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We study the distributional behavior for products and sums of Boolean independent random variables in a general infinitesimal triangular array. We show that the limit laws of Boolean convolutions are determined by the limit laws of free convolutions, and vice versa. We further use these results to demonstrate several connections between the limiting distributional behavior of classical convolutions and that of Boolean convolutions. The proof of our results is based on the analytical apparatus developed by Bercovici and Wang for free convolutions.

1. Introduction

Denote by $\mathcal{M}_{\mathbb{R}}$ the collection of all Borel probability measures on the real line $\mathbb{R}$, and by $\mathcal{M}_{\mathbb{T}}$ Borel probability measures on the unit circle $\mathbb{T}$. The classical convolution $*$ for elements in $\mathcal{M}_{\mathbb{R}}$ corresponds to the addition of independent real random variables, and the convolution $\circledast$ for measures in $\mathcal{M}_{\mathbb{T}}$ corresponds to the multiplication of independent circle-valued random variables. A binary operation $\uplus$ on $\mathcal{M}_{\mathbb{R}}$, called *additive Boolean convolution*, was introduced by Speicher and Woroudi [1997]. They also showed that it corresponds to the addition of random variables belonging to algebras that are Boolean independent. Later Franz [2004] introduced the concept of *multiplicative Boolean convolution* $\mathbin{\rotatebox[origin=c]{180}{$\Cap$}}$ for measures in $\mathcal{M}_{\mathbb{T}}$, which is a multiplicative counterpart of the additive Boolean convolution. Voiculescu [1986; 1987] showed there are two other convolutions defined respectively for measures on $\mathbb{R}$ and $\mathbb{T}$. These are additive free convolution $\boxplus$ and multiplicative free convolution $\boxtimes$.

This paper investigates the limiting distributional behavior for Boolean convolutions of measures in an infinitesimal triangular array. Let $\{k_n\}_{n=1}^{\infty}$ be a sequence of natural numbers. A triangular array $\{\mu_{nk} : n \in \mathbb{N},\ 1 \leq k \leq k_n\} \subset \mathcal{M}_{\mathbb{T}}$ is said to be *infinitesimal* if

$$\lim_{n\to\infty} \max_{1\leq k\leq k_n} \mu_{nk}(\{\zeta \in \mathbb{T} : |\zeta - 1| \geq \varepsilon\}) = 0,$$

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for every $\varepsilon > 0$. Given such an array and a sequence $\{\lambda_n\}_{n=1}^{\infty} \subset \mathbb{T}$, define

$$\mu_n = \delta_{\lambda_n} \utimes \mu_{n1} \utimes \mu_{n2} \utimes \cdots \utimes \mu_{nk_n}, \qquad \nu_n = \delta_{\lambda_n} \boxtimes \mu_{n1} \boxtimes \mu_{n2} \boxtimes \cdots \boxtimes \mu_{nk_n},$$

and

$$\sigma_n = \delta_{\lambda_n} \circledast \mu_{n1} \circledast \mu_{n2} \circledast \cdots \circledast \mu_{nk_n} \quad \text{for } n \in \mathbb{N},$$

where δ_{λ_n} is the point mass at λ_n. We first prove in this paper that any weak limit of such a sequence $\{\mu_n\}_{n=1}^{\infty}$ is an infinitely divisible measure. This result may be viewed as the multiplicative Boolean analogue of Hinčin's classical theorem [1937]. Note that the same result has been proved for $\boxplus$ in [Bercovici and Pata 2000], and for $\boxtimes$ in [Belinschi and Bercovici 2008]. Next, we find necessary and sufficient conditions for the weak convergence of μ_n to a given infinitely divisible measure. In particular, our results show that the sequence μ_n converges weakly if and only if the sequence ν_n converges weakly. As an application, we show that the measures σ_n have a weak limit if the measures μ_n (or ν_n) have a weak limit whose first moment is not zero. Moreover, the classical limits and the Boolean limits are related in an explicit manner. We also introduce the notion of Boolean normal distributions on $\mathbb{T}$, and we show that the sequence μ_n converges weakly to such a distribution if and only if the sequence σ_n converges weakly to a normal distribution (which is the push-forward measure of a Gaussian law on $\mathbb{R}$ via the natural homomorphism from $\mathbb{R}$ into $\mathbb{T}$.)

The additive version of our results was studied earlier by Bercovici and Pata in [1999] for arrays with identically distributed rows; see also [Stoica 2005] for a central limit theorem and a weak law of large numbers with weighted components. Thus, consider an infinitesimal array $\{\nu_{nk}\}_{n,k} \subset \mathcal{M}_{\mathbb{R}}$ with $\eta_n = \nu_{n1} = \nu_{n2} = \cdots = \nu_{nk_n}$ for $n \in \mathbb{N}$. The infinitesimality here means that

$$\lim_{n\to\infty} \max_{1\le k\le k_n} \nu_{nk}(\{t \in \mathbb{R} : |t| \ge \varepsilon\}) = 0,$$

for every $\varepsilon > 0$. Set

$$\rho_n = \underbrace{\eta_n * \eta_n * \cdots * \eta_n}_{k_n \text{ times}}, \qquad \tau_n = \underbrace{\eta_n \boxplus \eta_n \boxplus \cdots \boxplus \eta_n}_{k_n \text{ times}}$$

and

$$\omega_n = \underbrace{\eta_n \uplus \eta_n \uplus \cdots \uplus \eta_n}_{k_n \text{ times}} \quad \text{for } n \in \mathbb{N}.$$

The main result in [Bercovici and Pata 1999, Theorem 6.3] is the equivalences of weak convergence among the sequences ρ_n, τ_n and ω_n. The result concerning ρ_n and τ_n was first extended to an arbitrary infinitesimal array by Chistyakov and Götze [2008]; see also [Bercovici and Wang 2008a] for a different argument. In the last part of this paper, we show how to extend the result for τ_n and ω_n to an arbitrary infinitesimal array using the methods in [Bercovici and Wang 2008a].

This paper is organized as follows. In Section 2, we review the analytic tools needed for the calculation of Boolean convolutions. We also describe the analytic characterization of infinite divisibility related to the various convolutions. In Section 3, we prove the limit theorems for arrays on $\mathbb{T}$. Section 4 proves results regarding the classical convolution $\circledast$. In Section 5, we present the analogous results for arrays on $\mathbb{R}$.

2. Preliminaries

The analytic methods needed for the calculation of free convolutions were discovered by Voiculescu [1986; 1987]. Likewise, the additive Boolean convolution formula was found by Speicher and Woroudi [1997], and the basic analysis of the multiplicative Boolean convolution was done by Franz [2004], but see also [Bercovici 2006] for a different approach to the calculation of both Boolean convolutions. The details are as follows.

2.1. ***Multiplicative Boolean and free convolutions on the unit circle.*** Denote by $\mathbb{D}$ the open unit disk of the complex plane $\mathbb{C}$, and by $\overline{\mathbb{D}}$ the closed unit disk of $\mathbb{C}$. For a probability measure μ supported on $\mathbb{T}$, one defines the analytic function $B_\mu : \mathbb{D} \to \mathbb{C}$ by

$$B_\mu(z) = \frac{1}{z}\frac{\psi_\mu(z)}{1+\psi_\mu(z)} \quad \text{for } z \in \mathbb{D},$$

where the formula of ψ_μ is

$$\psi_\mu(z) = \int_{\mathbb{T}} \frac{\zeta z}{1-\zeta z}\, d\mu(\zeta).$$

Note that

$$B_\mu(0) = \psi'_\mu(0) = \int_{\mathbb{T}} \zeta \, d\mu(\zeta) \tag{2-1}$$

and that $B_{\delta_\lambda}(z) = \lambda$ for all $z \in \mathbb{D}$. As observed in [Belinschi and Bercovici 2005],

$$|B_\mu(z)| \le 1 \quad \text{for } z \in \mathbb{D},$$

and, conversely, any analytic function $B : \mathbb{D} \to \overline{\mathbb{D}}$ is of the form B_μ for a unique probability measure μ on $\mathbb{T}$.

Let μ_1 and μ_2 be two probability measures on $\mathbb{T}$. As shown in [Franz 2004; Bercovici 2006], the multiplicative Boolean convolution $\mu_1 \mathbin{\rlap{$\cup$}{\times}} \mu_2$ is characterized by the identity

$$B_{\mu_1 \mathbin{\rlap{$\cup$}{\times}} \mu_2}(z) = B_{\mu_1}(z) B_{\mu_2}(z) \quad \text{for } z \in \mathbb{D}. \tag{2-2}$$

It is easy to verify that weak convergence of probability measures can be translated in terms of the corresponding functions B . More precisely, given probability measures μ and $\{\mu_n\}_{n=1}^{\infty}$ on $\mathbb{T}$, the sequence μ_n converges weakly to μ if and only if the sequence $B_{\mu_n}(z)$ converges to $B_\mu(z)$ uniformly on the compact subsets of $\mathbb{D}$.

A probability measure ν on $\mathbb{T}$ is $\circledast$-*infinitely divisible* if, for each $n \in \mathbb{N}$, there exists a probability measure ν_n on $\mathbb{T}$ such that

$$\nu = \underbrace{\nu_n \circledast \nu_n \circledast \cdots \circledast \nu_n}_{n \text{ times}}.$$

We define analogously the notion of infinite divisibility for other convolutions.

The $\circledast$-infinite divisibility is characterized in [Franz 2004] as follows. A probability measure ν is $\circledast$-infinitely divisible if and only if either ν is Haar measure m (that is, normalized arclength measure on $\mathbb{T}$) or the function B_ν can be expressed as

$$B_\nu(z) = \gamma \exp\Big(-\int_{\mathbb{T}} \frac{1+\zeta z}{1-\zeta z}\, d\sigma(\zeta)\Big) \quad \text{for } z \in \mathbb{D}, \tag{2-3}$$

where $\gamma \in \mathbb{T}$, and σ is a finite positive Borel measure on $\mathbb{T}$. In other words, a measure ν is $\circledast$-infinitely divisible if and only if either $B_\nu(z) = 0$ for all $z \in \mathbb{D}$ or $0 \notin B_\nu(\mathbb{D})$. We use the notation $\nu_{\circledast}^{\gamma,\sigma}$ to denote the $\circledast$-infinitely divisible measure ν determined by γ and σ.

Free multiplicative convolution $\boxtimes$ for probability measures on the unit circle was introduced by Voiculescu [1987]. For the definition of $\boxtimes$, we refer to [Voiculescu et al. 1992]. Throughout, we will use the notation $\mathcal{M}_{\mathbb{T}}^{\times}$ to denote the collection of all Borel probability measures ν on $\mathbb{T}$ with nonzero first moment, that is, with $\int_{\mathbb{T}} \zeta \, d\nu(\zeta) \neq 0$.

We will require the following characterization [Bercovici and Voiculescu 1992] of $\boxtimes$-infinite divisibility. If a measure ν is in the class $\mathcal{M}_{\mathbb{T}}^{\times}$, then the function ψ_ν will have an inverse ψ_ν^{-1} in a neighborhood of zero. In this case one defines

$$\Sigma_\nu(z) = \frac{1}{z}\psi_\nu^{-1}\Big(\frac{z}{1-z}\Big)$$

for z near the origin, and the remarkable identity $\Sigma_{\mu\boxtimes\nu}(z) = \Sigma_\mu(z)\Sigma_\nu(z)$ holds for z in a neighborhood of zero where three involved functions are defined. A measure $\nu \in \mathcal{M}_{\mathbb{T}}^{\times}$ is $\boxtimes$-infinitely divisible if and only if the function Σ_ν can be expressed as

$$\Sigma_\nu(z) = \gamma \exp\Big(\int_{\mathbb{T}} \frac{1+\zeta z}{1-\zeta z}\, d\sigma(\zeta)\Big) \quad \text{for } z \in \mathbb{D},$$

where $|\gamma| = 1$, and σ is a finite positive Borel measure on $\mathbb{T}$. We will use the notation $\nu_{\boxtimes}^{\gamma,\sigma}$ to denote the $\boxtimes$-infinitely divisible measure ν in this case. The Haar measure m is the only $\boxtimes$-infinitely divisible probability measure on $\mathbb{T}$ with zero first moment.

2.2. ***Additive Boolean and free convolutions on the real line.*** Set

$$\mathbb{C}^+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$$

and $\mathbb{C}^- = -\mathbb{C}^+$. For $\alpha, \beta > 0$, define the cone $\Gamma_\alpha = \{z = x + iy \in \mathbb{C}^+ : |x| < \alpha y\}$ and the truncated cone $\Gamma_{\alpha,\beta} = \{z = x + iy \in \Gamma_\alpha : y > \beta\}$. We associate to every measure $\mu \in \mathcal{M}_{\mathbb{R}}$ its *Cauchy transform*

$$G_\mu(z) = \int_{-\infty}^{\infty} \frac{1}{z-t}\, d\mu(t) \quad \text{for } z \in \mathbb{C}^+,$$

and its reciprocal $F_\mu = 1/G_\mu : \mathbb{C}^+ \to \mathbb{C}^+$. Then we have $\operatorname{Im} z \le \operatorname{Im} F_\mu(z)$, so that the function $E_\mu(z) = z - F_\mu(z)$ takes values in $\mathbb{C}^- \cup \mathbb{R}$. The function E_μ is such that $E_\mu(z)/z \to 0$ as $z \to \infty$ nontangentially (that is, $|z| \to \infty$, but z stays within a cone Γ_α for some $\alpha > 0$). Conversely, any analytic function $E : \mathbb{C}^+ \to \mathbb{C}^- \cup \mathbb{R}$ such that $E_\mu(z)/z \to 0$ as $z \to \infty$ nontangentially is of the form E_μ for a unique probability measure μ on $\mathbb{R}$.

For $\mu_1, \mu_2 \in \mathcal{M}_{\mathbb{R}}$, the additive Boolean convolution $\mu_1 \uplus \mu_2$ is characterized [Speicher and Woroudi 1997; Bercovici 2006] by the identity

$$E_{\mu_1 \uplus \mu_2}(z) = E_{\mu_1}(z) + E_{\mu_2}(z) \quad \text{for } z \in \mathbb{C}^+.$$

Let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{M}_{\mathbb{R}}$. Bercovici and Pata [1999, Proposition 6.2] showed that the sequence μ_n converges weakly to a probability measure $\mu \in \mathcal{M}_{\mathbb{R}}$ if and only if there exists $\beta > 0$ such that $\lim_{n\to\infty} E_{\mu_n}(iy) = E_\mu(iy)$ for every $y > \beta$, and $E_{\mu_n}(iy) = o(y)$ uniformly in n as $y \to \infty$.

Every measure $\nu \in \mathcal{M}_{\mathbb{R}}$ is $\uplus$-infinitely divisible [Speicher and Woroudi 1997]. The function E_ν has a Nevanlinna representation [Akhiezer 1965]

$$E_\nu(z) = \gamma + \int_{-\infty}^{\infty} \frac{1+tz}{z-t}\, d\sigma(t) \quad \text{for } z \in \mathbb{C}^+,$$

where $\gamma \in \mathbb{R}$ and σ is a finite positive Borel measure on $\mathbb{R}$. We use the notation $\nu_{\uplus}^{\gamma,\sigma}$ to denote the ($\uplus$-infinitely divisible) measure ν.

The additive free convolution $\boxplus$ was first introduced by Voiculescu [1986] for compactly supported measures on the real line. Then it was extended by Maassen [1992] to measures with finite variance, and by Bercovici and Voiculescu [1993] to the whole class $\mathcal{M}_{\mathbb{R}}$. The book [Voiculescu et al. 1992] also contains a detailed description of the theory related to this convolution.

We require a result from [Bercovici and Voiculescu 1993] regarding a characterization of $\boxplus$-infinite divisibility. We have seen earlier that $E_\mu(z)/z \to 0$ as $z \to \infty$ nontangentially for a measure $\mu \in \mathcal{M}_{\mathbb{R}}$. It follows that for every $\alpha > 0$, there exists a

$\beta = \beta(\mu, \alpha) > 0$ such that the function F_μ has a right inverse F_μ^{-1} defined on $\Gamma_{\alpha,\beta}$. The *Voiculescu transform*

$$\phi_\mu(z) = F_\mu^{-1}(z) - z \quad \text{for } z \in \Gamma_{\alpha,\beta}$$

linearizes the free convolution in the sense that the identity $\phi_{\mu \boxplus \nu}(z) = \phi_\mu(z) + \phi_\nu(z)$ holds for z in a truncated cone where all functions involved are defined. A measure $\nu \in \mathcal{M}_\mathbb{R}$ is $\boxplus$-infinitely divisible if and only if there exist $\gamma \in \mathbb{R}$ and a finite positive Borel measure σ on $\mathbb{R}$ such that

$$\phi_\nu(z) = \gamma + \int_{-\infty}^{\infty} \frac{1+tz}{z-t}\, d\sigma(t) \quad \text{for } z \in \mathbb{C}^+.$$

We will denote the above measure ν by $\nu_\boxplus^{\gamma,\sigma}$.

The Lévy–Hinčin formula (see [Billingsley 1995]) characterizes the $*$-infinitely divisible measures in terms of their Fourier transform as follows: a measure $\rho \in \mathcal{M}_\mathbb{R}$ is $*$-infinitely divisible if and only if there exist $\gamma \in \mathbb{R}$ and a finite positive Borel measure σ on $\mathbb{R}$ such that the Fourier transform $\widehat{\rho}$ is given by

$$\widehat{\rho}(t) = \exp\Bigl(i\gamma t + \int_{-\infty}^{\infty} \Bigl(e^{itx} - 1 - \frac{itx}{1+x^2}\Bigr) \frac{1+x^2}{x^2}\, d\sigma(x)\Bigr) \quad \text{for } t \in \mathbb{R},$$

where the integrand $(e^{itx} - 1 - itx/(1+x^2))\,(1+x^2)/x^2$ is interpreted as $-t^2/2$ for $x = 0$. The notation $\nu_*^{\gamma,\sigma}$ will be used to denote the $*$-infinitely divisible measure determined by γ and σ.

We will require the following result which was already noted in a different form as [Bercovici and Wang 2008b, Lemma 2.3].

Lemma 2.1. *Consider a sequence of real numbers $\{r_n\}_{n=1}^\infty$ and triangular arrays*

$$\{z_{nk} \in \mathbb{C} : n \in \mathbb{N},\ 1 \le k \le k_n\}, \qquad \{w_{nk} \in \mathbb{C} : n \in \mathbb{N},\ 1 \le k \le k_n\},$$
$$\{s_{nk} \in \mathbb{R} : n \in \mathbb{N},\ 1 \le k \le k_n\}.$$

Suppose that

(i) *all the s_{nk} are nonnegative and* $\sup_{n \ge 1} \sum_{k=1}^{k_n} s_{nk} < +\infty$;

(ii) $\operatorname{Re} w_{nk} \le 0$ *and* $\operatorname{Re} z_{nk} \le 0$ *for every n and k*;

(iii) $z_{nk} = w_{nk}(1+\varepsilon_{nk})$, *where the sequence* $\varepsilon_n = \max_{1 \le k \le k_n} |\varepsilon_{nk}|$ *converges to zero as $n \to \infty$*;

(iv) *there exists a positive constant M such that*

$$|\operatorname{Im} w_{nk}| \le M |\operatorname{Re} w_{nk}| + s_{nk} \quad \textit{for } n \in \mathbb{N} \textit{ and } 1 \le k \le k_n.$$

Define sequences

$$Z := \{\exp(ir_n + \textstyle\sum_{k=1}^{k_n} z_{nk})\}_{n=1}^\infty \quad \textit{and} \quad W := \{\exp(ir_n + \textstyle\sum_{k=1}^{k_n} w_{nk})\}_{n=1}^\infty.$$

Then the sequence Z converges if and only if the sequence W converges, and the two sequences have the same limit.

Proof. From the assumptions on $\{z_{nk}\}_{n,k}$ and $\{w_{nk}\}_{n,k}$, we deduce that

$$\Big|\sum_{k=1}^{k_n}(z_{nk}-w_{nk})\Big| \leq (1+M)\varepsilon_n\Big(-\sum_{k=1}^{k_n}\operatorname{Re} w_{nk}\Big)+\varepsilon_n\sum_{k=1}^{k_n}s_{nk}, \tag{2-4}$$

and

$$(1-\varepsilon_n-M\varepsilon_n)\Big(-\sum_{k=1}^{k_n}\operatorname{Re} w_{nk}\Big) \leq \Big(-\sum_{k=1}^{k_n}\operatorname{Re} z_{nk}\Big)+\varepsilon_n\sum_{k=1}^{k_n}s_{nk}, \tag{2-5}$$

for sufficiently large n. Suppose that the sequence Z converges to a complex number z. If $z=0$, then we have $\lim_{n\to\infty}\sum_{k=1}^{k_n}\operatorname{Re} z_{nk}=-\infty$. Hence (2-4) implies that $\lim_{n\to\infty}\sum_{k=1}^{k_n}\operatorname{Re} w_{nk}=-\infty$ so that the sequence W converges to zero as well. If $z\neq 0$, then the sequence $\exp(\sum_{k=1}^{k_n}\operatorname{Re} z_{nk})$ converges to $|z|$ as $n\to\infty$. In particular, $\sum_{k=1}^{k_n}\operatorname{Re} z_{nk}$ is bounded. By (2-4) and (2-5), we conclude that $\lim_{n\to\infty}\exp(\sum_{k=1}^{k_n}\operatorname{Re} w_{nk})=|z|$, and that

$$\lim_{n\to\infty}\frac{\exp(i\sum_{k=1}^{k_n}\operatorname{Im} w_{nk})}{\exp(i\sum_{k=1}^{k_n}\operatorname{Im} z_{nk})}=1.$$

Therefore the sequence W also converges to z. The converse implication is proved in the same way. □

3. Multiplicative Boolean convolution on $\mathbb{T}$

Fix an infinitesimal array $\{\mu_{nk}: n\in\mathbb{N},\ 1\leq k\leq k_n\}$ of probability measures on $\mathbb{T}$. For any neighborhood of zero $\mathcal{V}\subset\mathbb{D}$, it was proved in [Belinschi and Bercovici 2008, Theorem 2.1] that

$$\lim_{n\to\infty}\psi_{\mu_{nk}}(z)=z/(1-z) \tag{3-1}$$

holds uniformly in k and $z\in\mathcal{V}$. It follows that, as n tends to infinity, the sequence $B_{\mu_{nk}}(z)$ converges to 1 uniformly in k and $z\in\mathcal{V}$. Thus, (2-1) implies that each μ_{nk} has nonzero first moment when n is large. Hence, for our purposes, we will always assume that each member in such an array belongs to the class $\mathcal{M}_{\mathbb{T}}^{\times}$. Another application of (3-1) is that the principal branch of $\log B_{\mu_{nk}}(z)$ is defined in $\mathcal{V}$ for large n.

Next, we introduce an auxiliary array $\{\mu_{nk}^{\circ}: n\in\mathbb{N},\ 1\leq k\leq k_n\}\subset\mathcal{M}_{\mathbb{T}}^{\times}$ as follows. Fix a constant $\tau\in(0,\pi)$. Define the measures μ_{nk}° by $d\mu_{nk}^{\circ}(\zeta)=d\mu_{nk}(b_{nk}\zeta)$, where the complex numbers b_{nk} are given by

$$b_{nk} = \exp\Big(i \int_{|\arg \zeta|<\tau} \arg \zeta \, d\mu_{nk}(\zeta)\Big).$$

Here $\arg \zeta$ is the principal value of the argument of ζ. Note that the array $\{\mu_{nk}^{\circ}\}_{n,k}$ is again infinitesimal, and

$$\lim_{n\to\infty} \max_{1\le k\le k_n} |\arg b_{nk}| = 0. \tag{3-2}$$

We associate each measure μ_{nk}° the function

$$h_{nk}(z) = -i \int_{\mathbb{T}} \operatorname{Im} \zeta \, d\mu_{nk}^{\circ}(\zeta) + \int_{\mathbb{T}} \frac{1+\zeta z}{1-\zeta z}(1-\operatorname{Re}\zeta)\, d\mu_{nk}^{\circ}(\zeta) \quad \text{for } z \in \mathbb{D}$$

and observe that $\operatorname{Re} h_{nk}(z) > 0$ for all $z \in \mathbb{D}$ unless the measure $\mu_{nk}^{\circ} = \delta_1$.

Lemma 3.1. *If $\varepsilon \in (0, 1/4)$, then we have, for sufficiently large n, that*

$$1 - B_{\mu_{nk}^{\circ}}(z) = h_{nk}(\overline{b_{nk}} z)(1 + v_{nk}(z)) \quad \textit{for } 1 \le k \le k_n,$$

where z is in $\mathcal{V}_{\varepsilon} = \{z \in \mathbb{D} : |z| < \varepsilon\}$. Moreover, $\lim_{n\to\infty} \max_{1\le k\le k_n} |v_{nk}(z)| = 0$ *uniformly on $\mathcal{V}_{\varepsilon}$.*

Proof. Applying (3-1) to the array $\{\mu_{nk}^{\circ}\}_{n,k}$, we deduce for large n that

$$\frac{z}{1+z} - \frac{\psi_{\mu_{nk}^{\circ}}(z/(1+z))}{1+\psi_{\mu_{nk}^{\circ}}(z/(1+z))} = \frac{1}{(1+z)^2}\Big(z - \psi_{\mu_{nk}^{\circ}}\Big(\frac{z}{1+z}\Big)\Big)(1 + u_{nk}(z)),$$

where $\lim_{n\to\infty} \max_{1\le k\le k_n} |u_{nk}(z)| = 0$ uniformly on $\{z : |z| < 1/3\}$. Introducing a change of variable $z \mapsto z/(1-z)$, we obtain

$$z - \frac{\psi_{\mu_{nk}^{\circ}}(z)}{1+\psi_{\mu_{nk}^{\circ}}(z)} = \Big(z \int_{\mathbb{T}} \frac{(1-z)(1-\zeta)}{1-\zeta z}\, d\mu_{nk}^{\circ}(\zeta)\Big)\Big(1 + u_{nk}\Big(\frac{z}{1-z}\Big)\Big) \quad \text{for } z \in \mathcal{V}_{\varepsilon}.$$

Using the identity

$$\frac{(1-z)(1-\zeta)}{1-\zeta z} = -i \operatorname{Im}\zeta + \frac{1+\zeta z}{1-\zeta z}(1-\operatorname{Re}\zeta),$$

we conclude, for sufficiently large n, that

$$1 - B_{\mu_{nk}^{\circ}}(z) = \frac{1}{z}\Big(z - \frac{\psi_{\mu_{nk}^{\circ}}(z)}{1+\psi_{\mu_{nk}^{\circ}}(z)}\Big) = h_{nk}(z)\Big(1 + u_{nk}\Big(\frac{z}{1-z}\Big)\Big) \quad \text{for all } z \in \mathcal{V}_{\varepsilon}.$$

To prove the result, it suffices to show that for every n and k, we have

$$h_{nk}(\overline{b_{nk}} z) = h_{nk}(z)(1 + w_{nk}(z)),$$

where $\lim_{n\to\infty}\max_{1\le k\le k_n}|w_{nk}(z)|=0$ uniformly in $\mathcal{V}_\varepsilon$. If the measure $\mu^\circ_{nk}=\delta_1$, then $h_{nk}(z)=0$ for all $z\in\mathbb{D}$. In this case, we define the function w_{nk} to be the zero function in $\mathbb{D}$. If $\mu^\circ_{nk}\ne\delta_1$, then we define the function

$$w_{nk}(z)=\frac{h_{nk}(\overline{b_{nk}}z)}{h_{nk}(z)}-1\quad \text{for } z\in\mathbb{D}.$$

Observe that

$$\begin{aligned}|h_{nk}(\overline{b_{nk}}z)-h_{nk}(z)| &= \left|(1-\overline{b_{nk}})\int_{\mathbb{T}}\Big(\frac{2\zeta z}{(1-\zeta z)(1-\zeta\overline{b_{nk}}z)}\Big)(1-\operatorname{Re}\zeta)\,d\mu^\circ_{nk}(\zeta)\right|\\ &\le |1-\overline{b_{nk}}|\int_{\mathbb{T}}\left|\frac{2\zeta z}{(1-\zeta z)(1-\zeta\overline{b_{nk}}z)}\right|(1-\operatorname{Re}\zeta)\,d\mu^\circ_{nk}(\zeta)\\ &\le \frac{2}{(1-\varepsilon)^2}|1-\overline{b_{nk}}|\int_{\mathbb{T}}(1-\operatorname{Re}\zeta)\,d\mu^\circ_{nk}(\zeta)\end{aligned}$$

for $z\in\mathcal{V}_\varepsilon$ and $\zeta\in\mathbb{T}$. Meanwhile, Harnack's inequality implies that there exists an $L=L(\varepsilon)>0$ such that

$$\left|\operatorname{Re}\Big(\frac{1+\zeta z}{1-\zeta z}\Big)\right|=\operatorname{Re}\Big(\frac{1+\zeta z}{1-\zeta z}\Big)\ge L\quad\text{for } z\in\mathcal{V}_\varepsilon \text{ and } \zeta\in\mathbb{T}.$$

Thus, we have

$$\begin{aligned}|h_{nk}(z)|\ge\operatorname{Re}h_{nk}(z)&=\int_{\mathbb{T}}\operatorname{Re}\Big(\frac{1+\zeta z}{1-\zeta z}\Big)(1-\operatorname{Re}\zeta)\,d\mu^\circ_{nk}(\zeta)\\ &\ge L\int_{\mathbb{T}}(1-\operatorname{Re}\zeta)\,d\mu^\circ_{nk}(\zeta).\end{aligned}$$

Combining the above inequalities, we get

$$\begin{aligned}|w_{nk}(z)|&\le\frac{|h_{nk}(\overline{b_{nk}}z)-h_{nk}(z)|}{|h_{nk}(z)|}\\ &\le\frac{2}{(1-\varepsilon)^2L}|1-\overline{b_{nk}}|\le\frac{2}{(1-\varepsilon)^2L}|\arg b_{nk}|,\end{aligned}$$

for $z\in\mathcal{V}_\varepsilon$. Hence the claim is proved by (3-2). □

A crucial property for the functions $h_{nk}(z)$ proved in [Bercovici and Wang 2008b, Lemma 4.1] is that for every neighborhood of zero $\mathcal{V}\subset\mathbb{D}$, there exists a constant $M=M(\mathcal{V},\tau)>0$ such that

$$|\operatorname{Im}h_{nk}(z)|\le M\operatorname{Re}h_{nk}(z)\quad\text{for } z\in\mathcal{V} \text{ and } 1\le k\le k_n, \tag{3-3}$$

for sufficiently large n.

Proposition 3.2. *Let $\{\lambda_n\}_{n=1}^\infty$ be a sequence in $\mathbb{T}$. Suppose that $B:\mathbb{D}\to\overline{\mathbb{D}}$ is an analytic function. Then $\lim_{n\to\infty}\lambda_n\prod_{k=1}^{k_n}B_{\mu_{nk}}(z)=B(z)$ uniformly on the compact* *subsets of $\mathbb{D}$ if and only if*

$$\lim_{n\to\infty} \exp\bigl(i \arg \lambda_n + i \textstyle\sum_{k=1}^{k_n} \arg b_{nk} - \sum_{k=1}^{k_n} h_{nk}(z)\bigr) = B(z) \tag{3-4}$$

uniformly on the compact subsets of $\mathbb{D}$.

Proof. Suppose that the convergence to $B(z)$ in (3-4) is uniform on the compact subsets of $\mathbb{D}$. Note that $\log w = w - 1 + o(|w-1|)$ as $w \to 1$, and $B_{\mu^{\circ}_{nk}}(z) = \overline{b_{nk}} B_{\mu_{nk}}(\overline{b_{nk}} z)$ for every $z \in \mathbb{D}$. For z near the origin, Lemma 3.1 shows that $\log(\overline{b_{nk}} B_{\mu_{nk}}(z)) = -h_{nk}(z)(1+o(1))$ uniformly in k as n tends to infinity. Then Lemma 2.1 implies that $\lim_{n\to\infty} \lambda_n \prod_{k=1}^{k_n} B_{\mu_{nk}}(z) = B(z)$ uniformly in a neighborhood of zero. Moreover, this convergence is actually uniform on the compact subsets of $\mathbb{D}$ since the family $\{\lambda_n \prod_{k=1}^{k_n} B_{\mu_{nk}}(z)\}_{n=1}^{\infty}$ is normal. The converse implication is proved in the same way. $\square$

Lemma 3.3. *Let $\{\nu_n\}_{n=1}^{\infty}$ be a sequence of $⋓$-infinitely divisible measures on $\mathbb{T}$. If the sequence ν_n converges weakly to a probability measure ν, then the measure ν is $⋓$-infinitely divisible.*

Proof. The weak convergence of ν_n implies that the sequence $B_{\nu_n}(z)$ converges to $B_\nu(z)$ uniformly on the compact subsets of $\mathbb{D}$. If the function B_ν is nonvanishing in $\mathbb{D}$, then the measure ν is $⋓$-infinitely divisible. On the other hand, if $B_\nu(z_0) = 0$ for some $z_0 \in \mathbb{D}$, then Rouché's theorem implies that there exists an $N = N(z_0) \in \mathbb{N}$ such that the function $B_{\nu_n}(z)$ also has a zero in the disk $\{z : |z - z_0| < 1 - |z_0|\}$ whenever $n \geq N$. Since each ν_n is $⋓$-infinitely divisible, we conclude in this case that ν_n is the Haar measure m for all $n \geq N$. Consequently, the measure ν must be m as well. $\square$

Our next result is the Boolean analogue of Hinčin's theorem.

Theorem 3.4. *Let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{T}$, and let $\{\mu_{nk}\}_{n,k}$ be an infinitesimal array in $\mathcal{M}_{\mathbb{T}}$. If the sequence of measures*

$$\delta_{\lambda_n} ⋓ \mu_{n1} ⋓ \mu_{n2} ⋓ \cdots ⋓ \mu_{nk_n}$$

converges weakly on $\mathbb{T}$ to a probability measure ν, then ν is $⋓$-infinitely divisible.

Proof. From (2-2) and the weak convergence of $\delta_{\lambda_n} ⋓ \mu_{n1} ⋓ \mu_{n2} ⋓ \cdots ⋓ \mu_{nk_n}$, we have $\lim_{n\to\infty} \lambda_n \prod_{k=1}^{k_n} B_{\mu_{nk}}(z) = B_\nu(z)$ uniformly on the compact subsets of $\mathbb{D}$. Observe that the function $-\sum_{k=1}^{k_n} h_{nk}(z)$ has negative real part in $\mathbb{D}$, and hence there exists a $⋓$-infinitely divisible measure ν_n on $\mathbb{T}$ such that

$$B_{\nu_n}(z) = \exp\bigl(i \arg \lambda_n + i \textstyle\sum_{k=1}^{k_n} \arg b_{nk} - \sum_{k=1}^{k_n} h_{nk}(z)\bigr) \quad \text{for } z \in \mathbb{D}.$$

Proposition 3.2 then implies that the sequence ν_n converges weakly to ν. The $⋓$-infinitely divisibility of the measure ν follows immediately by Lemma 3.3. $\square$

Fix $\gamma \in \mathbb{T}$ and a finite positive Borel measure σ on $\mathbb{T}$.

Theorem 3.5. *For an infinitesimal array $\{\mu_{nk}\}_{n,k} \subset \mathcal{M}_{\mathbb{T}}^{\times}$ and a sequence $\{\lambda_n\}_{n=1}^{\infty} \subset \mathbb{T}$, the following statements are equivalent*:

(i) *The sequence $\delta_{\lambda_n} \mathbin{\bigtriangledown\!\!\!\!\times} \mu_{n1} \mathbin{\bigtriangledown\!\!\!\!\times} \mu_{n2} \mathbin{\bigtriangledown\!\!\!\!\times} \cdots \mathbin{\bigtriangledown\!\!\!\!\times} \mu_{nk_n}$ converges weakly to $\nu_{\bigtriangledown\!\!\!\!\times}^{\gamma,\sigma}$.*

(ii) *The sequence $\delta_{\lambda_n} \boxtimes \mu_{n1} \boxtimes \mu_{n2} \boxtimes \cdots \boxtimes \mu_{nk_n}$ converges weakly to $\nu_{\boxtimes}^{\bar{\gamma},\sigma}$.*

(iii) *The sequence of measures*

$$d\sigma_n(\zeta) = \textstyle\sum_{k=1}^{k_n} (1 - \operatorname{Re} \zeta)\, d\mu_{nk}^{\circ}(\zeta)$$

converges weakly on $\mathbb{T}$ to σ, and the limit $\lim_{n\to\infty} \gamma_n = \gamma$ exists, where

$$\gamma_n = \exp\Bigl(i \arg \lambda_n + i \sum_{k=1}^{k_n} \arg b_{nk} + i \sum_{k=1}^{k_n} \int_{\mathbb{T}} \operatorname{Im} \zeta \, d\mu_{nk}^{\circ}(\zeta)\Bigr).$$

Proof. Bercovici and Wang [2008b] proved the equivalence of (ii) and (iii). We will focus on the equivalence of (i) and (iii). Assume that (i) holds. Then we have

$$\lim_{n\to\infty} \lambda_n \prod_{k=1}^{k_n} B_{\mu_{nk}}(z) = B_{\nu_{\bigtriangledown\!\!\!\!\times}^{\gamma,\sigma}}(z) = \gamma \exp\Bigl(-\int_{\mathbb{T}} \frac{1+\zeta z}{1-\zeta z}\, d\sigma(\zeta)\Bigr)$$

uniformly on the compact subsets of $\mathbb{D}$. Proposition 3.2 then shows that

$$\text{(3-5)}\quad \lim_{n\to\infty} \exp\Bigl(i \arg \lambda_n + i \sum_{k=1}^{k_n} \arg b_{nk} - \sum_{k=1}^{k_n} h_{nk}(z)\Bigr) = \gamma \exp\Bigl(-\int_{\mathbb{T}} \frac{1+\zeta z}{1-\zeta z} d\sigma(\zeta)\Bigr)$$

uniformly on the compact subsets of $\mathbb{D}$. Taking the absolute value on both sides, we conclude that

$$\text{(3-6)}\quad \lim_{n\to\infty} \exp\Bigl(-\sum_{k=1}^{k_n} \operatorname{Re} h_{nk}(z)\Bigr) = \exp\Bigl(-\int_{\mathbb{T}} \operatorname{Re}\Bigl(\frac{1+\zeta z}{1-\zeta z}\Bigr)\, d\sigma(\zeta)\Bigr) \quad \text{for } z \in \mathbb{D}.$$

Since

$$\exp\bigl(i \arg \lambda_n + i \textstyle\sum_{k=1}^{k_n} \arg b_{nk} - \sum_{k=1}^{k_n} h_{nk}(z)\bigr) = \gamma_n \exp\Bigl(-\displaystyle\int_{\mathbb{T}} \frac{1+\zeta z}{1-\zeta z}\, d\sigma_n(\zeta)\Bigr),$$

and the real part of the function $\sum_{k=1}^{k_n} h_{nk}(z)$ is the Poisson integral of the measure $d\sigma_n(\bar{\zeta})$, Equation (3-6) uniquely determines the measure σ that is the weak cluster point of $\{\sigma_n\}_{n=1}^{\infty}$. Hence σ_n must converge weakly to σ. The convergence property of the sequence γ_n follows immediately by letting $z = 0$ in (3-5) and (3-6).

For the converse implication from (iii) to (i), one can easily reverse the above steps to reach (i) by Proposition 3.2. The details are left to the reader. □

Bercovici and Wang [2008b, Theorem 4.4] gave the equivalent condition for the weak convergence of $\delta_{\lambda_n} \boxtimes \mu_{n1} \boxtimes \mu_{n2} \boxtimes \cdots \boxtimes \mu_{nk_n}$ to the Haar measure m. It turns out that the same condition is also equivalent to the weak convergence of $\delta_{\lambda_n} \between \mu_{n1} \between \mu_{n2} \between \cdots \between \mu_{nk_n}$ to m. We will not provide the details of this proof because they are entirely analogous to the free case. We only point out that the relevant fact needed in the proof is that $B_\mu(0) = \int_{\mathbb{T}} \zeta \, d\mu(\zeta)$ for all probability measure μ on $\mathbb{T}$.

Theorem 3.6. *For an infinitesimal array $\{\mu_{nk}\}_{n,k} \subset \mathcal{M}_{\mathbb{T}}^{\times}$ and a sequence $\{\lambda_n\}_{n=1}^{\infty} \subset \mathbb{T}$, the following statements are equivalent*:

(i) *The sequence $\delta_{\lambda_n} \between \mu_{n1} \between \mu_{n2} \between \cdots \between \mu_{nk_n}$ converges weakly to m.*

(ii) *The sequence $\delta_{\lambda_n} \boxtimes \mu_{n1} \boxtimes \mu_{n2} \boxtimes \cdots \boxtimes \mu_{nk_n}$ converges weakly to m.*

(iii) $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{k_n} \int_{\mathbb{T}} (1 - \operatorname{Re} \zeta) \, d\mu_{nk}^{\circ}(\zeta) = +\infty.$

We conclude this section by using Theorem 3.5 to determine the multiplicative Boolean analogues of Gaussian and Poisson laws on $\mathbb{R}$. The following result generates a measure analogous to the Gaussian distribution on the real line.

Corollary 3.7. *For every $t > 0$, the function*

$$B(z) = \exp\left(-\frac{t}{2}\left(\frac{1+z}{1-z}\right)\right) \quad \textit{for } z \in \mathbb{D},$$

is of the form $B = B_\nu$ for some $\between$-infinitely divisible measure $\nu \in \mathcal{M}_{\mathbb{T}}^{\times}$.

Proof. For $n > t$, we define $\mu_{nk} = \mu_n = \frac{1}{2}(\delta_{\xi_n} + \delta_{\overline{\xi_n}})$ for $1 \le k \le n$, where $\xi_n = \sqrt{1 - t/n} + i\sqrt{t/n}$. To apply Theorem 3.5, we choose $\tau = 1$, so that $b_{nk} = 1$ for every n and k. Hence we have $\mu_n^{\circ} = \mu_n$. As in the statement of Theorem 3.5, we define the measures $d\sigma_n(\zeta) = n(1 - \operatorname{Re}\zeta)\, d\mu_n(\zeta)$ and the numbers $\gamma_n = \exp(i\, n \int_{\mathbb{T}} \operatorname{Im} \zeta \, d\mu_n(\zeta))$. Note that $\gamma_n = 1$ for all $n \in \mathbb{N}$ and that the p-th Fourier coefficient $\widehat{\sigma_n}(p)$ of the measure σ_n is given by

$$\widehat{\sigma_n}(p) = \int_{\mathbb{T}} \zeta^{p}\, n(1 - \operatorname{Re}\zeta)\, d\mu_n(\zeta) = n \operatorname{Re} \xi_n^{p} (1 - \operatorname{Re} \xi_n),$$

where p is an integer. Since $\lim_{n\to\infty} \widehat{\sigma_n}(p) = t/2$ for all p, we conclude that the sequence σ_n converges weakly on $\mathbb{T}$ to the measure $\sigma = (t/2)\delta_1$. Theorem 3.5 then implies that the sequence $\mu_n \between \mu_n \between \cdots \between \mu_n$ (in which μ_n occurs n times) converges weakly to $\nu_{\between}^{1,\sigma}$ as $n \to \infty$. The desired result now follows from (2-3). $\square$

Definition. A $\between$-infinitely divisible measure $\nu_{\between}^{\gamma,\sigma} \in \mathcal{M}_{\mathbb{T}}^{\times}$ is said to be $\between$-*normal* if the measure σ is concentrated at the point 1 (that is, $\sigma = \sigma(\mathbb{T})\delta_1$).

Our next result produces a Boolean analogue of the Poisson distribution on $\mathbb{R}$.

Corollary 3.8. *For every $t > 0$ and $\lambda \in \mathbb{T}$, the function*

$$B(z) = \exp\Bigl(-t(1-\lambda)\Bigl(\frac{1-z}{1-\lambda z}\Bigr)\Bigr) \quad \textit{for } z \in \mathbb{D},$$

is of the form $B = B_\nu$ for some $\uplus$-infinitely divisible measure $\nu \in \mathcal{M}_{\mathbb{T}}^{\times}$.

Proof. Note that $B = B_{\delta_1}$ when $\lambda = 1$. Assume now $\lambda \neq 1$. This time we set $\mu_{nk} = \mu_n = (1-t/n)\,\delta_1 + t/n\delta_\lambda$ for $1 \le k \le n$, and we choose $\tau = |\arg\lambda|/2$ so that $\mu_n^\circ = \mu_n$. Meanwhile, we define the measures σ_n and the numbers γ_n as in the proof of Corollary 3.7. Then we have $\widehat{\sigma_n}(p) = t\lambda^p(1-\operatorname{Re}\lambda)$ and $\gamma_n = e^{it\operatorname{Im}\lambda}$ for all $p \in \mathbb{Z}$ and $n \in \mathbb{N}$. Thus, the measures σ_n converge weakly on $\mathbb{T}$ to the measure $\sigma = t(1-\operatorname{Re}\lambda)\delta_\lambda$, while the number $\gamma = e^{it\operatorname{Im}\lambda}$. Then the proof is completed by Theorem 3.5 and the observation that

$$\begin{aligned} it\operatorname{Im}\lambda - t(1-\operatorname{Re}\lambda)\frac{1+\lambda z}{1-\lambda z} &= -t\Bigl(-i\operatorname{Im}\lambda + (1-\operatorname{Re}\lambda)\frac{1+\lambda z}{1-\lambda z}\Bigr) \\ &= -t\Bigl(\frac{(1-\lambda)(1-z)}{1-\lambda z}\Bigr). \end{aligned}$$

□

4. Classical convolution on $\mathbb{T}$

Consider an infinitesimal array $\{\mu_{nk}\}_{n,k} \subset \mathcal{M}_{\mathbb{T}}^{\times}$ and a sequence $\{\lambda_n\}_{n=1}^\infty \subset \mathbb{T}$. We define

$$\mu_n = \delta_{\lambda_n} \uplus \mu_{n1} \uplus \mu_{n2} \uplus \cdots \uplus \mu_{nk_n} \quad \text{and} \quad \nu_n = \delta_{\lambda_n} \circledast \mu_{n1} \circledast \mu_{n2} \circledast \cdots \circledast \mu_{nk_n},$$

for every $n \in \mathbb{N}$. This section investigates connections between the asymptotic distributional behavior of $\{\mu_n\}_{n=1}^\infty$ and that of $\{\nu_n\}_{n=1}^\infty$. For our purposes, we introduce the complex numbers

$$b_{nk} = \exp\Bigl(i\int_{|\arg\zeta|<1} \arg\zeta\, d\mu_{nk}(\zeta)\Bigr),$$

and the centered measures $d\mu_{nk}^\circ(\zeta) = d\mu_{nk}(b_{nk}\zeta)$. Note that we have

$$\widehat{\mu_{nk}}(p) = b_{nk}^p\widehat{\mu_{nk}^\circ}(p)$$

for any integer p, and that the function $(\zeta^p - 1 - ip\operatorname{Im}\zeta)/(1-\operatorname{Re}\zeta)$ is continuous and bounded on $\mathbb{T}$. (The value of this function for $\zeta = 1$ is set to $-p^2$ in order to preserve its continuity at that point.)

Theorem 4.1. *Assume that $\gamma \in \mathbb{T}$ and that σ is a finite positive Borel measure on $\mathbb{T}$. If the sequence μ_n converges weakly to $\nu_{\uplus}^{\gamma,\sigma}$, then there exists a probability measure ν on $\mathbb{T}$ such that the sequence ν_n converges weakly to ν. Moreover, the Fourier coefficients of the limit law ν can be calculated by the formula*

$$\widehat{\nu}(p) = \gamma^p \exp\Bigl(\int_{\mathbb{T}} \frac{\zeta^p - 1 - ip\operatorname{Im}\zeta}{1-\operatorname{Re}\zeta}\, d\sigma(\zeta)\Bigr) \quad \textit{for } p \in \mathbb{Z}. \tag{4-1}$$

Proof. Observe that $\widehat{\nu_n}(0) = 1$ for all $n \in \mathbb{N}$, and the right side of (4-1) is 1 when $p = 0$. Fix now a nonzero integer p. To prove the theorem, it suffices to show that the sequence $\{\widehat{\nu_n}(p)\}_{n=1}^{\infty}$ has a limit and that this limit can be identified as the right side of (4-1). Since the array $\{\mu_{nk}^{\circ}\}_{n,k}$ is infinitesimal, the principal logarithm of $\widehat{\mu_{nk}^{\circ}}(p)$ exists when n is sufficiently large. Moreover, we have

$$\widehat{\nu_n}(p) = \exp\bigl(ip \arg \lambda_n + ip \textstyle\sum_{k=1}^{k_n} \arg b_{nk} + \sum_{k=1}^{k_n} \log \widehat{\mu_{nk}^{\circ}}(p)\bigr) \tag{4-2}$$

for large n. Define the complex numbers $A_{nk} = A_{nk}(p) = \widehat{\mu_{nk}^{\circ}}(p) - 1$, and set

$$d\sigma_n(\zeta) = \textstyle\sum_{k=1}^{k_n} (1 - \operatorname{Re} \zeta)\, d\mu_{nk}^{\circ}(\zeta)$$

and

$$\gamma_n = \exp\Bigl(i \arg \lambda_n + i \sum_{k=1}^{k_n} \arg b_{nk} + i \sum_{k=1}^{k_n} \int_{\mathbb{T}} \operatorname{Im} \zeta \, d\mu_{nk}^{\circ}(\zeta)\Bigr).$$

By Theorem 3.5, the measures σ_n converge weakly on $\mathbb{T}$ to the measure σ, and the limit of the sequence γ_n is γ. Note that

$$\begin{aligned}
x_n &:= \exp\bigl(ip \arg \lambda_n + ip \textstyle\sum_{k=1}^{k_n} \arg b_{nk} + \sum_{k=1}^{k_n} A_{nk}\bigr) \\
&= \gamma_n^p \exp\Bigl(\sum_{k=1}^{k_n} \Bigl(A_{nk} - \int_{\mathbb{T}} ip \operatorname{Im} \zeta \, d\mu_{nk}^{\circ}(\zeta)\Bigr)\Bigr) \\
&= \gamma_n^p \exp\Bigl(\sum_{k=1}^{k_n} \int_{\mathbb{T}} (\zeta^p - 1 - ip \operatorname{Im} \zeta)\, d\mu_{nk}^{\circ}(\zeta)\Bigr) \\
&= \gamma_n^p \exp\Bigl(\int_{\mathbb{T}} \frac{\zeta^p - 1 - ip \operatorname{Im} \zeta}{1 - \operatorname{Re} \zeta}\, d\sigma_n(\zeta)\Bigr).
\end{aligned}$$

Therefore, we deduce that

$$\lim_{n\to\infty} x_n = \gamma^p \exp\Bigl(\int_{\mathbb{T}} \frac{\zeta^p - 1 - ip \operatorname{Im} \zeta}{1 - \operatorname{Re} \zeta}\, d\sigma(\zeta)\Bigr).$$

The infinitesimality of the array $\{\mu_{nk}^{\circ}\}_{n,k}$ implies that $\max_{1\le k\le k_n} |A_{nk}| \to 0$ as $n \to \infty$. Hence for sufficiently large n, the expansion

$$\log \widehat{\mu_{nk}^{\circ}}(p) = \log(1 + A_{nk}) = A_{nk} - \tfrac{1}{2} A_{nk}^2 + \tfrac{1}{3} A_{nk}^3 - \cdots$$

holds. Thus, we deduce that $\log \widehat{\mu_{nk}^{\circ}}(p) = A_{nk}(1 + o(1))$ uniformly in k as $n \to \infty$.

Denote by $\mathcal{U}_p$ the set of all complex numbers $\zeta \in \mathbb{T}$ such that $3|\arg \zeta| < \min\{1, |\pi/p|\}$ and by $\mathcal{V}_p$ the set of all $\zeta \in \mathcal{U}_p$ such that $6|\arg \zeta| < \min\{1, |\pi/p|\}$.

We also introduce the sets $\mathcal{U}_p^\circ = \mathcal{U}_p^\circ(n,k) = \{b_{nk}\zeta : \zeta \in \mathcal{U}_p\}$. By (3-2), we have

$$\begin{aligned}
&\Big|\int_{\mathcal{U}_p} \arg\zeta \, d\mu_{nk}^\circ(\zeta)\Big| \\
&\qquad = \Big|\arg b_{nk} - \int_{\{|\arg\zeta|<1\}\setminus\mathcal{U}_p^\circ} \arg\zeta \, d\mu_{nk}(\zeta) - \arg b_{nk}\int_{\mathcal{U}_p^\circ} d\mu_{nk}(\zeta)\Big| \\
&\qquad = \Big|\arg b_{nk}\mu_{nk}^\circ(\mathbb{T}\setminus\mathcal{U}_p) - \int_{\{|\arg\zeta|<1\}\setminus\mathcal{U}_p^\circ} \arg\zeta \, d\mu_{nk}(\zeta)\Big| \\
&\qquad \le 2\mu_{nk}^\circ(\mathbb{T}\setminus\mathcal{V}_p)
\end{aligned}$$

for sufficiently large n. Hence we conclude for large n that

$$\begin{aligned}
|\mathrm{Im}\, A_{nk}| &\le \int_{\mathcal{U}_p} |\mathrm{Im}\,\zeta^p - p\arg\zeta| \, d\mu_{nk}^\circ(\zeta) \\
&\qquad + \Big|\int_{\mathcal{U}_p} p\arg\zeta \, d\mu_{nk}^\circ(\zeta)\Big| + \int_{\mathbb{T}\setminus\mathcal{U}_p} |\mathrm{Im}\,\zeta^p| \, d\mu_{nk}^\circ(\zeta) \\
&\le 2\int_{\mathcal{U}_p} (1-\mathrm{Re}\,\zeta^p) \, d\mu_{nk}^\circ(\zeta) + (2|p|+1)\mu_{nk}^\circ(\mathbb{T}\setminus\mathcal{V}_p) \\
&\le 2|\mathrm{Re}\, A_{nk}| + (2|p|+1)\mu_{nk}^\circ(\mathbb{T}\setminus\mathcal{V}_p).
\end{aligned}$$

Meanwhile, the weak convergence of σ_n implies that

$$\lim_{n\to\infty}\int_{\mathbb{T}\setminus\mathcal{V}_p} \frac{1}{1-\mathrm{Re}\,\zeta}\, d\sigma_n(\zeta) = \int_{\mathbb{T}\setminus\mathcal{V}_p} \frac{1}{1-\mathrm{Re}\,\zeta}\, d\sigma(\zeta).$$

Since

$$\sum_{k=1}^{k_n} \mu_{nk}^\circ(\mathbb{T}\setminus\mathcal{V}_p) = \int_{\mathbb{T}\setminus\mathcal{V}_p} \frac{1}{1-\mathrm{Re}\,\zeta}\, d\sigma_n(\zeta),$$

we conclude that $\sum_{k=1}^{k_n} \mu_{nk}^\circ(\mathbb{T}\setminus\mathcal{V}_p)$ is bounded.

Applying Lemma 2.1 to the arrays $\{A_{nk}\}_{n,k}$ and $\{\log\widehat{\mu_{nk}^\circ}(p)\}_{n,k}$, we conclude at once that the sequence $\widehat{\nu_n}(p)$ converges, and

$$\begin{aligned}
\lim_{n\to\infty}\widehat{\nu_n}(p) &= \lim_{n\to\infty}\exp\Big(ip\arg\lambda_n + ip\textstyle\sum_{k=1}^{k_n}\arg b_{nk} + \sum_{k=1}^{k_n} A_{nk}\Big) \\
&= \gamma^p\exp\Big(\int_{\mathbb{T}} \frac{\zeta^p - 1 - ip\,\mathrm{Im}\,\zeta}{1-\mathrm{Re}\,\zeta}\, d\sigma(\zeta)\Big). \qquad \square
\end{aligned}$$

Remark. Note that (4-1) implies that the limit law ν in Theorem 4.1 is $\circledast$-infinitely divisible. Indeed, for every $n \in \mathbb{N}$, there exists a probability measure ν_n on $\mathbb{T}$ such that

$$\widehat{\nu_n}(p) = \gamma^{p/n}\exp\Big(\frac{1}{n}\int_{\mathbb{T}} \frac{\zeta^p - 1 - ip\,\mathrm{Im}\,\zeta}{1-\mathrm{Re}\,\zeta}\, d\sigma(\zeta)\Big) \quad \text{for } p \in \mathbb{Z}.$$

It follows that $\nu = \nu_n \circledast \nu_n \circledast \cdots \circledast \nu_n$ (where ν_n occurs n times) and hence the measure ν is $\circledast$-infinitely divisible.

Suppose $a \in \mathbb{R}$ and $t > 0$. Denote by $N(a, t)$ the Gaussian distribution on $\mathbb{R}$ with mean a and variance t, that is,

$$dN(a,t)(x) = \tfrac{1}{\sqrt{2\pi t}} e^{-(x-a)^2/(2t)}\, dx \quad \text{for } -\infty < x < \infty.$$

Let τ be the continuous homomorphism $x \mapsto e^{ix}$ from $\mathbb{R}$ into the circle $\mathbb{T}$. A probability measure ν on $\mathbb{T}$ is called a normal distribution [Heyer 1977, Chapter V, Section 5.2] if ν is the push-forward measure of a Gaussian law $N(a, t)$ by the map τ. One computes its measure $\nu(S)$ of a Borel measurable set $S \subset \mathbb{T}$ as

$$\nu(S) = \int_{\arg S} \sum_{n \in \mathbb{Z}} \tfrac{1}{\sqrt{2\pi t}} e^{-(u-a+2n\pi)^2/(2t)}\, du,$$

where the set $\arg S = \{\arg \zeta : \zeta \in S\}$. Note that ν is normal if and only if

$$\widehat{\nu}(p) = \exp(iap - tp^2/2) \quad \text{for } p \in \mathbb{Z}.$$

It follows that each normal distribution on $\mathbb{T}$ is $\circledast$-infinitely divisible. In accordance with the established terminology, we shall call a central limit theorem any assertion that, under some conditions, convolutions of probability measures converge to a normal distribution. The next result shows that the Boolean (or free) central limit theorem holds if and only if the classical central limit theorem holds. Recall that a $\utimes$-normal distribution on $\mathbb{T}$ is a $\utimes$-infinitely divisible measure $\nu_{\utimes}^{\gamma,\sigma}$ such that the measure σ is concentrated at the point 1.

Corollary 4.2. *The sequence μ_n converges weakly on $\mathbb{T}$ to a $\utimes$-normal distribution if and only if the sequence ν_n converges weakly on $\mathbb{T}$ to a normal distribution.*

Proof. If the sequence μ_n converges weakly to a $\utimes$-normal distribution $\nu_{\utimes}^{\gamma,\sigma}$, then Theorem 4.1 shows that the sequence ν_n converges weakly to a probability measure ν such that $\widehat{\nu}(p) = \gamma^p \exp(-\sigma(\{1\})p^2)$ for all $p \in \mathbb{Z}$. Therefore, the measure ν is a normal distribution on $\mathbb{T}$.

Assume now that the sequence ν_n converges weakly on $\mathbb{T}$ to a normal distribution ν. Then we have

$$\lim_{n\to\infty} \widehat{\nu_n}(p) = \widehat{\nu}(p) = \gamma^p \exp(-tp^2/2) \quad \text{for } p \in \mathbb{Z}$$

and for some $\gamma \in \mathbb{T}$ and $t > 0$. Define the complex numbers $A_{nk}(p)$, γ_n and the measures σ_n as in the proof of Theorem 4.1, and note that

$$\exp\bigl(i \arg \lambda_n + i \textstyle\sum_{k=1}^{k_n} \arg b_{nk} + \sum_{k=1}^{k_n} A_{nk}(1)\bigr) = \gamma_n e^{-\sigma_n(\mathbb{T})} \quad \text{for } n \in \mathbb{N}.$$

Let us recall, from Section 3, the definition of functions

$$h_{nk}(z) = -i \int_{\mathbb{T}} \operatorname{Im}\zeta \, d\mu_{nk}^{\circ}(\zeta) + \int_{\mathbb{T}} \frac{1+\zeta z}{1-\zeta z}(1-\operatorname{Re}\zeta)\, d\mu_{nk}^{\circ}(\zeta) \quad \text{for } z \in \mathbb{D},$$

and observe that $|\operatorname{Im} A_{nk}(1)| = |\operatorname{Im} h_{nk}(0)|$ and $|\operatorname{Re} A_{nk}(1)| = |\operatorname{Re} h_{nk}(0)|$. Then (3-3) shows that there exists an $M > 0$ such that $|\operatorname{Im} A_{nk}(1)| \le M|\operatorname{Re} A_{nk}(1)|$ for large n. Since $\log \widehat{\mu_{nk}^{\circ}}(1) = A_{nk}(1)(1+o(1))$ uniformly in k as $n \to \infty$, Lemma 2.1 and (4-2) imply that

$$\begin{aligned}\lim_{n\to\infty} \gamma_n e^{-\sigma_n(\mathbb{T})} &= \lim_{n\to\infty} \exp\bigl(i \arg \lambda_n + i \textstyle\sum_{k=1}^{k_n} \arg b_{nk} + \sum_{k=1}^{k_n} A_{nk}(1)\bigr) \\ &= \lim_{n\to\infty} \widehat{\nu_n}(1) = \gamma \exp(-t/2)\end{aligned}.$$

Consequently, we have $\lim_{n\to\infty} \sigma_n(\mathbb{T}) = t/2$, and $\lim_{n\to\infty} \gamma_n = \gamma$. In particular, we deduce that the family $\{\sigma_n\}_{n=1}^{\infty}$ is tight. Let σ be a weak cluster point of $\{\sigma_n\}_{n=1}^{\infty}$, and suppose that a subsequence σ_{n_j} converges weakly to σ as $j \to \infty$. Then we have $\sigma(\mathbb{T}) = t/2$. Moreover, Theorems 3.5 and 4.1 yield that

$$\gamma^p \exp(-tp^2/2) = \widehat{\nu}(p) = \gamma^p \exp\Bigl(\int_{\mathbb{T}} \frac{\zeta^p - 1 - ip \operatorname{Im}\zeta}{1-\operatorname{Re}\zeta}\, d\sigma(\zeta)\Bigr) \quad \text{for } p \in \mathbb{Z}.$$

Taking the absolute value of both sides, we have

$$\begin{aligned} 0 &= \frac{t}{2}p^2 - \int_{\mathbb{T}} \frac{1-\operatorname{Re}\zeta^p}{1-\operatorname{Re}\zeta}\, d\sigma(\zeta) \\ &= \sigma(\mathbb{T})p^2 - \int_{\mathbb{T}} \frac{1-\operatorname{Re}\zeta^p}{1-\operatorname{Re}\zeta}\, d\sigma(\zeta) = \int_{\mathbb{T}} \Bigl(p^2 - \frac{1-\operatorname{Re}\zeta^p}{1-\operatorname{Re}\zeta}\Bigr)\, d\sigma(\zeta), \end{aligned}$$

for every $p \in \mathbb{Z}$. Therefore, we deduce that $p^2 = (1-\operatorname{Re}\zeta^p)/(1-\operatorname{Re}\zeta)$ for σ-almost all $\zeta \in \mathbb{T}$. Since the function $\zeta \mapsto (1-\operatorname{Re}\zeta^p)/(1-\operatorname{Re}\zeta)$ achieves its maximum p^2 only at $\zeta = 1$, we conclude $\sigma = (t/2)\delta_1$. Hence, the full sequence σ_n must converge weakly to σ because σ is unique. The result now follows by Theorem 3.5. □

Remark. The attentive reader might have noticed that a crucial step in the proof of Corollary 4.2 is that (4-1) uniquely determines the measure σ. The following example inspired by [Parthasarathy 1967, Chapter IV, Section 8] shows that this phenomenon does not happen in general. Consider the function

$$f(\zeta) = 4\pi \operatorname{Im}\zeta \quad \text{for } \zeta \in \mathbb{T}.$$

Note that $\int_{\mathbb{T}} \zeta^p f(\zeta)\, dm(\zeta) = 2p\pi i$ when $p = \pm 1$, and $\int_{\mathbb{T}} \zeta^p f(\zeta)\, dm(\zeta) = 0$ for other p's. Denote by f^+ the positive part of f and by f^- the negative part of f. Let us introduce measures

$$d\sigma_1(\zeta) = (1-\operatorname{Re}\zeta) f^+(\zeta)\, dm(\zeta)$$

and

$$d\sigma_2(\zeta) = (1 - \operatorname{Re}\zeta) f^-(\zeta)\, dm(\zeta).$$

Then $\sigma_1 \neq \sigma_2$, and yet

$$\exp\Big(\int_{\mathbb{T}} \frac{\zeta^p - 1 - ip\operatorname{Im}\zeta}{1 - \operatorname{Re}\zeta}\, d\sigma_1(\zeta)\Big) = \exp\Big(\int_{\mathbb{T}} \frac{\zeta^p - 1 - ip\operatorname{Im}\zeta}{1 - \operatorname{Re}\zeta}\, d\sigma_2(\zeta)\Big)$$

for every $p \in \mathbb{Z}$.

We conclude this section by showing a result concerning the weak convergence to the Haar measure m.

Theorem 4.3. *The sequence*

$$\lambda_n \prod_{k=1}^{k_n} \int_{\mathbb{T}} \zeta\, d\mu_{nk}(\zeta)$$

converges to zero as $n \to \infty$ if and only if the sequence μ_n converges weakly to m as $n \to \infty$.

Proof. Define the measures σ_n and the complex numbers γ_n as in the proof of Theorem 4.1. Then Lemma 2.1 and the proof of Corollary 4.2 show that the sequence $\gamma_n e^{-\sigma_n(\mathbb{T})}$ converges if and only if the sequence

$$\widehat{\nu_n}(1) = \lambda_n \prod_{k=1}^{k_n} \int_{\mathbb{T}} \zeta\, d\mu_{nk}(\zeta)$$

converges. Moreover, the two sequences have the same limit. Therefore, the result follows at once by Theorem 3.6. □

Remark. Theorem 4.3 shows that if the measures ν_n converge weakly to the Haar measure m, then the measures μ_n converge weakly to m as well. The example below indicates that the converse of this fact may not be true in general. Define

$$\rho_n = (1 - 1/n)\delta_1 + (1/n)\delta_{-1} \quad \text{for } n \in \mathbb{N}.$$

Note that

$$\widehat{\rho_n}(p) = \begin{cases} 1 & \text{if } p \text{ is even,} \\ 1 - 2/n & \text{if } p \text{ is odd.} \end{cases}$$

Theorem 3.6 shows that the sequence $\rho_n \boxtimes \rho_n \boxtimes \cdots \boxtimes \rho_n$, in which ρ_n occurs n^2 times, converges weakly to m as $n \to \infty$. However, the sequence $\rho_n \circledast \rho_n \circledast \cdots \circledast \rho_n$, in which ρ_n again occurs n^2 times, converges weakly to the probability measure $\nu = \frac{1}{2}(\delta_1 + \delta_{-1})$ as $n \to \infty$. Note that the limit law ν is $\circledast$-infinitely divisible because $\nu \circledast \nu = \nu$. However, the measure ν is neither $\boxtimes$-infinitely divisible nor $\boxtimes$-infinitely divisible.

5. Measures on $\mathbb{R}$

Let $\{\nu_{nk} : n \in \mathbb{N},\ 1 \le k \le k_n\} \subset \mathcal{M}_{\mathbb{R}}$ be an infinitesimal array. Define the probability measures ν_{nk}° by $d\nu_{nk}^{\circ}(t) = d\nu_{nk}(t + a_{nk})$, where the numbers $a_{nk} \in [-1, 1]$ are given by $a_{nk} = \int_{|t|<1} t\, d\nu_{nk}(t)$. Note that the array $\{\nu_{nk}^{\circ}\}_{n,k}$ is infinitesimal, and that $\lim_{n\to\infty} \max_{1\le k\le k_n} |a_{nk}| = 0$. We introduce analytic functions

$$f_{nk}(z) = \int_{-\infty}^{\infty} \frac{t}{1+t^2}\, d\nu_{nk}^{\circ}(t) + \int_{-\infty}^{\infty} \left(\frac{1+tz}{z-t}\right) \frac{t^2}{1+t^2}\, d\nu_{nk}^{\circ}(t) \quad \text{for } z \in \mathbb{C}^+$$

and note that

$$f_{nk}(z) = \int_{-\infty}^{\infty} \frac{tz}{z-t}\, d\nu_{nk}^{\circ}(t)$$

for every n and k. Observe that $\operatorname{Im} f_{nk}(z) < 0$ for all $z \in \mathbb{C}^+$ unless the measure $\nu_{nk}^{\circ} = \delta_0$, and that $f_{nk}(z) = o(|z|)$ as $z \to \infty$ nontangentially. The following result is analogous to Lemma 3.1.

Lemma 5.1. *Let $\Gamma_{\alpha,\beta}$ be a truncated cone. Then for sufficiently large n, we have*

$$E_{\nu_{nk}^{\circ}}(z) = f_{nk}(z + a_{nk})(1 + v_{nk}(z)),$$

where the sequence

$$v_n(z) = \max_{1\le k\le k_n} |v_{nk}(z)|$$

has properties that $\lim_{n\to\infty} v_n(z) = 0$ for all $z \in \Gamma_{\alpha,\beta}$, and that $v_n(z) = o(1)$ uniformly in n as $|z| \to \infty$ for $z \in \Gamma_{\alpha,\beta}$.

Proof. It was shown in [Bercovici and Pata 1999, Proposition 6.1] that the function $E_{\nu_{nk}^{\circ}}(z)$ can be approximated by the function $f_{nk}(z)$ in the way we stated in the present lemma for sufficiently large n. To prove the lemma, we only need to show that the function $f_{nk}(z + a_{nk})$ can be approximated by the function $f_{nk}(z)$ in the same way. As in Lemma 3.1, we may assume that $\operatorname{Im} f_{nk}(z) < 0$ for all n, k, and $z \in \Gamma_{\alpha,\beta}$. Then it suffices to show that the sequence

$$u_n(z) = \max_{1\le k\le k_n} \left| \frac{f_{nk}(z+a_{nk})}{f_{nk}(z)} - 1 \right|$$

converges to zero as $n \to \infty$ for every $z \in \Gamma_{\alpha,\beta}$, and that $u_n(z) = o(1)$ uniformly in n as $z \to \infty$ for $z \in \Gamma_{\alpha,\beta}$. Indeed, we have, for all n, k, and $z \in \Gamma_{\alpha,\beta}$, that

$$\begin{aligned} |f_{nk}(z + a_{nk}) - f_{nk}(z)| &\le |a_{nk}| \int_{-\infty}^{\infty} \frac{t^2}{|z+a_{nk}-t||z-t|}\, d\nu_{nk}^{\circ}(t) \\ &= |a_{nk}| \int_{-\infty}^{\infty} \frac{t^2}{|z-t|^2} \frac{|z-t|}{|z+a_{nk}-t|}\, d\nu_{nk}^{\circ}(t) \\ &\le 2\sqrt{1+\alpha^2}|a_{nk}| \int_{-\infty}^{\infty} \frac{t^2}{|z-t|^2}\, d\nu_{nk}^{\circ}(t), \end{aligned}$$

while

$$|f_{nk}(z)| \geq |\mathrm{Im}\, f_{nk}(z)| > \mathrm{Im}\, z \int_{-\infty}^{\infty} \frac{t^2}{|z-t|^2}\, d\nu_{nk}^{\circ}(t).$$

Hence, we conclude that

$$\left|\frac{f_{nk}(z+a_{nk})}{f_{nk}(z)} - 1\right| \leq \frac{|f_{nk}(z+a_{nk}) - f_{nk}(z)|}{|\mathrm{Im}\, f_{nk}(z)|} \leq 2\sqrt{1+\alpha^2}\,\frac{|a_{nk}|}{\mathrm{Im}\, z}.$$

The result follows since $\lim_{n\to\infty} \max_{1\leq k\leq k_n} |a_{nk}| = 0$. □

As shown in [Bercovici and Wang 2008a, Lemma 3.1], the functions $f_{nk}(z)$ possess remarkable features: For $y \geq 1$, and for sufficiently large n, we have

$$\begin{aligned} |\mathrm{Re}\, f_{nk}(iy)| &\leq (3+6y)|\mathrm{Im}\, f_{nk}(iy)| && \text{for } 1 \leq k \leq k_n,\\ |\mathrm{Re}(f_{nk}(iy) - b_{nk}(y))| &\leq 2|\mathrm{Im}\, f_{nk}(iy)| && \text{for } 1 \leq k \leq k_n, \end{aligned}$$

where the real-valued function $b_{nk}(y)$ is given by

$$b_{nk}(y) = \int_{|t|\geq 1} \left(a_{nk} + \frac{(t-a_{nk})y^2}{y^2+(t-a_{nk})^2}\right) d\nu_{nk}(t).$$

Proposition 5.2. *Let $\{c_n\}_{n=1}^{\infty}$ be a sequence of real numbers.*

(i) *For any $y \geq 1$, the sequence $\{c_n + \sum_{k=1}^{k_n} E_{\nu_{nk}}(iy)\}_{n=1}^{\infty}$ converges if and only if the sequence $\{c_n + \sum_{k=1}^{k_n} [a_{nk} + f_{nk}(iy)]\}_{n=1}^{\infty}$ converges. The two sequences have the same limit.*

(ii) *If*

$$L = \sup_{n\geq 1} \sum_{k=1}^{k_n} \int_{-\infty}^{\infty} \frac{t^2}{1+t^2}\, d\nu_{nk}^{\circ}(t) < +\infty,$$

then $c_n + \sum_{k=1}^{k_n} E_{\nu_{nk}}(iy) = o(y)$ uniformly in n as $y \to \infty$ if and only if $c_n + \sum_{k=1}^{k_n} (a_{nk} + f_{nk}(iy)) = o(y)$ uniformly in n as $y \to \infty$.

Proof. Fix $y \geq 1$. Since $E_{\nu_{nk}^{\circ}}(z) = E_{\nu_{nk}}(z+a_{nk}) - a_{nk}$, we obtain from Lemma 5.1 that

$$-E_{\nu_{nk}}(iy) + a_{nk} = -f_{nk}(iy)(1+u_{nk}(iy)),$$

where the sequence $u_n(iy) = \max_{1\leq k\leq k_n} |u_{nk}(iy)|$ converges to zero as $n \to \infty$. Thus, (i) follows from (2-4) and (2-5) by setting $z_{nk} = -iE_{\nu_{nk}}(iy) + ia_{nk}$, $w_{nk} = -if_{nk}(iy)$ and $s_{nk} = 0$.

Now, let us prove (ii). Since $\lim_{n\to\infty} \max_{1\leq k\leq k_n} |a_{nk}| = 0$ and $u_n(iy) = o(1)$ uniformly in n as $y \to \infty$, we may assume that $|a_{nk}| \leq 1/2$ and that $u_n(iy) < 1/6$

for all n and k and for sufficiently large y. Observe that

$$\begin{aligned}\sum_{k=1}^{k_n}|b_{nk}(y)| &= \sum_{k=1}^{k_n}\left|\int_{|t|\geq 1}\left(a_{nk}+\frac{(t-a_{nk})y^2}{y^2+(t-a_{nk})^2}\right)d\nu_{nk}(t)\right| \\ &\leq (1+y)\sum_{k=1}^{k_n}\int_{|t|\geq 1}\tfrac{1}{2}\,d\nu_{nk}(t) \leq 5y\sum_{k=1}^{k_n}\int_{|t|\geq 1}\tfrac{1}{5}\,d\nu_{nk}(t) \\ &\leq 5y\sum_{k=1}^{k_n}\int_{|t|\geq 1}\frac{(t-a_{nk})^2}{1+(t-a_{nk})^2}\,d\nu_{nk}(t) \leq 5yL.\end{aligned}$$

Then (2-4) and (2-5) imply that

$$\left|\left(\sum_{k=1}^{k_n}E_{\nu_{nk}}(iy)\right)-\left(\sum_{k=1}^{k_n}(a_{nk}+f_{nk}(iy))\right)\right| \leq \frac{1}{2}\left|\sum_{k=1}^{k_n}\operatorname{Im} f_{nk}(iy)\right| + 5yLu_n(iy),$$

and

$$\frac{1}{2}\left|\sum_{k=1}^{k_n}\operatorname{Im} f_{nk}(iy)\right| \leq \left|\sum_{k=1}^{k_n}\operatorname{Im} E_{\nu_{nk}}(iy)\right| + 5yLu_n(iy),$$

for $n \in \mathbb{N}$. Then (ii) follows since $u_n(iy) = o(1)$ uniformly in n as $y \to \infty$. □

We are now ready for the main result of this section. With Proposition 5.2 in hand, it follows by applying almost word-for-word the argument of [Bercovici and Wang 2008a, Theorem 3.3]. Therefore, we will not repeat this rather lengthy proof here but instead refer to that paper for its details.

Theorem 5.3. *Fix a real number γ and a finite positive Borel measure σ on $\mathbb{R}$. Let $\{\nu_{nk}\}_{n,k}$ be an infinitesimal array in $\mathcal{M}_{\mathbb{R}}$, and let $\{c_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Then the following statements are equivalent*:

(i) *The sequence $\delta_{c_n} * \nu_{n1} * \nu_{n2} * \cdots * \nu_{nk_n}$ converges weakly to $\nu_{*}^{\gamma,\sigma}$.*

(ii) *The sequence $\delta_{c_n} \boxplus \nu_{n1} \boxplus \nu_{n2} \boxplus \cdots \boxplus \nu_{nk_n}$ converges weakly to $\nu_{\boxplus}^{\gamma,\sigma}$.*

(iii) *The sequence $\delta_{c_n} \uplus \nu_{n1} \uplus \nu_{n2} \uplus \cdots \uplus \nu_{nk_n}$ converges weakly to $\nu_{\uplus}^{\gamma,\sigma}$.*

(iv) *The sequence of measures*

$$d\sigma_n(t) = \sum_{k=1}^{k_n}\frac{t^2}{1+t^2}\,d\nu_{nk}^{\circ}(t)$$

converges weakly on $\mathbb{R}$ to σ, and the sequence of numbers

$$\gamma_n = c_n + \sum_{k=1}^{k_n}\left(a_{nk}+\int_{-\infty}^{\infty}\frac{t}{1+t^2}\,d\nu_{nk}^{\circ}(t)\right)$$

converges to γ as $n \to \infty$.

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