Pacific Journal of Mathematics

THE STRAIGHTENING THEOREM FOR TANGENT-LIKE MAPS

Piotr Gałązka and Janina Kotus

THE STRAIGHTENING THEOREM FOR TANGENT-LIKE MAPS

PIOTR GAŁĄZKA AND JANINA KOTUS

By analogy to polynomial-like maps, we introduce a notion of tangent-like maps. The main result of this paper is the straightening theorem. It says that a tangent-like map is quasiconformally equivalent to some tangent-type function $f : \mathbb{C} \to \overline{\mathbb{C}} \setminus \{a, b\}$ for $a \neq b$, which is unique up to an affine map. We also prove that quasiconformal conjugacy is conformal on the interior of the filled Julia set.

1. Introduction

In [1985], A. Douady and J. H. Hubbard introduced a notion of the polynomial-like maps. A polynomial-like map of degree $d \geq 2$ is a triple (f, U, V) where U and V are open subsets of $\mathbb{C}$ isomorphic to discs, with U relatively closed in V and $f : U \to V$ an analytic map, proper of degree d. Of course, every polynomial of degree $d \geq 2$ is locally polynomial-like of the same degree. It appears that some transcendental meromorphic functions are locally polynomial-like of finite degree (for example, $f(z) = \cos(z) - 2$ in some neighborhood of zero). Small perturbations of polynomial-like maps of degree d stay in the same class of maps. The filled Julia set of a polynomial-like map $f : U \to V$ is defined as $K(f) = \bigcap_{n \geq 0} f^{-n}(U)$. The set $K(f)$ is a compact subset of U. Douady and Hubbard proved the straightening theorem [1985, Theorem 1], which says that every polynomial-like map $f : U \to V$ is hybrid equivalent to a polynomial of degree d. This result has many applications, mainly in the renormalization theory of rational functions initiated by D. Sullivan. For other applications see [McMullen 2000].

Nevertheless, the concept of polynomial-like maps does not fit with those meromorphic functions whose inverses have at least one transcendental singularity at some point in $\mathbb{C}$ (for example, for $\arctan(z)$ at $\pm i$, $\log(z)$ at 0, and so on).

The aim of this paper is to introduce a definition of tangent-like maps and prove an analogue of the Straightening Theorem for this class.

The paper is organized as follows. In Section 2, we describe the dynamics of some family of transcendental meromorphic functions with two omitted values. In Section 3, we introduce a notion of tangent-like maps. In Section 4, we prove the main result of our paper, that is, the straightening theorem.

2. A model family

The *Fatou set $F(f)$* of a meromorphic function $f : \mathbb{C} \to \overline{\mathbb{C}}$ is defined in exactly the same manner as for rational functions: $F(f)$ is the set of points $z \in \mathbb{C}$ such that all the iterates are defined and form a normal family on a neighborhood of z. The *Julia set $J(f)$* is the complement of $F(f)$ in $\overline{\mathbb{C}}$. Thus, $F(f)$ is open, $J(f)$ is closed, $F(f)$ is completely invariant while $f^{-1}(J(f)) \subset J(f)$, and $f(J(f) \setminus \{\infty\}) = J(f)$. For a general description of the dynamics of meromorphic functions see for example [Bergweiler 1993]. It follows from Montel's criterion of normality that if $f : \mathbb{C} \to \overline{\mathbb{C}}$ has at least one pole that is not an omitted value, then

$$J(f) = \overline{\bigcup_{n \geq 0} f^{-n}(\infty)}.$$

See [Baker et al. 1991b]. Let $\mathrm{Crit}(f) = \{z \in \mathbb{C} : f'(z) = 0\}$ be the set of critical points of f. We note that we do not consider multiple poles as critical points. Let $\mathrm{Sing}(f^{-1})$ denote the set of *singular values* of f, that is, $c \in \mathrm{Sing}(f^{-1})$ if $c \in \mathbb{C}$ and c is a critical or an asymptotic value of f.

Consider a family of transcendental meromorphic functions of the form

$$T_\beta(z) = \frac{e^{z/2} - e^{\beta z/2}}{e^{z/2} - \beta e^{\beta z/2}}$$

for $\beta \in \mathbb{C} \setminus \{0, 1\}$. It is easy to see that $T_\beta(z)$ has no critical points. It has two omitted values 1 and $1/\beta$ (which are always the asymptotic values), and its poles are simple. Thus $T_\beta : \mathbb{C} \to \overline{\mathbb{C}} \setminus \{1, 1/\beta\}$ is a *regular holomorphic cover* if we define a derivative of f by the means of the spherical metric. One can easily check for all $\beta \in \mathbb{C} \setminus \{0, 1\}$ that $z_0 = 0$ is an attracting fixed point, that is,

$$T_\beta(0) = 0 \quad \text{and} \quad T_\beta'(0) = 1/2,$$

where T_β' denotes a derivative of T_β defined with respect to the Euclidean metric. We note that the choice of a multiplier of the attracting fixed point is arbitrary. It could be any complex number α such that $0 < |\alpha| < 1$.

The next proposition shows why the family of functions T_β is called a model family.

Proposition 2.1. *For every $\beta \in \mathbb{C} \setminus \{0, 1\}$ the map $T_\beta \circ A$, in which $A(z) = z + 4k\pi i/(1-\beta)$ for $k \in \mathbb{Z}$, is a regular holomorphic covering f from $\mathbb{C}$ onto $\overline{\mathbb{C}} \setminus \{1, 1/\beta\}$ such that $f(0) = 0$ and $f'(0) = 1/2$.*

Proof. Suppose that $f : \mathbb{C} \to \overline{\mathbb{C}} \setminus \{1, 1/\beta\}$ is an another regular covering. Then there exists a Möbius map $A : \mathbb{C} \to \mathbb{C}$ such that $f = T_\beta \circ A$. Then $A(z) = az + b$ for $a \neq 0$, and

$$f(z) = T_\beta(az + b) = \frac{e^{(az+b)/2} - e^{\beta(az+b)/2}}{e^{(az+b)/2} - \beta e^{\beta(az+b)/2}}.$$

It implies

$$0 = f(0) = T_\beta(b) = \frac{e^{(1-\beta)b/2} - 1}{e^{(1-\beta)b/2} - \beta}$$

if and only if $b = 4k\pi i/(1-\beta)$ for $k \in \mathbb{Z}$. But $\frac{1}{2} = f'(0) = T_\beta'(b)A'(0) = \frac{1}{2}a$; thus $a = 1$. Therefore $f(z) = T_\beta(z + 4k\pi i/(1-\beta))$. □

It follows from the classification of the periodic components of the Fatou set (see [Baker et al. 1991a]) that at least one of the asymptotic values $\{1, 1/\beta\}$ must belong to the basin of attraction $B(0)$ of $z_0 = 0$. It is easy to see that for large β the asymptotic value $1/\beta$ is close to $z_0 = 0$. Now we show that, for sufficiently large β, the asymptotic value $1/\beta \in B(0)$. We rewrite $T_\beta(z)$ as

$$T_\beta(z) = 1 + \frac{\beta e^{z(\beta-1)/2} - e^{z(\beta-1)/2}}{1 - \beta e^{z(\beta-1)/2}}.$$

For a given $\beta \in \mathbb{C} \setminus \{0, 1\}$, we choose disc $D(0, 3/|2\beta|)$. We want to prove that for large β

(1) $$\overline{T_\beta\left(D\left(0, 3/|2\beta|\right)\right)} \subset D\left(0, 3/|2\beta|\right).$$

Let $\beta = |\beta|e^{i\alpha}$. We consider $z = 3e^{i\theta}/|2\beta|$ for $\theta \in [0, 2\pi)$. Then

$$\frac{z(\beta-1)}{2} = \frac{3}{2}\frac{e^{i\theta}}{|\beta|}\frac{(|\beta|e^{i\alpha}-1)}{2} = \frac{3}{4}e^{i(\theta+\alpha)} - \frac{3}{4}\frac{e^{i\theta}}{|\beta|}.$$

For sufficiently large β we can approximate $z(\beta-1)/2$ by $\frac{3}{4}e^{i(\theta+\alpha)}$. Observe that for $\theta = -\alpha$, we have $\frac{3}{4}e^{i(\theta+\alpha)} = \frac{3}{4}$, while for $\theta = \pi - \alpha$ we have $\frac{3}{4}e^{i(\theta+\alpha)} = -\frac{3}{4}$. Consequently

$$T_\beta\left(\frac{3}{2}\frac{e^{-i\alpha}}{|\beta|}\right) = 1 + \frac{\beta e^{3/4} - e^{3/4}}{1 - \beta e^{3/4}} = 1 + \frac{e^{3/4}(\beta-1)}{1 - \beta e^{3/4}}$$

$$= 1 - \frac{e^{3/4}(\beta-1)}{e^{3/4}(\beta-e^{-3/4})} = \frac{1 - e^{-3/4}}{\beta - e^{-3/4}} \approx \frac{1}{2(\beta-0.5)}.$$

Analogously

$$T_\beta\Big(\frac{3}{2}\frac{e^{i(\pi-\alpha)}}{|\beta|}\Big) = 1 + \frac{\beta e^{-3/4} - e^{-3/4}}{1 - \beta e^{-3/4}} = 1 + \frac{e^{-3/4}(\beta-1)}{1 - \beta e^{-3/4}}$$

$$= 1 - \frac{e^{-3/4}(\beta-1)}{e^{-3/4}(\beta - e^{3/4})} = \frac{1 - e^{3/4}}{\beta - e^{3/4}} \approx -\frac{1}{\beta-2}.$$

Consequently, for $z \in \partial D(0, 3/|2\beta|)$ the maximum of $|T_\beta(z)| \le 1/|\beta-2|$. Enlarging β if necessary, we may say that $1/|\beta-2| < 3e^{i\theta}/|2\beta|$. Thus we may claim that (1) is satisfied. It proves that $D(0, 3/|2\beta|) \subset B(0)$. Since $1/\beta \in D(0, 3/|2\beta|)$, we have $1/\beta \in B(0)$.

From now we assume that β is so large that $1/\beta \in B(0)$.

Let D be a component of the Fatou set $F(T_\beta)$ containing $z_0 = 0$. Then D is forward invariant. Of course $D \subset B(0)$ and $1/\beta \in D$. Take a component D_0 of $T_\beta^{-1}(D)$ containing a fixed point $z_0 = 0$. Since $1/\beta$ is an omitted value, D_0 must contain an asymptotic tract corresponding to $1/\beta$. Thus D must be a completely invariant unbounded domain and $D = B(0)$. As $B(0)$ is completely invariant, $J(T_\beta) = \partial B(0)$. One can easily prove that $B(0)$ is simply connected if and only if the second asymptotic value $1 \notin B(0)$. By analogy to polynomials we define the filled Julia set

$$K(T_\beta) = \{z \in \mathbb{C} : T_\beta^n(z) \notin \overline{B(0)} \text{ for all } n \in \mathbb{N} \cup \{0\}\}.$$

Then the Julia set can be written as

$$J(T_\beta) = \partial K(T_\beta) \cup \{\infty\}.$$

Observe that the interior of $K(T_\beta)$ is nonempty if and only if the second asymptotic value 1 is related to some periodic component of the Fatou set $F(T_\beta)$ different from $B(0)$. Since $\text{Sing}(T_\beta^{-1})$ consists of two asymptotic values $\{1, 1/\beta\}$, the map T_β belongs to the class $\mathscr{S}$ defined in [Bergweiler 1993]. Then the classification of periodic components of the $F(T_\beta)$ is the same as for rational map. That is, T_β does not have Baker domains; see [Bergweiler 1993]. Moreover, T_β does not have Herman rings. The proof is same for tangent family; see [Keen and Kotus 1997]. Thus the interior $K(T_\beta)$ is nonempty if $F(T_\beta)$ has either some attracting periodic cycle, some parabolic periodic cycle, or some Siegel periodic point. In the last case $1 \in J(T_\beta)$.

Finally we would like to explain why we prefer to work with the family $T_\beta(z)$ instead of the tangent family $f_\lambda(z) = \lambda \tan(z)$ with $\lambda \in \mathbb{C}^*$, which was described [Keen and Kotus 1997]. The map f_λ is a regular holomorphic cover of $\overline{\mathbb{C}} \setminus \{\pm\lambda i\}$ and the orbits of two asymptotic values $\pm\lambda i$ are symmetric — they belong either to the same or to two symmetric periodic cycles of the Fatou set; see [Keen and Kotus 1997]. To get some analogy with quadratic polynomials, we should have

an attracting fixed point and associated with it some asymptotic value, while the second asymptotic value should be "free". But this never happens in the classical tangent family. On the other hand, the proposed family admits phenomena discovered in [Keen and Kotus 1997] for tangent maps, for example, collisions of repelling periodic points with poles, if some iterate of the asymptotic value 1 is eventually mapped onto a pole.

3. Tangent-like maps

Definition 3.1. The triple (f, U, V) is a tangent-like map if

(i) U and V are Jordan domains such that $\overline{U} \subset V$,

(ii) there exist $v \in V$ and $u \in \partial U$ such that $f : \overline{U} \setminus \{u\} \to \overline{V} \setminus \{v\}$ is a cover,

(iii) f is holomorphic on U, and

(iv) U is a quasidisc.

Here u plays a role of essential singularity, and v is an asymptotic value of f.

Definition 3.2. The set

$$K(f, U, V) := \bigcap_{n \geq 0} \overline{f^{-n}(U)}$$

is called the *filled Julia set* of (f, U, V), while $J(f, U, V) := \partial K(f, U, V)$ is called the *Julia set* of (f, U, V).

Proposition 3.3. *For every sufficiently large $\beta \in \mathbb{C} \setminus \{0, 1\}$, the function $T_\beta(z)$ is a tangent-like map.*

Proof. We know that if β sufficiently large, the asymptotic value $1/\beta$ belongs to the basin of the attraction of the fixed point $z_0 = 0$. Since T_β is locally linearizable in some disc $D_0 := D(0, \varepsilon)$ (shrinking ε if necessary) we can say that there exist a quasidisc D_0 and an integer $N \in \mathbb{N}$ such that

$$0 \in D_0, \qquad D_0 \subset \mathbb{C} \setminus K(T_\beta), \qquad T_\beta(D_0) \subset D_0,$$
$$T_\beta^N(1/\beta) \in D_0, \qquad T_\beta^{N-1}(1/\beta) \notin \overline{D}_0, \qquad T_\beta^N(1) \notin \overline{D}_0.$$

Let D_{i+1} denotes a component of $T^{-1}(D_i)$ that contains D_i for $i = 0, \ldots, N-1$. Then D_N contains an asymptotic value $1/\beta$, and a branch of the inverse function T_β^{-1} that maps $D_N \setminus \{1/\beta\}$ onto D_{N+1} has a logarithmic singularity. Thus D_{N+1} is a simply connected unbounded domain, and $T_\beta : D_{N+1} \to D_N \setminus \{1/\beta\}$ is a universal cover. Let $U := \mathbb{C} \setminus \overline{D}_{N+1}$ and $V := \mathbb{C} \setminus \overline{D}_N$. Set also $u := \infty$ and $v = 1$. Then $T_\beta : \overline{U} \setminus \{u\} \to \overline{V} \setminus \{v\}$ is a universal cover. Since D_0 is a quasidisc, D_{N+1} and U are also quasidiscs. It proves that (T_β, U, V) is a tangent-like map. $\square$

Definition 3.4. We say that two tangent-like maps (f_1, U_1, V_1) and (f_2, U_2, V_2) are *hybrid equivalent* if there exists a quasiconformal homeomorphism ϕ mapping a neighborhood of $K(f_1, U_1, V_1)$ onto a neighborhood of $K(f_2, U_2, V_2)$ in $\mathbb{C}$ such that $\phi \circ f_1(z) = f_2 \circ \phi$ on a neighborhood of $K(f_1, U_1, V_1)$ and $(\partial \phi / \partial z)(z) = 0$ for $z \in K(f_1, U_1, V_1)$.

4. The straightening theorem

Definition 4.1. Suppose that γ is a Jordan curve. A map $p : \mathbb{R} \to \gamma$ is called a unit speed parametrization of γ if for all $t \in \mathbb{R}$, $p'(t)$ is well defined, $|p'(t)| = 1$, and $p(t)$ winds anticlockwise as t increases.

Definition 4.2. If γ_1 and γ_2 are two Jordan curves, p_1 and p_2 are their respective unit speed parametrizations. Let $f : \gamma_1 \to \gamma_2$ be a homeomorphism. Then there exists a lift $\tilde{f} : \mathbb{R} \to \mathbb{R}$ such that $f \circ p_1 = p_2 \circ \tilde{f}$. We say that f is a C^r diffeomorphism with respect to the parametrization if and only if $\tilde{f}$ is a C^r diffeomorphism.

We leave to the reader a proof of the next lemma.

Lemma 4.3. *Let $D \subset \mathbb{C}$ be a Jordan domain such that $\partial D = \gamma(S^1)$ and γ is C^2 in some neighborhood of S^1. Then ∂D has a constant speed parametrization. If $\phi : \mathbb{D} \to D$ is a conformal isomorphism with homeomorphic extension $\tilde{\phi} : \overline{\mathbb{D}} \to \overline{D}$, then the restriction $\tilde{\phi} : S^1 \to \partial D$ is a C^1 diffeomorphism with respect to the parametrization.*

For a given conformal annulus $A \subset \mathbb{C}$ such that each component of the boundary is a Jordan curve, let ∂A^+ and ∂A^- denote respectively the outer and the inner component. Let A_1 and A_2 be two annuli. We say that a homeomorphism $h : \partial A_1 \to A_2$ is sense preserving if $h : \partial A_1^+ \to \partial A_2^+$ and $h : \partial A_1^- \to \partial A_2^-$ are sense preserving homeomorphisms.

Let
$$\mathbb{A}_t := \{z \in \mathbb{C} : 1 < |z| < e^t\} \quad \text{for } t > 0.$$

Lemma 4.4. *Let $\phi : \partial \mathbb{A}_1 \to \partial \mathbb{A}_1$ be a sense preserving homeomorphism such that its restriction to components of $\partial \mathbb{A}_1$ is a C^1 diffeomorphism with respect to the parametrization. Then there exists a homeomorphic extension $\phi : \overline{\mathbb{A}}_1 \to \overline{\mathbb{A}}_1$ that is quasiconformal on $\mathbb{A}_1$.*

Proof. There must exist a C^1 lift of ϕ that commutes with the translation $T(x) = x + 2\pi$. Suppose that $a : \mathbb{R} \to \mathbb{R}$ is a lift of $\phi|_{\partial A^-}$ and b is a lift of $\phi|_{\partial A^+}$, with a and b both increasing maps. Then $\phi(e^i t) = e^{ia(t)}$ and $\phi(e^{1+it}) = e^{1+ib(t)}$ for all $t \in \mathbb{R}$. Then the interpolation $\phi(e^{s+it}) := e^{(1-s)ia(t)+s(1+ib(t))}$ is quasiconformal. $\qquad\square$

Lemma 4.5. *Let A_1 and $A_2 \subset$ be two annuli whose boundary components are images of S^1 by C^2 diffeomorphisms. Let $h : \partial A_1 \to \partial A_2$ be a sense preserving*

homeomorphism that extends to a C^2 diffeomorphism on a neighborhood of ∂A_1. Then there exists a homeomorphic extension $h : \overline{A}_1 \to \overline{A}_2$ that is quasiconformal on A_1

Proof. For $t > 0$, the map $L_t : \mathbb{C}^* \to \mathbb{C}^*$ defined as $L_t(re^{it}) = r^t e^{it}$ is a real-analytic quasiconformal homeomorphism. It maps $\mathbb{A}_1$ homeomorphically onto $\mathbb{A}_t$. For $i = 1, 2$, choose a conformal isomorphism $\phi_{i,1}$ from the interior of ∂A_i^+ to $\mathbb{D}$, and let $A_{i,1} := \phi_{i,1}(A_i)$. Next, choose a conformal isomorphism $\phi_{i,2}$ from the exterior of ∂A_i^- to $\overline{\mathbb{C}} \setminus \mathbb{D}$, and let $A_{i,2} := \phi_{i,2}(A_{i,1})$. Finally choose a conformal isomorphism $\phi_{i,3}$ from $A_{i,2}$ to $\mathbb{A}_{t_i}$ for some $t_i > 0$. (Notice that the boundary components of $A_{i,2}$ are both conformal images of S^1, so by the Schwartz reflection principle, $\phi_{i,3}$ extends holomorphically to a neighborhood of the closure of $A_{i,2}$.) Now define $\phi_i : A_i \to \mathbb{A}_1$ by $\phi_i := L_{1/t_i} \circ \phi_{i,3} \circ \phi_{i,2} \circ \phi_{i,1}$. This is a conformal isomorphism that extends holomorphically to a map $\phi_i : \overline{A}_i \to \overline{\mathbb{A}}_1$. We can define $H : \partial \mathbb{A}_1 \to \partial \mathbb{A}_1$ by $H = \phi_2 \circ h \circ \phi_1$. This is a sense preserving homeomorphism. It follows from Lemma 4.3 and the Schwartz reflection principle that H restricts on the boundary components of $\partial \mathbb{A}_1$ to give C^1 diffeomorphic maps with respect to parametrization. By Lemma 4.4, H admits a homeomorphic extension of $H : \overline{\mathbb{A}}_1 \to \overline{\mathbb{A}}_1$ that is quasiconformal on $\mathbb{A}_1$. This induces an extension $h : \overline{A}_1 \to \overline{A}_2$. $\qquad\square$

Lemma 4.6. *If D is a quasidisc and $\phi : \mathbb{D} \to D$ is conformal, then ϕ has a quasiconformal extension $\overline{\phi} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ that is real analytic on $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$.*

Proof. Suppose that $h : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a quasiconformal homeomorphism such that $h(\mathbb{D}) = D$. Consider a reflection $j : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ in the unit circle. Let $\psi : \overline{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ be a conformal isomorphism. Then ϕ and ψ has a homeomorphic extension to $\overline{\mathbb{D}}$ and $\overline{\mathbb{C}} \setminus \mathbb{D}$. Thus $h^{-1} \circ \phi : S^1 \to S^1$ is a homeomorphism. Define a map $H = j \circ \psi^{-1} \circ h \circ j \circ h^{-1} \circ \phi$. Then H is a quasiconformal extension of $h^{-1} \circ \phi$ to $\overline{\mathbb{D}}$. Then $h^{-1} \circ \phi$ must be quasisymmetric. If follows from [Douady and Earle 1986] that $h^{-1} \circ \phi$ has a real analytic extension $E_{h^{-1} \circ \phi} : \mathbb{D} \to \mathbb{D}$. Take $\tilde{\phi}$ on $\overline{\mathbb{D}}$ and $\tilde{\phi} := \psi \circ j \circ E_{h^{-1} \circ \phi} \circ j$ on $\mathbb{C} \setminus \mathbb{D}$. $\qquad\square$

Proposition 4.7. *Suppose that (f_1, U_1, V_1) and (f_2, U_2, V_2) are two tangent-like maps. Then there exists a quasiconformal homeomorphism $\phi : \overline{V}_1 \to \overline{V}_2$ such that $f_2 \circ \phi(z) = \phi \circ f_1(z)$ for $z \in \overline{U}_1 \setminus \{u_1\}$ and such that ϕ is holomorphic on U_1.*

Proof. For simplicity, we may assume that $V_1 = V_2 = \mathbb{D}$ and $v_1 = v_2 = 0$. To get a general case one must conjugate f_1 and f_2 with Riemann maps. Define $A_i = V_i \setminus \overline{U}_i$ for $i = 1, 2$. Since f_1 and f_2 are universal coverings of $\mathbb{D}^*$, there must exist a conformal isomorphism $h : U_1 \to U_2$ such that

$$f_2 \circ h(z) = f_1(z) \quad \text{for } z \in U_1.$$

Since U_1, U_2 are quasidiscs there exist an extension $\tilde{h} : \overline{U}_1 \to \overline{U}_2$ such that $f_2 \circ \tilde{h}(z) = f_1(z)$ for $z \in \partial U_1 \setminus \{u_1\}$. We want to construct a quasiconformal homeomorphism $H : A_1 \to A_2$ such that $H(z) = \text{Id}(z)$ if $z \in \partial V_1$ and $H(z) = \tilde{h}(z)$ for $z \in \partial U_1$. Let's take two conformal isomorphisms $g_i : \mathbb{D} \to U_i$ for $i = 1, 2$ such that $g_2 \circ g_1^{-1} = h$. Lemma 4.6 implies that g_i extends to a quasiconformal self-map of $\overline{\mathbb{C}}$ that is real-analytic on $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$. Let $\tilde{A}_i := g_i^{-1}(A_i)$ for $i = 1, 2$ be two annuli. By Lemma 4.5, there exists a quasiconformal homeomorphism $\tilde{\phi} : \tilde{A}_1 \to \tilde{A}_2$ that agrees with $g_2^{-1} \circ \tilde{h} \circ g_1 = \text{Id}$ on $\partial \tilde{A}_1^-$ and with $g_2^{-1} \circ \text{Id} \circ g_1$ on $\partial \tilde{A}_1^+$. Then $\phi := g_2 \circ \tilde{\phi} \circ g_1^{-1}$ is a quasiconformal homeomorphism agreeing with $\tilde{h}$ on ∂A_1^- and with Id on ∂A_1^+. Thus $\phi \circ f_1 = f_2 \circ \phi$ on $\partial U_1 \setminus \{u_1\}$. We extend ϕ to U_1 by setting $\phi = h$. □

Theorem 4.8. *Every tangent-like map is hybrid equivalent to some $T_\beta \circ A$ with sufficiently large $\beta \in \mathbb{C} \setminus \{0, 1\}$ and $A(z) = z + 4k\pi i/(1 - \beta)$ for $k \in \mathbb{Z}$.*

Proof. It follows from Proposition 3.3 that we can choose a map T_β having a tangent-like restriction (F_0, U_0, V_0). This is possible if β is sufficiently large. Then the asymptotic value $1/\beta$ belongs to the basin of attraction of $z_0 = 0$. Let (f, U, V) be a tangent-like map. It follows from Proposition 4.7 that there is a homeomorphism $\phi : \overline{V}_0 \to \overline{V}$ that is quasiconformal on V_0 and holomorphic on U_0 and that has $\phi \circ F_0(z) = f \circ \phi(z)$ for $z \in \overline{U}_0 \setminus \{u_0\}$. Now we define a map $F(z) := \phi^{-1} \circ f \circ \phi(z)$ for $z \in U_0$ and $F(z) = T_\beta(z)$ if $z \notin U_0$. Then F is well defined in $\mathbb{C}$ and holomorphic outside the closure of $F^{-1}(V_0 \setminus \overline{U}_0)$. For any $z \in \mathbb{C}$, the forward trajectory $F^n(z)$ lands at most once in $F^{-1}(V_0 \setminus \overline{U}_0)$. Let σ_0 denote the standard conformal structure on $\overline{\mathbb{C}}$. We define a new structure σ by $\sigma = (F^n)^* \sigma_0$ on $F^{-n}(V_0 \setminus U_0)$ for $n \geq 1$, by $\sigma = \sigma_0$ on $K(F) = \phi^{-1}(K(f))$, and by $\sigma = \sigma_0$ on $\overline{\mathbb{C}} \setminus U_0$. The structure σ is F invariant, that is, $F^*(\sigma) = \sigma$. There exists a quasiconformal homeomorphism $\Phi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ fixing $0, 1, \infty$ and such that $\Phi^*(\sigma) = \sigma_0$; see [Boyarskiĭ 1955]. Then

$$g = \Phi \circ F \circ \Phi^{-1} : \mathbb{C} \to \overline{\mathbb{C}} \setminus \{1, w\}$$

for some $w \neq 1$. Since σ agrees with σ_0 on a neighborhood of 0 (more precisely on $\mathbb{C} \setminus U_0$), Φ is holomorphic on this neighborhood, and $g'(0) = F_0'(0) = 1/2$. So g is a regular holomorphic covering from $\mathbb{C}$ onto $\overline{\mathbb{C}} \setminus \{1, w\}$. By Proposition 2.1, $g = T_\beta \circ A$, where $A(z) = z + 4k\pi i/(1 - \beta)$ for $k \in \mathbb{Z}$ and $\beta = 1/w$. □

Acknowledgments

References

[Baker et al. 1991a] I. N. Baker, J. Kotus, and Y. N. Lü, "Iterates of meromorphic functions, III: Preperiodic domains", *Ergodic Theory Dynam. Systems* **11**:4 (1991), 603–618. MR 92m:58115 Zbl 0774.30023

[Baker et al. 1991b] I. N. Baker, J. Kotus, and L. Yinian, "Iterates of meromorphic functions, I", *Ergodic Theory Dynam. Systems* **11**:2 (1991), 241–248. MR 92m:58113 Zbl 0711.30024

[Bergweiler 1993] W. Bergweiler, "Iteration of meromorphic functions", *Bull. Amer. Math. Soc. (N.S.)* **29**:2 (1993), 151–188. MR 94c:30033 Zbl 0791.30018

[Boyarskiĭ 1955] B. V. Boyarskiĭ, "Homeomorphic solutions of Beltrami systems", *Dokl. Akad. Nauk SSSR (N.S.)* **102** (1955), 661–664. In Russian. MR 17,157a

[Douady and Earle 1986] A. Douady and C. J. Earle, "Conformally natural extension of homeomorphisms of the circle", *Acta Math.* **157**:1-2 (1986), 23–48. MR 87j:30041 Zbl 0615.30005

[Douady and Hubbard 1985] A. Douady and J. H. Hubbard, "On the dynamics of polynomial-like mappings", *Ann. Sci. École Norm. Sup.* (4) **18**:2 (1985), 287–343. MR 87f:58083 Zbl 0587.30028

[Keen and Kotus 1997] L. Keen and J. Kotus, "Dynamics of the family $\lambda \tan z$", *Conform. Geom. Dyn.* **1** (1997), 28–57. MR 98h:58159 Zbl 0884.30019

[McMullen 2000] C. T. McMullen, "The Mandelbrot set is universal", pp. 1–17 in *The Mandelbrot set, theme and variations*, edited by L. Tan, London Math. Soc. Lecture Note Ser. **274**, Cambridge Univ. Press, 2000. MR 2002f:37081 Zbl 1062.37042

PIOTR GALĄZKA
FACULTY OF MATHEMATICS AND INFORMATION SCIENCES
WARSAW UNIVERSITY OF TECHNOLOGY
PLAC POLITECHNIKI 1
00 661 WARSAW
POLAND
T.Galazka@mini.pw.edu.pl

JANINA KOTUS
FACULTY OF MATHEMATICS AND INFORMATION SCIENCES
WARSAW UNIVERSITY OF TECHNOLOGY
PLAC POLITECHNIKI 1
00 661 WARSAW
POLAND
J.Kotus@impan.gov.pl