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# LIPSCHITZ SOLUTIONS TO THE ISOMETRY RELATION FOR PAIRS OF RIEMANNIAN METRICS

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**Let $M$ be a smooth manifold of dimension $n$ with two Riemannian metrics $g_1, g_2$ which are related by $a^2 g_1 < g_2 < b^2 g_1$. Let $\mathbb{R}^q$ be the Euclidean space with two Euclidean metrics $h_1$, $h_2$ such that $h_1 - h_2$ has distinct eigenvalues. Further, suppose that $c^2 h_1 - h_2$ is nondegenerate for each $c \in (a, b)$, and $r_\pm(a^2 h_1 - h_2) \geq 2n$, where $r_+$ and $r_-$ denote respectively the positive and the negative ranks of an indefinite metric. Under these conditions we show that there exists an almost everywhere differentiable (Lipschitz) map $f : M \longrightarrow \mathbb{R}^q$ satisfying $(df_x)^* h_i = g_i$ for $i = 1, 2$ for almost all $x \in M$.**

## 1. Introduction

It is a classical result due to Nash and Kuiper that a Riemannian manifold $(M, g)$ admitting a $C^\infty$-immersion in $\mathbb{R}^q$ also admits a $C^1$-immersion $f : M \to \mathbb{R}^q$ such that $f^* h = g$ provided $q > n$, where $h$ is the canonical metric on $\mathbb{R}^q$. Gromov generalised this result via the method of convex integration by showing that if there exists a strictly short immersion of $(M, g)$ into another Riemannian manifold $(N, h)$ then there exists an isometric $C^1$-immersion $f : M \to N$, when $\dim N > \dim M$. He further proved that in the equidimensional case, there are almost everywhere differentiable (Lipschitz) maps whose derivatives $df$ are isometric almost everywhere on $M$. By an abuse of language, such maps will be referred as the *Lipshcitz isometric maps*; classically, the Lipschitz maps which preserve the lengths of all rectifiable curves relative to the given metrics are referred as isometric maps. Our notion of Lipschitz isometric maps satisfy a much weaker condition; in fact, such an $f$ may collapse a submanifold of positive codimension in $M$ to a single point.

In this paper we generalise the above mentioned result of Gromov when both the manifolds $M$ and $N$ come with a pair of Riemannian metrics.

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*Keywords:* Lipschitz map, isometric immersion, convex integration.

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Let $M$ be a smooth manifold of dimension $n$. Let $\mathbb{R}^q$ be the $q$-dimensional Euclidean space with two Euclidean metrics $h_1$ and $h_2$ which satisfy the following conditions: There exist two numbers $0 < a < b$ such that

(1) $c^2h_1 - h_2$ is a nondegenerate indefinite form for each real number $c$ lying in $[a, b]$;

(2) $r_+(a^2h_1 - h_2) \geq 2n$ and $r_-(b^2h_1 - h_2) \geq 2n$, where $r_+$ and $r_-$ denote respectively the positive and the negative ranks of an indefinite metric; and

(3) if $A : \mathbb{R}^q \to \mathbb{R}^q$ is the unique linear isomorphism given by $h_2(v, w) = h_1(Av, w)$ for all $v, w \in \mathbb{R}^q$, then $A$ has distinct eigenvalues.

**Theorem 1.1.** *Let $g_1, g_2$ be two Riemannian metrics which are related by $a^2g_1 < g_2 < b^2g_1$. Then under assumptions (1)–(3) mentioned above, there exists an almost everywhere differentiable (Lipschitz) map $f : M \to \mathbb{R}^q$ satisfying $(df_x)^*h_i = g_i$ for $i = 1, 2$ for almost all $x \in M$. Moreover, such maps are $C^0$-dense in the space of strictly $(g_1, g_2)$-short maps* (see Definition 5.1).

We further observe that if $M$ is a *one-dimensional* manifold, then (under the same hypothesis) there exists a $C^1$-map $f : M \to \mathbb{R}^q$ such that $f^*h_i = g_i$ for $i = 1, 2$.

The maps $f$ obtained in Theorem 1.1 will be referred as Lipschitz isometric maps for pairs of metrics. If $\mathbb{R}^q$ is replaced by a general manifold $N$ in Theorem 1.1 we may have to presuppose the existence of strictly $(g_1, g_2)$-short maps in order to conclude the existence of Lipschitz isometric maps [Gromov 1986, 2.4.9 (A)]. It may be observed that $(g_1, g_2)$-short maps always exist for $N = \mathbb{R}^q$ (see Proposition 5.2).

In our earlier paper [D'Ambra and Datta 2002] we proved the existence of isometric $C^1$-immersions $M \to \mathbb{R}^q$ for pairs of Riemannian metrics when

$$r_\pm(c^2h_1 - h_2) \geq 3n + 2$$

for all $c \in [a, b]$, generalizing the Nash–Kuiper $C^1$-immersion theorem. The proof was based on Nash's technique for obtaining isometric $C^1$-immersions.

In the present paper, we have substantially relaxed the restrictions on $r_\pm$, however, at the cost of $C^1$-regularity of solutions. Our study of Lipschitz isometric maps $f : (M, g_1, g_2) \to (\mathbb{R}^q, h_1, h_2)$ relies extensively on the convex integration theory which incorporates the essence of the approach of Kuiper [1955]. The key idea of the method of convex integration can be stated as follows: If $A$ is a connected subset of $\mathbb{R}^q$ such that the interior of the convex hull of $A$ contains the origin then there is a $C^1$-map $f : S^1 \to \mathbb{R}^q$ whose derivative maps $S^1$ into $A$. This can be viewed as the convex integration over a circle. However, in this paper we obtain only Lipschitz solutions in contrast with $C^1$-solutions in

[D'Ambra and Datta 2002]. The reason behind this is that we are unable to solve the connectivity problem for the subsets of the form $S_1 \cap S_2 \cap T$, where $S_1$ and $S_2$ are two spheres in $\mathbb{R}^q$ relative to the metrics $h_1$ and $h_2$ respectively and $T$ is an affine subspace in $\mathbb{R}^q$.

We organize the paper as follows. We devote Section 2 to review the basic language of $h$-principle theory and convex integration techniques to deal with open first order partial differential relations. In Section 3 we introduce the notion of $(h_1, h_2)$-regularity for $C^1$-maps $f : M \to \mathbb{R}^q$ and study the geometry underlying the regularity condition which plays a crucial role in our treatment. In Section 4 we prove the Main Lemma (Lemma 4.1) leading to Theorem 1.1 and in Section 5 we prove the existence of an approximate solution to our problem. The proof of the Main Theorem (Theorem 1.1) is given in Section 6. The one-dimensional case is separately studied in Section 7 where we show that there exists, in fact, a $C^1$-solution.

## 2. Review of convex integration techniques

In this section we recall the terminology of the theory of $h$-principle and discuss in brief the main result of convex integration technique following [Eliashberg and Mishachev 2002].

Let $f$ be the germ of some local $C^r$-map at $x \in M$. The $r$-jet of $f$ at $x$ is by definition the ordered tuple

$$j^r_f(x) = \big(x, f(x), Df(x), \dots, D^r f(x)\big),$$

where $D^k f$ denotes the derivative map of $f$ of order $k$. The collection of all such $r$-jets constitutes the total space of a fibre bundle over $M$ which is denoted by $p^r : J^r(M, N) \to M$. The bundle is referred as the $r$-jet bundle associated with the space of $C^r$-maps from $M$ to $N$.

If $r = 1$ then

$$j^1_f(x) = \big(x, f(x), Df(x)\big)$$

and $J^1(M, N)$ can be identified with the total space of the bundle $\operatorname{Hom}(TM, TN)$.

A continuous map $\sigma : M \to J^1(M, N)$ is said to be a *section* if $p^r \circ \sigma = \mathrm{id}_M$. If $f : M \to N$ is a $C^r$-map then its $r$-jet map $j^r_f$ defined by

$$j^r_f(x) = \big(x, f(x), Df(x), \dots, D^r f(x)\big)$$

is a section of $p^r$.

**Definition 2.1.** An *$r$-th order partial differential relation* is a subset $\mathcal{R}$ of $J^r(M,N)$. A $C^r$-map $f : M \longrightarrow N$ is said to be a *solution* of $\mathcal{R}$ if its $r$-jet map $j^r_f$ maps $M$ into $\mathcal{R}$.

A section of $p^r$ whose image is contained in $\mathcal{R}$ is called a *formal solution* of the differential relation. A formal solution of $\mathcal{R}$ is said to be *holonomic* if it is the $r$-jet map of some $C^r$-map $f : M \to N$.

A differential relation $\mathcal{R}$ is said to satisfy the *h-principle* if every formal solution $\sigma$ can be homotoped to a holonomic section in the space of all formal solutions.

**Definition 2.2.** Let $\Omega$ be an open subset of a manifold $M$. A continuous map $f$ from $\Omega$ into a manifold $N$ is said to be *piecewise $C^r$* if there exists a countable system of mutually disjoint open sets $\Omega_j \subset \Omega$ which cover $\Omega$ up to a set of measure zero and the restriction of $f$ to each $\Omega_j$ is $C^r$.

Let $\mathcal{R} \subset J^r(M, N)$ be an $r$-th order differential relation. A piecewise $C^r$-map $f : M \to N$ is said to be a *piecewise $C^r$-solution of $\mathcal{R}$* if $j^r_f(x) \in \mathcal{R}$ for all $x \in M$ where the $r$-th derivative of $f$ exists.

The convex integration technique gives solutions to $h$-principle for some differential relations which satisfy certain convexity condition. The key idea of the convex integration technique is stated in the following lemma.

**Lemma 2.3** [Gromov 1986, 2.4.1]. *Let $A$ be a connected subset of $\mathbb{R}^q$ and let $\mathbf{0}$ belong to the interior of the convex hull of $A$. Then there exists a $C^1$-map $f : [0, 1] \to \mathbb{R}^q$ such that $f'(t) \in A$ for all $t \in [0, 1]$. Moreover, $f$ can be made to lie in an arbitrary small neighbourhood of $\mathbf{0}$.*

If the connectivity condition on $A$ is dropped in the above lemma then it delivers a piecewise linear map $f$ such that $f(0) = f(1) = \mathbf{0}$ and $f'(t) \in A$ whenever $f$ is differentiable [Eliashberg and Mishachev 2002, §17.4(D)]. More generally we obtain:

**Proposition 2.4.** *Let $\mathcal{R}$ be an open subset of $J^1(\mathbb{R}, \mathbb{R}^q)$ and let $f : [0, 1] \to \mathbb{R}^q$ be a continuous function which is $C^1$ on $(0, 1)$. Suppose that $j^1_f(x)$ lies in the convex hull of $\mathcal{R}_{b(x)}$ for all $x \in (0, 1)$, where*

$$b(x) = (x, f(x)) \in J^0(\mathbb{R}, \mathbb{R}^q).$$

*Then $f$ can be homotoped to a piecewise $C^1$-solution $f_1$ of $\mathcal{R}$ in any $C^0$-neighbourhood of $f$ such that $f_1(0) = f(0)$ and $f_1(1) = f(1)$.*

*Proof.* Consider any $\varepsilon > 0$. Appealing to one-dimensional convex integration [Eliashberg and Mishachev 2002, §17.3] we can construct a piecewise linear map $f^1$ on the interval $[\varepsilon, 1 - \varepsilon]$ which coincides with $f$ at the boundary points and is a piecewise $C^1$-solution of $\mathcal{R}$ on $(\varepsilon, 1 - \varepsilon)$. Next consider, for each $n \geq 1$, a pair of disjoint intervals $I_n = [\varepsilon/2^n, \varepsilon/2^{n-1}]$ and $J_n = [1 - \varepsilon/2^{n-1}, 1 - \varepsilon/2^n]$. The interior of these sets cover $[0, 1]$ up to a set of measure zero. Now, applying [Eliashberg and Mishachev 2002, §17.3] again to the restriction of $f$ to $I_n \cup J_n$ we obtain a piecewise linear map $f^n$ on $I_n \cup J_n$ which coincides with $f$ at the endpoints and satisfies the differential relation except at the points where the derivative does not exist. Further, we can choose $f^n$ to be $\varepsilon/2^n$-close to $f$ on the set. Now all these maps patch together to give a piecewise linear map $f_1$ on $(0, 1)$. Further, this map extends continuously to the closed interval $[0, 1]$ and the extended map satisfies the desired conditions. □

**Remark.** If $f$ is a solution of $\mathcal{R}$ on a neighbourhood of some closed subset $K$, then the homotopy remains constant on some (possibly smaller) open neighbourhood of $K$.

The result above may be generalised to a parametric version following [Eliashberg and Mishachev 2002, §17.5.1].

**Proposition 2.5.** *Let $\mathcal{R}$ be an open subset of $I^l \times J^1(\mathbb{R}, \mathbb{R}^q)$ and*

$$\mathcal{R}_p = p \times J^1(\mathbb{R}, \mathbb{R}^q) \cap \mathcal{R}.$$

*Let $f : I^l \times I \to \mathbb{R}^q$ be a continuous function which is $C^1$ in the interior of $I^l \times I$. Let $f_p$ denote the restriction of $f$ to $p \times I$ and suppose that for each $p$, the pair $(f_p, \mathcal{R}_p)$ satisfies the hypothesis of Proposition 2.4. Then $f$ can be homotoped to a piecewise $C^1$-map $f_1$ in any $C^0$-neighbourhood of $f$ such that*

1. *$(f_1)_p$ is a piecewise $C^1$-solution of $\mathcal{R}_p$;*
2. *$f_1 = f$ on $I^l \times \{0, 1\}$;*
3. *the first order derivatives of $f_1(p, t)$ with respect to $p$ are arbitrarily $C^0$ close to the respective derivatives of $f(p, t)$.*

*Further, if $f_p$ is a genuine solution of $\mathcal{R}_p$ for $p \in \mathcal{O}p\, \partial I^l$ then the homotopy can be kept constant for $p \in \mathcal{O}p\, \partial I^l$. (The notation $\mathcal{O}p\, \partial I^l$ is used to denote a nonspecified open neighbourhood of $\partial I^l$ which may become smaller in the course of the argument.)*

We shall now state the main result on convex integration which yields piecewise $C^1$-solutions to certain open relations. Before stating it we need to recall the basic language of $\perp$-jets.

Let $\tau$ be an integrable hyperplane field on $\mathbb{R}^n$. With respect to this $\tau$ we define an equivalence relation $\sim$ on $J^1(\mathbb{R}^n, \mathbb{R}^q)$ as follows: If $(x, y, \alpha)$, $(x, y, \beta)$ lie in the same fibre over $(x, y) \in J^0(\mathbb{R}^n, \mathbb{R}^q)$, then

$$\alpha \sim \beta \quad \text{if and only if} \quad \alpha|_\tau = \beta|_\tau.$$

The equivalence class of $(x, y, \alpha)$, denoted as $P_\alpha$, is an affine subspace of dimension $q$ in the jet space. Indeed, if we fix a vector field $\mathbf{v}$ on $\mathbb{R}^n$ transversal to $\tau$, then $(x, y, \beta) \in P_\alpha$ is completely determined by $\beta(\mathbf{v}) \in \mathbb{R}^q$. Thus relative to $\tau$ we can slice the 1-jet space into $q$-dimensional affine subspaces $P_\alpha$. $P_\alpha$ is called the *principal subspace* through $(x, y, \alpha)$ corresponding to $\tau$. The set of equivalence classes is denoted by $J^{\perp}(\mathbb{R}^n, \mathbb{R}^q)$ and there is a canonical projection $p : J^1(\mathbb{R}^n, \mathbb{R}^q) \to J^{\perp}(\mathbb{R}^n, \mathbb{R}^q)$ which takes a 1-jet onto its equivalence class. Let $j_f^{\perp} = p \circ j_f^1$.

Identifying $j_f^1(x)$ with $(j_f^{\perp}(x), df_x(\mathbf{v}))$ we can write

$$J^1(\mathbb{R}^n, \mathbb{R}^q) = J^{\perp}(\mathbb{R}^n, \mathbb{R}^q) \times \mathbb{R}^q.$$

Note that when $n = 1$, $J^{\perp}(\mathbb{R}^1, \mathbb{R}^q) = J^0(\mathbb{R}^1, \mathbb{R}^q) = \mathbb{R} \times \mathbb{R}^q$.

**Theorem 2.6.** *Let $\mathcal{R}$ be an open subset of $J^1(\mathbb{R}^n, \mathbb{R}^q)$. Let $f_0 : I^n \to \mathbb{R}^q$ be a piecewise $C^1$-function such that $j_{f_0}^1(x)$ lies in the convex hull of $\mathcal{R}_{b(x)}$ whenever the derivative exists, where $b(x) = j_{f_0}^{\perp}(x)$. Then there exists a piecewise $C^1$-solution of $\mathcal{R}$, $f_1 : I^n \to \mathbb{R}^q$, which is homotopic to $f_0$. Moreover, the homotopy can be made to lie in an arbitrary $C^0$-neighbourhood of $f_0$.*

*Further, if $f_0$ is a piecewise $C^1$-solution of $\mathcal{R}$ on some open neighbourhood of a compact set $K \subset I^n$, then the homotopy remains constant on some (possibly smaller) neighbourhood of $K$.*

For the sake of completeness we include the proof from [Eliashberg and Mishachev 2002].

*Proof.* Consider the splitting of the cube $I^n$ as $I^{n-1} \times I$. Form a relation

$$\mathcal{R}^1 \subset I^{n-1} \times J^1(\mathbb{R}, \mathbb{R}^q)$$

fibred over $I^{n-1}$ as follows:

For each $x \in I^n$ let $P\big(j_f^{\perp}(x)\big)$ denote the principal subspace through $j_f^1(x)$ corresponding to the splitting $I^{n-1} \times I$. Let $\Omega(f(p,t))$ be the subset defined by

$$\big\{j_f^{\perp}(p,t)\big\} \times \Omega(f(p,t)) = P\big(j_f^{\perp}(p,t)\big) \cap \mathcal{R}.$$

By the given hypothesis, $\partial_t f(p,t)$ belongs to the convex hull of $\Omega(f(p,t))$ in $P\big(j_f^{\perp}(x)\big)$. Since $\mathcal{R}$ is open there is an open neighbourhood $D_\varepsilon^q(f(p,t))$ of $f(p,t)$ in $\mathbb{R}^q$ and an open subset $\Omega'(f(p,t))$ contained in $\Omega(f(p,t))$ such that

(1) $\Omega'(f(p,t))$ contains $\partial_t f(p,t)$ in its convex hull and

(2) $\{(p,t)\} \times D_\varepsilon^q(f(p,t)) \times \{\partial_p f(p,t)\} \times \Omega'(f(p,t)) \subset \mathcal{R}$ for all $(p,t) \in I^{n-1} \times I$.

In the above, $\partial_t$ and $\partial_p$ respectively denote the derivatives of the function with respect to the coordinates $t$ and $p$.

For each $p \in I^{n-1}$ define a relation $\mathcal{R}_p^1 \subset J^1(\mathbb{R}, \mathbb{R}^q)$ as

$$\mathcal{R}_p^1 = \big\{(t, y, v) \in I \times \mathbb{R}^q \times \mathbb{R}^q : y \in D_\varepsilon^q(f(p,t)),\, v \in \Omega'(f(p,t))\big\}.$$

Then $\mathcal{R}^1 = \bigcup_p \{p\} \times \mathcal{R}_p^1$ is a fibred relation in $I^{n-1} \times J^1(\mathbb{R}, \mathbb{R}^q)$ which is defined over an open neighbourhood of the graph of the section $f$ in $I^n \times \mathbb{R}^q$.

Further for an appropriate choice of $\Omega'(f(p,t))$ we may assume that $\mathcal{R}^1$ is an open fibred relation in $I^{n-1} \times J^1(\mathbb{R}, \mathbb{R}^q)$.

Also note that for a fixed $p \in I^{n-1}$, $t \mapsto f(p,t)$ is a short solution of $\mathcal{R}^1_p$.

We now apply the parametric one-dimensional convex integration to obtain a piecewise $C^1$-homotopy $f_\tau$ of fibrewise "short" (see [Eliashberg and Mishachev 2002] for the definition) solutions of $\mathcal{R}^1$ which is $C^0$ close to $f$ and satisfies

$$f_\tau(p,0) = f(p,0) \quad \text{and} \quad f_\tau(p,1) = f(p,1)$$

for all $p \in I^{n-1}$. Furthermore, the first order derivatives of $f_1(p,t)$ with respect to the parameter $p$ (wherever exist) are arbitrarily $C^0$ close to the respective derivatives of $f(p,t)$. Hence,

$$\big(f_1(p,t), \partial_p f(p,t), \partial_t f_1(p,t)\big) \in \mathcal{R}.$$

Since $\mathcal{R}$ is open and since the derivatives of $f_1$ with respect to $p$ are arbitrarily close to the respective derivatives of $f$ it follows that

$$\big(f_1(p,t), \partial_p f_1(p,t), \partial_t f_1(p,t)\big) \in \mathcal{R}.$$

Thus $f_1$ is a solution of $\mathcal{R}$ with the desired properties. □

**Remark 2.7.** We refer the reader to [Gromov 1986, p. 218] for a general result on the existence of (almost everywhere differentiable) Lipschitz solutions to some differential relations.

## 3. $(h_1, h_2)$ regularity and underlying geometry

Throughout this section $h_1$ and $h_2$ will denote two positive definite symmetric bilinear forms on $\mathbb{R}^q$. For any subspace $V$ of $\mathbb{R}^q$, we shall denote its orthogonal complement with respect to $h_i$ by $V^{\perp_i}$ for $i = 1, 2$.

**Definition 3.1.** A subspace $V$ of $\mathbb{R}^q$ is said to be $(h_1, h_2)$-*regular* if $V^{\perp_1}$ is transversal to $V^{\perp_2}$.

Observe that if $A : \mathbb{R}^q \longrightarrow \mathbb{R}^q$ is the (unique) linear transformation defined by $h_2(v, w) = h_1(Av, w)$ for all $v, w \in \mathbb{R}^q$, then a subspace $V$ in $\mathbb{R}^q$ is regular if and only if $V + A(V)$ has the maximum dimension.

**Definition 3.2.** A vector $v \in \mathbb{R}^q$ is said to be $(h_1, h_2)$-*regular* provided the one-dimensional subspace $\langle v \rangle$ spanned by $v$ is a $(h_1, h_2)$-regular subspace of $\mathbb{R}^q$.

**Observation 1.** A vector $v$ is $(h_1, h_2)$-regular if and only if $v$ and $Av$ are linearly independent, $A : \mathbb{R}^q \longrightarrow \mathbb{R}^q$ being the unique linear map defined above. Consequently, the set of nonregular vectors precisely consists of the eigen-vectors of $A$.

The following observation brings out the underlying geometry of the $(h_1, h_2)$-regular vectors.

**Observation 2.** Let $(\mathbb{R}^q, h_1, h_2)$ be as in the above. We shall denote the norms of a vector $w \in \mathbb{R}^q$ relative to $h_1$ and $h_2$ by $\|w\|_1$ and $\|w\|_2$ respectively. Let

$$S_r = \{w \in \mathbb{R}^q \mid \|w\|_1 = r\} \quad \text{and} \quad E_r = \{w \in \mathbb{R}^q \mid \|w\|_2 = r\}$$

denote the spheres of radius $r$ in $\mathbb{R}^q$ relative to the two metrics. Observe that, a vector $v \in S_r \cap E_{r'}$ is $(h_1, h_2)$-regular if and only if $S_r$ and $E_{r'}$ intersect transversally at $v$. Indeed, $v$ is a regular vector if and only if $v^{\perp_1}$ is transversal to $v^{\perp_2}$. If $v \in S_r \cap E_{r'}$, then $v^{\perp_1}$ is tangent to $S_r$ at $v$ and $v^{\perp_2}$ is tangent to $E_{r'}$ at $v$. Therefore it follows that $S_r$ is transversal to $E_{r'}$ at $v$.

**Observation 3.** Let $V$ be a $(h_1, h_2)$-regular subspace of $\mathbb{R}^q$ of dimension $(n-1)$ and let

$$X = V^{\perp_1} \cap V^{\perp_2} = (V \oplus A(V))^{\perp_1}.$$

Then $X$ has codimension $2(n-1)$ in $\mathbb{R}^q$. For any vector $w \in \mathbb{R}^q$, $\tau \oplus \langle w\rangle$ is an $(h_1, h_2)$-regular subspace if and only if $w^{\perp_1} \cap X$ is transversal to $w^{\perp_2} \cap X$ in $X$. Indeed, $V \oplus \langle w\rangle$ is a $(h_1, h_2)$-regular subspace if and only if $(V \oplus \langle w\rangle)^{\perp_1}$ is transversal to $(V \oplus \langle w\rangle)^{\perp_2}$, that is, if and only if

$$\operatorname{codim}\big((V \oplus \langle w\rangle)^{\perp_1} \cap (V \oplus \langle w\rangle)^{\perp_2}\big) = 2n.$$

This is equivalent to saying $X \cap w^{\perp_1} \cap w^{\perp_1}$ has codimension 2 in $X$. Thus $w^{\perp_1} \cap X$ is transversal to $w^{\perp_2} \cap X$.

Let $T$ be a translate of $X$ through $w$. Suppose that $r = \|w\|_1$ and $r' = \|w\|_2$. Since $w^{\perp_1} \cap X$ is the tangent space of $S_r \cap T$ at $w$ and $w^{\perp_2} \cap X$ is the tangent space of $E_{r'} \cap T$ at $w$, it follows from the above that the sets $S_r \cap T$ and $E_r' \cap T$ intersect transversally in $T$ at $w$.

In particular, we can show that if $w$ is in $X$, then $V \oplus \langle w\rangle$ is $(h_1, h_2)$-regular if and only if $w$ is $(\bar{h}_1, \bar{h}_2)$-regular, where $\bar{h}_1$ and $\bar{h}_2$ denote the restrictions of $h_1$ and $h_2$ respectively to $X$.

Let $\bar{A}$ denote the unique linear transformation $X \to X$ such that

$$\bar{h}_2(v, w) = \bar{h}_1(\bar{A}v, w) \quad \text{for} \quad v, w \in V.$$

If $w \in X$ is $(\bar{h}_1, \bar{h}_2)$-regular then $w$ and $\bar{A}(w)$ are linearly independent. Let $A(w) = x + x^{\perp}$, where $x \in X$ and $x^{\perp} \in X^{\perp_1}$. Then

$$h_2(w, v) = h_1(Aw, v) = h_1(x + x^{\perp}, v) = h_1(x, v)$$

for all $v \in X$. Hence $x = \bar{A}(w)$. This proves that $Aw = \bar{A}w + x^{\perp}$. Since $w$, $\bar{A}w$ are linearly independent in $X$ and $x^{\perp} \notin X$ it follows that $w$ and $Aw$ are linearly independent and consequently, $V \oplus \langle w\rangle$ is $(h_1, h_2)$-regular.

**Definition 3.3.** Let $N$ be a smooth manifold with two Riemannian metrics $h_1$ and $h_2$. A smooth map $f : M \to N$ will be called $(h_1, h_2)$-*regular* if for each $x \in M$, $df_x(T_x M)$ is a $(h_1, h_2)$-regular subspace of $T_{f(x)}N$.

**Proposition 3.4** [D'Ambra and Datta 2002]. *Let $h_1$, $h_2$ be two positive definite symmetric bilinear forms on $\mathbb{R}^q$ such that the eigen-values of $A$ (as defined above) are all distinct. Then a generic map $f : M \longrightarrow \mathbb{R}^q$ is $(h_1, h_2)$-regular if $q$ exceeds* $3 \dim M - 1$.

## 4. The Main Lemma

Let $M$ be a smooth manifold of dimension $n$. Let $\mathbb{R}^q$ be the $q$-dimensional Euclidean space. In what follows $h_1$ and $h_2$ will denote two Euclidean metrics on $\mathbb{R}^q$ which satisfy the following conditions:

There exist two numbers $0 < a < b$, such that

(1) $c^2 h_1 - h_2$ is a nondegenerate indefinite form for each real number $c$ lying in $[a, b]$;

(2) $r_+(a^2 h_1 - h_2) \geq 2n$ and $r_-(b^2 h_1 - h_2) \geq 2n$, where $r_+$ and $r_-$ denote respectively the positive and the negative ranks of an indefinite metric; and

(3) if $A : \mathbb{R}^q \to \mathbb{R}^q$ is the unique linear isomorphism given by $h_2(v, w) = h_1(Av, w)$ for all $v, w \in \mathbb{R}^q$, then $A$ has distinct eigenvalues.

**Lemma 4.1.** *Let $g_1$ and $g_2$ be two Riemannian metrics on $M$ which are related by $a^2 g_1 < g_2 < b^2 g_2$. Let $f : M \to \mathbb{R}^q$ be an $(h_1, h_2)$-regular immersion such that*

$$g_1 - f^* h_1 = \phi^2 d\psi^2 \quad \text{and} \quad g_2 - f^* h_2 = c^2 \phi^2 d\psi^2,$$

*where $\phi$, $\psi$ are smooth functions on $M$, $\phi$ has compact support contained in an open set $U$ of $M$ and $a < c < b$.*

*Then there exists a piecewise $C^1$-map $\bar{f}$ which is a fine $C^0$-approximation of $f$ and has the following properties*:

(1) *$\bar{f}$ coincides with $f$ outside $U$;*

(2) *$\bar{f}^* h_i$ is arbitrarily close to $g_i$ ($\bar{f}^* h_i \approx g_i$) for $i = 1, 2$ relative to the fine $C^0$-topology on each component where $\bar{f}$ is $C^1$.*

*Proof.* Let $\mathcal{I}$ denote the subset of $J^1(M, \mathbb{R}^q)$ consisting of all 1-jets $(x, y, \alpha)$ such that $\alpha^* h_1 = g_1$ and $\alpha^* h_2 = g_2$. Let $\tau$ be the hyperplane field over $U$ defined by $\ker d\psi$. Then $\tau$ is integrable and its integral submanifolds are precisely the level sets of the function $\psi$.

Consider the bundle

$$p^1_\perp : J^{(1)}(U, \mathbb{R}^q) \to J^\perp(U, \mathbb{R}^q)$$

relative to the hyperplane distribution $\tau$ on $U$. An element $b$ of $J^{\perp}(M, \mathbb{R}^q)$ is of the form $b=(x, y, \beta)$, where $x \in U$, $y \in \mathbb{R}^q$ and $\beta : \tau_x \to \mathbb{R}^q$ is a linear map. The fibre over $b$ consists of all linear maps $\alpha : T_x M \to \mathbb{R}^q$ which restricts to $\beta$ on $\tau_x$.

To describe the intersection of the relation $\mathcal{I}$ with the principal subspaces of the fibration $p^1_{\perp}$, we choose a vector field $\mathbf{v}_0$ on $TU$ such that

$$\|\mathbf{v}_0\|_1 = \sqrt{g_1(\mathbf{v}_0, \mathbf{v}_0)} = 1 \quad \text{and} \quad g_1(\mathbf{v}_0, \tau) = 0$$

on $U \supset \operatorname{supp} \phi$. Let $\|\mathbf{v}_0\|_2 = \sqrt{g_2(\mathbf{v}_0, \mathbf{v}_0)} = r$; then $r$ is a smooth function on $U$ satisfying the inequality $a < r(x) < b$ for all $x \in U$. Let $p' : \mathcal{I} \to J^{\perp}(M, \mathbb{R}^q)$ denote the restriction of $p^1_{\perp}$ to $\mathcal{I}$. Recall that a 1-jet $(x, y, \alpha)$ in a principal subspace $J^1_b(U, \mathbb{R}^q)$ is completely determined by its value at $\mathbf{v}_0$. Moreover, if

$$(x, y, \alpha) \in \mathcal{I}_b = J^{(1)}_b(U, \mathbb{R}^q) \cap \mathcal{I},$$

then $\alpha(\mathbf{v}_0)$ is contained in the unique affine space

$$T_b = \left\{ w \in \mathbb{R}^q \;\middle|\; h_1(w, \beta(\tau)) = 0 \text{ and } h_2(w, \beta(v)) = g_2(\mathbf{v}_0, v) \text{ for all } v \in \tau \right\},$$

where $b = (x, y, \beta) \in J^{\perp}(U, \mathbb{R}^q)$. Note that the equation $h_2(w, \beta(v)) = g_2(v_0, v)$ defines an affine subspace of $\mathbb{R}^q$ which is a translate of $\beta(\tau)^{\perp_2}$. If $\alpha$ is $(h_1, h_2)$-regular then, in particular, $\beta(\tau_x)^{\perp_1}$ is transversal to $\beta(\tau_x)^{\perp_2}$ and the same is true for any translates of these spaces. Thus $T_b$ is an affine plane of codimension $2(n-1)$. Moreover, this is the translate of the vector subspace $X_b = \beta(\tau_x)^{\perp_1} \cap \beta(\tau_x)^{\perp_2}$ in $\mathbb{R}^q$.

Thus $J^{(1)}(U, \mathbb{R}^q) \cap \mathcal{I}$ is contained in an affine subbundle of codimension $2(n-1)$ (over some open subset of $J^{\perp}(U, \mathbb{R}^q)$). Further, it follows that if

$$\alpha \in J^{(1)}_b(U, \mathbb{R}^q) \cap \mathcal{I}$$

then $\|\alpha(v_0)\|_1 = 1$ and $\|\alpha(v_0)\|_2 = r$. Therefore we can characterize $J^{(1)}_b(U, \mathbb{R}^q) \cap \mathcal{I}$ as

$$J^{(1)}_b(U, \mathbb{R}^q) \cap \mathcal{I} = \{w \in T_b : \|w\|_1 = 1, \|w\|_2 = r\}.$$

We shall now show that the pair $(f, \mathcal{I})$ satisfies the conditions stated in the hypothesis of Theorem 2.6 except that $\mathcal{I}$ is *not* an open relation.

**Notation.** We fix the following notations for the subsequent discussion:

$$S = \{w \in \mathbb{R}^q : \|w\|_1 = 1\}, \qquad E = \{w \in \mathbb{R}^q : \|w\|_2 = r\}.$$

**Sublemma 4.2.** *$j^1_f(x)$ lies in the convex hull of $\mathcal{I}_{b(x)}$ if $r_{\pm}(c^2 h_1 - h_2) \geq 2n$. In other words, $df_x(\mathbf{v}_0)$ lies in the convex hull of the set*

$$\{w \in T_{b(x)} : \|w\|_1 = 1, \|w\|_2 = r\},$$

*where $b(x) = j^{\perp}_f(x)$.*

*Proof of Sublemma 4.2.* Observe that

(1) $df_x(\mathbf{v}_0)$ lies in $T_{b(x)}$, and

(2) $df_x(\mathbf{v}_0)$ satisfies the equation

$$c^2(1 - \|w\|_1^2) = r^2 - \|w\|_2^2$$

since $g_2 - f^*h_2 = c^2(g_1 - f^*h_1)$.

The above equation can be equivalently expressed as $(c^2h_1 - h_2)(w, w) = c^2 - r^2$. This represents a generalised hyperboloid $H$ since $r_\pm(c^2h_1 - h_2) \geq 2n$. It may be seen easily that $H \cap S = E \cap S = H \cap E$.

Since $r_\pm(c^2h_1 - h_2) \geq 2n$, $H$ is generated by affine subspaces of dimension $2n - 1$. To see this, let $h$ be a nondegenerate symmetric bilinear form on $\mathbb{R}^q$ of signature $(q_+, q_-)$. Let $v \in H$ be such that $h(v, v) = d \neq 0$ and let $V$ denote the $h$-orthogonal complement of the subspace generated by $v$. Then $V$ has dimension $n - 1$ and

$$r_+(h|_V) \geq q_+ - 1, \quad r_-(h|_V) \geq q_- - 1.$$

Consequently, $V$ admits a regular $h$-isotropic subspace $I$ of dimension

$$\min(q_+ - 1, q_- - 1).$$

Here regularity means that $I$ does not intersect the kernel of $h|_V$. Consider the affine subspace $W = I + v$. It is easy to see that $h(w, w) = d$ for every $w \in W$. This proves the above assertion.

Let $A_x$ be an affine subspace in $H$ which passes through $df_x(\mathbf{v}_0)$. Since

$$\operatorname{codim} T_x = 2(n - 1) < 2n - 1 = \dim A_x,$$

the intersection $T_x \cap A_x$ is an affine subspace of dimension at least 1. Since $df_x(\mathbf{v}_0) \in T_x \cap A_x$ and $\|df_x(\mathbf{v}_0)\|_1 < 1$, $T_x \cap A_x \cap S$ contains at least two points and $df_x(\mathbf{v}_0)$ lies in the convex hull of this intersection. Noting that

$$T_x \cap A_x \cap S \subset T_x \cap E \cap S,$$

we conclude that $df_x(\mathbf{v}_0)$ lies in the convex hull of $T_x \cap E \cap S$. This completes the proof of Sublemma 4.2. □

Now we conclude the proof of the Main Lemma (Lemma 4.1). Since $\mathcal{I}$ is not an open relation we cannot directly apply Theorem 2.6 to the pair $(f, \mathcal{I})$. We take an arbitrary small open neighbourhood $\tilde{\mathcal{I}}$ of $\mathcal{I}$ and apply Theorem 2.6 to the pair $(f, \tilde{\mathcal{I}})$. Thus we obtain a fine $C^0$-approximation of $f$ by a piecewise $C^1$-solution $\bar{f}$ of $\tilde{\mathcal{I}}$. Choosing $\tilde{\mathcal{I}}$ sufficiently small, we can make $\bar{f}^*h_1$ and $\bar{f}^*h_2$ arbitrarily $C^0$ close to the pair $(g_1, g_2)$ as desired. This completes the proof. □

## 5. Approximate solution

We recall the definition of short maps from [D'Ambra and Datta 2002].

**Definition 5.1.** Let $M$ be a manifold with two Riemannian metrics $g_1$ and $g_2$. A $C^1$-map $f_0 : M \longrightarrow (\mathbb{R}^q, h_1, h_2)$ is $(g_1, g_2)$-*short* if the metrics $g_1 - f_0^*(h_1)$ and $g_2 - f_0^*(h_2)$ on $M$ are positive definite. This will be expressed by $g_i - f_0^*(h_i) > 0$ or $g_i > f_0^*(h_i)$, for $i = 1, 2$.

**Proposition 5.2.** *Let $M$ be a $C^\infty$-manifold with two Riemannian metrics $g_1$ and $g_2$ which are related by $a^2 g_1 < g_2 < b^2 g_1$. Then there exists a $(g_1, g_2)$-short $C^\infty$-immersion $f_0 : M \longrightarrow (\mathbb{R}^q, h_1, h_2)$ which also satisfies the inequalities*

$$a^2(g_1 - f_0^* h_1) < (g_2 - f_0^* h_2) < b^2(g_1 - f_0^* h_1),$$
$$a^2 f_0^* h_1 < f_0^* h_2 < b^2 f_0^* h_1. \tag{5-1}$$

*Proof.* For any number $c$ with $a < c < b$, consider the nondegenerate form $\bar{h} = c^2 h_1 - h_2$. By the hypothesis of Theorem 1.1, $r_+(\bar{h}) \geq 2n$ and $r_-(\bar{h}) \geq 2n$. Therefore, there exists a $C^1$-immersion $f : M \to \mathbb{R}^q$ such that $f^*(\bar{h}) = 0$. (This follows from an exercise in [Gromov 1986, 2.4.9, Corollary (2′)]). Such an $f$ clearly satisfies the relation $a^2 f^* h_1 < f^* h_2 < b^2 f^* h_1$. Moreover, without any loss of generality we may assume that the map $f$ satisfying the above inequality is smooth, because if that is not the case we replace $f$ by a $C^\infty$-immersion which is sufficiently $C^1$ close to $f$.

Now, if $M$ is a closed manifold, then starting with an $f$ as above we can obtain the required $f_0$ by scaling the map $f$ with a suitable scalar (see the corresponding result in [D'Ambra and Datta 2002]). To obtain such an $f_0$ in the case of open manifolds we have to employ the partition of unity techniques. □

Let $\mathcal{F}$ denote the set of all piecewise $C^1$-maps $f : M \to \mathbb{R}^q$ which satisfy the following conditions at each point $x \in M$ where $f$ is differentiable:

F1. $f$ is $(h_1, h_2)$-regular;

F2. $f$ is $(g_1, g_2)$-short;

F3. $a^2(g_1 - f^* h_1) < g_2 - f^* h_2 < b^2(g_1 - f^* h_1)$;

F4. $a^2 f^* h_1 < f^* h_2 < b^2 f^* h_1$.

**Proposition 5.3.** *Let $f_0 : M \to \mathbb{R}^q$ belong to $\mathcal{F}$ and let $0 < \varepsilon < 1$ be any positive number. Then there exists a piecewise $C^1$-map $f_1 \in \mathcal{F}$ such that the following conditions are satisfied*:

(1) $\varepsilon g_1 < f_1^* h_1 < g_1$ *on the set of points where $f$ is differentiable*;

(2) $f_1$ *is arbitrarily close to $f_0$ in the fine $C^0$-topology.*

**Remark 5.4.** Condition (1) in the above proposition implies that $f_1$ is strictly $g_1$-short and the induced metric $f_1^*h_1$ is sufficiently close to $g_1$ when $\varepsilon$ is close to 1.

*Proof.* Fix a locally finite open covering $\{U_i\}$ of $M$ by coordinate neighbourhoods. Since the metrics $g_1 - f^*h_1$ and $g_2 - f^*h_2$ are related by the inequalities (5-1) we can get simultaneous decomposition of $g_1 - f^*h_1$ and $g_2 - f^*h_2$ as

$$\varepsilon(g_1 - f^*h_1) = \sum_i \phi_i^2 d\psi_i^2 \quad \text{and} \quad \varepsilon(g_2 - f^*h_2) = \sum_i c_i^2 \phi_i^2 d\psi_i^2$$

where $c_i$'s are constants which lie between $a$ and $b$, and $\phi_i$'s and $\psi_i$'s are smooth real valued functions. Further, for each $i$, the function $\phi_i$ has compact support contained in $U_i$ [D'Ambra and Datta 2002, Decomposition Lemma]. Let us define two sequences of Riemannian metrics $\{g_1^i\}$ and $\{g_2^i\}$ as

$$g_1^i = g_1^{i-1} + \phi_i^2 d\psi_i^2 \quad \text{and} \quad g_2^i = g_2^{i-1} + c_i^2 \phi_i^2 d\psi_i^2,$$

where $g_1^0 = f^*h_1$ and $g_2^0 = f^*h_2$. Clearly, $g_1^i < g_1$ and $g_2^i < g_2$ for each $i$. Further, since $a^2 f^*h_1 < f^*h_2 < b^2 f^*h_1$ and $a < c_i < b$ for each $i$, $a^2 g_1^i < g_2^i < b^2 g_1^i$ for each $i$.

By applying the Main Lemma (Lemma 4.1) successively (with an appropriate choice of $\tilde{\mathcal{I}}$ for each $i$) we obtain a sequence of piecewise $C^1$-maps such that

$$\bar{f}_i^* h_\alpha \approx g_\alpha^i,$$

for $\alpha = 1, 2$, $i = 1, 2, \ldots$ and $\bar{f}_i$ lies in a given neighbourhood of $f$ in the fine $C^0$-topology. Note that each $\bar{f}_i$ satisfies conditions F2 and F4. Since supp $\phi_i \subset U_i$ for each $i$, where $\{U_i\}$ is a locally finite open covering of $M$, the sequence $\bar{f}_i$ is eventually constant near any point $x \in M$. Therefore the sequence converges to a piecewise $C^1$-map on $V$. Let

$$f_1 = \lim_{i \to \infty} \bar{f}_i.$$

If ${\bar{f}_i}^* h_\alpha$ are sufficiently close to $g_\alpha^i$ for $\alpha = 1, 2$ and for all $i$, then $f_1$ can be made to satisfy F2, F3 and F4. Further,

$$g_1 - f_1^*h_1 \approx g_1 - (f^*h_1 + \varepsilon(g_1 - f^*h_1)) = (1-\varepsilon)(g_1 - f^*h_1) < (1-\varepsilon)g_1.$$

Hence $f_1$ satisfies $\varepsilon g_1 < f_1^*h_1 < g_1$. □

## 6. Proof of the Main Theorem

We begin this section with some preliminaries on Lipschitz maps.

**Definition 6.1.** Let $(X, d)$ and $(Y, d')$ be two metric spaces and let $f : X \to Y$ be a continuous map. The map $f$ is said to be *Lipschitz* if there is a constant $K > 0$

such that $d'(f(x), f(x')) < Kd(x, x')$ for all $x, x' \in X$. $K$ is called the *Lipscitz constant* for $f$.

A Riemannian metric $g$ on a $C^\infty$-manifold $M$ induces a canonical metric space structure on $M$. If we denote this metric by $d_g$, then the *distance* $d_g(x, x')$ between two points $x, x' \in M$ is defined to be the infimum of the lengths of all piecewise $C^1$-paths in $M$ joining $x$ and $x'$.

**Definition 6.2.** A continuous map $f : (M, g) \to (N, h)$ from a Riemannian manifold $(M, g)$ into another Riemannian manifold $(N, h)$ will be called *Lipschitz* if it is a Lipschitz map relative to the metrics $d_g$ and $d_h$ on $M$ and $N$ respectively.

**Example 6.3.** A $C^1$-isometric map $f : (M, g) \to (N, h)$ between Riemannian manifolds is a Lipschitz map with a Lipschitz constant equal to 1. Hence, every $g$-short map is also a Lipschitz map.

A Riemannian metric $g$ on a manifold $M$ induces a canonical volume measure which we denote by $\mu_g$. Measurability on $(M, g)$ is therefore to be understood in terms of this $\mu_g$. Observe that if $g'$ is another Riemannian metric on $M$ then a set $A$ in $M$ has measure zero relative to $\mu_g$ if and only if it has measure zero relative to $\mu_{g'}$.

We recall the following facts about Lipschitz maps between Riemannian manifolds from [Weaver 1999].

- Every Lipschitz map between Riemannian manifolds is almost everywhere differentiable, since a Lipschitz map $f : \Omega \to \mathbb{R}^q$ defined on some open subset of $\mathbb{R}^n$ is almost everywhere differentiable.
- The Lipschitz functions on a Riemannian manifold are precisely those which have bounded measurable exterior derivative $df$.

**Definition 6.4.** A Lipschitz map $f : (M, g) \to (N, h)$ from a Riemannian manifold $(M, g)$ into another Riemannian manifold $(N, h)$ will be called *Lipschitz isometric* if $df_x : T_x M \to T_{f(x)} N$ is isometric for almost all $x \in M$.

- If $g_1$ and $g_2$ are two Riemannian metrics on a manifold $M$ satisfying $a^2 g_1 < g_2 < b^2 g_1$ then a map $f : M \to \mathbb{R}^q$ is Lipschitz with respect to the pair $(g_1, h_1)$ if and only if it is Lipschitz with respect to the pair $(g_2, h_2)$, where $h_1, h_2$ are two linear metrics on $\mathbb{R}^q$. Therefore, there is no ambiguity when we speak of almost everywhere differentiable Lipschitz maps in the context of Theorem 1.1.

*Proof of Theorem 1.1.* Since $(h_1, h_2)$-regular immersions are generic for $q \geq 3 \dim M$, it follows from Proposition 5.2 that there is a $(h_1, h_2)$-regular immersion $f_0 : M \to \mathbb{R}^q$ which satisfies the inequalities in (5-1).

Let $\mathcal{R}$ denote the set of all 1-jets $(x, y, \alpha)$ which satisfy the following properties:

(1) $\alpha$ is short relative to both $(g_1, h_1)$ and $(g_2, h_2)$;

(2) $a^2(g_1 - \alpha^* h_1) < g_2 - \alpha^* h_2 < b^2(g_1 - \alpha^* h_1)$;

(3) $a^2\alpha^* h_1 < \alpha^* h_2 < b^2 \alpha^* h_1$.

For every $\eta > 0$ define relations $\mathcal{R}_\eta$ by

$$\mathcal{R}_\eta = \mathcal{R} \cap \{(x, y, \alpha) : (1-\eta) g_1 < \alpha^* h_1 < g_1\}.$$

Let $\mathcal{I}$ denote the isometry relation

$$\mathcal{I} = \{(x, y, \alpha) \in J^1(M, \mathbb{R}^q) : \alpha^* h_1 = g_1, \alpha^* h_2 = g_2\},$$

then:

- Each $\mathcal{R}_\eta$ is an open relation.
- The fibres of $\mathcal{I}$ over $J^0(M, \mathbb{R}^q)$ are compact sets. Hence, the relations $\mathcal{R}_\eta$ are uniformly bounded over compact sets in $M$.
- Let $\eta_i$ be a sequence of positive numbers such that $\eta_i \to 0$. If $\alpha_i \in \mathcal{R}_{\eta_i}$ and $\alpha_i \to \alpha$, then $\alpha \in \mathcal{I}$. (Compare with [Gromov 1986, p. 218].)

Let $\eta_i$ be a sequence of constants converging to zero and $\delta_i$ be a sequence of positive continuous functions on $M$ such that the series $\sum_i \delta_i$ converges pointwise on $M$. By applying Proposition 5.3 we obtain a sequence of piecewise $C^1$-maps $f_i : M \to \mathbb{R}^q$ for $i = 1, 2, \ldots$ such that $f_i$ is a piecewise $C^1$-solution of the relation $\mathcal{R}_{\eta_i}$ and the distance between $f_i(x)$ and $f_{i+1}(x)$ is less than $\delta_i(x)$ for all $x \in M$. Thus the sequence $\{f_i\}$ converges (in the $C^0$ compact open topology) to a continuous function $f$ on $M$. Since $f_i$ is a piecewise $C^1$-solution of the relation $\mathcal{R}_{\eta_i}$, it is Lipschitz (relative to $(g_1, h_1)$) and the Lipschitz constants of $f_i$ are uniformly bounded. Hence the limit function $f$ is also a Lipschitz map [Weaver 1999]. Consequently, $f$ is almost everywhere differentiable and the $L^\infty$ norm of $df$ is finite on any coordinate neighbourhood of $M$.

We would further like to show that the sequence $df_i$, $i = 1, 2, \ldots$, converges to $df$ in $L^1(\Omega)$ for any compact coordinate neighbourhood $\Omega$. Since $L^1$ convergence of a sequence of functions guarantees the almost everywhere convergence of a subsequence of the original sequence to $df$, this would imply that $f$ is a Lipschitz solution of $\mathcal{I}$ on all of $M$ (by a property of $\mathcal{R}_\eta$ discussed above).

However, to prove the desired $L^1$ convergence we need to choose the functions $\delta_i$ appropriately. First we fix a locally finite open covering of $M$ by coordinate neighbourhoods $\{\Omega_\alpha : \alpha = 1, 2, \ldots\}$. For our convenience we choose each $\Omega_\alpha$ to be compact. Suppose we have already constructed $\delta_i$ and $f_i$ for $i = 1, 2, \ldots, k$. Let $\{\varepsilon_\alpha\}$ be a sequence of positive numbers with $0 < \varepsilon_\alpha < 2^{-\alpha}$ such that

$$\| df_i * \rho_{\varepsilon_\alpha} - df_i \|_{L^1(\Omega_\alpha)} \leq 2^{-\alpha}.$$

The functions $\rho_\varepsilon$ are defined as in [Müller and Šverák 2003] by $\rho_\varepsilon = \varepsilon^{-n}\rho(x/\varepsilon)$, where $\rho : \mathbb{R}^n \to \mathbb{R}$ is the mollifying kernel, that is, a smooth nonnegative function supported in the open unit disc in $\mathbb{R}^n$ with $\int \rho\, dx = 1$.

Observing that there exists a positive continuous function $\varepsilon$ on $M$ which is strictly less than $\varepsilon_\alpha$ on $\Omega_\alpha$ for each $\alpha = 1, 2, \ldots,$ define

$$\delta_{i+1} = \varepsilon\delta_i.$$

Now we apply Proposition 5.3 to obtain a piecewise $C^1$-solution of $\mathcal{R}_{\eta_{i+1}}$ such that $|f_{i+1} - f_i| < \delta_{i+1}$. Proceeding this way we construct a sequence $\{f_i\}$, $i = 1, 2, \ldots,$ which has all the desired property.

Now, arguing exactly as in [Müller and Šverák 2003, Theorem 3.2] we can prove that $df_i$ converges to the derivative map of $f$ in $L^1(\Omega_\alpha)$ for each $\alpha$. This completes the proof of the theorem. □

**Remark 6.5.** The proof of the main theorem begins with an immersion $f_0 : M \to \mathbb{R}^q$ satisfying the inequalities (5-1). If $\mathbb{R}^q$ is replaced by a general manifold $N$ then such maps are no longer guaranteed. This is the main obstruction to generalise the result for arbitrary manifold $N$ in the place of $\mathbb{R}^q$. However, assuming the existence of such maps we may possibly prove the existence of Lipschitz isometric maps for pairs of Riemannian metrics [Gromov 1986, 2.4.9 (A)].

## 7. One-dimensional case

In this section we discuss the one-dimensional case which is the motivation to the general problem.

Let $M = S^1$ be the unit circle and let $g_1 = d\theta^2$ be the canonical metric on $S^1$. Let $g_2 = c^2 g_1$. If $f : S^1 \to \mathbb{R}^q$ is a $C^1$-immersion such that $f^*h_i = g_i$ for $i = 1, 2$ then

$$\left\| \frac{\partial f}{\partial \theta} \right\|_1 = 1 \quad \text{and} \quad \left\| \frac{\partial f}{\partial \theta} \right\|_2 = c,$$

where $\|.\|_i$ denote the norms relative to the metric $h_i$ for $i = 1, 2$. In other words, $\frac{\partial f}{\partial \theta} \in A$, where $A$ is given by

$$A = \left\{ \mathbf{y} = (y_1, \ldots, y_q) \in \mathbb{R}^q : \sum y_i^2 = 1 \text{ and } \sum \lambda_i^2 y_i^2 = c^2 \right\}.$$

**Lemma 7.1.** *Let $h_1$ and $h_2$ be two inner products on $\mathbb{R}^q$ such that $h_1 - h_2$ is nondegenerate. Let $S_1$ and $S_2$ denote the unit spheres relative to the metrics $h_1$ and $h_2$ respectively. Then $S_1 \cap S_2$ has the same homotopy type as $S^{r_+ - 1} \times S^{r_- - 1}$, where $r_+$ and $r_-$ are respectively the positive and the negative ranks of $h_1 - h_2$. Consequently, if $r_\pm \geq 2$ then $S_1 \cap S_2$ is connected. Further the interior of the convex hull of $S_1 \cap S_2$ contains the origin.*

*Proof.* Let $h_1 - h_2$ be nondegenerate. Note that a nonzero vector $v$ satisfies

$$(h_1 - h_2)(v, v) = 0$$

if and only if $\lambda v$ satisfies the same equation for all $\lambda$. This means that the one-dimensional subspace $\ell_v$ containing $v$ lies completely inside the solution space $C$ of $h_1 - h_2 = 0$. In other words, the solution space of this equation in $\mathbb{R}^q$ is a cone. Now, if $h$ is an arbitrary positive definite quadratic form on $\mathbb{R}^q$ then $\ell_v$ intersects the unit sphere relative to $h$ in exactly two points. Thus we see that $S_1 \cap S_2$ has the same homotopy type as the space of nonzero solutions of the equation $h_1 - h_2 = 0$. Choose basis vectors in $\mathbb{R}^q$ so that both $h_1$ and $h_2$ are in the diagonal form. The set $S_1 \cap S_2$ has the same homeomorphism type as the solution space of the system of equations

$$\begin{aligned} x_1^2 + \cdots + x_{r_+}^2 + y_1^2 + \cdots + y_{r_-}^2 &= 1, \\ x_1^2 + \cdots + x_{r_+}^2 - y_1^2 - \cdots - y_{r_-}^2 &= 0, \end{aligned}$$

which is further equivalent to

$$\begin{aligned} x_1^2 + x_2^2 + \cdots + x_{r_+}^2 &= \tfrac{1}{2}, \\ y_1^2 + y_2^2 + \cdots + y_{r_-}^2 &= \tfrac{1}{2}. \end{aligned}$$

Therefore, $S_1 \cap S_2$ has the homeomorphism type of $S^{r_+ - 1} \times S^{r_- - 1}$, which is $k$-connected for $k \leq \min(r_+ - 2, r_- - 2)$. Thus if $r_\pm \geq 2$ then $S_1 \cap S_2$ is connected and nowhere flat. (Note that in the lowest admissible dimension the intersection is topologically equivalent to a torus embedded in $S^3$.) Also note that if $(\bar{x}_1, \dots, \bar{x}_{r_+}, \bar{y}_1, \dots, \bar{y}_{r_-}) \in S_1 \cap S_2$ then $(\pm\bar{x}_1, \dots, \pm\bar{x}_{r_+}, \pm\bar{y}_1, \dots, \pm\bar{y}_{r_-}) \in S_1 \cap S_2$, so that the convex hull of $S_1 \cap S_2$ has nonempty interior and 0 belongs to the interior convex hull of $S_1 \cap S_2$. □

It follows from the above lemma that if $r_\pm(c^2 h_1 - h_2) \geq 2$, then $A$ is connected and the interior of the convex hull of $A$ contains the origin. Thus, by Lemma 2.3 there exists a $C^1$-immersion $f : S^1 \to \mathbb{R}^q$ such that $f^* h_i = g_i$ for $i = 1, 2$ when $r_\pm(c^2 h_1 - h_2) \geq 2$.

On the other hand there does not exist any such isometric immersion if $q \leq 3$ since it is observed in [Gromov 1986, 2.4.1(A) Example] that if $f : S^1 \to \mathbb{R}^q$ is a $C^1$-map whose derivative takes the unit circle $S^1$ into a (connected) subset $A$, then the convex hull of $A$ must contain the origin. Indeed, if $q = 3$ and $h_1 - h_2$ is a nondegenerate indefinite form, then $A$ is a disjoint union of two circles none of which contains the origin in its convex hull, thereby ruling out the existence of $C^1$-immersion with the desired isometry property.

We conclude the paper with a conjecture:

**Conjecture.** *If* $r_{\pm}(c^2h_1 - h_2) \geq 2n+1$ *for all* $c \in [a, b]$, *then it is possible to obtain a* $C^1$*-solution of the general problem.*

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