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SER PEOW TAN, YAN LOI WONG AND YING ZHANG

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In 1998, Greg McShane demonstrated a remarkable identity for the lengths of simple closed geodesics on cusped hyperbolic surfaces. In 2006, we generalized this to hyperbolic cone-surfaces, possibly with cusps and/or geodesic boundary. In this paper, we generalize the identity further to the case of classical Schottky groups. As a consequence, we obtain some surprising new identities in the case of Fuchsian Schottky groups. For classical Schottky groups of rank 2, we also give generalizations of the Weierstrass identities, given by McShane in 2004.

1. Introduction

Greg McShane [1998] demonstrated a striking identity for the lengths of simple closed geodesics on cusped hyperbolic surfaces. In [Tan et al. 2006a], we gave a generalization of McShane's identity to hyperbolic cone-surfaces, possibly with cusps and/or geodesic boundary; a version for compact hyperbolic surfaces with geodesic boundary was also given independently by M. Mirzakhani [2007]. To state the generalization, we first define two functions $G(x, y, z)$ and $S(x, y, z)$.

Definition 1.1. For $x, y, z \in \mathbb{C}$, we define

$$G(x, y, z) := 2 \tanh^{-1} \left(\frac{\sinh(x)}{\cosh(x) + \exp(y+z)} \right),$$
$$S(x, y, z) := \tanh^{-1} \left(\frac{\sinh(x) \sinh(y)}{\cosh(z) + \cosh(x) \cosh(y)} \right).$$

Note that here $\tanh^{-1}(x)$ for a complex number x is defined to have imaginary part in $(-\pi/2, \pi/2]$, and hence the functions $G(x, y, z)$ and $S(x, y, z)$ are analytic in the arguments x, y and z in their respective domains of definition.

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Recall, as defined in [Tan et al. 2006a], that a *compact hyperbolic cone-surface* is a compact topological surface M equipped with a hyperbolic cone structure so that each boundary component of M is a smooth geodesic, and there is a finite number of points in the topological interior of M , which form the set of cusps and cone points. The *geometric boundary* of M consists of all the cusps, cone points and boundary geodesics. The complement of the geometric boundary in M is its *geometric interior*. Recall also that (a) a geometric boundary component of a compact hyperbolic cone-surface M is either a cusp, a cone point or a boundary geodesic; (b) an interior generalized simple closed geodesic is either a simple closed geodesic in the geometric interior of M or a degenerate one which is the double cover of a simple geodesic arc connecting two angle π cone points; (c) a generalized simple closed geodesic is either an interior one as in (b) or a boundary one, namely, a geometric boundary component.

The generalized identity in [Tan et al. 2006a] for compact hyperbolic cone-surfaces can be stated as follows.

Theorem 1.2 [Tan et al. 2006a, Theorem 10.3]. *For a compact hyperbolic cone-surface M with all cone angles in $(0, \pi]$, let $\Delta_0, \Delta_1, \dots, \Delta_m$ be its geometric boundary components with complex lengths L_0, L_1, \dots, L_m , respectively. If Δ_0 is a cone point or a boundary geodesic then*

$$(1) \quad \sum_{\alpha, \beta} G\left(\frac{1}{2}L_0, \frac{1}{2}|\alpha|, \frac{1}{2}|\beta|\right) + \sum_{j=1}^m \sum_{\beta} S\left(\frac{1}{2}L_0, \frac{1}{2}L_j, \frac{1}{2}|\beta|\right) = \frac{1}{2}L_0,$$

where the first sum is taken over all unordered pairs of generalized simple closed geodesics α and β on M such that α and β bound with Δ_0 an embedded pair of pants on M (note that one of α and β might be a geometric boundary component) and the subsum in the second sum is taken over all interior simple closed geodesics β such that β bounds with Δ_j and Δ_0 an embedded pair of pants on M . Furthermore, all the infinite series in (1) converge absolutely.

Remark 1.3. (i) $|\alpha|$ denotes the complex length of the generalized simple closed geodesic α , which is the usual hyperbolic length if α is a geodesic, is equal to $i\theta$ if α is a cone point of cone angle θ , and is 0 if α is a cusp point. Similarly, L_i is strictly positive, pure imaginary, or 0 depending on whether Δ_i is a geodesic boundary, cone point, or cusp, respectively.

(ii) As explained in [Tan et al. 2006a], the above identity unifies all the various known versions of McShane's identity in the real case by using the complex lengths of the geometric boundary components. A version for surfaces with geodesic boundary and no cone points was also given by Mirzakhani [2007]. See also [Akiyoshi et al. 2004; 2006; Bowditch 1996; 1997; 1998; McShane

2004; Sakuma 1999] for other variations and generalizations of the original identity.

In this paper, motivated by the idea of the complexification of the boundary lengths, we extend [Theorem 1.2](#) to classical Schottky groups by analytic continuation. This is possible because, first, the marked classical Schottky space, appropriately parametrized, is a connected open subset of the parametrization space. Second, both sides of the identity [\(1\)](#) are analytic functions of the parameters for the marked classical Schottky space. Finally, all the infinite series on the left side of [\(1\)](#) converge absolutely in the marked classical Schottky space. However, we will need to reinterpret $|\alpha|$ as the complex length of α in the formula, and there are some subtleties involved as we will need a specific choice of the half lengths (recall that the complex length is defined up to multiples of $2\pi i$ and that there are two possible choices for the half lengths, up to multiples of $2\pi i$). The exact statement, [Theorem 3.4](#), requires a choice of a lift from a representation of the free group F_n into $\mathrm{PSL}(2, \mathbb{C})$ to $\mathrm{SL}(2, \mathbb{C})$ (the exact choice of the lift is not important); a reformulation of [Theorem 1.2](#) to a more algebraic form; precise definitions for a marked classical Schottky group, the marked classical Schottky space; and a Fuchsian marking for the marked classical Schottky space. The basic idea is that a Fuchsian marking corresponds to a hyperbolic surface M with geodesic boundary. After we fix a boundary component Δ_0 for M , the identity [\(1\)](#) holds. Now as we deform away from the Fuchsian marking to an arbitrary point in the marked classical Schottky space, the identity continues to hold by analytic continuation, provided we can show that the infinite series converge absolutely and uniformly on compact subsets of the marked classical Schottky space. In particular, if we deform to a different Fuchsian-marked classical Schottky group, the original identity still holds even though the corresponding hyperbolic surface M' may be of a different topological type from the original surface M (or it may be of the same topological type but with a completely different marking). This gives new identities for the surface M' . For example, in the case when the Schottky group is of rank 2, M may be the hyperbolic one-holed torus and M' the hyperbolic three-holed sphere (pair of pants), so that we obtain new identities for the hyperbolic pair of pants with geodesic boundary, which are different from the trivial identity obtained by a direct application of [Theorem 1.2](#).

The rest of this paper is organized as follows. In [Section 2](#), we give some basic facts about classical Schottky groups and give a precise definition of marked classical Schottky space, as well as a parametrization for the space. In [Section 3](#), we state and prove the main result, [Theorem 3.4](#). In [Section 4](#), we state and discuss the Weierstrass identity for marked rank 2 classical Schottky groups ([Theorem 4.1](#)). Finally, in [Section 5](#), we analyze an example to show that our generalization of

McShane's identity to classical Schottky groups does give some surprising new identities for hyperbolic surfaces with geodesic boundary.

2. Marked classical Schottky groups

In this section we state some basic facts about marked classical Schottky groups. See [Marden 1974; Maskit 2000; 2002; Button 1998; 2003] for a more complete study of various Schottky spaces. Note that the terminology is not completely standardized; we use the terminology which is best suited for our purposes. In particular, we need to pay special attention to the marking in order to get a clean and precise statement of our result. Hence, we define a marked classical Schottky group and marked classical Schottky space in a way analogous to that of marked hyperbolic structures and Teichmüller space.

Fix $n \geq 2$, and let $F_n = \langle a_1, \dots, a_n \rangle$ be a free group of rank n , where $\{a_1, \dots, a_n\}$ is a (fixed) distinguished, ordered set of generators for F_n , which will provide the marking. Let \mathbb{C}_∞ be the extended complex plane, which we also identify with the Riemann sphere and also the ideal boundary of \mathbb{H}^3 .

Definition 2.1. A (marked) classical Schottky group (of rank n) is a discrete, faithful representation $\rho : F_n \rightarrow \mathrm{PSL}(2, \mathbb{C})$ such that there is a region $D \subset \mathbb{C}_\infty$, where D is bounded by $2n$ disjoint geometric circles $C_1, C'_1, \dots, C_n, C'_n$ in \mathbb{C}_∞ , such that $\rho(a_i)(C_i) = C'_i$ and $\rho(a_i)(D) \cap D = \emptyset$ for $i = 1, \dots, n$. It is *Fuchsian* if the representation can be conjugated to a representation into $\mathrm{PSL}(2, \mathbb{R})$.

Note that the circles C_i and C'_i are not uniquely determined by ρ . Also, $\rho(a_i)$ is strictly loxodromic, with an attracting and repelling ideal fixed point, denoted by $\mathrm{Fix}^+ \rho(a_i)$ and $\mathrm{Fix}^- \rho(a_i)$, respectively. The image $\Gamma := \rho(F_n)$ is often referred to in the literature as a classical Schottky group; it is a Schottky group if we only require C_i and C'_i to be disjoint simple closed curves for $i = 1, \dots, n$. Two (marked) classical Schottky groups ρ_1 and ρ_2 are equivalent if the representations are conjugate to each other.

If ρ is a marked classical Schottky group, we denote by $\tilde{\rho}$ any lift of ρ to a representation into $\mathrm{SL}(2, \mathbb{C})$. Note that since F_n is free, ρ can always be lifted, and there are 2^n possible lifts.

Remark 2.2. It was shown by Marden [1974] that there exist nonclassical Schottky groups for every $n \geq 2$. An explicit example of a nonclassical Schottky group was constructed by Yamamoto [1991]. On the other hand, Button [1998] has proved that all Fuchsian Schottky groups are classical (but in general not on every set of generators).

Definition 2.3. The space of equivalence classes of (marked) classical Schottky groups is called the *marked classical Schottky space*; we denote it by $\mathcal{S}_{\mathrm{alg}}^{\mathrm{mc}}$.

Note that we define (marked) classical Schottky groups in terms of representations rather than the subgroup $\rho(F_n)$ of $\mathrm{PSL}(2, \mathbb{C})$, and we define the space $\mathcal{G}_{\mathrm{alg}}^{\mathrm{mc}}$ as the space of such representations, modulo conjugation. In particular, two representations ρ_1 and ρ_2 may have the same image in $\mathrm{PSL}(2, \mathbb{C})$ modulo conjugation but still represent different points of $\mathcal{G}_{\mathrm{alg}}^{\mathrm{mc}}$ because of the marking. To simplify notation, we denote elements of $\mathcal{G}_{\mathrm{alg}}^{\mathrm{mc}}$ by ρ instead of the more cumbersome $[\rho]$; there should be no confusion since the traces and complex lengths are well defined on the equivalence classes.

Next we give a natural parametrization of the marked classical Schottky space $\mathcal{G}_{\mathrm{alg}}^{\mathrm{mc}}$ by the ideal fixed points and the square of the traces or the complex lengths of $\rho(a_i)$ for $i = 1, \dots, n$.

We may normalize ρ by conjugation so that

$$\mathrm{Fix}^- \rho(a_1) = 0, \quad \mathrm{Fix}^+ \rho(a_1) = \infty, \quad \mathrm{Fix}^- \rho(a_2) = 1.$$

Then it is not difficult to see that we can parameterize ρ by

$$(\mathrm{Fix}^+ \rho(a_2), \mathrm{Fix}^- \rho(a_3), \mathrm{Fix}^+ \rho(a_3), \dots, \mathrm{Fix}^+ \rho(a_n); \mathrm{tr}^2 \rho(a_1), \dots, \mathrm{tr}^2 \rho(a_n)),$$

which belongs to $\mathbb{C}_{\infty}^{2n-3} \times \mathbb{C}^n$, or, alternatively, by

$$(\mathrm{Fix}^+ \rho(a_2), \mathrm{Fix}^- \rho(a_3), \mathrm{Fix}^+ \rho(a_3), \dots, \mathrm{Fix}^+ \rho(a_n); l(\rho(a_1)), \dots, l(\rho(a_n))),$$

which belongs to $\mathbb{C}_{\infty}^{2n-3} \times (\mathbb{C}/2\pi i\mathbb{Z})^n$. Here the complex lengths $l(\rho(a))$ are related to the traces by the formula $l(\rho(a)) = 2 \cosh^{-1}(\mathrm{tr} \rho(a)/2)$ and can be chosen to have positive real part (since all elements are strictly loxodromic). Then the lengths are defined up to multiples of $2\pi i$, and depend only on $\pm \mathrm{tr} \rho(a)$ or $\mathrm{tr}^2 \rho(a)$. We shall see later that to define the half length we will be choosing a lift of the representation and using the negative of the trace on the right side of the formula. With this normalized parametrization we have the following.

Lemma 2.4 [Maskit 2002]. *The marked classical Schottky space $\mathcal{G}_{\mathrm{alg}}^{\mathrm{mc}}$ is a path connected open subset of $\mathbb{C}_{\infty}^{2n-3} \times (\mathbb{C}/2\pi i\mathbb{Z})^n$.*

Proof. We only sketch the proof; see Maskit [2002] for details. We use the conformal ball model of hyperbolic 3-space \mathbb{H}^3 . The ideal sphere is then the unit sphere S_{∞} . Consider a marked classical Schottky group ρ so that $\Gamma = \rho(F_n) \subset \mathrm{Isom}^+(\mathbb{H}^3)$. Then, for $j = 1, \dots, n$, there exists open disks D_j and D'_j on the ideal sphere whose boundary circles are denoted C_j and C'_j , respectively, such that $\rho(a_j)(C_j) = C'_j$ and $\rho(a_j)(D_j) \cap D'_j = \emptyset$. Now we may find a one parameter family ρ_t of deformations of ρ in $\mathcal{G}_{\mathrm{alg}}^{\mathrm{mc}}$ such that the ideal fixed points of all the $\rho_t(a_j)$ are unchanged but the real part of $l(\rho_t(a_j)) \rightarrow \infty$ (that is, the translation length of $\rho_t(a_j)$ approaches ∞). We may assume that, under the deformation, the circles C_j and C'_j shrink towards the respective ideal fixed points of $\rho(a_j)$, so that

their sizes become arbitrarily small. This gives us room to continue to deform the marked classical Schottky group in $\mathcal{S}_{\text{alg}}^{\text{mc}}$ continuously to one whose fixed points are in some fixed standard configuration, say, with all of them lying on a circle. Finally, keeping the ideal fixed points fixed, we can deform the lengths to some predetermined quantities, with sufficiently large real part. Since any marked classical Schottky group can be so deformed, the space $\mathcal{S}_{\text{alg}}^{\text{mc}}$ is path connected. That $\mathcal{S}_{\text{alg}}^{\text{mc}}$ is open is easily seen from the definition and parametrization. \square

With the parametrization, for each $g \in F_n$, $\rho(g) \in \text{PSL}(2, \mathbb{C})$ is completely determined by the parameters, where ρ is the normalized representation. Hence, $\text{tr}^2(\rho(g))$ and $l(\rho(g))$ are analytic functions of the parameters for each $g \in F_n$. Furthermore, if we start with a Fuchsian Schottky group, we may define all the lengths to be real and positive, and if we extend the definition of the complex lengths by analytic continuation on the space $\mathcal{S}_{\text{alg}}^{\text{mc}}$, then the following proposition states that $\Re(l(\rho(g))) > 0$ for all $\rho \in \mathcal{S}_{\text{alg}}^{\text{mc}}$ and $g \in F_n$.

Proposition 2.5. *If $\rho_0 \in \mathcal{S}_{\text{alg}}^{\text{mc}}$ is Fuchsian, we may define $l(\rho_0(g))$ so that $l(\rho_0(g))$ is positive real for all $g \in F_n$. Then if $l(\rho(g))$ is defined by analytic continuation along a path in the space $\mathcal{S}_{\text{alg}}^{\text{mc}}$, we have $\Re(l(\rho(g))) > 0$ for all $\rho \in \mathcal{S}_{\text{alg}}^{\text{mc}}$ in the path and all $g \in F_n$.*

Proof. If ρ_0 is Fuchsian, then it is well known that $|\text{tr } \rho_0(g)| > 2$ for all $g \in F_n$ so that $l(\rho_0(g))$ is positive real. If $\rho \in \mathcal{S}_{\text{alg}}^{\text{mc}}$ and ρ_t for $0 \leq t \leq 1$ is a path from ρ_0 to ρ , then we claim that $\Re(l(\rho_t(g))) > 0$ for all $g \in F_n$ and $0 \leq t \leq 1$. Otherwise, we will have $\Re(l(\rho_t(g))) = 0$ for some $g \in F_n$ and $t \in [0, 1]$, which is impossible as all elements are strictly loxodromic in a classical Schottky group. \square

Finally let us say a few words about the fundamental domain in \mathbb{H}^3 of a classical Schottky group. Let $\Gamma = \rho(F_n)$ be the image of a classical Schottky group, with the set of disjoint discs D_j and D'_j defined as in the proof of [Lemma 2.4](#). Suppose the circles C_j and C'_j bound respectively geodesic planes E_j and E'_j in \mathbb{H}^3 . For $j = 1, \dots, n$, let H_j be the open half space of \mathbb{H}^3 bounded by D_j and E_j , and similarly for H'_j . Then $\mathcal{D} := \mathbb{H}^3 - \bigcup_{j=1}^n H_j - \bigcup_{j=1}^n H'_j$ is a fundamental domain in \mathbb{H}^3 of Γ . Note that $D = \overline{\mathcal{D}} \cap \mathbb{C}_\infty$ is the fundamental domain in the extended complex plane \mathbb{C}_∞ of Γ as described at the beginning of this section. It is well known that the quotient hyperbolic 3-manifold $\mathcal{H} = \mathbb{H}^3 / \Gamma = \mathcal{D} / \Gamma$ is a handlebody of genus n . Note that $\mathbb{C}_\infty / \Gamma = D / \Gamma$ is a conformal surface of genus n and is the conformal boundary of the handlebody \mathcal{H} .

3. McShane’s identity for Schottky groups

In this section we state and prove our main theorem, [Theorem 3.4](#). We start with the following definition.

Definition 3.1. A Fuchsian marking in $\mathcal{G}_{\text{alg}}^{\text{mc}}$ is a representation $\rho_0 \in \mathcal{G}_{\text{alg}}^{\text{mc}}$ that is Fuchsian (see [Definition 2.1](#)), and the circles C_i and C'_i can all be taken to be circles that are orthogonal to \mathbb{R} .

For ρ_0 a Fuchsian marking, $\mathbb{H}^2/\rho_0(F_n)$ is a complete hyperbolic surface. Its convex core M_0 is a hyperbolic surface with geodesic boundary, which we call the hyperbolic surface corresponding to the Fuchsian marking. Let $\Delta_0, \Delta_1, \dots, \Delta_m$ be the boundary components of M_0 . The image $\rho(F_n)$, and hence F_n , can be identified with $\pi_1(M_0)$, and if we define an equivalence relation \sim on F_n by $g \sim h$ if g is conjugate to h or h^{-1} , then there is a bijection $\mathfrak{f}: F_n/\sim \rightarrow \mathcal{C}$ from F_n/\sim to the set \mathcal{C} of free homotopy classes of closed curves on M_0 . Note that there is a unique geodesic representative for each nontrivial element of \mathcal{C} .

Definition 3.2. For a fixed Fuchsian marking ρ_0 , let M_0 be the corresponding hyperbolic surface. Let $\Delta_0, \Delta_1, \dots, \Delta_n$ be the boundary components of M_0 , and let $[d_i] \in F_n/\sim$ for $i = 0, \dots, m$ be the equivalence class corresponding to the boundary component Δ_i , that is, $\mathfrak{f}[d_i] = \Delta_i$.

We define \mathcal{P} to be the set of all unordered pairs $\{[g], [h]\}$ of elements in F_n/\sim such that $\mathfrak{f}[g]$ and $\mathfrak{f}[h]$ are free homotopy classes of simple closed curves that bound together with Δ_0 an embedded pair of pants in M_0 (note that it is possible that $\mathfrak{f}[g] = \Delta_k$ for some $1 \leq k \leq m$).

For $j = 1, \dots, m$, we define \mathcal{B}_j to be the set of elements $[g] \in F_n/\sim$ such that $\mathfrak{f}[g]$ bounds together with Δ_0 and Δ_j an embedded pair of pants in M_0 .

We will also need to define the half lengths, for which we need representations into $\text{SL}(2, \mathbb{C})$ instead of $\text{PSL}(2, \mathbb{C})$.

Definition 3.3. If $\rho \in \mathcal{G}_{\text{alg}}^{\text{mc}}$ and $\tilde{\rho}$ is a lift of ρ to $\text{SL}(2, \mathbb{C})$, then for an element $g \in F_n$, we define the half length $l(\tilde{\rho}(g))/2 \in \mathbb{C}/2\pi i\mathbb{Z}$ of $\tilde{\rho}(g)$ by

$$(2) \quad \cosh \frac{1}{2}l(\tilde{\rho}(g)) = -\frac{1}{2} \text{tr } \tilde{\rho}(g),$$

with $\Re l(\rho(g)) > 0$.

Note that the real part of the half length is just half of the real part of the length, and both are positive, while the above choice fixes the imaginary part, up to multiples of $2\pi i$. The minus sign on the right side of (2) is crucial; see [Remark 3.5\(iv\)](#).

Our main theorem can then be stated as follows.

Theorem 3.4. *Let $\rho \in \mathcal{G}_{\text{alg}}^{\text{mc}}$, and let $\tilde{\rho}$ be any lift of ρ to $\text{SL}(2, \mathbb{C})$. Suppose ρ_0 is a Fuchsian marking, with corresponding hyperbolic surface M_0 , with boundary components $\Delta_0, \dots, \Delta_m$. For $j = 1, \dots, m$, let \mathcal{P} and \mathcal{B}_j be defined relative to*

M_0 as in [Definition 3.2](#). Then

$$(3) \quad \sum_{\{[g],[h]\} \in \mathcal{P}} G\left(\frac{1}{2}l(\tilde{\rho}(d_0)), \frac{1}{2}l(\tilde{\rho}(g)), \frac{1}{2}l(\tilde{\rho}(h))\right) + \sum_{j=1}^m \sum_{[g] \in \mathcal{B}_j} S\left(\frac{1}{2}l(\tilde{\rho}(d_0)), \frac{1}{2}l(\tilde{\rho}(d_j)), \frac{1}{2}l(\tilde{\rho}(g))\right) = \frac{1}{2}l(\tilde{\rho}(d_0)) \pmod{\pi i}.$$

Each series on the left side of (3) converges absolutely.

Remark 3.5. (i) In the case where $\rho = \rho_0$, the above is just a reformulation of [Theorem 1.2](#) for the case of a hyperbolic surface with geodesic boundary components, and is true without the modulo condition. In fact, the lift can be chosen so that the right side is real and positive.

- (ii) The identity (3) is true only modulo πi because we have fixed the choice of the \tanh^{-1} function in the definition of the functions $G(x, y, z)$ and $S(x, y, z)$ (see [Definition 1.1](#)); this may differ from the values obtained by analytic continuation by some multiple of πi . Indeed, as we will see in [Corollary 3.8](#) and in the example in [Section 5](#), there is a difference of $2\pi i$ in that example when $m = 0$, that is, when M_0 has a single boundary component.
- (iii) The result is independent of the lift chosen. This is because if $\tilde{\rho}$ and $\bar{\rho}$ are two different lifts of ρ , then for each of the summands on the first series, either $\text{tr } \tilde{\rho}(g)$, $\text{tr } \tilde{\rho}(h)$ and $\text{tr } \tilde{\rho}(d_0)$ are all equal to $\text{tr } \bar{\rho}(g)$, $\text{tr } \bar{\rho}(h)$ and $\text{tr } \bar{\rho}(d_0)$, or exactly two of them differ by their signs (and similarly for the summands in the second series). In the latter case, two of the half lengths differ by πi , but it can be easily checked that both $G(x, y, z)$ and $S(x, y, z)$ remain the same if πi is added to two of the arguments.
- (iv) The choice of the half length functions given above is not arbitrary but arises from the computation of $G(x, y, z)$ and $S(x, y, z)$ as “gap” functions (this is based on the convention adopted in [\[Fenchel 1989\]](#), see [\[Tan et al. 2006a\]](#) for details; see also Goldman [\[2003\]](#), who uses a similar convention). Roughly speaking, the relative positions of the axes for $\tilde{\rho}(g)$, $\tilde{\rho}(h)$ and $\tilde{\rho}(d_0)$ are completely determined by their traces. These axes form the nonadjacent sides of a right angled hexagon in \mathbb{H}^3 , and the half lengths basically arise as the lengths of these sides of the hexagon.
- (v) It can be shown that $G(x, y, z)$ and $S(x, y, z)$ can also be expressed as

$$(4) \quad G(x, y, z) = \log \frac{\exp(x) + \exp(y+z)}{\exp(-x) + \exp(y+z)},$$

$$(5) \quad S(x, y, z) = \frac{1}{2} \log \frac{\cosh(z) + \cosh(x+y)}{\cosh(z) + \cosh(x-y)},$$

as used by Mirzakhani [2007] (with different notation), where the function \log is in the principal branch, that is, the imaginary parts of its images are in $(-\pi, \pi]$. It would be interesting to see if her results on the Weil–Petersson volumes can be generalized to the classical Schottky space.

Proof. Let ρ_t for $0 \leq t \leq 1$ be a deformation from the Fuchsian marking to an arbitrary marked classical Schottky group ρ , where ρ_0 is the Fuchsian marking and $\rho_1 = \rho$; this is possible by Lemma 2.4. Let $\tilde{\rho}_t$ be a continuous lift of ρ_t . We shall then prove that the series on the left side of (3) converge uniformly on compact subsets of the marked classical Schottky space $\mathcal{S}_{\text{alg}}^{\text{mc}}$. Then each side of (3) is a holomorphic function modulo πi on the space $\mathcal{S}_{\text{alg}}^{\text{mc}}$. By Theorem 1.2, the identity (3) holds on the totally real subspace in a neighborhood of ρ_0 in $\mathcal{S}_{\text{alg}}^{\text{mc}}$. Note that for this to be true, the correct choice of the half length as given in (2) must be used; see [Tan et al. 2006a] for details. Hence the identity also holds modulo πi for each $t \in [0, 1]$ by analytic continuation. This proves the identity for a particular lift; that it holds for all lifts now follows from Remark 3.5(iii).

Given a compact subset \mathcal{K} of $\mathcal{S}_{\text{alg}}^{\text{mc}}$, we have a constant $\kappa > 0$ such that for each $\rho \in \mathcal{K}$, $\rho(F_n)$ has a fundamental domain \mathcal{D} in \mathbb{H}^3 as described at the end of Section 2, and the minimum hyperbolic distance between any pair of its bounding geodesic planes $E_1, E'_1, \dots, E_n, E'_n$ is $\geq \kappa$. Then we have the following length estimate lemma for $\rho \in \mathcal{K}$.

Lemma 3.6. *If $g \in F_n$ is a cyclically reduced word in the set of generators $a_1, a_1^{-1}, \dots, a_n, a_n^{-1}$ with word length $\|g\|$, then the closed geodesic γ that $\rho(g)$ represents in the quotient hyperbolic 3-manifold $\mathcal{D} = \mathbb{H}^3/\rho(F_n)$ has hyperbolic length $\geq \kappa\|g\|$.*

Proof. Choose in \mathbb{H}^3 an arbitrary lift $\tilde{\gamma}$ of the closed geodesic γ . Note that \mathbb{H}^3 is tiled by the images of a fundamental domain \mathcal{D} under the action of elements of $\rho(F_n)$, that is, $\mathbb{H}^3 = \bigcup_{g' \in F_n} \rho(g')(\mathcal{D})$. It can be shown that the line $\tilde{\gamma}$ in \mathbb{H}^3 passes through $\|g\|$ successive images of \mathcal{D} “periodically” dictated by the word g . Thus the hyperbolic length of γ , which equals the length of the part of $\tilde{\gamma}$ lying in the union of these $\|g\|$ successive images of \mathcal{D} , is at least $\kappa\|g\|$. \square

Now we prove the uniform convergence of the first series in (3) for $t \in [0, 1]$. By the above lemma there is a constant $\kappa > 0$ such that for every $t \in [0, 1]$ and every $g \in F_n$, we have $L(\tilde{\rho}_t(g)) \geq \kappa\|g\|$, where $L(\tilde{\rho}_t(g))$ is the hyperbolic length of the closed geodesic that $\tilde{\rho}_t(g)$ represents in the quotient hyperbolic 3-manifold $\mathbb{H}^3/\tilde{\rho}_t(F_n)$, and where $\|g\|$ is the cyclically reduced word length of g in the letters $a_1^{\pm 1}, \dots, a_n^{\pm 1}$.

Note that the image of the Fuchsian marking $\rho_0(F_n) \subset \text{PSL}(2, \mathbb{R})$ has a fundamental domain $\mathcal{D}(0)$ in \mathbb{H}^3 whose intersection with $\mathbb{H}^2 \subset \mathbb{H}^3$ is a fundamental domain of $\mathbb{H}^2/\rho_0(F_n)$ in \mathbb{H}^2 . Let \mathcal{P} be the set of unordered pairs of elements

$\{[g], [h]\}$ of F_n/\sim as defined in [Definition 3.2](#). Then the pairs $\{[g], [h]\} \in \mathcal{P}$ can be ordered by using the sum of their (cyclically reduced) word lengths $\|g\| + \|h\|$. We have the following version of [\[Birman and Series 1985, Lemma 2.2\]](#):

Lemma 3.7. *There is a polynomial $P(\cdot)$ such that the number of pairs $\{[g], [h]\}$ in \mathcal{P} with $\|g\| + \|h\| = n$ is no greater than $P(n)$.*

Proof. We follow the proof of Birman and Series. We start with the fundamental polygon $\mathcal{D}(0) \cap \mathbb{H}^2$. Then for each pair $\{[g], [h]\}$ in \mathcal{P} with $\|g\| + \|h\| = n$, we can associate a simple diagram on $\mathcal{D}(0) \cap \mathbb{H}^2$ consisting of n disjoint arcs (note that though the original argument is for just one simple geodesic, it works as well here for the pair of disjoint simple closed geodesics on the surface M_0 associated with $\{[g], [h]\}$). Conversely, the diagram determines the pair $\{[g], [h]\}$. The number of such simple diagrams is bounded by $P(n)$ for a certain polynomial $P(\cdot)$. \square

Note that for $t \in [0, 1]$ and a simple closed geodesic α on M_0 , the real part $\text{Re} l(\alpha(t))$ of the complex translation length $l(\alpha(t))$ is equal to the hyperbolic length $L(\alpha(t))$ of the closed geodesic that $\alpha(t)$ represents in the quotient hyperbolic manifold $\mathbb{H}^3/\Gamma(t)$. Note also that for each simple closed geodesic γ on M_0 , there exist constants $c_1(\gamma)$ and $c_2(\gamma) > 0$ such that $c_1(\gamma) \leq \text{Re} l(\gamma(t)) \leq c_2(\gamma)$ for all $t \in [0, 1]$.

Now for each pair $\{\alpha, \beta\}$ in \mathcal{G} , we have $L(\alpha(t)) + L(\beta(t)) \geq \kappa(\|\alpha\| + \|\beta\|)$ for all $t \in [0, 1]$, and hence $\text{Re} l(\alpha(t)) + \text{Re} l(\beta(t)) = L(\alpha(t)) + L(\beta(t)) \rightarrow +\infty$ uniformly as $n = \|\alpha\| + \|\beta\| \rightarrow \infty$.

From the definition of G , we have

$$\begin{aligned}
 & G\left(\frac{1}{2}l(d_0(t)), \frac{1}{2}l(\alpha(t)), \frac{1}{2}l(\beta(t))\right) \\
 &= \log\left(\frac{\exp\frac{1}{2}l(d_0(t)) + \exp\left(\frac{1}{2}l(\alpha(t)) + \frac{1}{2}l(\beta(t))\right)}{\exp\left(-\frac{1}{2}l(d_0(t))\right) + \exp\left(\frac{1}{2}l(\alpha(t)) + \frac{1}{2}l(\beta(t))\right)}\right) \\
 (6) \quad &= \log\left(1 + \frac{2 \sinh\frac{1}{2}l(d_0(t))}{\exp\left(-\frac{1}{2}l(d_0(t))\right) + \exp\left(\frac{1}{2}l(\alpha(t)) + \frac{1}{2}l(\beta(t))\right)}\right).
 \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
 & \left|\exp\left(-\frac{1}{2}l(d_0(t))\right) + \exp\left(\frac{1}{2}l(\alpha(t)) + \frac{1}{2}l(\beta(t))\right)\right| \\
 (7) \quad & \geq \left|\exp\left(\frac{1}{2}l(\alpha(t)) + \frac{1}{2}l(\beta(t))\right)\right| - \left|\exp\left(-\frac{1}{2}l(d_0(t))\right)\right| \\
 (8) \quad & = \exp\left(\frac{1}{2} \text{Re} l(\alpha(t)) + \frac{1}{2} \text{Re} l(\beta(t))\right) - \exp\left(-\frac{1}{2} \text{Re} l(d_0(t))\right) \\
 & \geq 1 - \exp\left(-\frac{1}{2} \text{Re} l(d_0(t))\right) \\
 (9) \quad & \geq 1 - \exp(-c_1(d_0)).
 \end{aligned}$$

Since $|\log(1 + u)| \leq 2|u|$ for all $u \in \mathbb{C}$ such that $|u| \leq 1/2$, it follows from [\(6\)](#) and [\(8\)](#) that there is a constant $C > 0$, depending only on the family $\{\Gamma(t)\}_{t \in [0, 1]}$, such

that for all but a finite number of pairs $\{\alpha, \beta\}$ in \mathcal{G} we have

$$(10) \quad \begin{aligned} |G(\tfrac{1}{2}l(d_0(t)), \tfrac{1}{2}l(\alpha(t)), \tfrac{1}{2}l(\beta(t)))| &\leq C \cdot \exp(-\tfrac{1}{2}(L(\alpha(t)) + L(\beta(t)))) \\ &\leq C \cdot \exp(-\tfrac{1}{2}(\kappa(\|\alpha\| + \|\beta\|))). \end{aligned}$$

The claim below tells us that the left side of (10) is always finite and hence, by continuity, bounded. Hence (10) actually holds for all pairs $\{\alpha, \beta\}$ in \mathcal{G} and with a slightly larger constant C . It then follows from Lemma 3.7 that the first series in (3) converges absolutely and uniformly for $t \in [0, 1]$.

Claim. For each pair $\{\alpha, \beta\}$ in \mathcal{G} and for all $t \in [0, 1]$, we have

$$(11) \quad \exp(\pm \tfrac{1}{2}l(d_0(t))) + \exp(\tfrac{1}{2}l(\alpha(t)) + \tfrac{1}{2}l(\beta(t))) \neq 0.$$

Proof. First notice that when \pm in (11) is $-$, the inequality follows from (9). The remaining case follows from the equivalent inequality:

$$(12) \quad \tfrac{1}{2}l(d_0(t)) + \pi i \neq \tfrac{1}{2}l(\alpha(t)) + \tfrac{1}{2}l(\beta(t)) \pmod{2\pi i}.$$

To prove (12), suppose $\tfrac{1}{2}l(d_0(t)) + \pi i = \tfrac{1}{2}l(\alpha(t)) + \tfrac{1}{2}l(\beta(t)) \pmod{2\pi i}$ holds for some $t = t_0 \in [0, 1]$. We may assume (by replacing α and β by their inverses and/or conjugates in $\Gamma(0)$ if necessary) that $d_0 = \alpha\beta$ and hence $d_0(t_0) = \alpha(t_0)\beta(t_0)$. Now it is easy to see (say, by the cosine rule of Fenchel [1989]) that $\alpha(t_0)$ and $\beta(t_0)$ have the same axis in \mathbb{H}^3 ; hence either $\Gamma(t_0)$ is not a discrete subgroup of $\text{SL}(2, \mathbb{C})$ or the representation $\rho(t_0) : \Gamma \rightarrow \text{SL}(2, \mathbb{C})$ is not faithful. In either case we have a contradiction. This proves (12) and hence the claim. \square

The absolute and uniform convergence for the other series in (3) can be similarly proved. This finishes the proof of Theorem 3.4. \square

Corollary 3.8. If $m = 0$ in Theorem 3.4 (that is, the surface M_0 has only one boundary component), then

$$(13) \quad \sum_{\{[g], [h]\} \in \mathcal{P}} G(\tfrac{1}{2}l(\tilde{\rho}(d_0)), \tfrac{1}{2}l(\tilde{\rho}(g)), \tfrac{1}{2}l(\tilde{\rho}(h))) = \tfrac{1}{2}l(\tilde{\rho}(d_0)) \pmod{2\pi i},$$

and the series converges absolutely.

Note that (13) holds modulo $2\pi i$ instead of πi . Also, if $m = 0$, then d_0 is actually a commutator, so $\text{tr}(\tilde{\rho}(d_0))$ is independent of the lift $\tilde{\rho}$; hence the right side of (13) is independent of the lift chosen. In fact, it can be shown, with a little bit of extra work, that for the Fuchsian marking ρ_0 , $\text{tr}(\tilde{\rho}_0(d_0))$ is always strictly negative in this case; see for example [Goldman 2003]. Furthermore, for the more general case where $m \neq 0$, we can always choose a lift $\tilde{\rho}_0$ such that $\text{tr}(\tilde{\rho}_0(d_0))$ is strictly negative.

4. The Weierstrass identities

In this section, we consider rank 2 classical Schottky groups $\rho : F_2 \rightarrow \text{PSL}(2, \mathbb{C})$. For ease of notation, we denote the marked generators of F_2 by a and b . Let ρ_0 be a Fuchsian marking such that the corresponding surface M_0 is the one-holed torus with geodesic boundary Δ_0 . Let \mathcal{S} be the set of nonperipheral simple closed geodesics on M_0 , and let w_1, w_2 and w_3 be the Weierstrass points on M_0 . Then each element of \mathcal{S} passes through exactly two of the Weierstrass points, and we define the Weierstrass classes to be the subsets \mathcal{A}_i for $i = 1, 2, 3$ of \mathcal{S} consisting of those geodesics that miss w_i . Then $\mathcal{S} = \bigsqcup_{i=1}^3 \mathcal{A}_i$. Let

$$\bar{\mathcal{F}} := \{[g] \in F_2/\sim \mid f[g] \in \mathcal{S}\} \quad \text{and} \quad \bar{\mathcal{A}}_i := \{[g] \in F_2/\sim \mid f[g] \in \mathcal{A}_i\}$$

be the corresponding sets in F_2/\sim . Then $[g] \in \bar{\mathcal{F}}$ if and only if any cyclically reduced representative g forms with another element h a generating set for F_2 . The set $\bar{\mathcal{F}}$ can be identified with $\mathbb{Q} \cup \infty$ by considering the slopes of the corresponding simple closed curves on the torus (without the hole). The subsets $\bar{\mathcal{A}}_i$ for $i = 1, 2$ and 3 can be identified with the subsets of $\mathbb{Q} \cup \infty$ with numerator and denominator odd, numerator odd and denominator even, and numerator even and denominator odd, respectively; see [Tan et al. 2008] for details.

We have the following extension of the Weierstrass identities proved by McShane [2004]; see also [Tan et al. 2006a].

Theorem 4.1. *For any rank 2 classical Schottky group $\rho \in \mathcal{S}_{\text{alg}}^{\text{mc}}$, if $\bar{\mathcal{A}}$ is a Weierstrass class and $\tilde{\rho}$ is a lift of ρ , then*

$$(14) \quad \sum_{[g] \in \bar{\mathcal{A}}} \tan^{-1} \left(\frac{\cosh \frac{1}{4}l(\tilde{\rho}(d_0))}{\sinh \frac{1}{2}l(\tilde{\rho}(g))} \right) = \frac{\pi}{2} \pmod{\pi},$$

where $[d_0] = [b^{-1}a^{-1}ba] \in F_2/\sim$ corresponds to the boundary Δ_0 of M_0 .

Note that there are two choices for the quarter-length $l(\tilde{\rho}(d_0))/4$. We can choose either, but the choice should be the same for all summands. The half length $l(\tilde{\rho}(g))/2$ depends on the choice of the lift ρ .

We skip the proof as it is essentially the same as the proof for Theorem 3.4. Here is a geometric interpretation of the above result. First note that the case of the two generator group is very special since for a lift $\tilde{\rho}$, each of $\tilde{\rho}(a_1)$ and $\tilde{\rho}(a_2)$ can be factored as a product of two involutions (half turns) as

$$\tilde{\rho}(a_1) = -H_3H_2 \quad \text{and} \quad \tilde{\rho}(a_2) = -H_1H_3,$$

where $H_i \in \text{SL}(2, \mathbb{C})$ with $H_i^2 = -I$ and the axes l_i for H_i are such that l_2 and l_3 are perpendicular to the axis for $\rho(a_1)$, and l_3 and l_1 are perpendicular to the axis for $\rho(a_2)$ (the minus sign is the convention adopted in [Fenchel 1989]). Furthermore,

in the case of the Fuchsian marking ρ_0 , the l_i are lines in \mathbb{H}^3 perpendicular to \mathbb{H}^2 and passing through lifts of the respective Weierstrass points w_i . If we call the l_i the Weierstrass axes, a deformation ρ_t will correspond to a deformation of the relative positions of the Weierstrass axes.

Recall that in McShane's geometric proof of his original identity for a once-cusped hyperbolic torus, he had considered the set of simple geodesics emanating from the cusp, which had a Cantor set structure. The summands of his identity measure the sizes of the gaps in the complement of this Cantor set. Furthermore, the boundary points of these gaps corresponded to simple geodesics spiraling around simple closed geodesics on the cusped hyperbolic torus. We can reinterpret the boundary of the gaps as simple geodesics emanating from the cusp and ending in a fixed point of $\rho(g)$, where ρ is the holonomy representation of the hyperbolic structure, $[g]$ varies over the set of simple closed geodesics, and the g are suitably chosen representatives of $[g]$. So, the summands of his identity measure the sizes of the gaps arising from the fixed points of $\rho(g)$ for suitably chosen representatives g of $[g]$.

In this sense, the summands of the left side of (14) are then the (complex) lengths of the gaps arising from the fixed points of $\rho(g)$ (for certain representatives g of $[g]$), but these lengths are now measured against the Weierstrass axis l corresponding to the Weierstrass class $\bar{\mathcal{A}}$. That is, we are now considering geodesics emanating normally from the Weierstrass axis l and ending at the fixed points of $\rho(g)$; see [Tan et al. 2006b, Corollary 1.10] for more details.

There are also generalizations of the variations and refinements of McShane's identity given by Bowditch [1997], Sakuma [1999], and Akiyoshi, Miyachi and Sakuma [2004; 2006]. In those cases, the variations of McShane's identity were defined relative to a cusp, and these can be generalized to the case of identities relative to a boundary geodesic. We can then study deformations (say in the space of discrete, faithful representations) where the trace of the boundary component remains real and with absolute value > 2 . Most of the identities can then be reinterpreted and generalized to this context.

5. An example: The three-holed sphere

As in the previous section, we consider rank 2 classical Schottky groups $\rho \in \mathcal{S}_{\text{alg}}^{\text{mc}}$, with the same notation: $F_2 = \langle a, b \rangle$; ρ_0 is a Fuchsian marking of $\mathcal{S}_{\text{alg}}^{\text{mc}}$ in which the corresponding hyperbolic surface M_0 is a one-holed torus with geodesic boundary Δ_0 ; and $\bar{\mathcal{P}} \subset F_2/\sim$ consists of the equivalence classes corresponding to the non-peripheral simple closed geodesics on M_0 . Let $\rho_1 \in \mathcal{S}_{\text{alg}}^{\text{mc}}$ correspond to another Fuchsian marking, with corresponding hyperbolic surface M_1 , a pair of pants with geodesic boundary, and let $\hat{\mathcal{P}}$ be the set of closed geodesics on M_1 corresponding

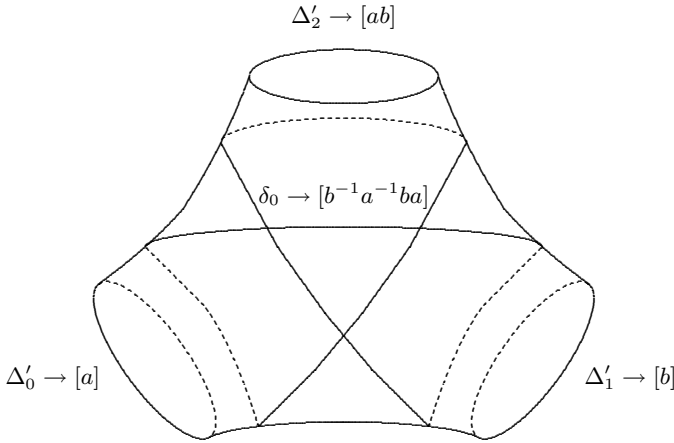


Figure 1. A commutator curve on M_1 .

to the elements of $\overline{\mathcal{F}}$. Note that the three boundary geodesics of M_1 are elements of $\hat{\mathcal{F}}$ corresponding to $[a]$, $[b]$ and $[ab]$ (with appropriate orientation). Apart from these, all other elements of $\hat{\mathcal{F}}$ are nonsimple geodesics on M_1 .

Suppose δ_0 is the geodesic on M_1 that corresponds to the commutator $[d_0] = [b^{-1}a^{-1}ba]$. Note that δ_0 is not a simple geodesic on M_1 — in fact, it has triple self-intersection; see Figure 1.

Let the three geodesic boundary components of M_1 be denoted by $\Delta'_0, \Delta'_1, \Delta'_2$ with hyperbolic lengths $L_0 = l(\Delta'_0) > 0, L_1 = l(\Delta'_1) > 0, L_2 = l(\Delta'_2) > 0$, respectively. Then for this hyperbolic surface, we have the trivial identity

$$(15) \quad G\left(\frac{1}{2}L_0, \frac{1}{2}L_1, \frac{1}{2}L_2\right) + S\left(\frac{1}{2}L_0, \frac{1}{2}L_1, \frac{1}{2}L_2\right) + S\left(\frac{1}{2}L_0, \frac{1}{2}L_2, \frac{1}{2}L_1\right) = \frac{1}{2}L_0.$$

There is, however, a nontrivial identity on M_1 derived from the Fuchsian marking ρ_0 . Recall that the trace $\text{tr } \rho[d_0]$ is well defined and independent of the lift of ρ , since d_0 is a commutator. In fact, for ρ_0 , we have $\text{tr } \rho_0[d_0] < -2$, and for ρ_1 , we have $\text{tr } \rho_1[d_0] > 18$ (see [Goldman 2003] for a detailed study of geometric structures arising from two generator subgroups of $\text{PSL}(2, \mathbb{C})$ with real character varieties). In particular, the half length $l(\delta_0)/2$ of δ_0 is well defined up to multiples of $2\pi i$, and $l(\delta_0)/2 = |\delta_0|/2 + \pi i$, where $|\delta_0|$ is the usual hyperbolic length of δ_0 on M_1 , since $\text{tr } \rho_1[d_0] > 2$ (recall the definition of the half length from (2)). From Corollary 3.8, we have

$$(16) \quad \sum_{\alpha \in \hat{\mathcal{F}}} G\left(\frac{1}{2}l(\delta_0), \frac{1}{2}l(\alpha), \frac{1}{2}l(\alpha)\right) = \frac{1}{2}l(\delta_0) \pmod{2\pi i},$$

or, equivalently,

$$(17) \quad \sum_{\alpha \in \hat{\mathcal{F}}} 2 \tanh^{-1} \left(\frac{\sinh \frac{1}{2} l(\delta_0)}{\cosh \frac{1}{2} l(\delta_0) + \exp l(\alpha)} \right) = \frac{1}{2} l(\delta_0) \pmod{2\pi i}.$$

We see from the second expression that it does not matter which choice of the two half lengths we use for α . It turns out that the three summands in the left side of (16) or (17) corresponding to the three boundary components of M_1 have positive real part and imaginary part equal to πi , whereas all the other summands are real and less than 0. As pointed out earlier, the right side is a complex number with positive real part and imaginary part equal to πi .

If, as in [Tan et al. 2008], we define $\nu := l(\delta_0)/2 = \cosh^{-1}(-\frac{1}{2} \operatorname{tr} \rho_1(d_0))$ and the functions $h(x)$ and $\mathfrak{h}(x)$ by

$$(18) \quad h(x) = \frac{1}{2} (1 - \sqrt{1 - 4/x^2}) \quad \text{and} \quad \mathfrak{h}(x) = \log \left(\frac{1 + (e^\nu - 1)h(x)}{1 + (e^{-\nu} - 1)h(x)} \right),$$

where the square root is always chosen to have nonnegative real part and we use the principal branch for the log function (that is, with the imaginary part of log in $(-\pi, \pi]$), then we can also express (17) as

$$(19) \quad \sum_{[g] \in \overline{\mathcal{F}}} \mathfrak{h}(\operatorname{tr} \rho_1(g)) = \nu \pmod{2\pi i}.$$

Remark 5.1. The identity (19) was derived in [Tan et al. 2008] with a different proof and under much more general conditions, and was expressed in terms of the μ -Markoff map corresponding to the μ -Markoff triple $(x, y, z) = (\operatorname{tr} a, \operatorname{tr} b, \operatorname{tr} ab)$. [Tan et al. 2006b] gave conditions necessary and sufficient for the identity to hold. For example, the identity still holds if some or all of the boundary components of M_1 degenerate to cusps.

We briefly describe the geometric interpretation of the above; see [Zhang 2004; Tan et al. 2008] for more details. For each $[g] \in \overline{\mathcal{F}}$, we may choose representatives g_1 and g_2 such that $g_1 g_2 = d_0$. Then each of the summands in the various versions of the identity above can be interpreted as gaps from the attracting fixed point of $\rho_1(g_1)$ to the repelling fixed point of $\rho_1(g_2)$ measured along the (directed) axis of $\rho_1(d_0)$. In the cases where $[g] \in \overline{\mathcal{F}}$ corresponds to the three boundary components of M_1 (that is, $[a]$, $[b]$ and $[ab]$), the above fixed points lie on the two different intervals of $\mathbb{R} \cup \infty$ separated by the fixed points of $\rho_1(d_0)$, which is why the summand has imaginary part πi in these three cases. For the other summands, the fixed points lie on the same interval and the summands are real. See Figure 2, where we have $\rho_1(a) = A$ and $\rho_1(b) = B$, where $A, B \in \operatorname{PSL}(2, \mathbb{R})$, and we use the notations $\bar{A} := A^{-1}$ and $\bar{B} := B^{-1}$. The picture is normalized so that the fixed points of the commutator $\bar{B}\bar{A}BA$ are 0 and ∞ . In the figure, the gap

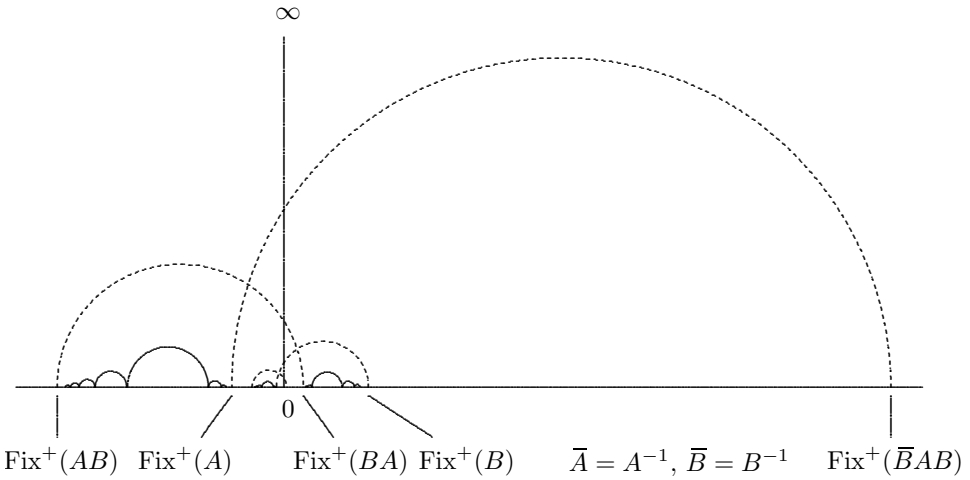


Figure 2. Gaps for M_1 .

arising from $[a] \in \mathcal{P}$ corresponding to one of the boundary components of M_1 is measured by dropping perpendiculars from $\text{Fix}^+(A)$ and $\text{Fix}^+(\overline{BAB})$ to the axis $[0, \infty]$ of $\overline{B\overline{A}BA}$; hence this gap has positive real part and imaginary part πi . Note that $(\overline{BAB})^{-1}A = \rho_1(d_0)$. Similarly, the other two middle-sized dotted semicircles in the figure show the gaps arising from $[b]$ and $[ab]$ corresponding to the other two boundary components. Gaps for the other elements of \mathcal{P} can be similarly obtained by a recursive process using the Farey construction of the rationals; all these gaps will be real and negative (they are represented by the solid semicircles in the figure). For a proof of these assertions, via generalized Markoff maps, see [Zhang 2004, Section 6.3].

Finally, we note that more generally, if the rank $n \geq 3$, it is possible to have two Fuchsian markings ρ_0 and ρ_1 such that the corresponding hyperbolic surfaces M_0 and M_1 are homeomorphic even though the markings are different. Then ρ_0 will induce identities for M_1 that are different from those obtained from ρ_1 ; for example, the gaps may be measured against a geodesic on M_1 that is not necessarily a boundary geodesic, or even simple.

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SER PEOW TAN
DEPARTMENT OF MATHEMATICS
NATIONAL UNIVERSITY OF SINGAPORE
2 SCIENCE DRIVE 2
SINGAPORE 117543
SINGAPORE
mattansp@nus.edu.sg
<http://www.math.nus.edu.sg/~mattansp/>

YAN LOI WONG
DEPARTMENT OF MATHEMATICS
NATIONAL UNIVERSITY OF SINGAPORE
2 SCIENCE DRIVE 2
SINGAPORE 117543
SINGAPORE
matwyl@nus.edu.sg

YING ZHANG
SCHOOL OF MATHEMATICAL SCIENCES
SOOCHOW UNIVERSITY
SUZHOU, JIANGSU 215006
CHINA
zhangyiing@gmail.com