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We derive the first and second variation formulas for P-harmonic maps between Finsler manifolds, and we prove that there is no nondegenerate stable P-harmonic map between a Riemannian unit sphere S^n for $n > P \geq 2$ and any compact Finsler manifold.

1. Introduction

Let M be an n-dimensional smooth manifold, and let $\pi : TM \to M$ be the natural projection from the tangent bundle. Let (x, Y) be a point of TM with $x \in M$ and $Y \in T_x M$, and let (x^i, Y^i) be the local coordinates on TM with $Y = Y^i \partial/\partial x^i$. A Finsler metric on M is a function $F : TM \to [0, +\infty)$ satisfying the following properties.

(i) Regularity: $F(x, Y)$ is smooth in $TM \backslash 0$.

(ii) Positive homogeneity: $F(x, \lambda Y) = \lambda F(x, Y)$ for $\lambda > 0$.

(iii) Strong convexity: the fundamental quadratic form $g = g_{ij}(x, Y) dx^i \bigotimes dx^j$ is positive definite.

Let $\phi : M \to \overline{M}$ be a nondegenerate (that is, $\ker(d\phi) = \varnothing$) smooth map between Finsler manifolds. Harmonic maps between Finsler manifolds are defined as the critical points of energy functionals. They are important in both classical and modern differential geometry. The first and second variation formulas of nondegenerate harmonic maps between Finsler manifolds were given in [He and Shen 2005; Shen and Zhang 2004]. As for stability of harmonic maps between Finsler manifolds, Q. He and Y. B. Shen [2005] proved that there is no nondegenerate stable harmonic map between a Riemannian unit sphere S^n for $n > 2$ and any compact Finsler manifold.

In this paper, we are concerned with a P-harmonic map between Finsler manifolds that is a natural generalization of a harmonic map. We derive the first and

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second variation formulas for P-harmonic maps between Finsler manifolds, and we prove the following:

Main Theorem. *There is no nondegenerate stable P-harmonic map between a Riemannian unit sphere S^n for $n > P \geq 2$ and any compact Finsler manifold.*

2. Preliminaries

We shall use the following convention of index ranges unless otherwise stated:

$$1 \leq i, j, \ldots \leq n; \quad 1 \leq \alpha, \beta, \ldots \leq m; \quad 1 \leq a, b, \ldots \leq n-1.$$

Let (M, F) be an n-dimensional Finsler manifold. F inherits the *Hilbert* form and *Cartan* tensor as follows:

$$\omega^n = \frac{\partial F}{\partial Y^i} dx^i,$$
$$A = A_{ijk} dx^i \otimes dx^j \otimes dx^k, \quad \text{with } A_{ijk} = F \frac{\partial g_{ij}}{\partial Y^k}.$$

It is well known that there exists a unique Chern connection ∇ on $\pi^* TM$ with $\nabla \partial/\partial x^i = \omega_i^j \partial/\partial x^j$ and $\omega_i^j = \Gamma_{ik}^j dx^k$ satisfying

$$dg_{ij} - g_{ik}\omega_j^k - g_{jk}\omega_i^k = 2A_{ijk} \frac{\delta Y^k}{F}, \tag{2-1}$$

where $\delta Y^i = dY^i + N_j^i dx^j$, $N_j^i = \gamma_{jk}^i Y^k - (1/F) A_{jk}^i \gamma_{st}^k Y^s Y^t$, and γ_{jk}^i are the formal Christoffel symbols of the second kind for g_{ij}.

On the other hand, by [Bao et al. 2000], $\nabla e_n = (\delta Y^k/F)(\partial/\partial x^k)$, where $e_n = (Y^i/F)(\partial/\partial x^i)$, so (2-1) is equivalent to

$$X\langle U, V\rangle = \langle \nabla_X U, V\rangle + \langle U, \nabla_X V\rangle + 2C(U, V, \nabla_X(F e_n)), \tag{2-2}$$

where $A_{ijk} = F C_{ijk}$ and $X, U, V \in \Gamma(\pi^* TM)$.

The curvature 2-forms of the Chern connection ∇ are

$$\omega_j^i - \omega_j^k \wedge \omega_k^i = \Omega_j^i = \tfrac{1}{2} R_{jkl}^i dx^k \wedge dx^l + \tfrac{1}{F} P_{jkl}^i dx^k \wedge \delta Y^l. \tag{2-3}$$

Similarly, (2-3) is equivalent to

$$\Omega_j^i(U, V) = R_j^i(U, V) + P_j^i(U, \nabla_V e_n) - P_j^i(V, \nabla_U e_n), \tag{2-4}$$

where $U, V \in \Gamma(\pi^* TM)$, $R_j^i = R_{jkl}^i dx^k \otimes dx^l$, and $P_j^i = P_{jkl}^i dx^k \otimes dx^l$.

Take a g-orthonormal frame $\{e_i = u_i^j(\partial/\partial x^j)\}$ with $e_n = (Y^i/F)(\partial/\partial x^i)$ for each fiber of $\pi^* TM$, and let $\{\omega^i\}$ be its dual coframe. The collection $\{\omega^i, \omega_n^i\}$ forms an orthonormal basis for $T^*(TM \backslash \{0\})$ with respect to the Sasaki-type metric

$g_{ij}dx^i \otimes dx^j + g_{ij}\delta Y^i \otimes \delta Y^j$. The pullback of the Sasaki metric from $TM\backslash\{0\}$ to SM is a Riemannian metric

$$\widehat{g} = g_{ij}dx^i \otimes dx^j + \delta_{ab}\omega_n^a \otimes \omega_n^b. \tag{2-5}$$

We quote several lemmas:

Lemma 2.1 [He and Shen 2005]. *For $\psi = \psi_i\omega^i \in \Gamma(\pi^*T^*M)$, we have*

$$\operatorname{div}_{\widehat{g}} \psi = \sum_i \psi_{i|i} + \sum_{a,b} \psi_a P_{bba} = (\nabla_{e_i^H}\psi)e_i + \sum_{a,b} \psi_a P_{bba},$$

where "$|$" denotes the horizontal covariant differential with respect to the Chern connection and $e_i^H = u_i^j(\delta/\delta x^j) = u_i^j((\partial/\partial x^j) - N_j^k(\partial/\partial Y^k))$ denotes the horizontal part of e_i.

For any fixed $x \in M$, $S_xM = \{Y \in T_xM \mid F(Y) = 1\}$ has a natural Riemannian metric

$$\widehat{r}_x = \sum_a \theta_n^a \otimes \theta_n^a, \quad \textit{where } \theta_n^a = \omega_n^a\big|_{S_xM}. \tag{2-6}$$

Lemma 2.2 [He and Shen 2005]. *Let (M, F) be a Finsler manifold. Then any function f on the sphere bundle SM satisfies*

$$\int_{S_xM} g^{ij}(F^2 f)_{Y^iY^j}\Omega\, d\tau = 2n\int_{S_xM} f\,\Omega\, d\tau. \tag{2-7}$$

where

$$(F^2f)_{Y^iY^j} = \frac{\partial}{\partial Y^j}\frac{\partial}{\partial Y^j}(F^2 f),$$
$$\Omega = \det(g_{ij}/F),$$
$$d\tau = \textstyle\sum_i (-1)^{i-1} Y^i dY^1 \wedge \cdots \wedge \widehat{dY^i} \wedge \cdots \wedge dY^n.$$

Lemma 2.3 [He and Shen 2005]. *For $\psi = \nu\psi_i dY^i$,*

$$\operatorname{div}_{\widehat{r_x}} \psi = F^2\nu g^{ij}(\psi_i)_{Y^j} - (n-2)\nu\psi_i Y^i, \quad \textit{where } \nu = \sqrt{\det(g_{ij})}.$$

3. The first variation formula

Let $\phi : M^n \to \overline{M}^m$ be a nondegenerate smooth map. The P-energy density of ϕ is the function $e_P(\phi) : SM \to R$ defined by

$$e_P(\phi)(x, Y) = \tfrac{1}{P}|d\phi|^P = \tfrac{1}{P}\big(g^{ij}(x, Y)\phi_i^\alpha\phi_j^\beta \bar{g}_{\alpha\beta}(\bar{x}, \overline{Y})\big)^{P/2}, \tag{3-1}$$

where $d\phi(\partial/\partial x^i) = \phi_i^\alpha \partial/\partial\bar{x}^\alpha$ and $\overline{Y} = \overline{Y}^\alpha\partial/\partial\bar{x}^\alpha = Y^i\phi_i^\alpha\partial/\partial\bar{x}^\alpha$.

We define the P-energy functional $E_P(\phi)$ by

$$E_P(\phi) = \frac{1}{C_{n-1}}\int_{SM} e_P(\phi)\, dV_{SM}, \tag{3-2}$$

where $dV_{SM} = \Omega d\tau \wedge dx$, $dx = dx^1 \wedge \cdots \wedge dx^n$, and C_{n-1} denotes the volume of the unit Euclidean sphere S^{n-1}.

Let $\widetilde{\nabla}$ be the pullback Chern connection on $\pi^*(\phi^{-1}T\overline{M})$ and $\widetilde{\Omega}$ be the curvature form of the pullback connection $\widetilde{\nabla}$. We have from (2-2), (2-4), and $d\phi(Fe_n) = \overline{F}\bar{e}_m$ two lemmas:

Lemma 3.1. *$X\langle d\phi U, d\phi V\rangle$ is equal to*

$$\langle \widetilde{\nabla}_X(d\phi U), d\phi V\rangle + \langle d\phi U, \widetilde{\nabla}_X(d\phi V)\rangle + 2\overline{C}(d\phi U, d\phi V, \widetilde{\nabla}_X(d\phi Fe_n)).$$

Lemma 3.2. *$\widetilde{\Omega}^\alpha_\beta(d\phi U, d\phi V)$ is equal to*

$$\overline{R}^\alpha_\beta(d\phi U, d\phi V) + \frac{F}{\overline{\overline{F}}}\,\overline{P}^\alpha_\beta(d\phi U, \widetilde{\nabla}_V d\phi e_n) - \frac{F}{\overline{\overline{F}}}\,\overline{P}^\alpha_\beta(d\phi V, \widetilde{\nabla}_U d\phi e_n).$$

We call ϕ a P-harmonic map if it is a critical point of the P-energy functional. Let ϕ_t be a smooth variation of ϕ with $\phi_0 = \phi$ and $\phi_t|_{\partial M} = \phi|_{\partial M}$. $\{\phi_t\}$ induces a vector field V along ϕ by

$$V = \frac{\partial \phi_t}{\partial t}\Big|_{t=0} = v^\alpha \frac{\partial}{\partial \bar{x}^\alpha} \quad \text{and} \quad V\big|_{\partial M} = 0.$$

Lemma 3.3. *$\displaystyle\sum_k \int_{S_xM} \Big(|d\phi|^{P-2}\frac{\partial}{\partial t}\langle d\phi e_k, d\phi e_k\rangle\Big)\Omega\, d\tau$ is equal to*

$$\int_{S_xM}\Big(n|d\phi|^{P-2}\frac{\partial}{\partial t}\langle d\phi e_n, d\phi e_n\rangle - 2g^{ij}|d\phi|^{P-2}_{Y^j}\frac{\partial}{\partial t}\langle d\phi\frac{\partial}{\partial x^i}, d\phi(Fe_n)\rangle - \tfrac{1}{2}g^{ij}|d\phi|^{P-2}_{Y^iY^j}\frac{\partial}{\partial t}\langle d\phi(Fe_n), d\phi(Fe_n)\rangle\Big)\Omega\, d\tau.$$

Proof. It can be seen from Lemma 3.1 that

$$\begin{aligned} &Y^l|d\phi|^{P-2}\widetilde{\nabla}_{\partial/\partial Y^i}\widetilde{\nabla}_{\partial/\partial t}\Big\langle d\phi\frac{\partial}{\partial x^s}, d\phi\frac{\partial}{\partial x^l}\Big\rangle \\ &\quad = Y^l|d\phi|^{P-2}\widetilde{\nabla}_{\partial/\partial t}\Big(2\overline{C}\Big(d\phi\frac{\partial}{\partial x^s}, d\phi\frac{\partial}{\partial x^l}, \widetilde{\nabla}_{\partial/\partial Y^i}d\phi Fe_n\Big)\Big) \\ &\quad = |d\phi|^{P-2}\widetilde{\nabla}_{\partial/\partial t}\Big(2\overline{C}\Big(d\phi\frac{\partial}{\partial x^s}, d\phi(Y^l\frac{\partial}{\partial x^l}), \widetilde{\nabla}_{\partial/\partial Y^i}d\phi Fe_n\Big)\Big) = 0. \end{aligned} \tag{3-3}$$

Let $f = |d\phi|^{P-2}\frac{\partial}{\partial t}\langle d\phi e_n, d\phi e_n\rangle$. By (3-3), we get

$$\begin{aligned} g^{ij}(F^2f)_{Y^iY^j} &= g^{ij}\Big(2Y^l|d\phi|^{P-2}\frac{\partial}{\partial t}\Big\langle d\phi\frac{\partial}{\partial x^i}, d\phi\frac{\partial}{\partial x^l}\Big\rangle + Y^kY^l|d\phi|^{P-2}_{Y^i}\frac{\partial}{\partial t}\Big\langle d\phi\frac{\partial}{\partial x^k}, d\phi\frac{\partial}{\partial x^l}\Big\rangle\Big)_{Y^j} \\ &= g^{ij}\Big(2|d\phi|^{P-2}\frac{\partial}{\partial t}\Big\langle d\phi\frac{\partial}{\partial x^i}, d\phi\frac{\partial}{\partial x^j}\Big\rangle \\ &\qquad + 4Y^l|d\phi|^{P-2}_{Y^i}\frac{\partial}{\partial t}\Big\langle d\phi\frac{\partial}{\partial x^j}, d\phi\frac{\partial}{\partial x^l}\Big\rangle + Y^kY^l|d\phi|^{P-2}_{Y^iY^j}\frac{\partial}{\partial t}\Big\langle d\phi\frac{\partial}{\partial x^k}, d\phi\frac{\partial}{\partial x^l}\Big\rangle\Big) \end{aligned}$$

$$= 2\sum_k |d\phi|^{P-2}\frac{\partial}{\partial t}\langle d\phi e_k, d\phi e_k\rangle$$
$$+4g^{ij}|d\phi|^{P-2}_{Y^i}\frac{\partial}{\partial t}\Big\langle d\phi\frac{\partial}{\partial x^j}, d\phi(Fe_n)\Big\rangle + g^{ij}|d\phi|^{P-2}_{Y^iY^j}\frac{\partial}{\partial t}\langle d\phi(Fe_n), (Fe_n)\rangle.$$

Lemma 3.3 follows immediately from this and Lemma 2.2. □

Lemma 3.4.

$$2\int_{S_xM} g^{ij}|d\phi|^{P-2}_{Y^j}\frac{\partial}{\partial t}\Big\langle d\phi\frac{\partial}{\partial x^i}, d\phi(Fe_n)\Big\rangle\Omega\, d\tau = -\int_{S_xM} g^{ij}|d\phi|^{P-2}_{Y^iY^j}\frac{\partial}{\partial t}\langle d\phi(Fe_n), d\phi(Fe_n)\rangle\Omega\, d\tau.$$

Proof. It is easy to see that

$$\begin{aligned} g^{ij}\frac{\partial F}{\partial Y^j}F|d\phi|^{P-2}_{Y^i} &= Y^i|d\phi|^{P-2}_{Y^i} \\ &= \frac{P-2}{2}Y^i|d\phi|^{P-4}\Big(-g^{ks}g^{lt}\frac{A_{kli}}{F}\phi^\alpha_k\phi^\beta_l\bar g^{\alpha\beta} + g^{kl}\phi^\alpha_k\phi^\beta_l\phi^\gamma_i\frac{\bar A_{\alpha\beta\gamma}}{\bar F}\Big) \\ &= 0. \end{aligned}$$

Let $\psi = |d\phi|^{P-2}_{Y^i}\frac{\partial}{\partial t}\langle d\phi e_n, d\phi e_n\rangle \nu dY^i$. Then ψ is a global section on $T^*(S_xM)$. It follows from the equation above and Lemma 2.3 that

$$\begin{aligned} \operatorname{div}_{\hat r_x}\psi &= F^2\nu g^{ij}\Big(|d\phi|^{P-2}_{Y^i}\frac{\partial}{\partial t}\Big\langle d\phi e_n, d\phi e_n\Big\rangle\Big)_{Y^j} \\ &= F^2\nu g^{ij}\Big(\frac{2Y^k}{F^2}|d\phi|^{P-2}_{Y^i}\frac{\partial}{\partial t}\Big\langle d\phi\frac{\partial}{\partial x^k}, d\phi\frac{\partial}{\partial x^j}\Big\rangle \\ &\qquad - \frac{2Y^k}{F^3}\frac{\partial F}{\partial Y^j}|d\phi|^{P-2}_{Y^i}\frac{\partial}{\partial t}\Big\langle d\phi\frac{\partial}{\partial x^k}, d\phi\frac{\partial}{\partial x^j}\Big\rangle \\ &\qquad + \frac{Y^kY^l}{F^2}|d\phi|^{P-2}_{Y^iY^j}\frac{\partial}{\partial t}\Big\langle d\phi\frac{\partial}{\partial x^k}, d\phi\frac{\partial}{\partial x^l}\Big\rangle \\ &\qquad + \frac{Y^kY^l}{F^2}|d\phi|^{P-2}_{Y^i}\frac{\partial}{\partial Y^j}\frac{\partial}{\partial t}\Big\langle d\phi\frac{\partial}{\partial x^k}, d\phi\frac{\partial}{\partial x^l}\Big\rangle\Big) \\ &= \Big(2g^{ij}|d\phi|^{P-2}_{Y^i}\frac{\partial}{\partial t}\Big\langle d\phi(Fe_n), d\phi\frac{\partial}{\partial x^j}\Big\rangle \\ &\qquad + g^{ij}|d\phi|^{P-2}_{Y^iY^j}\frac{\partial}{\partial t}\langle d\phi(Fe_n), d\phi(Fe_n)\rangle\Big). \end{aligned} \tag{3-4}$$

By integrating this result, we get Lemma 3.4. □

It follows from Lemmas 3.1, 3.3, and 3.4 that

$$\begin{aligned}\frac{d}{dt}E_P(\phi) &= \frac{1}{C_{n-1}}\int_{SM}\Big(\tfrac{1}{2}n|d\phi|^{P-2}\frac{\partial}{\partial t}\langle d\phi e_n, d\phi e_n\rangle \\ &\qquad\qquad + \tfrac{1}{4}g^{ij}|d\phi|^{P-2}_{Y^iY^j}\frac{\partial}{\partial t}\langle d\phi(Fe_n), d\phi(Fe_n)\rangle\Big)dV_{SM} \\ &= \frac{1}{C_{n-1}}\int_{SM}\frac{Q}{2}\frac{\partial}{\partial t}\langle d\phi e_n, d\phi e_n\rangle dV_{SM} \\ &= \frac{1}{C_{n-1}}\int_{SM}\langle \widetilde{\nabla}_{e_n^H} d\phi\frac{\partial}{\partial t}, Qd\phi e_n\rangle dV_{SM},\end{aligned} \tag{3-5}$$

where $Q = n|d\phi|^{P-2} + (F^2/2)g^{ij}|d\phi|^{P-2}_{Y^iY^j}$.

Similarly, we also have

$$\begin{aligned}\frac{1}{C_{n-1}}\int_{SM} Q\langle d\phi e_n, d\phi e_n\rangle &= \frac{1}{C_{n-1}}\int_{SM}\textstyle\sum_i |d\phi|^{P-2}\langle d\phi e_i, d\phi e_i\rangle \\ &= \frac{1}{C_{n-1}}\int_{SM}|d\phi|^P.\end{aligned} \tag{3-6}$$

Let $\psi = \langle d\phi\frac{\partial}{\partial t}, Qd\phi e_n\rangle\omega^n$. By Lemma 2.1, we get

$$\begin{aligned}\mathrm{div}_{\hat{g}}\,\psi =& \nabla_{e_n^H}\langle d\phi\tfrac{\partial}{\partial t}, Qd\phi e_n\rangle \\ =& \langle \nabla_{e_n^H} d\phi\tfrac{\partial}{\partial t}, Qd\phi e_n\rangle \\ &+ \langle d\phi\tfrac{\partial}{\partial t}, \nabla_{e_n^H}(Qd\phi e_n)\rangle + 2\bar{A}(d\phi\tfrac{\partial}{\partial t}, Qd\phi e_n\,(\widetilde{\nabla}_{e_n} d\phi)e_n).\end{aligned}$$

It follows from this and (3-5) that

$$\frac{d}{dt}E_P(\phi) = -\frac{1}{C_{n-1}}\int_{SM}\langle d\phi\tfrac{\partial}{\partial t}, \tau_P(\phi)\rangle dV_{SM}, \tag{3-7}$$

where $\tau_P(\phi) = \widetilde{\nabla}_{e_n^H}(Qd\phi e_n)$.

Remark. In the case $P = 2$, we have $Q = n$, so Equation (3-7) becomes the result in [He and Shen 2005].

Equation (3-7) has an immediate consequence:

Theorem 3.5. *ϕ is a P-harmonic map if and only if*

$$\int_{SM}\langle V, \tau_P(\phi)\rangle dV_{SM} = 0$$

for any vector $V \in \Gamma(\phi^{-1}T\overline{M})$.

4. The second variation formula

First, for Lemma 3.2 and $\nabla_{X^H} e_n = 0$ for all $X \in \Gamma(\pi^* TM)$, we obtain immediately a lemma:

Lemma 4.1. *$\big\langle \widetilde{\nabla}_{\partial/\partial t} \widetilde{\nabla}_{e_n^H} d\phi \frac{\partial}{\partial t}, d\phi e_n \big\rangle - \big\langle \widetilde{\nabla}_{e_n^H} \widetilde{\nabla}_{\partial/\partial t} d\phi \frac{\partial}{\partial t}, d\phi e_n \big\rangle$ is equal to*

$$-\big\langle \overline{R}(d\phi e_n, d\phi \tfrac{\partial}{\partial t}) d\phi \tfrac{\partial}{\partial t}, d\phi e_n \big\rangle + \frac{F}{\overline{F}} \big\langle \overline{P}(d\phi \tfrac{\partial}{\partial t}, (\widetilde{\nabla}_{e_n} d\phi) e_n) d\phi \tfrac{\partial}{\partial t}, d\phi e_n \big\rangle,$$

where

$$\overline{R} = \overline{R}^{\alpha}_{\beta\gamma\sigma} \frac{\partial}{\partial \overline{x}^{\alpha}} \otimes d\overline{x}^{\beta} \otimes d\overline{x}^{\gamma} \otimes d\overline{x}^{\sigma} \quad \text{and} \quad \overline{P} = \overline{P}^{\alpha}_{\beta\gamma\sigma} \frac{\partial}{\partial \overline{x}^{\alpha}} \otimes d\overline{x}^{\beta} \otimes d\overline{x}^{\gamma} \otimes d\overline{x}^{\sigma}.$$

On putting $\psi = \langle \widetilde{\nabla}_{\partial/\partial t} d\phi \frac{\partial}{\partial t}, (Q d\phi e_n) \rangle \omega^n$, we obtain from Lemma 2.1

$$\begin{aligned} \operatorname{div}_{\widehat{g}} \psi &= \widetilde{\nabla}_{e_n^H} \langle \widetilde{\nabla}_{\partial/\partial t} d\phi \tfrac{\partial}{\partial t}, (Q d\phi e_n) \rangle \\ &= \langle \widetilde{\nabla}_{e_n^H} (\widetilde{\nabla}_{\partial/\partial t} d\phi \tfrac{\partial}{\partial t}), (Q d\phi e_n) \rangle + \langle \widetilde{\nabla}_{\partial/\partial t} d\phi \tfrac{\partial}{\partial t}, \widetilde{\nabla}_{e_n^H} (Q d\phi e_n) \rangle. \end{aligned} \tag{4-1}$$

It can be seen from (4-1) that

$$\int_{SM} \langle \widetilde{\nabla}_{e_n^H} (\widetilde{\nabla}_{\partial/\partial t} d\phi \tfrac{\partial}{\partial t}), (Q d\phi e_n) \rangle dV_{SM} = -\int_{SM} \langle \widetilde{\nabla}_{d\phi \partial/\partial t} d\phi \tfrac{\partial}{\partial t}, \tau_P(\phi) \rangle dV_{SM}.$$

It follows from (3-4), Lemma 3.1 and Lemma 4.1 that

$$\begin{aligned}
\frac{d^2}{dt^2} E_P(\phi) &= \frac{1}{C_{n-1}} \int_{SM} \frac{d}{dt} \langle \widetilde{\nabla}_{e_n^H} d\phi \tfrac{\partial}{\partial t}, Q d\phi e_n \rangle dV_{SM} \\
&= \frac{1}{C_{n-1}} \int_{SM} \Big(\frac{dQ}{dt} \langle \widetilde{\nabla}_{e_n^H} d\phi \tfrac{\partial}{\partial t}, d\phi e_n \rangle \\
&\qquad + Q \langle \widetilde{\nabla}_{\partial/\partial t} \widetilde{\nabla}_{e_n^H} d\phi \tfrac{\partial}{\partial t}, d\phi e_n \rangle + Q \langle \widetilde{\nabla}_{e_n^H} d\phi \tfrac{\partial}{\partial t}, \widetilde{\nabla}_{\partial/\partial t} d\phi e_n \rangle \\
&\qquad\qquad + 2Q \overline{C}(\widetilde{\nabla}_{\partial/\partial t} d\phi F e_n, d\phi e_n, \widetilde{\nabla}_{e_n} \tfrac{\partial}{\partial t}) \Big) dV_{SM} \\
&= \frac{1}{C_{n-1}} \int_{SM} \Big(\frac{dQ}{dt} \langle \widetilde{\nabla}_{e_n^H} d\phi \tfrac{\partial}{\partial t}, d\phi e_n \rangle \\
&\qquad + Q \langle \widetilde{\nabla}_{e_n^H} \widetilde{\nabla}_{\partial/\partial t} d\phi \tfrac{\partial}{\partial t}, d\phi e_n \rangle - Q \langle \overline{R}(d\phi e_n, d\phi \tfrac{\partial}{\partial t}) d\phi \tfrac{\partial}{\partial t}, d\phi e_n \rangle \\
&\qquad + Q \frac{F}{\overline{F}} \langle \overline{P}(d\phi \tfrac{\partial}{\partial t}, (\widetilde{\nabla}_{e_n} d\phi) e_n) d\phi \tfrac{\partial}{\partial t}, d\phi e_n \rangle + Q \langle \widetilde{\nabla}_{e_n^H} d\phi \tfrac{\partial}{\partial t}, \widetilde{\nabla}_{e_n} d\phi \tfrac{\partial}{\partial t} \rangle \Big) dV_{SM} \\
&= \frac{1}{C_{n-1}} \int_{SM} \Big(\frac{dQ}{dt} \langle \widetilde{\nabla}_{e_n^H} d\phi \tfrac{\partial}{\partial t}, d\phi e_n \rangle \\
&\qquad - \langle \widetilde{\nabla}_{d\phi \partial/\partial t} d\phi \tfrac{\partial}{\partial t}, \tau_P \rangle - Q \langle \overline{R}(d\phi e_n, d\phi \tfrac{\partial}{\partial t}) d\phi \tfrac{\partial}{\partial t}, d\phi e_n \rangle \\
&\qquad + Q \frac{F}{\overline{F}} \langle \overline{P}(d\phi \tfrac{\partial}{\partial t}, (\widetilde{\nabla}_{F e_n} d\phi) e_n) d\phi \tfrac{\partial}{\partial t}, d\phi e_n \rangle + Q \langle \widetilde{\nabla}_{e_n^H} d\phi \tfrac{\partial}{\partial t}, \widetilde{\nabla}_{e_n} d\phi \tfrac{\partial}{\partial t} \rangle \Big) dV_{SM}.
\end{aligned}$$

On the other hand, we have

$$
\begin{aligned}
\frac{d}{dt}Q &= n\frac{d}{dt}|d\phi|^{P-2}+\tfrac{1}{2}F^2\Big(\frac{d}{dt}|d\phi|^{P-2}\Big)_{Y^iY^j}\\
&= n(P-2)|d\phi|^{P-4}\big(\langle\widetilde{\nabla}_{\partial/\partial t}d\phi e_i, d\phi e_i\rangle+\overline{C}(d\phi e_i, d\phi e_i, (\widetilde{\nabla}_{\partial/\partial t}d\phi Fe_n))\big)\\
&\qquad +\tfrac{1}{2}(P-2)F^2g^{ij}\Big(|d\phi|^{P-4}\big(\langle\widetilde{\nabla}_{\partial/\partial t}d\phi e_i, d\phi e_i\rangle\\
&\qquad\qquad +\overline{C}(d\phi e_i, d\phi e_i, (\widetilde{\nabla}_{\partial/\partial t}d\phi Fe_n))\big)\Big)_{Y^iY^j}\\
&= n(P-2)|d\phi|^{P-4}\big(\langle\widetilde{\nabla}_{e_i}d\phi\tfrac{\partial}{\partial t}, d\phi e_i\rangle+\overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n}d\phi\tfrac{\partial}{\partial t})\big)\\
&\quad +\tfrac{1}{2}(P-2)F^2g^{ij}\big(|d\phi|^{P-4}\big(\langle\widetilde{\nabla}_{e_i}d\phi\tfrac{\partial}{\partial t}, d\phi e_i\rangle+\overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n}d\phi\tfrac{\partial}{\partial t})\big)\big)_{Y^iY^j}
\end{aligned}
$$

Substituting this result into the previous equation, we get the second variation formula:

Theorem 4.2. . *Let $\phi : M^n \to \overline{M}^m$ be a nondegenerate smooth map. Let ϕ_t be a smooth variation of ϕ with $\phi_0=\phi$ and $V=\partial\phi_t/\partial t\big|_{t=0}$. Then the second variation of the P-energy functional for ϕ is*

$$
\begin{aligned}
I_P(V,V) &= \frac{d^2}{dt^2}E_P(\phi_t)\Big|_{t=0}\\
&= \frac{1}{C_{n-1}}\int_{SM}\Big(\big(-Q\langle\overline{R}(d\phi e_n, V)V, d\phi e_n\rangle-\langle\widetilde{\nabla}_V V, \tau_P(\phi)\rangle\\
&\qquad +QF/\overline{F}\langle\overline{P}(V,(\widetilde{\nabla}_{e_n}d\phi)e_n)V, d\phi e_n\rangle+Q\langle\widetilde{\nabla}_{e_n^H}V, \widetilde{\nabla}_{e_n}V\rangle\big)\\
&\qquad +n(P-2)|d\phi|^{P-4}\langle\widetilde{\nabla}_{e_n^H}V, d\phi e_n\rangle\\
&\qquad\qquad \times\big(\langle\widetilde{\nabla}_{e_i}V, d\phi e_i\rangle+\overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n}V)\big)\\
&\quad +\sum\nolimits_k\tfrac{1}{2}(P-2)F^2g^{ij}\langle\widetilde{\nabla}_{e_n^H}V, d\phi e_n\rangle\\
&\qquad \times\big(|d\phi|^{P-4}\big(\langle\widetilde{\nabla}_{e_k}V, d\phi e_k\rangle+\overline{C}(d\phi e_k, d\phi e_k, \widetilde{\nabla}_{Fe_n}V)\big)\big)_{Y^iY^j}\Big)dV_{SM}.
\end{aligned}
$$

5. Stability in the case that the source manifold is a Riemannian unit sphere

Let $\{e_i\}$ be an orthonormal frame of S^n, and let $\{\Lambda_1,\dots,\Lambda_{n+1}\}$ be the standard orthonormal basis in $\mathbb{R}^{n+1}$. Let $V_\lambda=\langle\Lambda_\lambda, e_i\rangle e_i$ for $\lambda=1,\dots,n+1$. Then we have [Xin 1980]

$$\nabla_X V_\lambda = -\langle\Lambda_\lambda, e_{n+1}\rangle X. \tag{5-1}$$

The second variation formula of the P-harmonic map $\phi : S^n\to\overline{M}^m$ can be written as

$$\sum\nolimits_\lambda I(d\phi V_\lambda, d\phi V_\lambda)=\Xi_1+\Xi_2+\Xi_3,$$

where

(5-2)
$$\Xi_1=\sum_\lambda\frac{1}{C_{n-1}}\int_{SM}\Big(-Q\langle\bar R(d\phi e_n,d\phi V_\lambda)d\phi V_\lambda,d\phi e_n\rangle-\langle\widetilde\nabla_{d\phi V_\lambda}d\phi V_\lambda,\tau_P(\phi)\rangle$$
$$+Q\frac{F}{\bar F}\langle\bar P(d\phi V_\lambda,(\widetilde\nabla_{e_n}d\phi)e_n)d\phi V_\lambda,d\phi e_n\rangle+Q\langle\widetilde\nabla_{e_n^H}d\phi V_\lambda,\widetilde\nabla_{e_n}d\phi V_\lambda\rangle\Big)dV_{SM},$$

(5-3)
$$\Xi_2=\sum_{\lambda,i}\frac{1}{C_{n-1}}\int_{SM}n(P-2)|d\phi|^{P-4}\langle\widetilde\nabla_{e_n^H}d\phi V_\lambda,d\phi e_n\rangle$$
$$\times\big(\langle\widetilde\nabla_{e_i}d\phi V_\lambda,d\phi e_i\rangle+\bar C(d\phi e_i,d\phi e_i,\widetilde\nabla_{Fe_n}d\phi V_\lambda)\big)dV_{SM},$$

(5-4)
$$\Xi_3=\sum_{\lambda,k}\frac{1}{C_{n-1}}\int_{SM}\tfrac12(P-2)F^2g^{ij}\langle\widetilde\nabla_{e_n^H}d\phi V_\lambda,d\phi e_n\rangle$$
$$\times\big(|d\phi|^{P-4}\big(\langle\widetilde\nabla_{e_k}d\phi V_\lambda,d\phi e_k\rangle+\bar C(d\phi e_k,d\phi e_k,\widetilde\nabla_{Fe_n}d\phi V_\lambda)\big)\big)_{Y^iY^j}dV_{SM}.$$

We have also [He and Shen 2005]

$$(5\text{-}5)\quad(\widetilde\nabla_{X^H}\widetilde\nabla_Zd\phi)Y=-d\phi R(X,Y)Z+(\widetilde\nabla_{Y^H}\widetilde\nabla_Zd\phi)X+(\widetilde\nabla_Yd\phi)(\nabla_{X^H}Z)$$
$$-(\widetilde\nabla_Xd\phi)(\nabla_{Y^H}Z)$$
$$+\bar R(d\phi X,d\phi Y)d\phi Z+(F/\bar F)\bar P(d\phi X,(\widetilde\nabla_{e_n}d\phi)Y)d\phi Z$$
$$-(F/\bar F)\bar P(d\phi Y,(\widetilde\nabla_{e_n}d\phi)X)d\phi Z.$$

Setting $X=Z=V$ and $Y=e_n$ in (5-5), we obtain

$$(5\text{-}6)\quad-\langle\bar R(d\phi e_n,d\phi V)d\phi V,d\phi e_n\rangle+(F/\bar F)\langle\bar P(d\phi V,(\widetilde\nabla_{e_n}d\phi)e_n)d\phi V,d\phi e_n\rangle$$
$$=-\langle d\phi R(e_n,V)V,d\phi e_n\rangle+\langle(\widetilde\nabla_{V^H}\widetilde\nabla_Vd\phi)e_n,d\phi e_n\rangle$$
$$-\langle(\widetilde\nabla_{e_n^H}\widetilde\nabla_Vd\phi)V,d\phi e_n\rangle+\langle(\widetilde\nabla_{e_n}d\phi)(\nabla_{V^H}V),d\phi e_n\rangle$$
$$+\langle(\widetilde\nabla_Vd\phi)(\nabla_{e_n^H}V),d\phi e_n\rangle.$$

We need the following lemma.

Lemma 5.1.

$$\sum_\lambda\int_{SM}\Big(Q\big(-\langle\bar R(d\phi e_n,d\phi V_\lambda)d\phi V_\lambda,d\phi e_n\rangle$$
$$+(F/\bar F)\langle\bar P(d\phi V_\lambda,(\widetilde\nabla_{e_n}d\phi)e_n)d\phi V_\lambda,d\phi e_n\rangle\big)-\langle\widetilde\nabla_{d\phi V_\lambda}d\phi V_\lambda,\tau_P(\phi)\rangle\Big)dV_{SM}$$
$$=\sum_\lambda\int_{SM}Q\big(-\langle d\phi R(e_n,V_\lambda)V_\lambda,d\phi e_n\rangle+\langle\widetilde\nabla_{V_\lambda^H}((\widetilde\nabla_{V_\lambda}d\phi)e_n),d\phi e_n\rangle\big)dV_{SM}.$$

Proof. It follows from (5-1) that

$$\textstyle\sum_\lambda \nabla_{V_\lambda^H} V_\lambda = \sum_\lambda \nabla_{V_\lambda} V_\lambda = -\sum_\lambda v_\lambda^i v_\lambda^{n+1} e_i = 0, \tag{5-7}$$

where $v_\lambda^i = \langle \Lambda_\lambda, e_i\rangle$. Next,

$$\begin{aligned}\textstyle\sum_\lambda (\widetilde{\nabla}_{V_\lambda} d\phi)(\nabla_{e_n^H} V_\lambda) &= \textstyle\sum_\lambda (\widetilde{\nabla}_{V_\lambda} d\phi)(\nabla_{e_n} V_\lambda)\\ &= \textstyle -\sum_\lambda v_\lambda^i v_\lambda^{n+1} (\widetilde{\nabla}_{e_i} d\phi) e_n = 0,\\ \textstyle\sum_\lambda (\widetilde{\nabla}_{e_n^H} \widetilde{\nabla}_{V_\lambda} d\phi) V_\lambda &= \textstyle\sum_\lambda \widetilde{\nabla}_{e_n^H}((\widetilde{\nabla}_{V_\lambda} d\phi) V_\lambda) - (\widetilde{\nabla}_{V_\lambda} d\phi)(\nabla_{e_n^H} V_\lambda)\\ &= \textstyle\sum_\lambda \widetilde{\nabla}_{e_n^H}((\widetilde{\nabla}_{V_\lambda} d\phi) V_\lambda),\\ \textstyle\sum_\lambda (\widetilde{\nabla}_{V_\lambda^H} \widetilde{\nabla}_{V_\lambda} d\phi) e_n &= \textstyle\sum_\lambda \widetilde{\nabla}_{V_\lambda^H}((\widetilde{\nabla}_{V_\lambda} d\phi) e_n) - (\widetilde{\nabla}_{V_\lambda} d\phi)(\nabla_{V_\lambda^H} e_n)\\ &= \textstyle\sum_\lambda \widetilde{\nabla}_{V_\lambda^H}((\widetilde{\nabla}_{V_\lambda} d\phi) e_n).\end{aligned} \tag{5-8}$$

Considering the 1-form $\psi = Q\langle (\widetilde{\nabla}_V d\phi) V, d\phi e_n\rangle \omega^n$, we have

$$\begin{aligned}\operatorname{div}_{\widehat{g}} \psi &= \nabla_{e_n^H} \langle (\widetilde{\nabla}_V d\phi) V, Q d\phi e_n\rangle\\ &= Q\langle \widetilde{\nabla}_{e_n^H}((\widetilde{\nabla}_V d\phi) V), d\phi e_n\rangle + \langle (\widetilde{\nabla}_V d\phi) V, \tau_P(\phi)\rangle.\end{aligned} \tag{5-9}$$

By integrating (5-9), we get

$$-\sum_\lambda \int_{SM} Q\langle \widetilde{\nabla}_{e_n^H}[(\widetilde{\nabla}_{V_\lambda} d\phi) V_\lambda], d\phi e_n\rangle dV_{SM} = \sum_\lambda \int_{SM} \langle \widetilde{\nabla}_{V_\lambda}(d\phi V_\lambda), \tau_P(\phi)\rangle dV_{SM}.$$

Because ϕ is a P-harmonic map, by $\widetilde{\nabla}_{d\phi V_\lambda}(d\phi V_\lambda) - \widetilde{\nabla}_{V_\lambda}(d\phi V_\lambda) \in \Gamma(\phi^{-1} T\overline{M})$ and the above, we have

$$\begin{aligned}-\sum_\lambda \int_{SM} \big(Q\langle \widetilde{\nabla}_{e_n^H}[(\widetilde{\nabla}_{V_\lambda} d\phi) V_\lambda], d\phi e_n\rangle + \langle \widetilde{\nabla}_{d\phi V_\lambda}(d\phi V_\lambda), \tau_P(\phi)\rangle\big) dV_{SM}&\\ = \sum_\lambda \int_{SM} \langle \widetilde{\nabla}_{V_\lambda}(d\phi V_\lambda) - \widetilde{\nabla}_{d\phi V_\lambda}(d\phi V_\lambda), \tau_P(\phi)\rangle dV_{SM} = 0.&\end{aligned}$$

Then this result together with (5-6)–(5-8) finishes the lemma. □

Lemma 5.2.

$$\begin{aligned}\sum_\lambda \int_{SM} \big(Q\langle \widetilde{\nabla}_{V_\lambda^H}\{(\widetilde{\nabla}_{V_\lambda} d\phi) e_n\}, d\phi e_n\rangle + Q\langle \widetilde{\nabla}_{e_n^H} d\phi V_\lambda, \widetilde{\nabla}_{e_n} d\phi V_\lambda\rangle\big) dV_{SM}&\\ = \int_{SM} \big(\textstyle -\sum_\lambda (\widetilde{\nabla}_{V_\lambda^H} Q)\langle (\widetilde{\nabla}_{V_\lambda} d\phi) e_n, d\phi e_n\rangle + Q\langle d\phi e_n, d\phi e_n\rangle\big) dV_{SM}.&\end{aligned}$$

Proof. Let $\psi = \sum_{\lambda,i} Q\langle(\widetilde{\nabla}_{V_\lambda} d\phi)e_n, d\phi e_n\rangle v^i_\lambda \omega^i$, where $v^i_\lambda = \langle \Lambda_\lambda, e_i\rangle$. We have

$$\begin{aligned}\operatorname{div}_{\widehat{g}} \psi &= \sum_\lambda \big((\nabla_{V^H_\lambda} Q)\langle(\widetilde{\nabla}_{V_\lambda} d\phi)e_n, d\phi e_n\rangle + Q\langle\nabla_{V^H_\lambda}\{(\widetilde{\nabla}_{V_\lambda} d\phi)e_n\}, d\phi e_n\rangle \\ &\qquad + Q\langle(\widetilde{\nabla}_{V_\lambda} d\phi)e_n, \nabla_{V^H_\lambda} d\phi e_n\rangle - Q\langle(\widetilde{\nabla}_{V_\lambda} d\phi)e_n, d\phi e_n\rangle v^{n+1}_\lambda\big) \\ &= \sum_\lambda \big((\nabla_{V^H_\lambda} Q)\langle(\widetilde{\nabla}_{V_\lambda} d\phi)e_n, d\phi e_n\rangle + Q\langle\nabla_{v^H_\lambda}((\widetilde{\nabla}_{V_\lambda} d\phi)e_n), d\phi e_n\rangle \\ &\qquad + Q\langle(\widetilde{\nabla}_{V_\lambda} d\phi)e_n, \nabla_{V^H_\lambda} d\phi e_n\rangle\big).\end{aligned}$$

By integrating this, we obtain

$$\begin{aligned}&\sum_\lambda \int_{SM} Q\langle\widetilde{\nabla}_{V^H_\lambda}((\widetilde{\nabla}_{V_\lambda} d\phi)e_n), d\phi e_n\rangle dV_{SM} \\ &= \sum_\lambda \int_{SM} \big(-(\nabla_{V^H_\lambda} Q)\langle(\widetilde{\nabla}_{V_\lambda} d\phi)e_n, d\phi e_n\rangle - Q\langle(\widetilde{\nabla}_{V_\lambda} d\phi)e_n, \nabla_{V^H_\lambda} d\phi e_n\rangle\big) dV_{SM}.\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}&\textstyle\sum_\lambda Q\langle\widetilde{\nabla}_{e^H_n} d\phi V_\lambda, \widetilde{\nabla}_{e_n} d\phi V_\lambda\rangle \\ &= \textstyle\sum_\lambda Q\langle(\widetilde{\nabla}_{e^H_n} d\phi)V_\lambda - d\phi(v^{n+1}_\lambda e_n), (\widetilde{\nabla}_{e_n} d\phi)V_\lambda - d\phi(v^{n+1}_\lambda e_n)\rangle \\ &= \textstyle\sum_\lambda Q\langle\widetilde{\nabla}_{V^H_\lambda}(d\phi e_n), (\widetilde{\nabla}_{V_\lambda} d\phi)e_n)\rangle + Q\langle d\phi e_n, d\phi e_n\rangle.\end{aligned}$$

The lemma follows from the last two results. □

Substituting Lemmas 5.1 and 5.2 into (5-2), we get

$$\begin{aligned}\Xi_1 = \frac{1}{C_{n-1}} \int_{SM} \big(&-Q \textstyle\sum_\lambda \langle d\phi R(e_n, V_\lambda)V_\lambda, d\phi e_n\rangle \\ &- \textstyle\sum_\lambda (\widetilde{\nabla}_{V^H_\lambda} Q)\langle(\widetilde{\nabla}_{V_\lambda} d\phi)e_n, d\phi e_n\rangle + Q\langle d\phi e_n, d\phi e_n\rangle\big) dV_{SM}.\end{aligned} \tag{5-10}$$

We have

$$\frac{\partial}{\partial Y^j}\frac{\partial}{\partial Y^i}\Big(N^s_k \frac{\partial}{\partial Y^s}\Big) = N^s_k \frac{\partial}{\partial Y^j}\frac{\partial}{\partial Y^i}\frac{\partial}{\partial Y^s} + \Gamma^s_{kj}\frac{\partial}{\partial Y^i}\frac{\partial}{\partial Y^s} + \Gamma^s_{ki}\frac{\partial}{\partial Y^j}\frac{\partial}{\partial Y^s}.$$

From this, we get

$$\begin{aligned}\nabla_{V^H_\lambda}\nabla_{\partial/\partial Y^i}\nabla_{\partial/\partial Y^j} = \nabla_{\partial/\partial Y^i}\nabla_{\partial/\partial Y^j}\nabla_{V^H_\lambda} &+ g^{kl}\langle\Lambda, \partial/\partial x^k\rangle\Gamma^s_{kj}\nabla_{\partial/\partial Y^i}\nabla_{\partial/\partial Y^s} \\ &+ g^{kl}\langle\Lambda, \partial/\partial x^k\rangle\Gamma^s_{ki}\nabla_{\partial/\partial Y^j}\nabla_{\partial/\partial Y^s},\end{aligned}$$

so we obtain

$$\begin{aligned}\nabla_{V^H_\lambda} Q &= \nabla_{V^H_\lambda}\big(n|d\phi|^{P-2} + \tfrac{1}{2}F^2 g^{ij}|d\phi|^{P-2}_{Y^iY^j}\big) \\ &= n|d\phi|^{P-2}_{V^H_\lambda} + \tfrac{1}{2}F^2(\nabla_{V^H_\lambda} g^{ij})|d\phi|^{P-2}_{Y^iY^j} + \tfrac{1}{2}F^2 g^{ij}(|d\phi|^{P-2}_{V^H_\lambda})_{Y^iY^j} \\ &\quad + \tfrac{1}{2}F^2 g^{ij} g^{kl}\langle\Lambda_\lambda, \partial/\partial x^l\rangle\Gamma^s_{kj}(|d\phi|^{P-2})_{Y^iY^s} \\ &\quad + \tfrac{1}{2}F^2 g^{ij} g^{kl}\langle\Lambda_\lambda, \partial/\partial x^l\rangle\Gamma^s_{ki}(|d\phi|^{P-2})_{Y^jY^s}.\end{aligned} \tag{5-11}$$

On the other hand, we also have

$$\nabla_{V_\lambda^H} g^{ij} = -g^{kl}\langle \Lambda_\lambda, \partial/\partial x^l\rangle g^{is} g^{jt}\frac{\partial g_{st}}{\partial x^k}. \tag{5-12}$$

and

$$\begin{aligned}\tfrac{1}{2}F^2 g^{ij} g^{kl}&\langle \Lambda_\lambda, \partial/\partial x^l\rangle \Gamma^s_{kj}(|d\phi|^{P-2})_{Y^iY^s}\\ +\tfrac{1}{2}F^2 g^{ij} g^{kl}&\langle \Lambda_\lambda, \partial/\partial x^l\rangle \Gamma^s_{ki}(|d\phi|^{P-2})_{Y^jY^s}\\ &= F^2 g^{ij} g^{kl}\langle \Lambda_\lambda, \partial/\partial x^l\rangle \Gamma^s_{kj}(|d\phi|^{P-2})_{Y^iY^s}\\ &= \tfrac{1}{2}F^2 g^{ij} g^{kl} g^{st}\langle \Lambda_\lambda, \partial/\partial x^l\rangle \frac{\partial g_{jt}}{\partial x^k}|d\phi|^{P-2}_{Y^iY^s}.\end{aligned} \tag{5-13}$$

Substituting (5-12) and (5-13) into (5-11), we obtain

$$\nabla_{V_\lambda^H} Q = n|d\phi|^{P-2}_{V_\lambda^H} + \tfrac{1}{2}F^2 g^{ij}(|d\phi|^{P-2}_{V_\lambda^H})_{Y^iY^j}. \tag{5-14}$$

Obviously, by Lemma 3.1, we can obtain

$$|d\phi|^{P-2}_{V_\lambda^H} = (P-2)|d\phi|^{P-4}\big(\langle\widetilde{\nabla}_{V_\lambda^H}(d\phi e_i), d\phi e_i\rangle + \overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{V_\lambda^H}(d\phi F e_n))\big).$$

It can be seen from this, together with (5-14) and (5-10), that

$$\Xi_1 = \Xi_{1-1} + \Xi_{1-2} + \Xi_{1-3},$$

where

$$\begin{aligned}\Xi_{1-1} &= \frac{1}{C_{n-1}}\int_{SM} Q\big(-\textstyle\sum_\lambda \langle d\phi R(e_n, V_\lambda)V_\lambda, d\phi e_n\rangle + Q\langle d\phi e_n, d\phi e_n\rangle\big) dV_{SM},\\ \Xi_{1-2} &= -\frac{1}{C_{n-1}}\sum_{\lambda,i}\int_{SM} n(P-2)|d\phi|^{P-4}\langle(\widetilde{\nabla}_{V_\lambda^H} d\phi)e_n, d\phi e_n\rangle\\ &\qquad\times\big(\langle\widetilde{\nabla}_{V_\lambda^H}(d\phi e_i), d\phi e_i\rangle + \overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{V_\lambda^H}(d\phi F e_n))\big) dV_{SM},\\ \Xi_{1-3} &= -\frac{1}{C_{n-1}}\sum_{\lambda,k}\int_{SM} \tfrac{1}{2}(P-2)F^2 g^{ij}\langle(\widetilde{\nabla}_{V_\lambda^H} d\phi)e_n, d\phi e_n\rangle\\ &\qquad\times\big(|d\phi|^{P-4}\big(\langle\widetilde{\nabla}_{V_\lambda^H}(d\phi e_k), d\phi e_k\rangle + \overline{C}(d\phi e_k, d\phi e_k, \widetilde{\nabla}_{V_\lambda^H}(d\phi F e_n))\big)\big)_{Y^iY^j} dV_{SM}.\end{aligned} \tag{5-15}$$

A direct calculation gives

$$\overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n}(d\phi V_\lambda)) = \overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{V_\lambda^H}(d\phi F e_n)),$$

so

$$\begin{aligned}(5\text{-}16)\quad &\textstyle\sum_\lambda \big(\langle \widetilde{\nabla}_{e_i} d\phi V_\lambda, d\phi e_i\rangle + \overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n} d\phi V_\lambda)\big)\langle \widetilde{\nabla}_{e_n^H} d\phi V_\lambda, d\phi e_n\rangle \\
&= \textstyle\sum_\lambda \big((\langle \widetilde{\nabla}_{e_i} d\phi) V_\lambda - v_\lambda^{n+1} d\phi e_i, d\phi e_i\rangle + \overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{V_\lambda^H}(d\phi F e_n))\big) \\
&\qquad \times \langle (\widetilde{\nabla}_{e_n^H} d\phi) V_\lambda - v_\lambda^{n+1} d\phi e_n, d\phi e_n\rangle \\
&= \textstyle\sum_\lambda \big((\langle \widetilde{\nabla}_{V_\lambda^H} d\phi) e_i, d\phi e_i\rangle + \overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{V_\lambda^H}(d\phi F e_n))\big) \\
&\qquad \times \langle (\widetilde{\nabla}_{V_\lambda^H} d\phi) e_n, d\phi e_n\rangle + |d\phi|^2 \langle d\phi e_n, d\phi e_n\rangle \\
&= \textstyle\sum_\lambda \big(\langle \widetilde{\nabla}_{V_\lambda^H}(d\phi e_i) - d\phi(\nabla_{V_\lambda^H} e_i), d\phi e_i\rangle + \overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{V_\lambda^H}(d\phi F e_n))\big) \\
&\qquad \times \langle \widetilde{\nabla}_{V_\lambda^H}(d\phi e_n), d\phi e_n\rangle + |d\phi|^2 \langle d\phi e_n, d\phi e_n\rangle.\end{aligned}$$

Putting $e_i = u_i^j (\partial/\partial x^j)$, we have

$$\begin{aligned}\sum_i \big\langle d\phi(\nabla_{V^H} e_i), d\phi e_i\big\rangle &= \sum_i \Big(g^{kl}\Big\langle \Lambda, \frac{\partial}{\partial X^k}\Big\rangle \frac{\partial u_i^s}{\partial X^l} u_i^t \Big\langle d\phi \frac{\partial}{\partial x^s}, d\phi \frac{\partial}{\partial x^t}\Big\rangle \\
&\quad + g^{kl}\Big\langle \Lambda, \frac{\partial}{\partial X^k}\Big\rangle u_i^s u_i^t \Gamma_{ls}^h \Big\langle d\phi \frac{\partial}{\partial x^h}, d\phi \frac{\partial}{\partial x^t}\Big\rangle \\
&\quad + g^{kl}\Big\langle \Lambda, \frac{\partial}{\partial X^k}\Big\rangle N_l^j \frac{\partial u_i^s}{\partial Y^j} u_i^t \Big\langle d\phi \frac{\partial}{\partial x^s}, d\phi \frac{\partial}{\partial x^t}\Big\rangle\Big) \\
&= \tfrac{1}{2} g^{kl}\Big\langle \Lambda, \frac{\partial}{\partial X^k}\Big\rangle \frac{\partial g^{st}}{\partial X^l} \Big\langle d\phi \frac{\partial}{\partial x^s}, d\phi \frac{\partial}{\partial x^t}\Big\rangle \\
&\quad + \textstyle\sum_i g^{kl}\Big\langle \Lambda, \frac{\partial}{\partial X^k}\Big\rangle u_i^h u_i^t \Gamma_{lh}^s \Big\langle d\phi \frac{\partial}{\partial x^s}, d\phi \frac{\partial}{\partial x^t}\Big\rangle \\
&\quad + \tfrac{1}{2} g^{kl}\Big\langle \Lambda, \frac{\partial}{\partial X^k}\Big\rangle N_l^j \frac{\partial g^{st}}{\partial Y^j} \Big\langle d\phi \frac{\partial}{\partial x^s}, d\phi \frac{\partial}{\partial x^t}\Big\rangle \\
&= 0.\end{aligned}$$

Substituting this into (5-16), we get

$$\begin{aligned}(5\text{-}17)\quad &\textstyle\sum_\lambda \big(\langle \widetilde{\nabla}_{e_i} d\phi V_\lambda, d\phi e_i\rangle + \overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n} d\phi V_\lambda)\big)\langle \widetilde{\nabla}_{e_n^H} d\phi V_\lambda, d\phi e_n\rangle \\
&= \textstyle\sum_\lambda \big(\langle \widetilde{\nabla}_{V_\lambda^H}(d\phi e_i), d\phi e_i\rangle + \overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{V_\lambda^H}(d\phi F e_n))\big) \\
&\qquad \times \langle \widetilde{\nabla}_{V_\lambda^H}(d\phi e_n), d\phi e_n\rangle + |d\phi|^2 \langle d\phi e_n, d\phi e_n\rangle.\end{aligned}$$

It follows from (5-3), (5-15) and (5-17) that

$$(5\text{-}18)\qquad \Xi_2 + \Xi_{1-2} = \frac{1}{C_{n-1}} \int_{SM} n(P-2)|d\phi|^{P-2} \langle d\phi e_n, d\phi e_n\rangle.$$

Similarly, we have

$$(5\text{-}19)\qquad \Xi_3 + \Xi_{1-3} = \frac{1}{C_{n-1}} \int_{SM} \tfrac{1}{2}(P-2) F^2 g^{ij} |d\phi|^{P-2}_{Y^i Y^j} \langle d\phi e_n, d\phi e_n\rangle.$$

It is easy to see from (3-7), (5-18), and (5-19) that

$$\Xi_2+\Xi_3+\Xi_{1-2}+\Xi_{1-3}=\frac{1}{C_{n-1}}\int_{SM}(P-2)Q\langle d\phi e_n, d\phi e_n\rangle = \frac{1}{C_{n-1}}\int_{SM}(P-2)|d\phi|^P. \tag{5-20}$$

Obviously,

$$\Xi_{1-1}=-\frac{1}{C_{n-1}}\int_{SM}(n-2)Q\langle d\phi e_n, d\phi e_n\rangle = -\frac{1}{C_{n-1}}\int_{SM}(n-2)|d\phi|^P. \tag{5-21}$$

Substituting (5-20) and (5-21) into (5-1), we obtain immediately

$$\textstyle\sum_\lambda I(d\phi V_\lambda, d\phi V_\lambda)=\frac{1}{C_{n-1}}\int_{SM}(P-n)|d\phi|^P,$$

which immediately implies this:

Theorem 5.3. *There is no nondegenerate stable P-harmonic map from a Riemannian unit sphere S^n for $n > P \geq 2$ to any Finsler manifold $\overline{M}$.*

6. Stability in the case that the target manifold is a Riemannian unit sphere

Theorem 6.1. *There is no nondegenerate stable P-harmonic map from a compact Finsler manifold $\overline{M}$ to a Riemannian unit sphere S^n for $n > P \geq 2$.*

Proof. Let $\{\bar{e}_\alpha\}$ and $\{e_i\}$ orthonormal frames of $\overline{M}$ and S^n, respectively, and let $\{\Lambda_1, \dots, \Lambda_{n+1}\}$ be the standard orthonormal basis in $\mathbb{R}^{n+1}$. Let $V_\lambda = \langle \Lambda_\lambda, e_i\rangle e_i$ for $\lambda = 1, \dots, n+1$. Then we have

$$\nabla_X V_\lambda = -\langle \Lambda_\lambda, e_{n+1}\rangle X. \tag{6-1}$$

Let $v_\lambda^\mu = \langle \Lambda_\lambda, e_\mu\rangle$ for $\lambda, \mu = 1, \dots, n+1$. By $\widetilde{\nabla}_{d\phi(\partial/\partial t)} d\phi \frac{\partial}{\partial t} \in \Gamma(\phi^{-1}TS^n)$, the second variation formula of the P-harmonic map $\phi : \overline{M}^m \to S^n$ can be written as

$$\begin{aligned} I_P(V_\lambda, V_\lambda) = \frac{1}{C_{n-1}}\int_{SM}\Big(\big(-Q\langle R(d\phi \bar{e}_m, V_\lambda)V_\lambda, d\phi\bar{e}_m\rangle + Q\langle \widetilde{\nabla}_{\bar{e}_m^H} V_\lambda, \widetilde{\nabla}_{\bar{e}_m} V_\lambda\rangle\big) \\ + n(P-2)|d\phi|^{P-4}\langle \widetilde{\nabla}_{\bar{e}_m^H} V_\lambda, d\phi e_n\rangle\langle \widetilde{\nabla}_{\bar{e}_\alpha} V_\lambda, d\phi \bar{e}_\alpha\rangle \\ + \tfrac{1}{2}(P-2)F^2 g^{ij}\langle \widetilde{\nabla}_{\bar{e}_m^H} V_\lambda, d\phi\bar{e}_m\rangle\big(|d\phi|^{P-4}\langle \widetilde{\nabla}_{\bar{e}_\alpha} V_\lambda, d\phi\bar{e}_\alpha\rangle\big)_{Y^iY^j}\Big) dV_{SM}. \end{aligned}$$

Substituting (6-1) into this, we get

$$\begin{aligned}
\textstyle\sum_\lambda I_P(V_\lambda, V_\lambda) &= \frac{1}{C_{n-1}} \int_{SM} \Big(-(n-2) Q \langle d\phi \bar{e}_m, d\phi \bar{e}_m \rangle \\
&\qquad + n(P-2) |d\phi|^{P-4} \langle d\phi \bar{e}_m, d\phi \bar{e}_m \rangle \langle d\phi \bar{e}_\alpha, d\phi \bar{e}_\alpha \rangle \\
&\qquad + \tfrac{1}{2}(P-2) F^2 g^{ij} \langle d\phi \bar{e}_m, d\phi \bar{e}_m \rangle \\
&\qquad\qquad \times \big(|d\phi|^{P-4} \langle d\phi \bar{e}_\alpha, d\phi \bar{e}_\alpha \rangle \big)_{Y^i Y^j} \Big) dV_{SM} \\
&= \frac{1}{C_{n-1}} \int_{SM} \Big(-(n-2) Q \langle d\phi \bar{e}_m, d\phi \bar{e}_m \rangle \\
&\qquad + (P-2) \langle d\phi \bar{e}_m, d\phi \bar{e}_m \rangle \\
&\qquad\qquad \times \big(n |d\phi|^{P-2} + \tfrac{1}{2} F^2 g^{ij} \big(|d\phi|^{P-2} \big)_{Y^i Y^j} \big) \Big) dV_{SM} \\
&= \frac{1}{C_{n-1}} \int_{SM} -(n-P) Q \langle d\phi \bar{e}_m, d\phi \bar{e}_m \rangle dV_{SM} \\
&= \frac{1}{C_{n-1}} \int_{SM} -(n-P) |d\phi|^P dV_{SM} < 0. \qquad \square
\end{aligned}$$

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