[G]$ defined by $F^\#(x_{ij}) = {c_{ij}}^l$ for all i, j. We also have the usual Frobenius map F for $\overline{G}$ taking x_{ij} to ${x_{ij}}^p$. Let $F^r = \mathrm{F}^{r-1}F$, and let G_r be the kernel of F^r. Then G_r is the subgroup of G with defining ideal of $k[G]$ generated by the elements ${c_{ij}}^{lp^{r-1}} - \delta_{ij}$ for $1 \leqslant i, j \leqslant n$ and ${\delta_q}^{lp^{r-1}} - 1$. Similarly, we may define M_r to be the submonoid of M with defining ideal of $k[M]$ generated by the elements ${c_{ij}}^{lp^{r-1}} - \delta_{ij}$ for $1 \leqslant i, j \leqslant n$. Let G_rT be the subgroup of G with defining ideal generated by the elements ${c_{ij}}^{lp^{r-1}}$ for $1 \leqslant i \neq j \leqslant n$. Note that $k[M_r]$ is isomorphic to $k[G_r]$ and $k[G_rT]$ is the localization of $k[M_rD]$ at the quantum determinant.

Let $\mathbb{G}_m$ be the multiplicative group with coordinate algebra $k[t, t^{-1}]$, and let $\mathbb{M}_m$ be the multiplicative monoid with coordinate algebra $k[t]$. Let

$$X(T) = \mathrm{Hom}(T, \mathbb{G}_m) \quad \text{and} \quad P(D) = \mathrm{Hom}(D, \mathbb{M}_m).$$

As usual, we identify $X(T)$ with $\mathbb{Z}^n$ and $P(D)$ with $\mathbb{N}^n$. Let

$$X^+(T) = \{\lambda \in X(T) \mid \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n\}, \qquad P^+(D) = P(D) \cap X^+(T),$$
$$\Lambda(n, d) = \{\lambda \in \mathbb{N}^n \mid \textstyle\sum_{1\leqslant i\leqslant n} \lambda_i = d\}, \qquad \Lambda^+(n, d) = \Lambda(n, d) \cap X^+(T).$$

For $\lambda \in \Lambda^+(n, d)$, let $\nabla(\lambda)$ and $\overline{\nabla}(\lambda))$ be the costandard modules for $S_q(n, d)$ and $S(n, d)$, respectively, with highest weight λ; see [Donkin 1998, Appendix]. Let $L(\lambda) = \mathrm{soc}_{S_q(n,d)}\nabla(\lambda)$ and $\overline{L}(\lambda) = \mathrm{soc}_{S(n,d)}\overline{\nabla}(\lambda)$. The sets $\{L(\lambda) \mid \lambda \in \Lambda^+(n, d)\}$

and $\{\bar{L}(\lambda) \mid \lambda \in \Lambda^+(n,d)\}$ form complete sets of inequivalent irreducible $S_q(n,d)$-modules and $S(n,d)$-modules, respectively. For $\lambda \in X(T)$, let $\widehat{L}_r(\lambda)$ be the corresponding irreducible G_rT-module; see [Donkin 1998; Cox 1997; Cox 2000].

Let $X_r(T) = P_r(D) = \{\lambda \in X(T) \mid 0 \leqslant \lambda_i - \lambda_{i+1} \leqslant lp^{r-1} - 1,\ 1 \leqslant i \leqslant n\}$, where $\lambda_{n+1} = 0$. By [Cox 1997; Cox 2000] the set $\{\widehat{L}_r(\lambda) \mid \lambda \in \Gamma_r^d(D)\}$ forms a complete set of nonisomorphic simple $s_q(n,d)_r$-modules, where

$$\Gamma_r(D) = P_r(D) + lp^{r-1}P(D) \quad \text{and} \quad \Gamma_r^d(D) = \{\lambda \in \Gamma_r(D) \mid \textstyle\sum_{i=1}^n \lambda_i = d\}.$$

By [Donkin 1998, 3.2] and [Cox 1997, 1.7], for $\alpha = \lambda + lp^{r-1}\mu \in \Lambda^+(n,d)$ with $\lambda \in P_r(D)$ and $\mu \in P^+(D)$, we have

$$L(\alpha)\big|_{s_q(n,d)_r} \cong \bigoplus_{j=1}^{s} \widehat{L}_r(\alpha^{(j)}). \tag{3-1}$$

where $\alpha^{(j)} = \lambda + lp^{r-1}\mu^{(j)}$ and $\{\mu^{(j)} : 1 \leqslant j \leqslant s\}$ is some enumeration of the weights of $\bar{L}(\mu)$.

Let $\Xi(n,d)$ be the set of $n \times n$ matrices with nonnegative integer entries summing to d. For $A \in \Xi(n,d)$, let $c^A = c_{11}^{a_{11}} c_{12}^{a_{12}} \cdots c_{1n}^{a_{1n}} c_{21}^{a_{21}} c_{22}^{a_{22}} \cdots c_{2n}^{a_{2n}} c_{n1}^{a_{n1}} c_{n2}^{a_{n2}} \cdots c_{nn}^{a_{nn}} \in A_q(n,d)$. By [Dipper and Donkin 1991, 1.1.8] the set $\{c^A \mid A \in \Xi(n,d)\}$ forms a k-basis for $A_q(n,d)$. For $A \in \Xi(n,d)$, we write ϕ_A for the element for $S_q(n,d) = A_q(n,d)^*$ dual to c^A. For $A \in \Xi(n,d)$, let

$$[A] = \upsilon^{-d_A}\phi_A \quad \text{with } d_A = \sum_{i \geqslant k,\ j<l} a_{ij}a_{kl}. \tag{3-2}$$

Then the set $\{[A] \mid A \in \Xi(n,d)\}$ forms a k-basis for $s_q(n,d)_r$. By [Cox 1997, 5.3.1], the set

$$\{[A] \mid A \in \Xi(n,d),\ a_{ij} < lp^{r-1} \text{ for } i \neq j\} \tag{3-3}$$

forms a k-basis for $s_q(n,d)_r$.

Lemma 3.1. *For any $\lambda, \mu \in \Gamma_r^d(D)$, we have*

$$\mathrm{Ext}^1_{s_q(n,d)_r}(\widehat{L}_r(\lambda), \widehat{L}_r(\mu)) = \mathrm{Ext}^1_{s_q(n,d)_r}(\widehat{L}_r(\mu), \widehat{L}_r(\lambda)).$$

Proof. By [Beĭlinson et al. 1990, 3.10], [Du 1992, A.1] and [Du et al. 1991, 5.7], there is an antiautomorphism Ψ on the q-Schur algebra $S_q(n,d)$ defined by sending $[A]$ to $[{}^tA]$ for all $A \in \Xi(n,d)$, where ${}^tA$ is the transpose of A. By (3-3), we have $\Psi(s_q(n,d)_r) = s_q(n,d)_r$. Using the antiautomorphism Ψ on the infinitesimal q-Schur algebra $s_q(n,d)_r$, we may construct from any $s_q(n,d)_r$-module M the contravariant dual module M^0. It is easy to see that $(\widehat{L}_r(\lambda))^0 \cong \widehat{L}_r(\lambda)$ for any $\lambda \in \Gamma_r^d(D)$. We can now imitate the proof of [Jantzen 1987, II, 2.12(4)] to get the result. □

For generalizing [Doty et al. 1997, 2.3] to the quantum case, we have to prove the following two lemmas.

Lemma 3.2. *For any $\lambda, \mu \in X_r(T)$, the restriction map*

$$\mathrm{res}_{G,G_r} : \mathrm{Ext}^1_G(L(\lambda), L(\mu)) \to \mathrm{Ext}^1_{G_r}(L(\lambda), L(\mu)) \quad \textit{is injective.}$$

Proof. Let $\overline{G}^r$ be the factor group of G whose coordinate algebra is the sub-Hopf algebra of $k[G]$ generated by the elements ${c_{ij}}^{lp^{r-1}}$ for $1 \leqslant i, j \leqslant n$ and ${\delta_q}^{-lp^{r-1}}$. Note that the factor group $\overline{G}^r$ is isomorphic to $\overline{G}$ via F^r.

By [Parshall and Wang 1991, (2.11.1) and (2.8.2)(3)], we have the five term exact sequence

$$\begin{aligned} 0 \to H^1(\overline{G}^r, \mathrm{Hom}_{G_r}(L(\lambda), L(\mu))) \\ \longrightarrow \mathrm{Ext}^1_G(L(\lambda), L(\mu)) \longrightarrow \mathrm{Ext}^1_{G_r}(L(\lambda), L(\mu))^{\overline{G}^r} \\ \longrightarrow H^2(\overline{G}^r, \mathrm{Hom}_{G_r}(L(\lambda), L(\mu))) \longrightarrow \mathrm{Ext}^2_G(L(\lambda), L(\mu)). \end{aligned}$$

Since $\overline{G}^r$ is isomorphic to $\overline{G}$, by [Jantzen 1987, II, 4.11] we have $H^i(\overline{G}^r, k) = 0$ for all $i > 0$. Hence we have $\mathrm{Ext}^1_G(L(\lambda), L(\mu)) \cong \mathrm{Ext}^1_{G_r}(L(\lambda), L(\mu))^{\overline{G}^r}$. □

Lemma 3.3. *For any G-module N, we have*

$$\mathrm{soc}_G N \cong \bigoplus_{\lambda \in X_r(T)} \mathrm{soc}_G \mathrm{Hom}_{G_r}(L(\lambda), N) \otimes L(\lambda).$$

Proof. By [Donkin 1998, 3.2(4)], the natural map $f : \mathrm{Hom}_{G_r}(L(\lambda), N) \otimes L(\lambda) \to N$ for $\lambda \in X_r(T)$ is a morphism of G-modules. In fact, the map f is injective, since $\mathrm{Hom}_{G_r}(L(\lambda), N) \otimes L(\lambda)$ is isomorphic to $(\mathrm{soc}_{G_r} N)_{L(\lambda)}$ via f. So we can view $\mathrm{Hom}_{G_r}(L(\lambda), N) \otimes L(\lambda)$ as a submodule of N for $\lambda \in X_r(T)$. By [Parshall and Wang 1991, 2.8.2(3) and 2.10.2], for $\lambda \in X_r(T)$ there exists a $\overline{G}$-module V such that

$$\mathrm{soc}_G(\mathrm{Hom}_{G_r}(L(\lambda), N)) \cong \mathrm{soc}_G((N \otimes L(\lambda)^*)^{G_r}) \cong \mathrm{soc}_G(V^{F^r}) \cong (\mathrm{soc}_{\overline{G}} V)^{F^r}.$$

It follows from [Donkin 1998, 3.2(5)] that $\mathrm{soc}_G \mathrm{Hom}_{G_r}(L(\lambda), N) \otimes L(\lambda)$ is a semisimple G-module for each $\lambda \in X_r(T)$. On the other hand, each simple G-submodule W of N is isomorphic to $\overline{L}(\mu)^{F^r} \otimes L(\lambda)$ for some $\lambda \in X_r(T)$ and $\mu \in X^+(T)$. By the proof of [Donkin 1998, 3.2(5)], $\overline{L}(\mu)^{F^r} \cong \mathrm{Hom}_{G_r}(L(\lambda), W) \subseteq \mathrm{Hom}_{G_r}(L(\lambda), N)$. Hence $W \subseteq \mathrm{soc}_G \mathrm{Hom}_{G_r}(L(\lambda), N) \otimes L(\lambda)$. The assertion follows. □

Now using the above two lemmas we can prove the following result, which gives information about the restriction of extensions of simple M-modules to $M_r D$.

Proposition 3.4. (1) *If $\lambda, \mu \in X_r(T)$, then the restriction map*

$$\mathrm{res}_{M,M_r D} : \mathrm{Ext}^1_M(L(\lambda), L(\mu)) \to \mathrm{Ext}^1_{M_r D}(L(\lambda), L(\mu)) \quad \textit{is injective.}$$

(2) *Let N be an M-module with two composition factors $L(\lambda)$ and $L(\mu)$, where $\lambda \in X_r(T)$ and $\mu \in P^+(D)$ with $\operatorname{soc}_M N \cong L(\lambda)$. Assume that $L(\mu) = \bigoplus_{j=1}^{s} \widehat{L}_r(\mu_j)$ is the decomposition of $L(\mu)$ as $M_r D$-modules. If $\widehat{L}_r(\lambda) \not\cong \widehat{L}_r(\mu_j)$ as G_r-modules for all j, then $\operatorname{soc}_{M_r D} N \cong L(\lambda)$.*

Proof. The proof is almost the same as [Doty et al. 1997, 2.3]. For $\lambda, \mu \in X_r(T)$, we have the commutative diagram

$$\begin{array}{ccc} \operatorname{Ext}^1_M(L(\lambda), L(\mu)) & \xrightarrow{\operatorname{res}_{M,M_r D}} & \operatorname{Ext}^1_{M_r D}(L(\lambda), L(\mu)) \\ \downarrow{\scriptstyle \operatorname{res}_{M,G}} & & \downarrow{\scriptstyle \operatorname{res}_{M_r D, G_r T}} \\ \operatorname{Ext}^1_G(L(\lambda), L(\mu)) & \xrightarrow[\operatorname{res}_{G,G_r T}]{} & \operatorname{Ext}^1_{G_r T}(L(\lambda), L(\mu)). \end{array}$$

By Lemma 3.2 we know that $\operatorname{res}_{G,G_r T}$ is injective, and by [Donkin 1996, 4(5)] the map $\operatorname{res}_{M,G}$ is an isomorphism. Hence the assertion (1) follows.

Now we consider part (2). If $\operatorname{soc}_{M_r D} N \not\cong L(\lambda)$, then $\widehat{L}_r(\mu_j)$ is a simple factor of $\operatorname{soc}_{M_r D} N$ for some $1 \leqslant j \leqslant s$. By [Cox 1997, bottom of page 76], [Cox 2000, §4] and [Donkin 1998, 3.1(18)], we have $\operatorname{soc}_{M_r D} N \cong \operatorname{soc}_{G_r T} N \cong \operatorname{soc}_{G_r} N$. Hence $\widehat{L}_r(\mu_j)$ is a factor of $\operatorname{soc}_{G_r} N$. It follows from Lemma 3.3 that $\operatorname{soc}_G N$ is not simple. This is a contradiction. □

Now we shall describe some results which will be used to reduce the general question of representation type of infinitesimal q-Schur algebras to that of finding the representation type of $s_q(n, d)_r$ for small n and small d. The first result relates the representation type of $s_q(n, d)_r$ with $s_q(n', d)_r$ where $n' \geqslant n$.

Theorem 3.5. *Assume $n' \geqslant n$. If $s_q(n, d)_r$ has infinite representation type, then so does $s_q(n', d)_r$.*

Proof. Let $e = \sum_{\lambda \in \Lambda(n,d)} [\operatorname{diag}(\lambda)] \in S_q(n', d)_r$. (See (3-2) for the definition of $[\operatorname{diag}(\lambda)]$.) Then we have $e s_q(n', d)_r e \cong s_q(n, d)_r$. Hence the assertion follows by [Erdmann 1990, I 4.7]. □

Lemma 3.6. *There is a surjective homomorphism φ_d from $s_q(n, d+n)_r$ to $s_q(n, d)_r$ for any d.*

Proof. By [Donkin 1998, 4.2(18)], there is a surjective homomorphism φ_d from $S_q(n, d+n)$ to $S_q(n, d)$. It is easy to check that restriction induces a surjective homomorphism φ_d from $s_q(n, d+n)_r$ to $s_q(n, d)_r$. □

By the above lemma, we get the following corollary which relates the representation type of $s_q(n, d)_r$ to that of $s_q(n, d+n)_r$.

Corollary 3.7. *If $s_q(n, d)_r$ has infinite representation type, then $s_q(n, d+n)_r$ does as well.*

Using the translation functor for G defined in [Erdmann and Nakano 2001, 2.4], we can define the translation functor for G_rT as follows. Let Φ^+ be a set of positive roots for the root system of type A_{n-1}, and let $\rho = (n-1, n-2, \dots, 0)$ and

$$\overline{C}_{\mathbb{Z}} = \{\lambda \in X(T) \mid 0 \leqslant \langle \lambda + \rho, \alpha^\vee \rangle \leqslant l \text{ for all } \alpha \in \Phi^+\}.$$

Let W_l be the affine Weyl group. For any G_rT-module V and $\lambda, \mu \in X(T)$, let $pr_\lambda V$ be the largest G_rT-submodule of V such that $\mu \in W_l \cdot \lambda$ for every composition factor $\widehat{L}_r(\mu)$ of V. For $\lambda, \mu \in \overline{C}_{\mathbb{Z}}$, let ν be the unique weight in $X^+(T) \cap W(\mu - \lambda)$. Then the translation functor $T_\lambda^\mu : \mathrm{mod}(G_rT) \to \mathrm{mod}(G_rT)$ is defined by

$$T_\lambda^\mu(V) = pr_\mu(\widehat{L}_r(\nu) \otimes pr_\lambda V).$$

For $V \in \mathrm{mod}(G_rT)$, let $\mathcal{F}_{M_rD}(V)$ be the unique maximal G_rT-submodule of V that lifts to an M_rD-module. For any $\lambda, \mu \in \overline{C}_{\mathbb{Z}}$, define a truncated translation functor $\overline{T}_\lambda^\mu$ to be the composite $\mathcal{F}_{M_rD} \circ T_\lambda^\mu$. By restriction, $\overline{T}_\lambda^\mu$ induces a functor from $\mathrm{mod}(M_rD)$ into itself.

For $\lambda \in \Gamma_r^d(D)$, let $\mathcal{B}_r^d(\lambda)$ be the block of $s_q(n,d)_r$ containing λ. Since a simple $s_q(n,d)_r$-module appears as a composition factor of exactly one block and the simple $s_q(n,d)_r$-modules are indexed by elements of $\Gamma_r^d(D) \subseteq \mathbb{Z}^n$, we may identify blocks for $s_q(n,d)_r$ with subsets of $\mathbb{Z}^n$.

Theorem 3.8. *Assume the block $\mathcal{B}_r^d(\lambda)$ of $s_q(n,d)_r$ has infinite representation type. Suppose that $\mu \in \Gamma_r^{d'}(D)$ is a weight in the same facet as λ with $\mu - \lambda \in P(D)$. Then $s_q(n,d')_r$ has infinite representation type.*

Proof. Since λ and μ lie in the same facet, there exist unique elements $\lambda', \mu' \in \overline{C}_{\mathbb{Z}}$ in the same facet and a unique $w \in W_l$ with $w \cdot \lambda' = \lambda$ and $w \cdot \mu' = \mu$. Since $\mu' - \lambda' \in W(\mu - \lambda)$ and $\mu - \lambda \in P(D)$, we have $\mu' - \lambda' \in P(D)$. View $\overline{T}_{\lambda'}^{\mu'}$ as a functor from $\{V \in \mathrm{mod}(s_q(n,d)_r) \mid pr_{\lambda'} V = V\}$ to $\{V \in \mathrm{mod}(s_q(n,d')_r) \mid pr_{\mu'} V = V\}$. Then one can prove $\overline{T}_{\mu'}^{\lambda'} \circ \overline{T}_{\lambda'}^{\mu'}$ is equivalent to identity functor as in the proof of [Doty et al. 1997, 4.2]. It follows that the functor $\overline{T}_{\lambda'}^{\mu'}$ preserves indecomposable modules and isomorphism classes. The assertion follows. □

4. Infinite representation type

In this section, we will prove that the infinitesimal q-Schur algebra has infinite representation type for the cases listed in Theorem 2.3.

Proposition 4.1. *The algebra $s_q(n,d)_r$ has infinitesimal representation type if one of the following holds*:

(1) $n \geqslant 3$, $d \geqslant 2l$ *and either* $r \geqslant 3$ *or both* $r = 2$ *and* $p \geqslant 5$;

(2) $n = 2$, $l \geqslant 3$, $r \geqslant 3$ *and* $d \geqslant lp$.

(3) $n = 2$, $l = 2$, $r \geqslant 3$ *and* d *is even with* $d \geqslant 2p$

(4) $n = 2,\ l = 2,\ r \geqslant 4$ *and* d *is odd with* $d \geqslant 2p^2 + 1$.

Proof. (1) Suppose either $r \geqslant 3$ or both $r = 2$ and $p \geqslant 5$. Then $lp^{r-1} \geqslant 4l > d$ for $d = 2l,\ 2l+1,\ 2l+2$. By (3-3) and Theorem 2.1, the algebra $s_q(3, d)_r = S_q(3, d)$ has infinite representation type for $d = 2l,\ 2l+1,\ 2l+2$. So by Corollary 3.7 and Theorem 3.5, we have $s_q(n, d)_r$ has infinite representation type for $n \geqslant 3$ and $d \geqslant 2l$.

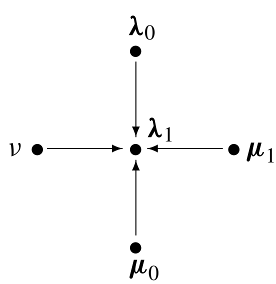
(2) Suppose $l \geqslant 3$ and $r \geqslant 3$. Then $lp^{r-1} \geqslant lp^2 > d$ for $d = lp, lp+1$. By (3-3) and Theorem 2.1, the algebra $s_q(2, d)_r = S_q(2, d)$ has infinite representation type for $d = lp, lp+1$. So by Corollary 3.7, we have $s_q(2, d)_r$ has infinite representation type for $d \geqslant lp$.

(3) Suppose $l = 2$ and $r \geqslant 3$. Then $lp^{r-1} \geqslant 2p^2 > 2p$. By (3-3) and Theorem 2.1, the algebra $s_q(2, 2p)_r = S_q(2, 2p)$ has infinite representation type. So by Corollary 3.7, we have $s_q(2, d)_r$ has infinite representation type for d even with $d \geqslant 2p$.

(4) Suppose $l = 2$ and $r \geqslant 4$. Then $lp^{r-1} \geqslant 2p^3 > 2p^2+1$. By (3-3) and Theorem 2.1, the algebra $s_q(2, 2p^2+1)_r = S_q(2, 2p^2+1)$ has infinite representation type. So by Corollary 3.7, we have $s_q(2, d)_r$ has infinite representation type for d odd with $d \geqslant 2p^2 + 1$. □

Proposition 4.2. *Assume* $l = 2$. *Then the algebra* $s_q(2, d)_3$ *has infinite representation type for* d *odd with* $d \geqslant 2p^2 + 1$.

Proof. Let $\boldsymbol{\lambda}_0 = (2p^2+1, 0)$, $\boldsymbol{\lambda}_1 = (2p^2+1-2p, 2p)$, $\boldsymbol{\mu}_0 = (2p^2-1, 2)$ and $\boldsymbol{\mu}_1 = (2p^2-2p-1, 2p+2)$. By [Erdmann and Nakano 2001, 3.2], the classical Schur algebra $S(2, p^2)$ is Morita equivalent to the principal block component of $S_q(2, 2p^2+1)$. It follows from [Erdmann 1993, 5.2] that $\mathrm{Ext}^1_M(L(\boldsymbol{\lambda}_1), L(\sigma)) \neq 0$ for $\sigma = \boldsymbol{\lambda}_0, \boldsymbol{\mu}_0, \boldsymbol{\mu}_1$. By (3-1) we have $L(\boldsymbol{\lambda}_0)|_{M_3D} \cong \widehat{L}_3(\boldsymbol{\lambda}_0) \oplus \widehat{L}_3(\nu)$, where $\nu = (1, 2p^2)$. Hence by Proposition 3.4, we have $\mathrm{Ext}^1_{M_3D}(\widehat{L}_3(\boldsymbol{\lambda}_1), \widehat{L}_3(\sigma)) \neq 0$ for $\sigma = \boldsymbol{\lambda}_0, \boldsymbol{\mu}_0, \boldsymbol{\mu}_1, \nu$. The Ext^1 quiver of $s_q(2, 2p^2+1)_3$ has a four subspace quiver as a subquiver, as illustrated.

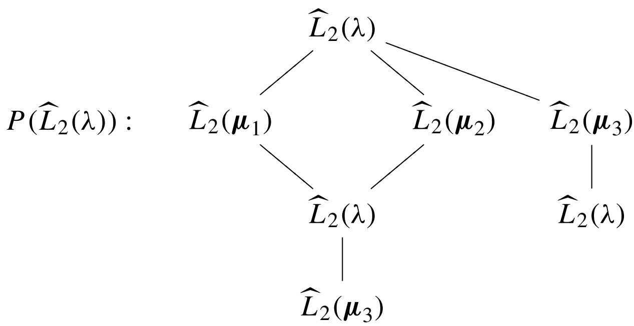


By [Gel'fand and Ponomarev 1972], $s_q(2, 2p^2+1)_3$ is of infinite type. Now the assertion follows from Corollary 3.7. □

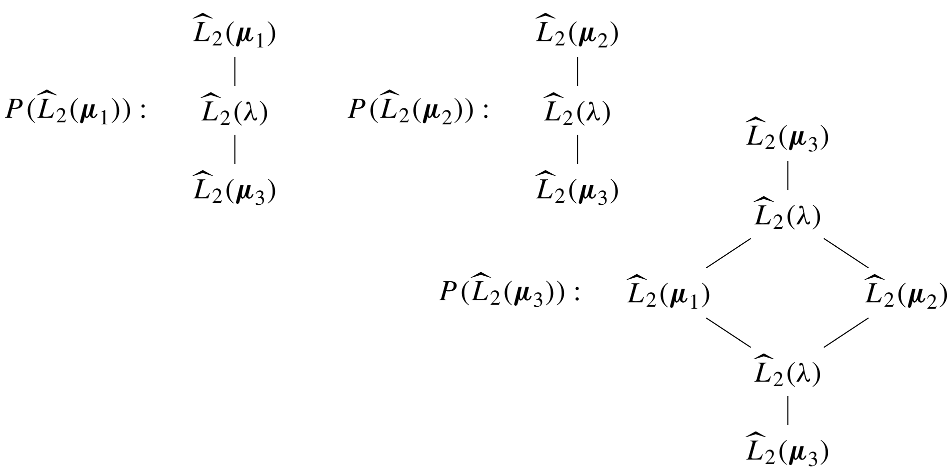
By Propositions 4.1 and 4.2, we know that the algebras listed in Theorem 2.3 for $r \geqslant 3$ have infinite representation type. It remains to check the algebras listed there have infinite representation type for $r = 1, 2$.

Proposition 4.3. *Assume $p = 2$. Then the algebra $s_q(n, d)_2$ has infinite representation type for $n \geqslant 2$ and $d \geqslant 2l$.*

Proof. Let $\boldsymbol{\mu}_1 = (2l, 0)$, $\boldsymbol{\mu}_2 = (0, 2l)$, $\boldsymbol{\mu}_3 = (2l-1, 1)$ and $\lambda = (l, l)$. By [Cox 1997, 6.2.13] and [Cox 2000, 5.12], the block $B_2^{2l}(\lambda)$ of $s_q(2, 2l)_2$ is equal to $\{\boldsymbol{\mu}_i, \lambda \mid 1 \leqslant i \leqslant 3\}$. By [Erdmann and Nakano 2001, Proposition 3.3(B)] and Proposition 3.4, the projective cover of $\widehat{L}_2(\lambda)$ has the following structure.



The vertices in this diagram correspond with composition factors, and the edges indicate a nonsplit extension. The structure of the projective covers for the other simple modules in the block $B_2^{2l}(\mu)$ are given below.



By [Doty et al. 1997, 5.2], the basic algebra of $B_2^{2l}(\lambda)$ is isomorphic to the basic algebra of the infinitesimal Schur algebra $s(2, 4)_2$. So $B_2^{2l}(\lambda)$ has infinite representation type. It follows that the algebra $s_q(2, 2l)_2$ has infinite representation type.

Since $p=2$ and l and p are coprime, we have $l \geqslant 3$. So the weight $\mu=(l+1, l)$ belongs to $\Gamma_2^{2l+1}(D)$, lies in the same facet as λ, and $\mu-\lambda \in P(D)$. It follows from Theorem 3.8 that $s_q(2, 2l+1)_2$ has infinite representation type. Now the assertion follows by Corollary 3.7 and Theorem 3.5. □

Proposition 4.4. *Assume $p \geqslant 3$. Then the algebra $s_q(2, lp+2j)_2$ has infinite representation type for $j \geqslant 0$.*

Proof. Let $\lambda=(lp-l, l)$, $\gamma=(lp, 0)$, $\beta=(lp-1, 1)$, $\tau=(0, lp)$ and $\eta=(lp-l-1, l+1)$. By [Thams 1994], the M-modules $\nabla(\lambda)$ and $\nabla(\gamma)$ have the following structure.

$$\nabla(\lambda): \quad \begin{array}{c} L(\eta) \\ | \\ L(\lambda) \end{array} \qquad\qquad \nabla(\gamma): \quad \begin{array}{c} L(\beta) \\ | \\ L(\lambda) \\ | \\ L(\gamma) \end{array}$$

By (3-1) we have $L(\gamma)|_{M_2D} \cong \widehat{L}_2(\gamma) \oplus \widehat{L}_2(\tau)$. By Proposition 3.4, it follows that $\operatorname{Ext}^1_{M_2D}(\widehat{L}_2(\lambda), \widehat{L}_2(\sigma)) \neq 0$ for $\sigma=\gamma, \beta, \tau, \eta$. The Ext^1-quiver for $s_q(2, lp)_2$ has a four subspace quiver as subquiver. Hence $s_q(2, lp)_2$ is of infinite type. Hence the assertion follows by Corollary 3.7. □

Corollary 4.5. *Assume $l \geqslant 3$. Then the algebra $s_q(2, d)_2$ has infinite representation type for $d \geqslant lp$.*

Proof. If $p=2$, then the assertion follows from Proposition 4.3. Now we assume $p \geqslant 3$. Let $\lambda=(lp-l, l)$ and $\mu=(lp-l+1, l)$. Then the weight μ belongs to $\Gamma_2^{lp+1}(D)$, it lies in the same facet as λ since $l \geqslant 3$, and $\mu-\lambda \in P(D)$. By the proof of Proposition 4.4, the block $B_2^{lp}(\lambda)$ of $s_q(2, lp)_2$ has infinite representation type. It follows from Theorem 3.8 that the algebra $s_q(2, lp+1)_2$ has infinite representation type. Hence the assertion follows by Corollary 3.7. □

Proposition 4.6. *Assume $p=3$. Then the algebra $s_q(n, d)_2$ has infinite representation type for $n \geqslant 3$ and $d \geqslant 2l$.*

Proof. There are two cases.

(1) Suppose $l>2$. Then $3l>d$ for $d=2l,\ 2l+1,\ 2l+2$. By (3-3) and Theorem 2.1, the algebra $s_q(3, d)_2=S_q(3, d)$ has infinite representation type for $d=2l,\ 2l+1,\ 2l+2$. So by Corollary 3.7 and Theorem 3.5, $s_q(n, d)_2$ has infinite representation type for $n \geqslant 3$ and $d \geqslant 2l$.

(2) Suppose $l=2$. By (3-3) and Theorem 2.1, the algebra $s_q(3, d)_2=S_q(3, d)$ has infinite representation type for $d=4, 5$. But by Proposition 4.4 and Theorem 3.5, the algebra $s_q(3, 6)_2$ has infinite representation type. So by Corollary 3.7 and Theorem 3.5, $s_q(n, d)_2$ has infinite representation type for $n \geqslant 3$ and $d \geqslant 4$. □

By Proposition 4.1(1) and Propositions 4.3–4.6, we know that the algebras listed in Theorem 2.3 for $r=2$ have infinite representation type. We can now concentrate on the situation when $r=1$.

Proposition 4.7. *Assume $l \geqslant 3$. Then the algebra $s_q(3,l)_1$ has infinite representation type.*

Proof. Let $\lambda=(l-1,1,0)$, $\gamma=(l,0,0)$, $\beta=(0,l,0)$, $\eta=(0,0,l)$ and $\tau=(l-2,1,1)$. By [Thams 1994], the M-module $\nabla(\gamma)$ has only two composition factors $L(\lambda)$ and $L(\gamma)$. So $\mathrm{Ext}^1_M(L(\lambda),L(\gamma))\neq 0$. It is clear that $L(\gamma)|_{M_1D}\cong\widehat{L}_1(\gamma)\oplus\widehat{L}_1(\beta)\oplus\widehat{L}_1(\eta)$. It follows from Proposition 3.4(2) that $\mathrm{Ext}_{M_1D}(\widehat{L}_1(\lambda),\widehat{L}_1(\sigma))\neq 0$ for $\sigma=\gamma,\beta,\eta$. By [Xi 1999], $\mathrm{Ext}^1_{M_1D}(\widehat{L}_1(\lambda),\widehat{L}_1(\tau))\neq 0$. The Ext^1 quiver for $s_q(3,l)_1$ has a four subspace quiver as subquiver. Hence the algebra $s_q(3,l)_1$ has infinite representation type. □

Corollary 4.8. *Assume $l\geqslant 4$. Then the algebra $s_q(3,d)_1$ has infinite representation type for $d\geqslant l$.*

Proof. Let $\tau=(l-2,1,1)$, $\boldsymbol{\mu}_1=(l-2,2,1)$ and $\boldsymbol{\mu}_2=(l-2,2,2)$. For $i=1,2$, we have $\boldsymbol{\mu}_i\in\Gamma_1^{l+i}(D)$, $\boldsymbol{\mu}_i$ lies in the same facet as τ since $l\geqslant 4$, and $\boldsymbol{\mu}_i-\tau\in P(D)$. From the proof of Proposition 4.7 we know that the block $B_1^l(\tau)$ of $s_q(3,l)_1$ has infinite representation type. It follows from Theorem 3.8 that $s_q(3,l+i)_1$ has infinite representation type for $i=1,2$. Hence by Proposition 4.7 and Corollary 3.7, the algebra $s_q(3,d)_1$ has infinite representation type for $d\geqslant l$. □

Lemma 4.9. *Let s_1 be one of the algebras $s_q(3,7)_1$ or $s_q(3,8)_1$ for $l=3$, or $s_q(3,4)_1$ or $s_q(3,5)_1$ for $l=2$. Then the algebra s_1 has infinite representation type.*

Proof. Let X, Y_1, Y_2 and Z denote the following simple s_1-modules.

Simple modules	$s_q(3,7)_1$ $l=3$	$s_q(3,8)_1$ $l=3$	$s_q(3,4)_1$ $l=2$	$s_q(3,5)_1$ $l=2$
X	$\widehat{L}_1(2,5,0)$	$\widehat{L}_1(3,5,0)$	$\widehat{L}_1(1,3,0)$	$\widehat{L}_1(3,0,2)$
Y_1	$\widehat{L}_1(4,3,0)$	$\widehat{L}_1(4,4,0)$	$\widehat{L}_1(2,2,0)$	$\widehat{L}_1(3,1,1)$
Y_2	$\widehat{L}_1(1,3,3)$	$\widehat{L}_1(1,4,3)$	$\widehat{L}_1(0,2,2)$	$\widehat{L}_1(1,1,3)$
Z	$\widehat{L}_1(4,2,1)$	$\widehat{L}_1(4,2,2)$	$\widehat{L}_1(2,1,1)$	$\widehat{L}_1(2,2,1)$

We consider the algebra $s_q(3,7)_1$ for $l=3$. By [Donkin 1998, 4.2(9) and 4.2 (15)] and [Thams 1994], we know that the M-modules $\nabla(4,3,0)$ and $\nabla(4,0,0)$ have the following structure.

$$\nabla(4,3,0):\quad \begin{array}{c} L(4,2,1)\\ |\\ L(4,3,0)\end{array}\qquad\qquad \nabla(4,0,0):\quad \begin{array}{c} L(2,2,0)\\ |\\ L(4,0,0)\end{array}$$

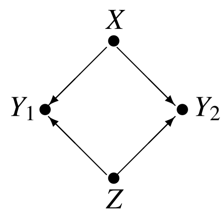
It is clear that $\bar{L}(1,1,0) = \bar{\nabla}(1,1,0)$. Hence by Weyl's character formula we know that the weights of $\bar{L}(1,1,0)$ are $(1,1,0)$, $(1,0,1)$ and $(0,1,1)$. It follows from (3-1) that $L(4,3,0)|_{M_1D} \cong \widehat{L}_1(4,3,0) \oplus \widehat{L}_1(1,3,3) \oplus \widehat{L}_1(4,0,3)$. Hence by Proposition 3.4 and Lemma 3.1, $\mathrm{Ext}^1_{M_1D}(\widehat{L}_1(\sigma), \widehat{L}_1(4,2,1)) \neq 0$ for $\sigma = (4,3,0)$ and $(1,3,3)$. Since

$$L(4,0,0)|_{M_1D} \cong \widehat{L}_1(4,0,0) \oplus \widehat{L}_1(1,3,0) \oplus \widehat{L}_1(1,0,3),$$

by Lemma 3.1, there exist nonsplit extensions of the form

$$0 \to \widehat{L}_1(4,0,0) \to M_1 \to \widehat{L}_1(2,2,0) \to 0,$$
$$0 \to \widehat{L}_1(1,0,3) \to M_2 \to \widehat{L}_1(2,2,0) \to 0.$$

Now, by tensoring these short exact sequences by the one-dimensional module $\widehat{L}_1(0,3,0)$, we have that $\mathrm{Ext}^1_{M_1D}(\widehat{L}_1(\sigma), \widehat{L}_1(2,5,0)) \neq 0$ for $\sigma = (4,3,0)$ and $(1,3,3)$. For the other algebras, we can prove the existence of the extensions in a similar manner. So one of the components of the separated quiver of the Ext^1-quiver of s_1 contains the subquiver given below.



The algebra s_1/J^2 has the same Ext^1 quiver as s_1, where $J = \mathrm{Rad}\, s_1$. By [Pierce 1982, 11.8], s_1/J^2 has infinite representation type. Hence s_1 has infinite representation type. □

Lemma 4.10. *The algebra $s_q(3,6)_1$ has infinite representation type for $l = 2$.*

Proof. Let $\lambda = (2,2,2)$, $\gamma = (4,1,1)$, $\beta = (2,3,1)$, $\tau = (2,1,3)$ and $\eta = (3,3,0)$. By [Donkin 1998, 4.2(9) and 4.2 (15)] and [Thams 1994], we know that the M-module $\nabla(\gamma)$ has two composition factors with top $L(\lambda)$. By [Erdmann and Nakano 2001, 5.6], we know that the M-module $\nabla(\eta)$ has two composition factors with top $L(\lambda)$. By (3-1), we have $L(\gamma)|_{M_1D} \cong \widehat{L}_1(\gamma) \oplus \widehat{L}_1(\beta) \oplus \widehat{L}_1(\tau)$. Upon restriction to M_1D, we have $\mathrm{Ext}^1_{M_1D}(\widehat{L}_1(\lambda), \widehat{L}_1(\sigma)) \neq 0$ for $\sigma = \gamma, \beta, \tau, \eta$ by Proposition 3.4. So the Ext^1-quiver for $s_q(3,6)_1$ has a four subspace quiver as a subquiver. The assertion follows. □

Proposition 4.11. *The algebra $s_q(3,d)_1$ has infinite representation type for*

(1) $l = 3$ *and* $d \geqslant 6$;

(2) $l = 2$ *and* $d \geqslant 4$.

Proof. By Proposition 4.7 and Corollary 3.7, the algebra $s_q(3,6)_1$ has infinite representation type for $l=3$. Hence the assertion follows by Lemmas 4.9 and 4.10 and Corollary 3.7. □

Proposition 4.12. *The algebra $s_q(n,d)_1$ has infinite representation type for $n \geqslant 4$ and $d \geqslant l$.*

Proof. If $l \geqslant 4$, then the assertion follows from Corollary 4.8 and Theorem 3.5. Now we assume $l<4$. Then $l=2$ or $l=3$. For $l=3$, set $s_1 = s_q(4,4)_1$ or $s_q(4,5)_1$, and for $l=2$, set $s_1 = s_q(4,2)_1$ or $s_q(4,3)_1$. For the algebra s_1 let α, β, λ, η and μ be the following weights.

	$s_q(4,4)_1$ $l=3$	$s_q(4,5)_1$ $l=3$	$s_q(4,2)_1$ $l=2$	$s_q(4,3)_1$ $l=2$
α	$(4,0,0,0)$	$(5,0,0,0)$	$(2,0,0,0)$	$(3,0,0,0)$
β	$(1,3,0,0)$	$(2,3,0,0)$	$(0,2,0,0)$	$(1,2,0,0)$
λ	$(1,0,3,0)$	$(2,0,3,0)$	$(0,0,2,0)$	$(1,0,2,0)$
η	$(1,0,0,3)$	$(2,0,0,3)$	$(0,0,0,2)$	$(1,0,0,2)$
μ	$(2,2,0,0)$	$(2,2,1,0)$	$(1,1,0,0)$	$(1,1,1,0)$

By [Thams 1994], we know that $\nabla(\alpha)$ has two composition factors with top $L(\mu)$. By (3-1), we have $L(\alpha)|_{M_1D} \cong L(\alpha)\oplus L(\beta)\oplus L(\lambda)\oplus L(\eta)$. It follows from Proposition 3.4(2) that $\mathrm{Ext}^1_{M_1D}(\widehat{L}_1(\mu), \widehat{L}_1(\sigma)) \neq 0$ for $\sigma = \alpha, \beta, \lambda, \eta$. The Ext^1 quiver of s_1 has a four subspace quiver as a subquiver. Hence the algebra s_1 has infinite representation type. By Propositions 4.7 and 4.11 and Theorem 3.5, the algebra $s_q(4,d)_1$ has infinite representation type for $l=3$ and $d=3,6$ or $l=2$ and $d=4,5$. Hence the assertion follows by Corollary 3.7 and Theorem 3.5. □

5. Finite representation type

In this section, we will prove the infinitesimal q-Schur algebra has finite representation type for the cases listed in Theorem 2.2.

Proposition 5.1. *The algebra $s_q(n,d)_r$ has finite representation type if one of the following holds*:

(1) $n \geqslant 3$, $r \geqslant 2$ *and* $d < 2l$;

(2) $n \geqslant 3$, $r = 1$ *and* $d < l$;

(3) $n = 2$, $r \geqslant 2$ *and* $d < lp$;

(4) $n = 2$, $l = 2$, $r \geqslant 3$ *and d is odd with* $2p+1 \leqslant d < 2p^2+1$;

(5) $n = 2$ *and* $r = 1$.

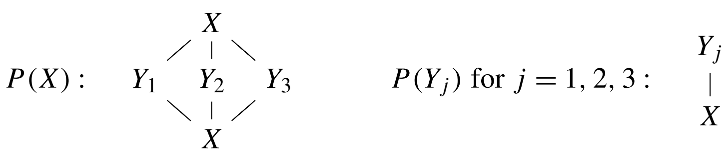
Proof. In cases (1)–(4), we have $s_q(n,d)_r = S_q(n,d)$. Hence the assertion in these cases follows from Theorem 2.1. In the last case, the assertion follows from [Erdmann and Fu 2008, 3.7]. □

Proposition 5.2. *Let s_1 be the algebra $s_q(3,4)_1$ or $s_q(3,5)_1$ for $l=3$, or $s_q(3,2)_1$ or $s_q(3,3)_1$ for $l=2$. Then the algebra s_1 has finite representation type.*

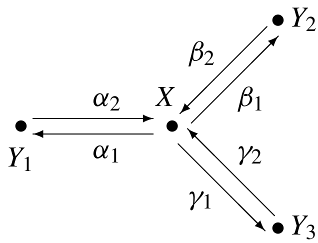
Proof. All nonsemsimple blocks of s_1 are Morita equivalent to a basic algebra B. The algebra B has four simple modules X, Y_1, Y_2, Y_3. The following table provides the correspondence between the simple modules for B and the simple modules for the blocks of s_1.

	$l=3$			$l=2$	
Simple modules	$s_q(3,4)_1$	$s_q(3,5)_1$ block 1	$s_q(3,5)_1$ block 2	$s_q(3,2)_1$	$s_q(3,3)_1$
Y_1	$\widehat{L}_1(4,0,0)$	$\widehat{L}_1(5,0,0)$	$\widehat{L}_1(4,1,0)$	$\widehat{L}_1(2,0,0)$	$\widehat{L}_1(3,0,0)$
Y_2	$\widehat{L}_1(1,3,0)$	$\widehat{L}_1(2,3,0)$	$\widehat{L}_1(1,4,0)$	$\widehat{L}_1(0,2,0)$	$\widehat{L}_1(1,2,0)$
Y_3	$\widehat{L}_1(1,0,3)$	$\widehat{L}_1(2,0,3)$	$\widehat{L}_1(1,1,3)$	$\widehat{L}_1(0,0,2)$	$\widehat{L}_1(1,0,2)$
X	$\widehat{L}_1(2,2,0)$	$\widehat{L}_1(2,2,1)$	$\widehat{L}_1(3,2,0)$	$\widehat{L}_1(1,1,0)$	$\widehat{L}_1(1,1,1)$

The projective covers of these modules have the following structure.



The Ext^1 quiver for B is illustrated below with relations $\alpha_1\alpha_2 = \beta_1\beta_2 = \gamma_1\gamma_2$ and all other products zero.



Hence by [Pierce 1982, 11.8], the algebra B/J^2 has finite representation type, where $J = \mathrm{Rad}(B)$. Since $P(X)$ is injective and $P(X)$ is the only indecomposable projective B-module of radical length greater than two, the algebra B has also finite representation type. □

We have now proved that the algebras listed in Theorem 2.2(1)–(6) and (8) are of finite type. It remains to check (7). Recall that δ_q is the q-determinant in

$A_q(n)$. We shall denote the corresponding 1-dimensional $G_q(n)$-module by the same symbol δ_q. For simplicity, we shall denote by δ_q the restriction to G_rT of the q-determinant module. We need the following reduction lemma.

Lemma 5.3. *Assume $\mathcal{B}$ is a block of $s_q(2,d)_r$ such that $\lambda - \mathbf{1} \in \Gamma_r(D)$ for any $\lambda \in \mathcal{B}$, where $\mathbf{1} = (1,1)$. Then $\mathcal{B}' := \{\lambda - \mathbf{1} \mid \lambda \in \mathcal{B}\}$ is a block of $s_q(2,d-2)_r$, and $\mathcal{B}'$ is Morita equivalent to $\mathcal{B}$.*

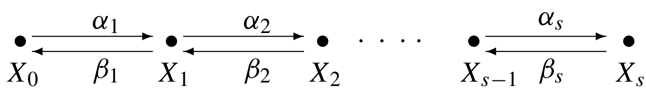
Proof. By [Cox 1997, 5.2] and [Cox 2000, Section 4], it is easy to check that $\mathcal{B}'$ is a block of $s_q(2,d-2)_r$. Since $P(\widehat{L}_r(\lambda)) \cong P(\widehat{L}_r(\lambda - \mathbf{1})) \otimes \delta_q$ for any $\lambda \in \mathcal{B}$, the assertion follows. □

By [Thams 1994], we have the following result.

Lemma 5.4. *Assume $l = 2$ and $0 \leqslant d < 2p$. Then*

$$\dim L(d,0) = \begin{cases} d+1 & \text{if } d \text{ is odd,} \\ d/2+1 & \text{if } d \text{ is even.} \end{cases}$$

Let $\mathcal{A}_s$ for $s \geqslant 2$ denote the quiver the figure below with relations $\alpha_1\beta_1 = 0 = \beta_s\alpha_s$, $\alpha_j\alpha_{j+1} = 0 = \beta_{j+1}\beta_j$ and $\beta_j\alpha_j = \alpha_{j+1}\beta_{j+1}$ for $j = 1, 2, \dots, s-1$.



For $k, t \geqslant 0$, let

$$\begin{aligned} \boldsymbol{\lambda}_{0,k}(1) &= (2p+2k+1, 0) + (t-k)\mathbf{1}, \\ \boldsymbol{\lambda}_{1,k}(1) &= (2p-1, 2k+2) + (t-k)\mathbf{1}, \\ \boldsymbol{\lambda}_{2,k}(1) &= (2k+1, 2p) + (t-k)\mathbf{1}. \end{aligned}$$

Let $\mathcal{B}_{k,t}(1) = \{\boldsymbol{\lambda}_{0,k}(1), \boldsymbol{\lambda}_{1,k}(1), \boldsymbol{\lambda}_{2,k}(1)\}$.

Proposition 5.5. *Assume $l = 2$. Then*

(1) *For $0 \leqslant t \leqslant p-2$, the nonsemisimple blocks of $s_q(2, 2p+2t+1)_2$ are $\mathcal{B}_{k,t}(1)$ for $0 \leqslant k \leqslant t$ and are Morita equivalent to $\mathcal{A}_2$.*

(2) *The nonsemisimple blocks of $s_q(2, 4p-1)_2$ are $\mathcal{B}_{k,p-1}(1)$ for $0 \leqslant k \leqslant p-2$ and are Morita equivalent to $\mathcal{A}_2$.*

Proof. We consider the algebra $s_q(2, 2p+1)_2$. Let

$$\boldsymbol{\lambda}_0 = (2p+1, 0), \qquad \boldsymbol{\lambda}_1 = (2p-1, 2), \qquad \boldsymbol{\lambda}_2 = (1, 2p).$$

By [Cox 1997, 6.2.13] and [Cox 2000, 5.12], we have $\mathcal{B}_2^{2p+1}(\boldsymbol{\lambda}_0) = \{\boldsymbol{\lambda}_0, \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2\}$. Since the classical Schur algebra $S(2,p)$ is Morita equivalent to the principal block component of $S_q(2, 2p+1)$ by [Erdmann and Nakano 2001, 3.2], we get the structure of projective covers for the simple modules in $\mathscr{B}_2^{2p+1}(\boldsymbol{\lambda}_0)$ as follows by [Erdmann 1993, 5.1] and Proposition 3.4.

$$P(\widehat{L}_2(\boldsymbol{\lambda}_0)): \quad \begin{array}{c}\widehat{L}_2(\boldsymbol{\lambda}_0)\\ | \\ \widehat{L}_2(\boldsymbol{\lambda}_1)\end{array} \qquad P(\widehat{L}_2(\boldsymbol{\lambda}_1)): \quad \begin{array}{ccc} & \widehat{L}_2(\boldsymbol{\lambda}_1) & \\ \diagup & & \diagdown \\ \widehat{L}_2(\boldsymbol{\lambda}_0) & & \widehat{L}_2(\boldsymbol{\lambda}_2) \\ \diagdown & & \diagup \\ & \widehat{L}_2(\boldsymbol{\lambda}_1) & \end{array}$$

$$P(\widehat{L}_2(\boldsymbol{\lambda}_2)): \quad \begin{array}{c}\widehat{L}_2(\boldsymbol{\lambda}_2)\\ | \\ \widehat{L}_2(\boldsymbol{\lambda}_1)\end{array}$$

Hence the basic algebra for $\mathscr{B}_2^{2p+1}(\boldsymbol{\lambda}_0)$ is isomorphic to $\mathscr{A}_2$. Since $s_q(2, 2p-1)_2 = S_q(2, 2p-1)$ is semisimple by [Erdmann and Nakano 2001, 1.3], the algebra $s_q(2, 2p+1)_2$ has only one nonsemisimple block $\mathscr{B}_2^{2p+1}(\boldsymbol{\lambda}_0)$ by Lemma 5.3. Now by induction on t, the assertion follows. □

Assume $a \geqslant 2$ and $0 \leqslant t \leqslant p-1$. For $0 \leqslant k \leqslant p-t-3$, let

$$\mathscr{B}_k(a-1) = \{\boldsymbol{\lambda}_{i,k}(a-1) \mid 0 \leqslant i \leqslant 2a-2\},$$

where

$$\boldsymbol{\lambda}_{2s,k}(a-1) = (2p(a-s)-p+k+t+1,\, 2ps+p-k+t),$$
$$\boldsymbol{\lambda}_{2s+1,k}(a-1) = (2p(a-s-1)+p-k+t-1,\, 2p(s+1)-p+k+t+2).$$

For $p-t-2 \leqslant k \leqslant p-2$, let

$$\mathscr{B}_k(a) = \{\boldsymbol{\lambda}_{i,k}(a) \mid 0 \leqslant i \leqslant 2a\},$$

where

$$\boldsymbol{\lambda}_{2s,k}(a) = (2p(a-s)-p+2t+3+k,\, 2ps+p-k-2),$$
$$\boldsymbol{\lambda}_{2s+1,k}(a) = (2p(a-s)+p-k-3,\, 2ps-p+2t+4+k).$$

Proposition 5.6. *Assume $l=2$. Then for $a \geqslant 2$ and $0 \leqslant t \leqslant p-1$, the nonsemisimple blocks of $s_q(2, 2pa+2t+1)_2$ are $\mathscr{B}_k(a-1)$ for $0 \leqslant k \leqslant p-t-3$ and $\mathscr{B}_k(a)$ for $p-t-2 \leqslant k \leqslant p-2$. Furthermore, the block $\mathscr{B}_k(a-1)$ is Morita equivalent to $\mathscr{A}_{2a-2}$ for each $0 \leqslant k \leqslant p-t-3$, and the block $\mathscr{B}_k(a)$ is Morita equivalent to $\mathscr{A}_{2a}$ for each $p-t-2 \leqslant k \leqslant p-2$.*

Proof. Let $d = 2pa+2t+1$. By induction on a and Proposition 5.5, one can prove for each $0 \leqslant k \leqslant p-t-3$ and $0 \leqslant i \leqslant 2a-2$ there exist indecomposable $s_q(2, d)_2$-modules $P(\boldsymbol{\lambda}_{i,k}(a-1))$ with the same structure as the projective cover $P(X_i)$ for the simple $\mathscr{A}_{2a-2}$-modules X_i, and for each $p-t-2 \leqslant k \leqslant p-2$

and $0 \leqslant i \leqslant 2a$, there exist indecomposable $s_q(2,d)_2$-modules $P(\boldsymbol{\lambda}_{i,k}(a))$ with the same structure as the projective cover $P(X_i)$ for the simple $\mathscr{A}_{2a}$-modules X_i. Let

$$A_1 = \{2p(a-i-1, i) + (2p+t, t+1) \mid 0 \leqslant i \leqslant a-1\},$$
$$A_2 = \{2p(a-i-2, i) + (3p+t, p+t+1) \mid 0 \leqslant i \leqslant a-2\}.$$

One can check

$$\Gamma_2^d(D) = \Bigg(\bigcup_{k=0}^{p-t-3} \mathscr{B}_k(a-1)\Bigg) \bigcup \Bigg(\bigcup_{k=p-t-2}^{p-2} \mathscr{B}_k(a)\Bigg) \bigcup A_1 \bigcup A_2.$$

By Lemma 5.4, for $\boldsymbol{\lambda}_{i,k}(a-1) \in B_k(a-1)$, we have

$$\dim(\widehat{L}_2(\boldsymbol{\lambda}_{i,k}(a-1))) = \begin{cases} 2p-2k-2 & \text{if } i \text{ is odd,} \\ 2k+2 & \text{if } i \text{ is even.} \end{cases}$$

Similarly, for $\boldsymbol{\lambda}_{i,k}(a) \in B_k(a)$, we have

$$\dim(\widehat{L}_2(\boldsymbol{\lambda}_{i,k}(a))) = \begin{cases} 4p-2k-2t-6 & \text{if } i \text{ is odd,} \\ -2p+2k+2t+6 & \text{if } i \text{ is even,} \end{cases}$$

and $\dim(\widehat{L}_2(\mu)) = 2p$ for each $\mu \in A_1 \bigcup A_2$. It follows from [Cox 1997, 5.3.2] that

$$\begin{aligned} \sum_{k=0}^{p-t-3} \sum_{\mu \in \mathscr{B}_k(a-1)} \dim P(\mu) \dim \widehat{L}_2(\mu) + \sum_{k=p-t-2}^{p-2} \sum_{\mu \in \mathscr{B}_k(a)} \dim P(\mu) \dim \widehat{L}_2(\mu) \\ + \sum_{\mu \in A_1 \bigcup A_2} (\dim \widehat{L}_2(\mu))^2 &= 4p^2(2ap-2p+2t+3) \\ &= \dim s_q(2,d)_2. \end{aligned}$$

Hence for $\mu \in A_1 \cup A_2$, the $s_q(2,d)_2$-module $\widehat{L}_2(\mu)$ is projective, and for other $\mu \in \Gamma_2^d(D)$, the projective cover of $\widehat{L}_2(\mu)$ is isomorphic to $P(\mu)$. The assertion follows. □

Proposition 5.7. *Assume $l=2$. Then the algebra $s_q(2,d)_2$ has finite representation type for d odd with $d \geqslant 2p+1$.*

Proof. By [Doty et al. 1997, 6.3], the algebra $\mathscr{A}_s$ has finite representation type for s even with $s \geqslant 2$. Hence the assertion follows from Propositions 5.5 and 5.6. □

Acknowledgments

The author thanks the Mathematical Institute at the University of Oxford for its hospitality during his visit, K. Erdmann for many useful discussions, and the referee for some helpful comments.

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Received September 9, 2007. Revised April 4, 2008.

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