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We give a proof of the DDVV conjecture, which is a pointwise inequality involving the scalar curvature, the normal scalar curvature and the mean curvature on a submanifold of a real space form. We also solve the problem of its equality case.

1. Introduction

Let $f : M^n \to N^{n+m}(c)$ be an isometric immersion of an n-dimensional submanifold M into the $(n+m)$-dimensional real space form $N^{n+m}(c)$ of constant sectional curvature c. The normalized scalar curvature ρ and normal scalar curvature $\rho^\perp$ are defined by [DDVV 1999]

$$\rho = \frac{2}{n(n-1)} \sum_{1=i<j}^{n} R(e_i, e_j, e_j, e_i),$$

$$\rho^\perp = \frac{2}{n(n-1)} \Big(\sum_{1=i<j}^{n} \sum_{1=r<s}^{m} \langle R^\perp(e_i, e_j)\xi_r, \xi_s \rangle^2 \Big)^{1/2},$$

where $\{e_1, \dots, e_n\}$ is an orthonormal basis of the tangent space, and R is the curvature tensor of the tangent bundle. Similarly, $\{\xi_1, \dots, \xi_m\}$ is an orthonormal basis of normal space, and $R^\perp$ is the curvature tensor of the normal bundle.

Let h be the second fundamental form and let $H = (1/n)\,\mathrm{Tr}\, h$ be the mean curvature vector field. The DDVV conjecture, of De Smet, Dillen, Verstraelen, and Vrancken [1999], says that there's a pointwise inequality among ρ, $\rho^\perp$, and $|H|^2$ given by

$$\rho + \rho^\perp \le |H|^2 + c.$$

Since this is a pointwise inequality, one can see using the Gauss and Ricci identities that it's equivalent to the following algebraic inequality (see [Dillen et al. 2007]):

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Conjecture A. *Let $B_1, \dots, B_m$ be $(n \times n)$ real symmetric matrices. Then*

$$\sum_{r,s=1}^{m} \|[B_r, B_s]\|^2 \le \Big(\sum_{r=1}^{m} \|B_r\|^2\Big)^2,$$

where $\|\cdot\|^2$ denotes the sum of the squares of the entries of the matrix and $[A, B] = AB - BA$ is the commutator of the matrices A and B.

The main purpose of this paper is to prove Conjecture A and also to give the equality condition:

Theorem 1.1. *Let $B_1, \dots, B_m$ be $(n \times n)$ real symmetric matrices. Then*

$$\sum_{r,s=1}^{m} \|[B_r, B_s]\|^2 \le \Big(\sum_{r=1}^{m} \|B_r\|^2\Big)^2,$$

where the equality holds if and only if under some rotation[1] *all B_r's are zero except two matrices which can be written as PH_1P^t and PH_2P^t, where P is an $(n \times n)$ orthogonal matrix, and*

$$H_1 = \operatorname{diag}(\mu, -\mu, 0, \dots), \qquad H_2 = \operatorname{diag}\left(\begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix}, 0, \dots\right).$$

Therefore, we can solve the DDVV conjecture also with its equality conditions in terms of shape operators:

Corollary 1.2. *Let $f : M^n \to N^{n+m}(c)$ be an isometric immersion. Then*

$$\rho + \rho^{\perp} \le |H|^2 + c,$$

where the equality holds at some point $p \in M$ if and only if there exist an orthonormal basis $\{e_1, \dots, e_n\}$ of T_pM and an orthonormal basis $\{\xi_1, \dots, \xi_m\}$ of $T_p^{\perp}M$, such that

$$A_{\xi_1} = \operatorname{diag}(\lambda_1 + \mu, \lambda_1 - \mu, \lambda_1, \dots, \lambda_1), \qquad A_{\xi_2} = \operatorname{diag}\left(\begin{pmatrix} \lambda_2 & \mu \\ \mu & \lambda_2 \end{pmatrix}, \lambda_2, \dots, \lambda_2\right),$$

and all other shape operators $A_{\xi_r} = \lambda_r I_n$, where $\mu, \lambda_1, \dots, \lambda_m$ are real numbers.

Remark. By the same method, one can see that Conjecture A also holds (though not optimally) for antisymmetric matrices. However, the following example shows that Conjecture A fails, as conjectured in [Lu 2007c], when the set $\{B_1, \dots, B_m\}$ contains both symmetric and antisymmetric matrices.

Example. The conclusion of Conjecture A fails when

$$B_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

[1] An orthogonal $m \times m$ matrix $R = (R_{rs})$ acts as a rotation on $(B_1, \dots, B_m)$ by $R(B_r) = \sum_{s=1}^{m} R_{sr} B_s$.

We point out that the inequality and its equality condition of Theorem 1.1 (respectively Corollary 1.2) for $m = 2$ were given in [Chern 1968] (respectively [DDVV 1999]). There are also some studies about classifying submanifolds that satisfy the equality everywhere, for example [Choi and Lu 2008; DDVV 1999; Dajczer and Florit 2001; Dillen et al. 2007]. The first nontrivial case, $n = 3$, of the DDVV conjecture was proved in [Choi and Lu 2008]. For general n and m, a weaker version was proved in [Dillen et al. 2007]. After we solved the conjecture and its equality case, we found very recently that Zhiqin Lu had proved the inequality without the equality case (the inequality's key step is contained in [Lu 2007a; Lu 2007c]), as seen on his homepage [Lu 2007b]. Since we use a quite different method and work out the equality condition in addition to the inequality, we'd like it to be part of the literature.

2. Notations and preparatory lemmas

Throughout this paper, we denote by $\mathrm{M}(m,n)$ the space of $m \times n$ real matrices, $\mathrm{M}(n)$ the space of $n \times n$ real matrices, and $\mathrm{SM}(n)$ the subspace of symmetric matrices in $\mathrm{M}(n)$, which has dimension $N := n(n+1)/2$.

For every (i, j) with $1 \le i \le j \le n$, let

$$\hat{E}_{ij} := \begin{cases} E_{ii} & \text{if } i = j, \\ (E_{ij} + E_{ji})/\sqrt{2} & \text{if } i < j, \end{cases}$$

where $E_{ij} \in \mathrm{M}(n)$ is the matrix with 1 in position (i, j) and 0 elsewhere. Clearly $\{\hat{E}_{ij}\}_{i \le j}$ is an orthonormal basis of $\mathrm{SM}(n)$. Let us put an ordering on the index set $S := \{(i, j) \mid 1 \le i \le j \le n\}$ by saying

$$(i, j) < (k, l) \text{ if and only if } i < k \text{ or } i = k \text{ and } j < l. \tag{2-1}$$

We use this ordering to index elements of S with a single (Greek) index in the range $\{1, \dots, N\}$.

For $\alpha = (i, j) < (k, l) = \beta$ in S, direct calculations imply

$$\|[\hat{E}_\alpha, \hat{E}_\beta]\|^2 = \begin{cases} 1 & \text{if } i = j = k < l \text{ or } i < j = k = l, \\ 1/2 & \text{if } i < j = k < l \text{ or } i = k < j < l \text{ or } i < k < j = l, \\ 0 & \text{otherwise,} \end{cases} \tag{2-2}$$

and for any $\alpha, \beta \in S$,

$$\sum\nolimits_{\gamma \in S} \langle [\hat{E}_\alpha, \hat{E}_\gamma], [\hat{E}_\beta, \hat{E}_\gamma] \rangle = n\delta_{\alpha\beta} - \delta_\alpha \delta_\beta, \tag{2-3}$$

where $\delta_{\alpha\beta} = \delta_{ik}\delta_{jl}$, $\delta_\alpha = \delta_{ij}$, $\delta_\beta = \delta_{kl}$, and $\langle \cdot, \cdot \rangle$ is the standard inner product of $\mathrm{M}(n)$.

Let $\{\hat{Q}_\alpha\}_{\alpha\in S}$ be any orthonormal basis of $\mathrm{SM}(n)$. Then there exists a unique orthogonal matrix $Q \in O(N)$ such that $(\hat{Q}_1, \dots, \hat{Q}_N) = (\hat{E}_1, \dots, \hat{E}_N)Q$, that is, $\hat{Q}_\alpha = \sum_\beta q_{\beta\alpha}\hat{E}_\beta$ for $Q = (q_{\alpha\beta})_{N\times N}$, and if $\hat{Q}_\alpha = (\hat{q}^\alpha_{ij})_{n\times n}$,

$$\hat{q}^\alpha_{ij} = \hat{q}^\alpha_{ji} = \begin{cases} q_{\beta\alpha} & \text{if } \beta = (i, j) \text{ and } i = j, \\ q_{\beta\alpha}/\sqrt{2} & \text{if } \beta = (i, j) \text{ and } i < j. \end{cases}$$

Let $\lambda_1, \dots, \lambda_n$ be n real numbers satisfying $\sum_i \lambda_i^2 = 1$ and $\lambda_1 \geq \dots \geq \lambda_n$. Define $I_1 := \{j \mid \lambda_1 - \lambda_j > 1\}$, $I_2 := \{i \mid \lambda_i - \lambda_n > 1\}$, and $I := \{(i, j) \mid \lambda_i - \lambda_j > 1\}$. Let n_0 be the number of elements of I. Then $(\{1\}\times I_1) \cup (I_2 \times \{n\}) \subset I \subset S$.

Lemma 2.1. *Either $I = \{1\}\times I_1$ or $I = I_2 \times \{n\}$.*

Proof. If $n_0 = 0$, the three sets are all empty. If $n_0 = 1$, the single element must be $(1, n)$, and the three sets are equal. Now let $(1, n)$ and (i_1, j_1) be two different elements of I, that is, $\lambda_1 - \lambda_n \geq \lambda_{i_1} - \lambda_{j_1} > 1$ and $(1, n) \neq (i_1, j_1)$. We assert that either $i_1 = 1$ and $j_1 \neq n$ or $i_1 \neq 1$ and $j_1 = n$, which shows exactly that $I = \{1\} \times I_1 \bigcup I_2 \times \{n\}$. Otherwise, $1, i_1, j_1$, and n will be four different elements in $\{1, \dots, n\}$, and thus

$$1 \geq \lambda_1^2 + \lambda_{i_1}^2 + \lambda_{j_1}^2 + \lambda_n^2 \geq \tfrac{1}{2}(\lambda_1 - \lambda_n)^2 + \tfrac{1}{2}(\lambda_{i_1} - \lambda_{j_1})^2 > 1$$

is a contradiction. Without loss of generality, we can assume $(i_1, j_1) \in \{1\} \times I_1$. Then it'll be seen that $I_2 \times \{n\} = \{(1, n)\}$ and thus $I = \{1\} \times I_1$, which completes the proof. Otherwise, if there's another element, say (i_2, n), in $I_2 \times \{n\}$, then $i_1 = 1$, j_1, i_2, and n are four different elements in $\{1, \dots, n\}$, and we come to the same contradiction as above. □

Lemma 2.2. *We have $\sum_{(i,j)\in I}[(\lambda_i - \lambda_j)^2 - 1] \leq 1$, where the equality holds in the case when $I = \{1\} \times I_1$ if and only if $1 \leq n_0 < n$ and $\lambda_1 = \sqrt{n_0/(n_0+1)}$, $\lambda_{n-n_0+1} = \dots = \lambda_n = -1/\sqrt{n_0^2 + n_0}$ and all other $\lambda_k = 0$.*

Proof. Without loss of generality, we can assume that $I = \{1\} \times I_1$ by Lemma 2.1. Then

$$\begin{aligned}\sum\nolimits_{(i,j)\in I}[(\lambda_i - \lambda_j)^2 - 1] &= \sum\nolimits_{j\in I_1}(\lambda_1^2 + \lambda_j^2 - 2\lambda_1\lambda_j) - n_0 \\ &= n_0\lambda_1^2 + \sum\nolimits_{j\in I_1}\lambda_j^2 - 2\lambda_1\sum\nolimits_{j\in I_1}\lambda_j - n_0 \\ &\leq (n_0+1)\lambda_1^2 + \sum\nolimits_{j\in I_1}\lambda_j^2 + (\sum\nolimits_{j\in I_1}\lambda_j)^2 - n_0 \\ &\leq (n_0+1)(\lambda_1^2 + \sum\nolimits_{j\in I_1}\lambda_j^2) - n_0 \\ &\leq (n_0+1)\sum\nolimits_i \lambda_i^2 - n_0 = 1,\end{aligned}$$

where the equality condition is easily seen from the proof. □

Lemma 2.3. *We have $\sum_{\beta\in J_\alpha}(\|[\hat{Q}_\alpha, \hat{Q}_\beta]\|^2 - 1) \leq 1$ for any $Q \in O(N)$, $\alpha \in S$ and $J_\alpha \subset S$.*

Proof. For $\alpha \in S$, we can assume without loss of generality $\hat{Q}_\alpha = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ with $\sum_i \lambda_i^2 = 1$ and $\lambda_1 \geq \cdots \geq \lambda_n$. Then by Lemma 2.2,

$$\begin{aligned}\sum_{\beta \in J_\alpha} (\|[\hat{Q}_\alpha, \hat{Q}_\beta]\|^2 - 1) &= \sum_{\beta \in J_\alpha} \sum_{i,j=1}^{n} ((\lambda_i - \lambda_j)^2 - 1)(\hat{q}_{ij}^{\beta})^2 \\ &= \sum_{\beta \in J_\alpha} \sum_{(i,j)=\gamma \in S} ((\lambda_i - \lambda_j)^2 - 1) q_{\gamma\beta}^2 \\ &\leq \sum_{(i,j)=\gamma \in I} ((\lambda_i - \lambda_j)^2 - 1) \sum_{\beta \in J_\alpha} q_{\gamma\beta}^2 \\ &\leq \sum_{(i,j) \in I} ((\lambda_i - \lambda_j)^2 - 1) \leq 1. \qquad \square\end{aligned}$$

Lemma 2.4. *We have $\sum_{\beta \in S} \|[\hat{Q}_\alpha, \hat{Q}_\beta]\|^2 \leq n$ for any $Q \in O(N)$ and $\alpha \in S$.*

Proof. It follows from Equation (2-3) that

$$\begin{aligned}\sum_{\beta \in S} \|[\hat{Q}_\alpha, \hat{Q}_\beta]\|^2 &= \sum_{\beta\gamma\tau\xi\eta} q_{\gamma\alpha} q_{\xi\alpha} q_{\tau\beta} q_{\eta\beta} \langle [\hat{E}_\gamma, \hat{E}_\tau], [\hat{E}_\xi, \hat{E}_\eta] \rangle \\ &= \sum_{\gamma\xi} q_{\gamma\alpha} q_{\xi\alpha} \textstyle\sum_\tau \langle [\hat{E}_\gamma, \hat{E}_\tau], [\hat{E}_\xi, \hat{E}_\tau] \rangle \\ &= \sum_{\gamma\xi} q_{\gamma\alpha} q_{\xi\alpha} (n\delta_{\gamma\xi} - \delta_\gamma \delta_\xi) = n \textstyle\sum_\gamma q_{\gamma\alpha}^2 - (\sum_i \hat{q}_{ii}^{\alpha})^2 \leq n. \qquad \square\end{aligned}$$

Now let $\varphi : \mathrm{M}(m,n) \to \mathrm{M}(C_m^2, C_n^2)$ be the map defined by $\varphi(A)_{(i,j)(k,l)} := A\binom{kl}{ij}$, where $1 \leq i < j \leq m,\ 1 \leq k < l \leq n$, and $A\binom{kl}{ij} = a_{ik}a_{jl} - a_{il}a_{jk}$ is the discriminant of the 2×2 submatrix of A that is the intersection of rows i and j with columns k and l, arranged with the same order as in (2-1). We have easily $\varphi(I_n) = I_{C_n^2}$, $\varphi(A)^t = \varphi(A^t)$, and the following lemma.

Lemma 2.5. *The map φ preserves the matrix product, that is, $\varphi(AB) = \varphi(A)\varphi(B)$ holds for $A \in \mathrm{M}(m,k)$ and $B \in \mathrm{M}(k,n)$.*

We'll also need a result of linear algebra for proving the equality case.

Lemma 2.6. *Let A, B be two matrices in $\mathrm{M}(m,n)$. Then $AA^t = BB^t$ if and only if $A = BR$ for some $R \in O(n)$.*

3. Proof of the main results

Let $B_1, \ldots, B_m$ be any real symmetric $n \times n$ matrices. Their coefficients in the standard basis $\{\hat{E}_\alpha\}_{\alpha \in S}$ of $\mathrm{SM}(n)$ are determined by a matrix $B \in \mathrm{M}(N,m)$ as $(B_1, \ldots, B_m) = (\hat{E}_1, \ldots, \hat{E}_N)B$. Taking the same ordering as in (2-1) for $1 \leq r <$

$s \le m$ and $1 \le \alpha < \beta \le N$, we arrange $\{[B_r, B_s]\}_{r<s}$ and $\{[\hat{E}_\alpha, \hat{E}_\beta]\}_{\alpha<\beta}$ into C_m^2 and C_N^2 vectors, respectively. We first observe that

$$([B_1, B_2], \dots, [B_{m-1}, B_m]) = ([\hat{E}_1, \hat{E}_2], \dots, [\hat{E}_{N-1}, \hat{E}_N]) \cdot \varphi(B).$$

Define a matrix $C(E)$ in $\mathrm{M}(C_N^2)$ by $C(E)_{(\alpha,\beta)(\gamma,\tau)} := \langle [\hat{E}_\alpha, \hat{E}_\beta], [\hat{E}_\gamma, \hat{E}_\tau] \rangle$ for $1 \le \alpha < \beta \le N$ and $1 \le \gamma < \tau \le N$. We use the same notation for $\{B_r\}$ and $\{\hat{Q}_\alpha\}$, that is, we write $C(B)$ and $C(Q)$, respectively. Then it's obvious that

$$C(B) = \varphi(B^t)C(E)\varphi(B) \quad \text{and} \quad C(Q) = \varphi(Q^t)C(E)\varphi(Q).$$

Since BB^t is a semi positive definite matrix in $\mathrm{SM}(N)$, there exists an orthogonal matrix $Q \in \mathrm{SO}(N)$ such that $BB^t = Q\,\mathrm{diag}(x_1, \dots, x_N)Q^t$ with $x_\alpha \ge 0$ for $1 \le \alpha \le N$. Thus $\sum_{r=1}^m \|B_r\|^2 = \|B\|^2 = \sum_{\alpha=1}^N x_\alpha$, and hence by Lemma 2.5

$$\begin{aligned}\sum_{r,s=1}^m \|[B_r, B_s]\|^2 &= 2\,\mathrm{Tr}\, C(B) = 2\,\mathrm{Tr}\,\varphi(B^t)C(E)\varphi(B) = 2\,\mathrm{Tr}\,\varphi(BB^t)C(E)\\ &= 2\,\mathrm{Tr}\,\varphi(\mathrm{diag}(x_1, \dots, x_N))C(Q) = \sum_{\alpha,\beta=1}^N x_\alpha x_\beta \|[\hat{Q}_\alpha, \hat{Q}_\beta]\|^2. \qquad (3\text{-}1)\end{aligned}$$

Proof of Theorem 1.1. For the inequality, the arguments above show that it is equivalent to prove

$$\sum_{\alpha,\beta=1}^N x_\alpha x_\beta \|[\hat{Q}_\alpha, \hat{Q}_\beta]\|^2 \le \Big(\sum_{\alpha=1}^N x_\alpha\Big)^2 \quad \text{for any } x \in \mathbb{R}_+^N \text{ and } Q \in \mathrm{SO}(N), \qquad (3\text{-}2)$$

where $\mathbb{R}_+^N := \{0 \ne x = (x_1, \dots, x_N) \in \mathbb{R}^N \mid x_\alpha \ge 0,\ 1 \le \alpha \le N\}$ is the cone spanned by the positive axes of $\mathbb{R}^N$.

Let $f_Q(x) = F(x, Q) := \sum_{\alpha,\beta=1}^N x_\alpha x_\beta \|[\hat{Q}_\alpha, \hat{Q}_\beta]\|^2 - (\sum_{\alpha=1}^N x_\alpha)^2$. Then F is a continuous function defined on $\mathbb{R}^N \times \mathrm{SO}(N)$ and thus uniformly continuous on any compact subset of $\mathbb{R}^N \times \mathrm{SO}(N)$. Let $\triangle := \{x \in \mathbb{R}_+^N \mid \sum_\alpha x_\alpha = 1\}$ and for any sufficiently small $\varepsilon > 0$, let $\triangle_\varepsilon := \{x \in \triangle \mid x_\alpha \ge \varepsilon,\ 1 \le \alpha \le N\}$. Also let

$$G := \{Q \in \mathrm{SO}(N) \mid f_Q(x) \le 0 \text{ for all } x \in \triangle\},$$
$$G_\varepsilon := \{Q \in \mathrm{SO}(N) \mid f_Q(x) < 0 \text{ for all } x \in \triangle_\varepsilon\}.$$

We claim that $G = \lim_{\varepsilon\to 0} G_\varepsilon = \mathrm{SO}(N)$. Note that this implies (3-2) and thus proves the inequality. In fact we can show

$$G_\varepsilon = \mathrm{SO}(N) \text{ for any sufficiently small } \varepsilon > 0. \qquad (3\text{-}3)$$

To prove (3-3), we use the continuity method, in which we must prove the following three properties:

(i) $I_N \in G_\varepsilon$ (and thus $G_\varepsilon \ne \varnothing$);

(ii) G_ε is open in $\mathrm{SO}(N)$;

(iii) G_ε is closed in $\mathrm{SO}(N)$.

Since F is uniformly continuous on $\triangle_\varepsilon \times \mathrm{SO}(N)$, (ii) is obvious.

Proof of (i). For any $x \in \triangle_\varepsilon$, $f_{I_N}(x) = \sum_{\alpha,\beta=1}^N x_\alpha x_\beta \|[\hat{E}_\alpha, \hat{E}_\beta]\|^2 - 1$.

It follows from (2-2) that

$$\begin{aligned} f_{I_N}(x) &= 2\Big\{\sum_{i<j}(x_{ii}x_{ij} + x_{ij}x_{jj}) + \tfrac{1}{2}\sum_{i<j<k}(x_{ij}x_{jk} + x_{ij}x_{ik} + x_{ik}x_{jk})\Big\} - 1 \\ &= 2\sum_{i<j}(x_{ii}x_{ij} + x_{ij}x_{jj}) + \sum_{i<j<k}(x_{ij}x_{jk} + x_{ij}x_{ik} + x_{ik}x_{jk}) - \big(\textstyle\sum_{i\le j}^N x_{ij}\big)^2 \\ &< 0, \end{aligned}$$

which means $I_N \in G_\varepsilon$. □

Proof of (iii). We only need to prove the following a priori estimate: Suppose $f_Q(x) \le 0$ for every $x \in \triangle_\varepsilon$. Then $f_Q(x) < 0$ for every $x \in \triangle_\varepsilon$.

The proof of this estimate is as follows: If there is a point $y \in \triangle_\varepsilon$ such that $f_Q(y) = 0$, we can assume without loss of generality that

$$y \in \triangle_\varepsilon^\gamma := \{x \in \triangle_\varepsilon \mid x_\alpha > \varepsilon \text{ for } \alpha \le \gamma \text{ and } x_\beta = \varepsilon \text{ for } \beta > \gamma\}$$

for some $1 \le \gamma \le N$. Then y is a maximum point of $f_Q(x)$ in the cone spanned by $\triangle_\varepsilon$ and an interior maximum point in $\triangle_\varepsilon^\gamma$. Hence there exist numbers $b_{\gamma+1}, \dots, b_N$ and a number a such that

$$\begin{aligned} \Big(\frac{\partial f_Q}{\partial x_1}(y), \dots, \frac{\partial f_Q}{\partial x_\gamma}(y)\Big) &= 2a(1, \dots, 1), \\ \Big(\frac{\partial f_Q}{\partial x_{\gamma+1}}(y), \dots, \frac{\partial f_Q}{\partial x_N}(y)\Big) &= 2(b_{\gamma+1}, \dots, b_N) \end{aligned} \tag{3-4}$$

or equivalently

$$\sum_{\beta=1}^N y_\beta(\|[\hat{Q}_\alpha, \hat{Q}_\beta]\|^2) - 1 = \begin{cases} a & \text{if } \alpha \le \gamma, \\ b_\alpha & \text{if } \alpha > \gamma. \end{cases} \tag{3-5}$$

Hence

$$f_Q(y) = \big(\textstyle\sum_{\alpha=1}^\gamma y_\alpha\big)a + \big(\sum_{\alpha=\gamma+1}^N b_\alpha\big)\varepsilon = 0 \quad \text{and} \quad \sum_{\alpha=1}^\gamma y_\alpha + (N-\gamma)\varepsilon = 1.$$

Meanwhile, we see $\partial f_Q/\partial \nu(y) = 2(a\gamma + \sum_{\alpha=\gamma+1}^N b_\alpha) \le 0$, where $\nu = (1, \dots, 1)$ is the vector normal to $\triangle$ in $\mathbb{R}^N$. For any sufficiently small ε (such as $\varepsilon < 1/N$), it follows from the above three formulas that $a \ge 0$. Without loss of generality, we assume $y_1 = \max\{y_1, \dots, y_\gamma\} > \varepsilon$. Let $J := \{\beta \in S \mid \|[\hat{Q}_1, \hat{Q}_\beta]\|^2 \ge 1\}$, and let

n_1 be the number of elements of J. Now combining Lemma 2.3, Lemma 2.4 and Equation (3-5) will give a contradiction as follows:

$$\begin{aligned}
1 \le 1+a &= \textstyle\sum_{\beta=2}^{N} y_\beta \|[\hat{Q}_1, \hat{Q}_\beta]\|^2 \\
&= \textstyle\sum_{\beta\in J} y_\beta(\|[\hat{Q}_1, \hat{Q}_\beta]\|^2 - 1) + \sum_{\beta\in J} y_\beta + \sum_{\beta\in S/J} y_\beta \|[\hat{Q}_1, \hat{Q}_\beta]\|^2 \\
&\le y_1 \textstyle\sum_{\beta\in J}(\|[\hat{Q}_1, \hat{Q}_\beta]\|^2 - 1) + \sum_{\beta\in J} y_\beta + \sum_{\beta\in S/J} y_\beta \|[\hat{Q}_1, \hat{Q}_\beta]\|^2 \\
&\le y_1 + \textstyle\sum_{\beta\in J} y_\beta + \sum_{\beta\in S/J} y_\beta \|[\hat{Q}_1, \hat{Q}_\beta]\|^2 \le \sum_{\beta=1}^{N} y_\beta = 1. \qquad (3\text{-}6)
\end{aligned}$$

Thus

$$y_\beta = y_1 \quad \text{for } \beta \in J \quad \text{and} \quad \textstyle\sum_{\beta\in J} \|[\hat{Q}_1, \hat{Q}_\beta]\|^2 = n_1 + 1 \le n < N. \qquad (3\text{-}7)$$

Hence $S/(J \cup \{1\}) \neq \varnothing$, and the second "$\le$" in line (3-6) should be "$<$" by the definition of J and the positivity of y_β for $\beta \in S/(J \cup \{1\})$. □

Now we consider the equality condition of Conjecture A in view of the proof of the a priori estimate.

If there's an orthogonal matrix Q and a point $y \in \triangle$ such that $f_Q(y) = 0$, we can assume without loss of generality that

$$y \in \triangle^\gamma := \{x \in \triangle \mid x_\alpha > 0 \text{ for all } \alpha \le \gamma \text{ and } x_\beta = 0 \text{ for all } \beta > \gamma\}$$

for some $1 \le \gamma \le N$. Then y is a maximum point of $f_Q(x)$ in $\mathbb{R}_+^N$ and an interior maximum point in $\triangle^\gamma$. Therefore, we have the same conclusions as (3-4), (3-5), (3-6), and (3-7) when $\gamma = n_1 + 1$, and all inequalities in the proof of Lemma 2.3 can be replaced by equalities. So $\sum_{\beta\in J} q_{\gamma\beta}^2 = 1$ implies $q_{\gamma\beta} = 0$ for all $\gamma \in I$ and $\beta \in S/J$. And also it follows from Lemma 2.2 that $(\lambda_i - \lambda_j)^2 - 1 < 0$ for all $\gamma = (i, j) \in S/I$, and thus $q_{\gamma\beta} = 0$ for all $\beta \in J$ and $\gamma \in S/I$. Hence $n_0 = n_1$, and it follows from Lemma 2.1 that all $\hat{Q}_\beta$ for $\beta \in J$ have rank 2. On the other hand, we know from the first formula of (3-7) that $\hat{Q}_1$ can be replaced in the above arguments by $\hat{Q}_\beta$ for some $\beta \in J$, which implies that $\hat{Q}_1$ has rank 2 and thus $n_0 = n_1 = 1$ and $\gamma = 2$. Finally we can conclude the equality case of Theorem 1.1 from Lemmas 2.2 and 2.6. □

Proof of Corollary 1.2. The inequality case is equivalent to that of Theorem 1.1; see also [Dillen et al. 2007, Theorem 3.1]. When the equality holds at some point $p \in M$, we can choose an orthonormal basis $\{u_1, \dots, u_m\}$ of $T_p^\perp M$ such that $u_1 = H/|H|$ if $H \neq 0$ or arbitrarily if $H = 0$. Put $B_1 = A_{u_1} - |H| I_n$ and $B_r = A_{u_r}$ for $2 \le r \le m$. Applying the equality case of Theorem 1.1 and choosing an orthonormal basis $\{e_1, \dots, e_n\}$ of $T_p M$, we get $(B_1, \dots, B_m) = (H_1, H_2, 0, \dots, 0)R$ for some $m \times m$ orthogonal matrix $R = (R_{rs})$. Therefore $(A_{u_1}, \dots, A_{u_m})R^t = (H_1 + |H|R_{11}I_n, H_2 + |H|R_{21}I_n, |H|R_{31}I_n, \dots, |H|R_{m1}I_n)$. Then taking the orthonormal basis $(\xi_1, \dots, \xi_m) = (u_1, \dots, u_m)R^t$ of $T_p^\perp M$ completes the proof. □

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