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We study the equation $\Delta_g u - (n-2)/(4(n-1))R(g)u + Ku^p = 0$ for p in $1+\zeta \leq p \leq (n+2)/(n-2)$ on locally conformally flat compact manifolds (M^n, g). We prove that when the scalar curvature $R(g) \equiv 0$ and $n \geq 5$, under suitable conditions on K, all positive solutions u with bounded energy have uniform upper and lower bounds. In our previous 2007 paper, we also assumed this energy bound condition for the uniform estimates in the lower-dimensional case. We now give an example showing that this condition is necessary.

1. Introduction

Let (M^n, g) be an n-dimensional compact manifold with metric g, and denote by $R(g)$ the scalar curvature of g. Let u be a positive function defined on M. The scalar curvature of the conformally deformed metric $u^{4/(n-2)}g$ is given by

$$R(u^{4/(n-2)}g) = -c(n)^{-1}u^{-(n+2)/(n-2)}(\Delta_g u - c(n)R(g)u),$$

where $c(n) = (n-2)/(4(n-1))$.

The Yamabe theorem, which was proved by Trudinger [1968], Aubin [1976] and Schoen [1984], says that there exists a $u > 0$ such that $R(u^{4/(n-2)}g)$ is equal to some constant K. The PDE formulation of this theorem is that the equation

$$\Delta_g u - c(n)R(g)u + c(n)Ku^{(n+2)/(n-2)} = 0$$

has a positive solution for some constant K.

J. Escobar and R. Schoen [1986] extended this result to the case when K is a function on M. They proved that, under certain conditions on K, the above equation has a positive solution u when $R(g) > 0$ or $R(g) \equiv 0$.

In fact, the solution in those existence results minimizes the associated constraint variational problem and can be obtained as a limit of a sequence of solutions of the corresponding subcritical equations. Therefore, a natural question is whether

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nonminimal solutions can also be produced from solutions of the subcritical equations. We would like to know if there are uniform estimates for solutions of the equation

$$\Delta_g u - c(n)R(g)u + Ku^p = 0, \qquad \text{where } 1+\zeta \le p \le (n+2)/(n-2). \tag{1}$$

This was proved to be true by Schoen [1988a; 1991] when K is a positive constant, $R(g) > 0$, and (M^n, g) is locally conformally flat and not conformally diffeomorphic to S^n. When the manifold has dimension $n = 3$, the work of Y. Li and M. Zhu [1999] shows that the same estimates hold when the locally conformally flat condition on M is dropped. This result was extended to dimensions $n = 4, 5$ by O. Druet [2003; 2004], and then to dimensions $n \le 7$ independently by Y. Li and L. Zhang [Li and Zhang 2005] and F. C. Marques [2005]; when the dimension $n \ge 8$, it was proved by Li and Zhang [2005] under an additional assumption on the Weyl tensor of the backgroud metric g.

When K is a positive function, Y. Li and M. Zhu [1999] obtained uniform estimates for the solutions when (M, g) is a 3-dimensional compact manifold with $R(g) > 0$ and is not conformally diffeomorphic to S^3. When the dimension $n \ge 4$, the same estimates hold on a locally conformally flat, scalar positive, compact manifold that is not conformally diffeomorphic to S^n, under the following additional flatness condition on K: near each critical point of K, there exists a neighborhood and a constant C_0 such that in that neighborhood

$$|\nabla^p K| \le C_0 |\nabla K|^{(n-2+\epsilon-p)/(n-3+\epsilon)} \quad \text{for } 2 \le p \le n-2,$$

where $\epsilon > 0$ and $\nabla^p K$ is the p-th covariant derivative of K. This result is proved by fine blow-up analysis similar to the analyses in [Li 1995] and [Schoen 1988a].

In [Yan 2007], we studied this problem on 3 and 4 dimensional locally conformally flat compact manifolds with zero scalar curvature. In this paper, we consider the higher-dimensional case $n \ge 5$.

When the scalar curvature $R(g) \equiv 0$ on the manifold M, Equation (1) becomes

$$\Delta_g u + Ku^p = 0, \quad \text{where } 1+\zeta \le p \le (n+2)/(n-2). \tag{2}$$

The necessary conditions for the existence of a solution $u > 0$ are that K changes sign on M and $\int_M K\,dv_g < 0$.

The corresponding existence result is as follows.

Theorem 1.1 [Escobar and Schoen 1986]. *Suppose M is locally conformally flat with zero scalar curvature. Suppose K is a nonzero smooth function on M satisfying the condition that there is a maximum point $P_0 \in M$ of K at which all derivatives of K of order less than or equal to $(n-3)$ vanish. Then K is the scalar curvature of a metric $\bar{g} = u^{4/(n-2)} g$ for some $u > 0$ on M if and only if K is* *such that K changes sign and $\int_M K\,dv_g < 0$. When the dimension $n = 3$ or 4, the flatness condition on K is automatically satisfied, and the locally conformally flat assumption on M can be removed.*

There is a compactness theorem when the dimension of M is equal to 3 or 4:

Theorem 1.2 [Yan 2007]. *Let (M, g) be a 3- or 4-dimensional locally conformally flat compact manifold with $R(g) \equiv 0$. Let $\mathcal{K} := \{K \in C^3(M) : K > 0$ somewhere on M, $\int_M K\,dv_g \le -C_K{}^{-1} < 0$, and $\|K\|_{C^3(M)} \le C_K\}$ for some constant C_K, and let $S_\Lambda := \{u : u > 0$ solves (2) with $K \in \mathcal{K}$, and $E(u) := \int_M |\nabla u|^2 dv_g \le \Lambda\}$. Then there exists $C = C(M, g, C_K, \Lambda, \zeta) > 0$ such that $u \in S_\Lambda$ satisfies $\|u\|_{C^3(M)} \le C$ and $\min_M u \ge C^{-1}$.*

In Section 2, we will give an example that shows that these estimates *cannot* be improved to be independent of the energy $E(u)$.

Next we give a similar theorem on manifolds of dimension $n \ge 5$. We first need to define a flatness condition on K as follows.

Definition 1.3. A function $K \in C^{n-2}(M)$ is said to satisfy the *flatness condition* $(*)$ if near each critical point P of K where $K(P) > 0$, there exist a neighborhood and a constant C_0 such that, in that neighborhood,

$$|\nabla^p K| \le C_0 |\nabla K|^{(n-2-p)/(n-3)} \quad \text{for } 2 \le p \le n-3,$$

where $\nabla^p K$ is the p-th covariant derivative of K.

In particular, this implies that all partial derivatives of K up to order $n-3$ vanish at those critical points, and the order of flatness is the same as that in Theorem 1.1. A simple example of a function satisfying this condition is a function that can be expressed near the critical points as $K(z) = a + b|z|^{n-2}$, where a and b are constants and z is a local coordinate system centered at the critical point. This type of flatness condition also appeared when Y. Li [1995; 1996] studied the problem of prescribing scalar curvature functions on S^n.

We are ready to state the theorem:

Theorem 1.4. *Let (M^n, g) be a locally conformally flat compact manifold with $R(g) \equiv 0$ and dimension $n \ge 5$. Let $K \in C^{n-2}(M)$ be a function satisfying the flatness condition $(*)$; assume K is positive somewhere on M and $\int_M K\,dv_g < 0$. If u is a positive solution of Equation (2) with bounded energy*

$$E(u) := \int_M |\nabla u|^2 dv_g \le \Lambda,$$

then there exists a positive constant C such that $\|u\|_{C^3(M)} \le C$ and $\min_M u \ge C^{-1}$, where C depends on M, g, $\|K\|_{C^{n-2}(M)}$, $\int_M K\,dv_g$, Λ, and ζ.

2. The example and some notations

Let (M^n, g) be a compact manifold with $R(g) \equiv 0$ and $n = 3$ or 4. (In fact in this example, M does not need to be locally conformally flat.) We choose $K \in C^3(M)$ satisfying the conditions

- $K > 0$ somewhere on M;
- $\int_M K\, dv_g \le -C_K^{-1} < 0$ and $\|K\|_{C^3(M)} \le C_K$, where C_K is a positive constant;,
- the set $\{x \in M : K(x) = 0\} = \overline{U}$ for some open set $U \subset M$.

We define

$$K_i(x) = \begin{cases} K(x)/i & \text{if } K(x) > 0, \\ K(x) & \text{if } K(x) \le 0. \end{cases}$$

Because on ∂U all derivatives of K up to order 3 are zero, it follows that $K_i \in C^3(M)$. Furthermore, this definition means that $K_i \in \mathcal{K}$, where $\mathcal{K}$ is as defined in Theorem 1.2. Then by Theorem 1.1, there exists a $u_i > 0$ that satisfies $\Delta_g u_i + K_i u_i^{(n+2)/(n-2)} = 0$.

Now suppose there is a constant C independent of i such that $\max_M u_i \le C$. As proved in [Yan 2007, Section 2], this implies that $\{u_i\}$ is uniformly bounded away from 0 and $\|u_i\|_{C^3(M)}$ is bounded above uniformly. Then, passing to a subsequence, $\{u_i\}$ converges in the C^2 norm to a function $u > 0$, and u satisfies $\Delta_g u + \tilde{K} u^{(n+2)/(n-2)} = 0$, where

$$\tilde{K}(x) = \lim_{i\to\infty} K_i(x) = \begin{cases} 0 & \text{if } K(x) > 0, \\ K(x) & \text{if } K(x) \le 0. \end{cases}$$

However, because $\tilde{K}$ is nowhere positive and somewhere negative, the equation $\Delta_g u + \tilde{K} u^{(n+2)/(n-2)} = 0$ cannot have a positive solution by Theorem 1.1. This contradiction shows that estimates like the ones in Theorem 1.2 cannot be true without the energy bound assumption on u.

Next we prove Theorem 1.4. We first give some definitions and a lemma which will be used in the proof.

Definition 2.1. We call a point $\bar{x}$ on a manifold M a *blow-up point* of a sequence $\{u_i\}$ if $\bar{x} = \lim_{i\to\infty} x_i$ for some $\{x_i\} \subset M$ and $u_i(x_i) \to \infty$.

Definition 2.2. Suppose u_i satisfies $\Delta_{g_i} u_i - c(n)R(g_i)u_i + K_i u_i^{p_i} = 0$, where $\{g_i\}$ converges to some metric g_0. A point $\bar{x} \in M$ is called an *isolated blow-up point* of $\{u_i\}$ corresponding to $\{g_i\}$ if there exist local maximum points x_i of u_i and a fixed radius $r_0 > 0$ such that

- $x_i \to \bar{x}$,
- $u_i(x_i) \to \infty$, and

- $u_i(x) \le C(d_{g_i}(x, x_i))^{-2/(p_i-1)}$ for any $x \in B_{r_0}(x_i)$, where the constant C is independent of i.

Lemma 2.3. *If $\bar{x} = \lim_{i\to\infty} x_i$ is an isolated blow-up point of $\{u_i\}$ corresponding to $\{g_i\}$, and K_i is uniformly bounded, then there exists a constant C independent of i and r such that, for any $0 < r \le r_0$,*

$$\max_{\partial B_r(x_i)} u_i(x) \le C \min_{\partial B_r(x_i)} u_i(x).$$

This can be proved as in the proof of [Yan 2007, Lemma 5.2].

Definition 2.4. We say $\bar{x}$ is a *simple blow-up point* of $\{u_i\}$ if it is an isolated blow-up point and there exists an $\bar{r} > 0$ independent of i such that $\overline{w}_i(r)$ has only one critical point for $r \in (0, \bar{r})$. Here

$$\overline{w}_i(r) := r^{2/(p_i-1)}\bar{u}_i(r) = \mathrm{Vol}(S_r)^{-1}\int_{S_r} |z|^{2/(p_i-1)} u_i(z)\, d\Sigma_g$$

and z is the conformally flat coordinate system centered at each x_i.

3. Initial steps of the proof of Theorem 1.4

The proof of Theorem 1.4 follows along the same line of reasoning as the proof of Theorem 1.2, which is done in [Yan 2007]. As proved in [Section 2] there, a lower bound on u follows directly if there is a uniform upper bound on u. By standard elliptic theory and the Sobolev embedding theorem, a bound on the C^0 norm of u easily implies a bound on its C^3 norm. Therefore, to prove Theorem 1.4 we only need to show that there is a uniform upper bound on u.

By an argument identical to that in [Yan 2007, Section 3], we can show that there exists a positive constant $\eta = \eta(M, g, n, \|K\|_{C^{n-2}(M)}, \Lambda)$ such that, on the set $K_\eta := \{x \in M : K(x) < \eta\}$, u has a uniform upper bound depending only on $M, g, n, \|K\|_{C^{n-2}(M)}$, and Λ. Thus it is left to show that u is uniformly bounded on the set where $K \ge \eta$. We have the following proposition.

Proposition 3.1. *Given $\epsilon > 0$ and $R \gg 0$, there exists $C = C(\epsilon, R)$ such that, if u is a solution of Equation (2) and*

$$\max_{x\in M}\big((d_g(x, K_{\eta/2}))^{2/(p-1)} u(x)\big) > C,$$

then there exists $\{x_1, \ldots, x_N\} \subset M \setminus K_{\eta/2}$ with N depending on u satifying the following:

- *Each x_i locally maximizes u, and the geodesic balls $\{B_{Ru(x_i)^{-(p-1)/2}}(x_i)\}$ are disjoint.*

- $|(n+2)/(n-2)-p|<\epsilon$, *and, in the coordinate system* y *chosen so that* $z=y/u(x_i)^{(p-1)/2}$ *is the conformally flat coordinate system centered at* x_i, *we have*
$$\left\|u(x_i)^{-1}u\big(y/u(x_i)^{(p-1)/2}\big)-\bar{v}(y)\right\|_{C^2(B_{2R}(0))}<\epsilon$$
on the ball $B_{2R}(0)\subset\mathbb{R}^n(y)$, *where*
$$\bar{v}(y)=\left(1+\frac{K(x_i)}{n(n-2)}|y|^2\right)^{-(n-2)/2}.$$
- *There exists* $C=C(\epsilon,R)$ *such that*
$$u(x)\le C\big(d_g(x,\overline{K_{\eta/2}\bigcup\{x_1,\dots,x_N\}})\big)^{-2/(p-1)}.$$

The proof is like that of [Yan 2007, Proposition 4.2], so we omit the details.

Now we are going to prove that u is uniformly bounded on $M\setminus K_\eta$. Suppose it is not. Then there are sequences $\{u_i\}$ and $\{p_i\}$ such that
$$\Delta_g u_i+Ku_i^{p_i}=0\quad\text{and}\quad\max_{M\setminus K_\eta}u_i\to\infty\ \text{as } i\to\infty.$$
Therefore $\max_{M\setminus K_\eta}((d_g(x,K_{\eta/2}))^{2/(p_i-1)}u_i(x))\to\infty$ as $i\to\infty$. Then for fixed $\epsilon>0$ and $R\gg0$ we can apply Proposition 3.1 to each u_i and find $x_{1,i},\dots,x_{N(i),i}$ such that

- each $x_{j,i}$ for $1\le j\le N(i)$ is a local maximum point of u_i, and
- the balls $B_{R/u_i(x_{j,i})^{(p_i-1)/2}}(x_{j,i})$ are disjoint.

For coordinates y centered at $x_{j,i}$ such that $y/u_i(x_{j,i})^{(p_i-1)/2}$ is the conformally flat coordinate system,
$$\left\|u_i(x_{j,i})^{-1}u_i\left(\frac{y}{u_i(x_{j,i})^{(p_i-1)/2}}\right)-\left(1+\frac{K(x_{j,i})}{n(n-2)}|y|^2\right)^{-(n-2)/2}\right\|_{C^2(B_{2R}(0))}<\epsilon,$$
$$u_i(x)\le C\big(d_g(x,\overline{K_{\eta/2}\bigcup\{x_{1,i},\dots,x_{N(i),i}\}})\big)^{-2/(p_i-1)}$$
for a constant $C=C(\epsilon,R)$.

Let $\sigma_i=\min\{d_g(x_{\alpha,i},x_{\beta,i}):\alpha\ne\beta,\ 1\le\alpha,\ \beta\le N(i)\}$. Without loss of generality we can assume $\sigma_i=d_g(x_{1,i},x_{2,i})$. There are two possibilities:

Case I. $\sigma_i\ge\varepsilon>0$. In this case, the points $x_{j,i}$ have isolated limiting points $x_1,x_2,\dots,$ which are isolated blow-up points of $\{u_i\}$ as defined above.

Case II. $\sigma_i\to0$. In this case, we rescale the coordinates to make the minimal distance 1: let $y=\sigma_i^{-1}z$, where z is the conformally flat coordinate system centered at $x_{1,i}$. We also rescale the function by defining $v_i(y)=\sigma_i^{2/(p_i-1)}u_i(\sigma_i y)$, which satisfies
$$\Delta_{g^{(i)}}v_i+K(\sigma_i y)v_i^{p_i}=0,$$

where the metric $g^{(i)}(y) = g_{\alpha\beta}(\sigma_i y)dy^\alpha dy^\beta$. As proved in [Yan 2007, Section 4], 0 is an isolated blow-up point of $\{v_i\}$.

In Sections 4 and 5, we will prove that neither Case I nor Case II can happen.

4. Ruling out Case I

If the blow-up points are all isolated, the argument of [Yan 2007, Section 6] shows that among the isolated blow-up points $\{x_1, x_2, \dots\}$, there must be one that is not a simple blow-up point; without loss of generality we assume it to be x_1. To simplify notation, we are going to rename it x_0. Let x_i be the local maximum point of u_i such that $\lim_{i\to\infty} x_i = x_0$.

Let $z = (z_1, \dots, z_n)$ be the conformally flat coordinates centered at each x_i. Since x_0 is not a simple blow-up point, $|z|^{2/(p_i-1)}\bar{u}_i(|z|)$ has, as a function of $|z|$, a second critical point at $|z| = r_i$ where $r_i \to 0$. Let $y = z/r_i$ and define $v_i(y) = r_i^{2/(p_i-1)}u_i(r_i y)$. Then $v_i(y)$ satisfies

$$\Delta_{g^{(i)}} v_i(y) + K_i(y)v_i(y)^{p_i} = 0,$$

where $g^{(i)}(y) = g_{\alpha\beta}(r_i y)dy^\alpha dy^\beta$ and $K_i(y) = K(r_i y)$.

By this definition $|y| = 1$ is the second critical point of $|y|^{2/(p_i-1)}\bar{v}_i(|y|)$. As shown in [Yan 2007, Section 6], 0 is a simple blow-up point of $\{v_i\}$.

4.1. *Estimates for v_i.* The following estimates are essentially the same as [Yan 2007, Proposition 5.3], except for a slightly different choice of parameters. However, we repeat the proof for completeness.

Proposition 4.1. *There exists a constant C independent of i such that*

- *if $0 \le |y| \le 1$, then*

$$v_i(y) \ge C v_i(0)\Big(1 + \frac{K_i(0)}{n(n-2)} v_i(0)^{4/(n-2)}|y|^2\Big)^{-(n-2)/2};$$

- *if $0 \le |y| \le R v_i(0)^{-(p_i-1)/2}$, then*

$$v_i(y) \le C v_i(0)\Big(1 + \frac{K_i(0)}{n(n-2)} v_i(0)^{p_i-1}|y|^2\Big)^{-(n-2)/2};$$

- *if $R v_i(0)^{-(p_i-1)/2} \le |y| \le 1$, then $v_i(y) \le C v_i(0)^{t_i}|y|^{-l_i}$, where l_i and t_i are chosen so that $(2n-5)/2 < \lim_{i\to\infty} l_i < n-2$ and $t_i = 1 - (p_i - 1)l_i/2$.*

Proof. Let $\rho_i := Rv_i(0)^{-(p_i-1)/2}$. By Proposition 3.1, when $0 \le |y| \le \rho_i$,

$$\begin{aligned}(1+\epsilon)v_i(0)\Big(1+\frac{K_i(0)}{n(n-2)}v_i(0)^{p_i-1}|y|^2\Big)^{-(n-2)/2} \\ &\ge v_i(y) \\ &\ge (1-\epsilon)v_i(0)\Big(1+\frac{K_i(0)}{n(n-2)}v_i(0)^{p_i-1}|y|^2\Big)^{-(n-2)/2} \\ &\ge (1-\epsilon)v_i(0)\Big(1+\frac{K_i(0)}{n(n-2)}v_i(0)^{4/(n-2)}|y|^2\Big)^{-(n-2)/2}.\end{aligned}$$

So we only need to find the upper and lower bounds on $v_i(y)$ when $\rho_i \le |y| \le 1$.

The lower bound. Let G_i be the Green's function of $\Delta_{g^{(i)}}$ that is singular at 0 and vanishes on ∂B_1. Since $\{g^{(i)}\}$ converges uniformly to the Euclidean metric, there exist constants C_1 and C_2 independent of i such that $C_1|y|^{2-n} \le G_i(y) \le C_2|y|^{2-n}$.

When $|y| = \rho_i$,

$$\begin{aligned} v_i(y) &\ge (1-\epsilon)v_i(0) \Big/ \Big(1+\frac{K_i(0)}{n(n-2)}v_i(0)^{p_i-1}|y|^2\Big)^{(n-2)/2} \\ &= (1-\epsilon)v_i(0) \Big/ \Big(1+\frac{K_i(0)}{n(n-2)}R^2\Big)^{(n-2)/2} \\ &= (1-\epsilon)\Big(R^{-2}+\frac{K_i(0)}{n(n-2)}\Big)^{-(n-2)/2}R^{2-n}v_i(0) \\ &\ge CR^{2-n}v_i(0) \\ &\ge CR^{2-n}v_i(0)^{(n-2)(p_i-1)/2-1} \qquad \text{(since } (n-2)(p_i-1)/2-1 \le 1) \\ &= Cv_i(0)^{-1}|y|^{2-n} \\ &\ge Cv_i(0)^{-1}G_i(y).\end{aligned}$$

With this constant C, we have $Cv_i(0)^{-1}G_i(y) = 0 < v_i(y)$ when $|y| = 1$,

We know that

$$\Delta_{g^{(i)}}(v_i(y) - Cv_i(0)^{-1}G_i(y)) = \Delta_{g^{(i)}}v_i(y) = -K_i(y)v_i(y)^{p_i} < 0$$

on $B_1 \setminus B_{\rho_i}$. Therefore, by the maximal principle, when $\rho_i \le |y| \le 1$,

$$v_i(y) > Cv_i(0)^{-1}G_i(y) \ge Cv_i(0)^{-1}|y|^{2-n}.$$

Now we need to compare $|y|^{2-n}v_i(0)^{-1}$ with

$$v_i(0)\cdot\Big(1+\frac{K_i(0)}{n(n-2)}v_i(0)^{4/(n-2)}|y|^2\Big)^{-(n-2)/2}$$

in order to get the desired lower bound:

$$v_i(0)^2|y|^{n-2}\Big(1+\frac{K_i(0)}{n(n-2)}v_i(0)^{4/(n-2)}|y|^2\Big)^{-(n-2)/2} \leq v_i(0)^2\Big(\frac{K_i(0)}{n(n-2)}v_i(0)^{4/(n-2)}\Big)^{-(n-2)/2} \leq C$$

for a constant C independent of i. Therefore

$$v_i(0)^{-1}|y|^{2-n} \geq C v_i(0)\Big(1+\frac{K_i(0)}{n(n-2)}v_i(0)^{4/(n-2)}|y|^2\Big)^{-(n-2)/2},$$

and consequently

$$v_i(y) \geq C v_i(0)\Big(1+\frac{K_i(0)}{n(n-2)}v_i(0)^{4/(n-2)}|y|^2\Big)^{-(n-2)/2}$$

when $\rho_i \leq |y| \leq 1$.

The upper bound. We are going to apply the same strategy of constructing a comparison function and using the maximal principle.

Define $\mathscr{L}_i\varphi := \Delta_{g^{(i)}}\varphi + K_i v_i^{p_i-1}\varphi$. By this definition $\mathscr{L}_i v_i = 0$. Let $M_i = \max_{\partial B_1} v_i$ and $C_i = (1+\epsilon)(K_i(0)/(n(n-2)))^{-(n-2)/2}$. Note that C_i is bounded above and below by constants independent of i. Consider the function

$$M_i|y|^{-n+2+l_i} + C_i v_i(0)^{t_i}|y|^{-l_i}.$$

When $|y| = \rho_i$,

$$\begin{aligned} v_i(y) &\leq (1+\epsilon)v_i(0) \Big/ \Big(1+\frac{K_i(0)}{n(n-2)}v_i(0)^{p_i-1}|y|^2\Big)^{(n-2)/2} \\ &= (1+\epsilon)v_i(0) \Big/ \Big(1+\frac{K_i(0)}{n(n-2)}R^2\Big)^{(n-2)/2} \\ &\leq C_i v_i(0) R^{-(n-2)} \\ &\leq C_i v_i(0) R^{-l_i} \\ &= C_i v_i(0)^{t_i}|y|^{-l_i}. \end{aligned}$$

When $|y| = 1$, the definition of M_i gives $v_i(y) \leq M_i = M_i|y|^{-n+2+l_i}$. Thus on $\{|y|=1\}\bigcup\{|y|=\rho_i\}$, we have $v_i(y) \leq M_i|y|^{-n+2+l_i} + C_i v_i(0)^{t_i}|y|^{-l_i}$.

In Euclidean coordinates,

$$\Delta|y|^{-l_i} = -l_i(n-2-l_i)|y|^{-l_i-2} \quad \text{and} \quad \Delta|y|^{-n+2+l_i} = -l_i(n-2-l_i)|y|^{-n+l_i}.$$

When i is sufficiently large, $g^{(i)}$ is close to the Euclidean metric. Therefore

$$\Delta_{g^{(i)}}|y|^{-l_i} \leq -\tfrac{1}{2}l_i(n-2-l_i)|y|^{-l_i-2} \tag{3}$$

and

$$\Delta_{g^{(i)}}|y|^{-n+2+l_i} \le -\tfrac{1}{2} l_i(n-2-l_i)|y|^{-n+l_i}.$$

Thus

$$\begin{aligned}\mathscr{L}_i(C_i v_i(0)^{t_i}|y|^{-l_i}) &= C_i v_i(0)^{t_i}\Delta_{g^{(i)}}|y|^{-l_i} + C_i v_i(0)^{t_i} K_i v_i(y)^{p_i-1}|y|^{-l_i}\\ &\le -Cl_i(n-2-l_i)v_i(0)^{t_i}|y|^{-l_i-2} + C' v_i(0)^{t_i} v_i(y)^{p_i-1}|y|^{-l_i}\end{aligned}$$

for some constants C and C' independent of i.

Lemma 2.3 and the upper bound on $v_i(y)$ when $|y| \le \rho_i$ imply that

$$\bar{v}_i(\rho_i) \le C \frac{(1+\epsilon)v_i(0)}{\left(1+\dfrac{K_i(0)}{n(n-2)} v_i(0)^{p_i-1}(\rho_i)^2\right)^{(n-2)/2}} \le C v_i(0) R^{2-n}.$$

Then since 0 is a simple blow-up point and $r^{2/(p_i-1)}\bar{v}_i(r)$ is decreasing from ρ_i to 1, we have

$$\begin{aligned}|y|^{2/(p_i-1)}\bar{v}_i(|y|) &\le {\rho_i}^{2/(p_i-1)} \cdot \bar{v}_i(\rho_i)\\ &\le C R^{2/(p_i-1)+2-n}.\end{aligned}$$

Thus again by Lemma 2.3, we have

(4) $$v_i(y)^{p_i-1} \le C\bar{v}_i(|y|)^{p_i-1} \le C|y|^{-2} R^{2-(n-2)(p_i-1)},$$

and hence $v_i(y)^{p_i-1}|y|^{-l_i} \le C|y|^{-2-l_i} R^{2-(n-2)(p_i-1)}$. Therefore

$$\mathscr{L}_i(C_i v_i(0)^{t_i}|y|^{-l_i}) \le \big(-Cl_i(n-2-l_i) + C' R^{2-(n-2)(p_i-1)}\big) v_i(0)^{t_i}|y|^{-l_i-2}.$$

Our choice of l_i means that $l_i(n-2-l_i)$ is always bounded below by some positive constant independent of i. When i is sufficiently large, $2-(n-2)(p_i-1) < 0$, so we can choose R big enough so that $-Cl_i(n-2-l_i) + C' R^{2-(n-2)(p_i-1)} < 0$, which implies $\mathscr{L}_i(C_i v_i(0)^{t_i}|y|^{-l_i}) < 0$.

Similarly,

$$\begin{aligned}\mathscr{L}_i(M_i|y|^{-n+2+l_i}) &= M_i\Delta_{g^{(i)}}|y|^{-n+2+l_i} + M_i K_i v_i^{p_i-1}|y|^{-n+2+l_i}\\ &\le -\tfrac{1}{2} l_i(n-2-l_i) M_i|y|^{-n+l_i} + K_i M_i R^{2-(n-2)(p_i-1)}|y|^{-n+l_i}\end{aligned}$$

by Equations (3) and (4). We can choose R large enough so that

$$-\tfrac{1}{2} l_i(n-2-l_i) + K_i R^{2-(n-2)(p_i-1)} < 0,$$

and hence $\mathscr{L}_i(M_i|y|^{-n+2+l_i}) < 0$. Therefore, when $\rho_i \le |y| \le 1$,

$$\mathscr{L}_i(M_i|y|^{-n+2+l_i} + C_i v_i(0)^{t_i}|y|^{-l_i}) < 0.$$

Then, by the maximal principle, $v_i(y) \le M_i|y|^{-n+2+l_i} + C_i v_i(0)^{t_i}|y|^{-l_i}$.

By Lemma 2.3 and because 0 is a simple blow-up point, we have

$$M_i \le C\theta^{2/(p_i-1)}\bar{v}_i(\theta) \le C\theta^{2/(p_i-1)}\left(M_i\theta^{-n+2+l_i} + C_i v_i(0)^{t_i}\theta^{-l_i}\right)$$
$$= C\theta^{2/(p_i-1)-n+2+l_i} M_i + C\theta^{2/(p_i-1)} \cdot C_i v_i(0)^{t_i}\theta^{-l_i}$$

for $\rho_i \le \theta \le 1$ and some constant C independent of i.

Note that

$$\lim_{i\to\infty}\left(\frac{2}{p_i-1} - n + 2 + l_i\right) = -\frac{n-2}{2} + \lim_{i\to\infty} l_i > -\frac{n-2}{2} + \frac{2n-5}{2} > 0$$

because $n \ge 5$.

Since $\rho_i \to 0$, we can choose θ small enough (fixed and independent of i) to absorb the first term on the right side of the above inequality into the left side. We then get $M_i \le 2C\theta^{2/(p_i-1)} \cdot C_i v_i(0)^{t_i}\theta^{-l_i} \le C v_i(0)^{t_i}$.

Therefore

$$v_i(y) \le M_i|y|^{-n+2+l_i} + C_i v_i(0)^{t_i}|y|^{-l_i}$$
$$\le M_i|y|^{-l_i} + C_i v_i(0)^{t_i}|y|^{-l_i} \le C v_i(0)^{t_i}|y|^{-l_i}. \qquad \square$$

4.2. ***A preliminary estimate for $\delta_i := (n+2)/(n-2) - p_i$.*** First we prove a technical lemma.

Lemma 4.2. *When $\sigma < 1$ and $0 \le \kappa \le n-2$,*

$$\int_{|y|\le\sigma} |y|^\kappa v_i(y)^{p_i+1} dy \le C v_i(0)^{-2\kappa/(n-2)+(n-2+\kappa)/2\delta_i},$$

where C is independent of i.

Proof. Let $\rho_i := R v_i(0)^{-(p_i-1)/2}$. By Proposition 4.1

$$\int_{|y|\le\rho_i} |y|^\kappa v_i(y)^{p_i+1}\, dy \le C v_i(0)^{p_i+1}\int_{|y|\le\rho_i} |y|^\kappa\, dy$$
$$\le C v_i(0)^{p_i+1-(n+\kappa)(p_i-1)/2} = C v_i(0)^{-2\kappa/(n-2)+(n-2+\kappa)/2\delta_i}.$$

Since $n \ge 5$, our choice of l_i gives

$$\lim_{i\to\infty}(n+\kappa - l_i(p_i+1)) = n+\kappa - \frac{2n}{n-2}\lim_{i\to\infty} l_i$$
$$< n+\kappa - \frac{2n}{n-2}\cdot\frac{2n-5}{2} \le n+(n-2) - \frac{n(2n-5)}{n-2} < 0.$$

Therefore

$$\int_{\rho_i\le|y|\le\sigma} |y|^\kappa v_i(y)^{p_i+1} dy \le C\int_{\rho_i\le|y|\le\sigma} |y|^\kappa (v_i(0)^{t_i}|y|^{-l_i})^{p_i+1}\, dy$$
$$\le C v_i(0)^{t_i(p_i+1)-(p_i-1)(n-l_i(p_i+1)+\kappa)/2}$$
$$= C v_i(0)^{p_i+1-(n+\kappa)(p_i-1)/2} \quad \text{(by the definition of } t_i)$$
$$= C v_i(0)^{-2\kappa/(n-2)+(n-2+\kappa)\delta_i/2}.$$

Thus

$$\int_{|y|\le\sigma} |y|^{\kappa} v_i(y)^{p_i+1}\,dy \le C v_i(0)^{-2\kappa/(n-2)+(n-2+\kappa)\delta_i/2}. \qquad \square$$

The next proposition is a preliminary estimate for $\delta_i := (n+2)/(n-2) - p_i$. We will also derive a refined estimate in a later part of this paper.

Proposition 4.3. $\lim_{i\to\infty} v_i(0)^{\delta_i} = 1$.

Proof. Since the original metric is locally conformally flat, it can be written locally as $\lambda(z)^{4/(n-2)}dz^2$. Let $\lambda_i(y) = \lambda(r_i y)$. Then $g^{(i)}(y) = \lambda_i(y)^{4/(n-2)}dy^2$. Let $\sigma < 1$. The Pohozaev identity in [Schoen 1988b] says that, for a conformal Killing field X on B_σ,

$$\frac{n-2}{2n}\int_{B_\sigma} X(R_i)\,dv_{g_i} = \int_{\partial B_\sigma} T_i(X,\nu_i)\,d\Sigma_i, \tag{5}$$

where the notations are as follows:

$g_i = v_i^{4/(n-2)} g^{(i)} = (\lambda_i v_i)^{4/(n-2)}dy^2$;	$dv_{g_i} = (\lambda_i v_i)^{2n/(n-2)}dy$;
$R_i = R(g_i) = c(n)^{-1}K_i v_i^{-\delta_i}$;	
$\nu_i = (\lambda_i v_i)^{-2/(n-2)}\sigma^{-1}\sum_j y^j\partial/\partial y^j$	is the unit outer normal vector on ∂B_σ with respect to g_i;
$d\Sigma_i = (\lambda_i v_i)^{2(n-1)/(n-2)}d\Sigma_\sigma$,	where $d\Sigma_\sigma$ is the surface element of the standard $S^{n-1}(\sigma)$;
$T_i = \mathrm{Ric}(g_i) - n^{-1}R(g_i)g_i$	is the traceless Ricci tensor with respect to g_i.

According to [Schoen 1989], T_i can also be expressed as

$$(n-2)(\lambda_i v_i)^{2/(n-2)}\Big(\mathrm{Hess}((\lambda_i v_i)^{-2/(n-2)}) - \tfrac{1}{n}\Delta((\lambda_i v_i)^{-2/(n-2)})dy^2\Big),$$

where Hess and Δ are taken with respect to the Euclidean metric dy^2.

We choose $X = \sum_{j=1}^n y^j\partial/\partial y^j$. Up to the constant $2(n-1)/n$, the integral in the left side of (5) is equal to

$$\int_{B_\sigma} X(K_i v_i^{-\delta_i})(\lambda_i v_i)^{2n/(n-2)}dy$$
$$= \int_{B_\sigma} X(K_i)v_i^{p_i+1}\lambda_i^{2n/(n-2)}dy - \delta_i\int_{B_\sigma} K_i v_i^{p_i}X(v_i)\lambda_i^{2n/(n-2)}dy.$$

By the divergence theorem, this is equal to

$$\int_{B_\sigma} |y| \frac{\partial K_i}{\partial r} v_i^{p_i+1} \lambda_i^{2n/(n-2)} dy$$
$$+ \frac{\delta_i}{p_i+1}\Big(\int_{B_\sigma} r \frac{\partial K_i}{\partial r} \lambda_i^{2n/(n-2)} v_i^{p_i+1} dy + \int_{B_\sigma} K_i v_i^{p_i+1} r \frac{\partial \lambda_i^{2n/(n-2)}}{\partial r} dy$$
$$+ \int_{B_\sigma} K_i v_i^{p_i+1} \lambda_i^{2n/(n-2)} \operatorname{div} X dy \Big)$$
$$- \frac{\delta_i}{p_i+1} \int_{\partial B_\sigma} K_i v_i^{p_i+1} \lambda_i^{2n/(n-2)} X \cdot \big((\textstyle\sum y^j \partial/\partial y^j)/\sigma \big)\, d\Sigma_\sigma .$$

Restoring the factor $2(n-1)/n$, we can now write the left side of (5) as

$$\tag{6} \frac{2(n-1)}{n}\Big(1+\frac{\delta_i}{p_i+1}\Big) \int_{B_\sigma} |y| \frac{\partial K_i}{\partial r} v_i^{p_i+1} \lambda_i^{2n/(n-2)} dy$$
$$+ \frac{2(n-1)}{n} \frac{\delta_i}{p_i+1} \int_{B_\sigma} |y| K_i v_i^{p_i+1} \frac{\partial \lambda_i^{2n/(n-2)}}{\partial r} dy + \frac{2(n-1)}{n} \frac{\delta_i n}{p_i+1} \int_{B_\sigma} K_i v_i^{p_i+1} \lambda_i^{2n/(n-2)} dy$$
$$- \frac{2(n-1)}{n} \frac{\delta_i}{p_i+1} \int_{\partial B_\sigma} \sigma K_i v_i^{p_i+1} \lambda_i^{2n/(n-2)} d\Sigma_\sigma .$$

The right side of (5) is

$$\int_{\partial B_\sigma} (n-2)(\lambda_i v_i)^{2/(n-2)} \Big(\operatorname{Hess}((\lambda_i v_i)^{-2/(n-2)}) \Big(r\frac{\partial}{\partial r}, (\lambda_i v_i)^{-2/(n-2)} \sigma^{-1} r \frac{\partial}{\partial r} \Big)$$
$$- \tfrac{1}{n} \Delta((\lambda_i v_i)^{-2/(n-2)}) \Big\langle r\frac{\partial}{\partial r}, (\lambda_i v_i)^{-2/(n-2)} \sigma^{-1} r \frac{\partial}{\partial r} \Big\rangle \Big) (\lambda_i v_i)^{2(n-1)/(n-2)} d\Sigma_\sigma$$

(where $\langle \cdot , \cdot \rangle$ is the Euclidean metric)

$$= (n-2) \int_{\partial B_\sigma} \Big(\sigma^{-1} \operatorname{Hess}((\lambda_i v_i)^{-2/(n-2)}) \Big(r\frac{\partial}{\partial r}, r\frac{\partial}{\partial r} \Big)$$
$$- \frac{\sigma}{n} \Delta((\lambda_i v_i)^{-2/(n-2)}) \Big) (\lambda_i v_i)^{2(n-1)/(n-2)} d\Sigma_\sigma$$
$$\tag{7} = (n-2) \int_{\partial B_\sigma} \Big(\sigma^{-1} \Big(-\frac{2}{n-2} (\lambda_i v_i) \sum_{j,k} y^j y^k \frac{\partial}{\partial y^k} \frac{\partial}{\partial y^j} (\lambda_i v_i)$$
$$+ \frac{2n}{(n-2)^2} \sum_{j,k} y^j y^k \frac{\partial(\lambda_i v_i)}{\partial y^k} \frac{\partial(\lambda_i v_i)}{\partial y^j} \Big)$$
$$- \sigma \cdot \Big(-\frac{2}{n(n-2)} (\lambda_i v_i) \sum_j \frac{\partial^2(\lambda_i v_i)}{(\partial y^j)^2} + \frac{2}{(n-2)^2} \sum_j \Big(\frac{\partial(\lambda_i v_i)}{\partial y^j} \Big)^2 \Big) \Big) d\Sigma_\sigma .$$

Next we are going to study the decay rate of each term in (6) and (7).

On ∂B_σ, Proposition 4.1 implies $v_i \le C v_i(0)^{t_i}$. Then by the elliptic regularity theory [Gilbarg and Trudinger 2001], we have $\|v_i\|_{C^2(\partial B_\sigma)} \le C v_i(0)^{t_i}$. Thus we know (7) decays at a rate of $v_i(0)^{2t_i}$.

The fourth term in (6) decays in the order of $\delta_i v_i(0)^{t_i(p_i+1)}$ by Proposition 4.1. By Lemma 4.2, we know that the second term in (6) is bounded above by

$$C\delta_i \int_{B_\sigma} |y| v_i^{p_i+1} dy \le C\delta_i v_i(0)^{-2/(n-2)+(n-1)\delta_i/2}.$$

Therefore the sum of the first and the third terms in (6), which is

$$\frac{n}{2(n-1)}\Big(1+\frac{\delta_i}{p_i+1}\Big)\int_{B_\sigma} |y| \frac{\partial K_i}{\partial r} v_i^{p_i+1} \lambda_i^{2n/(n-2)} dy \\ + \frac{n}{2(n-1)} \frac{\delta_i}{p_i+1} n \int_{B_\sigma} K_i v_i^{p_i+1} \lambda_i^{2n/(n-2)} dy,$$

is bounded above by $C v_i(0)^{2t_i} + C\delta_i v_i(0)^{t_i(p_i+1)} + C\delta_i v_i(0)^{-2/(n-2)+(n-1)\delta_i/2}$.

By our choice of l_i and t_i, we have, as $i \to \infty$,

$$t_i = 1 - \frac{(p_i-1)l_i}{2} \to 1 - \frac{2}{n-2} \lim_{i\to\infty} l_i < 1 - \frac{2}{n-2}\cdot\frac{2n-5}{2} < 0.$$

Thus $C v_i(0)^{2t_i} + C\delta_i v_i(0)^{t_i(p_i+1)} \le C v_i(0)^{2t_i} + C v_i(0)^{t_i(p_i+1)} \le C v_i(0)^{2t_i}$.

On the other hand,

$$\frac{\delta_i}{p_i+1} n \int_{B_\sigma} K_i v_i^{p_i+1} \lambda_i^{2n/(n-2)} dy \ge C\delta_i \int_{B_\sigma} v_i^{p_i+1} dy.$$

When $|y| \le \rho_i$, Proposition 4.1 gives

$$\begin{aligned} v_i(y) &\ge (1-\epsilon) v_i(0) \Big/ \Big(1 + \frac{K_i(0)}{n(n-2)} v_i(0)^{p_i-1} |y|^2\Big)^{(n-2)/2} \\ &\ge (1-\epsilon) v_i(0) \Big/ \Big(1 + \frac{K_i(0)}{n(n-2)} R^2\Big)^{(n-2)/2} \ge C v_i(0), \end{aligned}$$

so

$$\begin{aligned} \int_{B_\sigma} v_i^{p_i+1} dy > \int_{|y|\le\rho_i} v_i^{p_i+1} dy \ge C v_i(0)^{p_i+1-n(p_i-1)/2} &= C v_i(0)^{(n-2)\delta_i/2} \\ &\ge C. \end{aligned} \tag{8}$$

This implies that the third term in (6) is bounded below by $C\delta_i$.

Then by comparing the decay rates of the terms in (6) and (7),

$$\delta_i \le C\Big(v_i(0)^{2t_i} + \delta_i v_i(0)^{-2/(n-2)+(n-1)\delta_i/2} + \Big|\int_{B_\sigma} \frac{\partial K_i}{\partial r} |y| v_i^{p_i+1} \lambda_i^{2n/(n-2)} dy\Big|\Big).$$

Since $v_i(0)^{-2/(n-2)+(n-1)\delta_i/2} \to 0$, the second term on the right side can be absorbed into the left side. Thus we conclude that

$$\delta_i \le C\Big(v_i(0)^{2t_i} + \Big|\int_{B_\sigma} \frac{\partial K_i}{\partial r}|y| v_i^{p_i+1} \lambda_i^{2n/(n-2)} dy\Big|\Big). \tag{9}$$

By Lemma 4.2,

$$\Big|\int_{B_\sigma} \frac{\partial K_i}{\partial r}|y| v_i^{p_i+1} \lambda_i^{2n/(n-2)} dy\Big| \le C v_i(0)^{-2/(n-2)+(n-1)\delta_i/2}.$$

Thus $\delta_i \le C(v_i(0)^{-2/(n-2)+(n-1)\delta_i/2} + v_i(0)^{2t_i})$. This implies that

$$\delta_i \ln v_i(0) \le C\big(v_i(0)^{-2/(n-2)+(n-1)\delta_i/2} + v_i(0)^{2t_i}\big) \ln v_i(0) \to 0$$

as $i \to \infty$. Therefore $\lim_{i\to\infty} v_i(0)^{\delta_i} = 1$. Consequently, we have

$$\delta_i \le C(v_i(0)^{-2/(n-2)} + v_i(0)^{2t_i}). \tag{10}$$

□

4.3. ***A preliminary estimate for*** $|\nabla K_i|$**.** We will again study the Pohozaev identity (5), but with a different choice of the conformal Killing field $X = \partial/\partial y^1$.

Direct calculation, like that in the proof of Proposition 4.3, shows that the right side of the identity is equal to

$$(n-2)\int_{\partial B_\sigma}\Big(\sum_j \frac{y^j}{\sigma}\Big(-\frac{2}{n-2}(\lambda_i v_i)\frac{\partial^2(\lambda_i v_i)}{\partial y^1 \partial y^j} + \frac{2n}{(n-2)^2}\frac{\partial(\lambda_i v_i)}{\partial y^1}\frac{\partial(\lambda_i v_i)}{\partial y^j}\Big)$$
$$- \frac{y^1}{\sigma}\sum_j\Big(-\frac{2}{n(n-2)}(\lambda_i v_i)\frac{\partial^2(\lambda_i v_i)}{(\partial y^j)^2} + \frac{2}{(n-2)^2}\Big(\frac{\partial(\lambda_i v_i)}{\partial y^j}\Big)^2\Big)\Big) d\Sigma_\sigma,$$

and decays at a rate of $v_i(0)^{2t_i}$.

The left side of this identity is $(n-2)/(2n)$ times

$$\begin{aligned}\int_{B_\sigma} \frac{\partial}{\partial y^1}(R_i) dv_{g_i} &= c(n)^{-1}\int_{B_\sigma} \frac{\partial}{\partial y^1}(K_i v_i^{-\delta_i})(\lambda_i v_i)^{2n/(n-2)} dy \\ &= c(n)^{-1}\int_{B_\sigma}\Big(1+\frac{\delta_i}{p_i+1}\Big)\lambda_i^{2n/(n-2)} v_i^{p_i+1}\frac{\partial K_i}{\partial y^1} dy \\ &\quad + c(n)^{-1}\int_{B_\sigma}\frac{\delta_i}{p_i+1}K_i v_i^{p_i+1}\frac{\partial \lambda_i^{2n/(n-2)}}{\partial y^1} dy \\ &\quad - c(n)^{-1}\frac{\delta_i}{p_i+1}\int_{\partial B_\sigma}\lambda_i^{2n/(n-2)} K_i v_i^{p_i+1}\frac{y^1}{\sigma} d\Sigma_\sigma.\end{aligned} \tag{11}$$

By Proposition 4.1, the last term in (11) is bounded from above by

$$C\delta_i \cdot v_i(0)^{t_i(p_i+1)} \le C\delta_i v_i(0)^{2t_i},$$

since $t_i < 0$ and $v_i(0) \to \infty$.

Note that since $\lambda_i(y) = \lambda(r_i y)$, the second term in (11) is bounded from above by

$$C\delta_i r_i \int_{|y|\le\sigma} v_i(y)^{p_i+1} dy,$$

which is further bounded by $C\delta_i r_i v_i(0)^{(n-2)/2\delta_i} \le C\delta_i r_i$ by Lemma 4.2 and Proposition 4.3.

Therefore the first term in (11), which is

$$c(n)^{-1} \int_{B_\sigma} \Bigl(1 + \frac{\delta_i}{p_i+1}\Bigr) \lambda_i^{2n/(n-2)} v_i^{p_i+1} \frac{\partial K_i}{\partial y^1} dy,$$

is bounded from above by $C(v_i(0)^{2t_i} + \delta_i v_i(0)^{2t_i} + \delta_i r_i) \le C(\delta_i r_i + v_i(0)^{2t_i})$.

This shows that

$$\Bigl| \int_{B_\sigma} \lambda_i^{2n/(n-2)} v_i^{p_i+1} \frac{\partial K_i}{\partial y^1} dy \Bigr| \le C(\delta_i r_i + v_i(0)^{2t_i}). \tag{12}$$

By the Taylor expansion,

$$\frac{\partial K_i}{\partial y^1}(y) = \frac{\partial K_i}{\partial y^1}(0) + \nabla\Bigl(\frac{\partial K_i}{\partial y^1}\Bigr)(\varsigma) \cdot y \qquad \text{for some } |\varsigma| \le |y|.$$

Note that $K_i(y) = K(r_i y)$. By Lemma 4.2 and Proposition 4.3,

$$\begin{aligned}
\int_{B_\sigma} \lambda_i^{2n/(n-2)} v_i^{p_i+1} \Bigl| \nabla\Bigl(\frac{\partial K_i}{\partial y^1}\Bigr)(\varsigma) \cdot y \Bigr| dy &\le C r_i \int_{B_\sigma} v_i^{p_i+1} |y| dy \\
&\le C r_i v_i(0)^{-2/(n-2)+(n-1)\delta_i/2} \\
&\le C r_i v_i(0)^{-2/(n-2)}.
\end{aligned}$$

Thus we know

$$\begin{aligned}
\Bigl| \frac{\partial K_i}{\partial y^1}(0) \Bigr| \int_{B_\sigma} v_i^{p_i+1} dy &\le C \Bigl| \int_{B_\sigma} \lambda_i^{2n/(n-2)} v_i^{p_i+1} \frac{\partial K_i}{\partial y^1}(0) dy \Bigr| \\
&\le C(r_i v_i(0)^{-2/(n-2)} + (\delta_i r_i + v_i(0)^{2t_i})) \\
&\le C(r_i v_i(0)^{-2/(n-2)} + r_i v_i(0)^{2t_i} + v_i(0)^{2t_i}) \qquad \text{(by (10))} \\
&\le C(r_i v_i(0)^{-2/(n-2)} + v_i(0)^{2t_i}).
\end{aligned}$$

Then by (8),

$$\Bigl| \frac{\partial K_i}{\partial y^1}(0) \Bigr| \le C \left(r_i v_i(0)^{-2/(n-2)} + v_i(0)^{2t_i} \right). \tag{13}$$

The same estimate holds for $|\partial K_i / \partial y^j(0)|$ for $j = 2, \dots, n$ as well, since we can also choose $X = \partial/\partial y^j$ in the above calculation.

4.4. ***Location of the blow-up.*** Choose a point $\bar{y}$ with $|\bar{y}| = 1$. It is proved in [Yan 2007, Section 6] that $v_i/v_i(\bar{y})$ converges in C^2 norm to a function h on any compact subset of $\mathbb{R}^n \setminus \{0\}$, and $h = \frac{1}{2} + \frac{1}{2}|y|^{2-n}$.

Recall that we chose the coordinate systems $z = (z^1, \dots, z^n)$ and $y = z/r_i$ to be centered at each $x_i \in M$; thus $\nabla K_i(0) = r_i \nabla K(x_i)$. Here we write $\nabla K(x_i)$ instead of $\nabla K(0)$ to emphasize the fact that ∇K is evaluated at different points x_i as $i \to \infty$. We claim that this blow-up must occur at a critical point of K:

Proposition 4.4. $\nabla K(x_0) = \lim_{i\to\infty} \nabla K(x_i) = 0$.

Proof. Suppose this is not true. Then there exists some $j \in \{1, \dots, n\}$ such that $|\partial K/\partial z^j(x_i)| \geq \varepsilon$ for a constant ε independent of i. Without loss of generality we assume $j = 1$. Then from inequality (13) we know that

$$\varepsilon r_i \leq C \left(r_i v_i(0)^{-2/(n-2)} + v_i(0)^{2t_i} \right).$$

Therefore

$$r_i \leq C v_i(0)^{2t_i} \tag{14}$$

when $v_i(0)^{-2/(n-2)}$ is sufficiently small.

Once more we look at the Pohozaev identity (5) with $X = \sum_j y^j \partial/\partial y^j$. We divide both sides of it by $v_i^2(\bar{y})$ so that it becomes

$$\frac{n-2}{2n} \frac{1}{v_i^2(\bar{y})} \int_{B_\sigma} X(R_i)\, dv_{g_i} = \frac{1}{v_i^2(\bar{y})} \int_{\partial B_\sigma} T_i(X, \nu_i)\, d\Sigma_i. \tag{15}$$

Its right side is

$$\begin{aligned}
&\frac{1}{v_i^2(\bar{y})} \int_{\partial B_\sigma} T_i(X, \nu_i) d\Sigma_i \\
&\qquad = \frac{1}{v_i^2(\bar{y})} \int_{\partial B_\sigma} (\mathrm{Ric}(g_i) - n^{-1} R(g_i) g_i)(X, \nu_i) d\Sigma_i \\
&\qquad = \frac{1}{v_i^2(\bar{y})} \int_{\partial B_\sigma} \big(\mathrm{Ric}((\lambda_i v_i)^{4/(n-2)} dy \otimes dy) \\
&\qquad\qquad - n^{-1} R((\lambda_i v_i)^{4/(n-2)} dy \otimes dy)(\lambda_i v_i)^{4/(n-2)} dy \otimes dy\big)(X, \nu_0)(\lambda_i v_i)^2 d\Sigma_\sigma
\end{aligned}$$

$$\begin{aligned}
&= \int_{\partial B_\sigma} \left(\frac{\lambda_i v_i}{v_i(\bar{y})}\right)^2 \left(\mathrm{Ric}\left(\left(\frac{\lambda_i v_i}{v_i(\bar{y})}\right)^{4/(n-2)} dy \otimes dy \right) \right. \\
&\qquad \left. - n^{-1} R\left(\left(\frac{\lambda_i v_i}{v_i(\bar{y})}\right)^{4/(n-2)} dy \otimes dy \right) \left(\frac{\lambda_i v_i}{v_i(\bar{y})}\right)^{4/(n-2)} dy \otimes dy \right)(X, \nu_0) d\Sigma_\sigma,
\end{aligned} \tag{16}$$

where $\nu_0 = \sigma^{-1} \sum_j y^j \partial/\partial y^j$ is the unit outer normal on ∂B_σ with respect to the Euclidean metric $dy \otimes dy$.

When $i \to \infty$, $\lambda_i(y) = \lambda(r_i y) \to \lambda(x_0)$ for $|y| = \sigma$. Thus when i goes to ∞, (16) converges (up to a constant) to

$$\begin{aligned}
&\int_{\partial B_\sigma} h^2 \big(\mathrm{Ric}(h^{4/(n-2)} dy \otimes dy) \\
&\qquad\qquad - n^{-1} R(h^{4/(n-2)} dy \otimes dy) h^{4/(n-2)} dy \otimes dy\big)(X, \nu_0)\, d\Sigma_\sigma \\
&\quad = \int_{\partial B_\sigma} h^2 \cdot (n-2) h^{2/(n-2)} \big(\mathrm{Hess}(h^{-2/(n-2)})(X, \nu_0) \\
&\qquad\qquad - \tfrac{1}{n} \Delta(h^{-2/(n-2)}) \langle X, \nu_0 \rangle\big)\, d\Sigma_\sigma \\
&\quad = (n-2)\sigma^{-1} \int_{\partial B_\sigma} h^{2(n-1)/(n-2)} \cdot \big(\mathrm{Hess}(h^{-2/(n-2)})(X, X) \\
&\qquad\qquad - \tfrac{1}{n} \Delta(h^{-2/(n-2)}) \sigma^2\big) d\Sigma_\sigma. \qquad (17)
\end{aligned}$$

We know that

$$h^{-2/(n-2)} = \big(\tfrac{1}{2}(1 + |y|^{2-n})\big)^{-2/(n-2)} = 2^{2/(n-2)} |y|^2 - \frac{2^{n/(n-2)}}{n-2} |y|^n + O(|y|^{2(n-1)}),$$

and by direct computation,

$$\begin{aligned}
\mathrm{Hess}\Big(2^{2/(n-2)} |y|^2 - \frac{2^{n/(n-2)}}{n-2} |y|^n\Big)(X, X) - \frac{1}{n} \Delta\Big(2^{2/(n-2)} |y|^2 - \frac{2^{n/(n-2)}}{n-2} |y|^n\Big) \sigma^2 \\
= -2^{n/(n-2)} (n-1) \sigma^n.
\end{aligned}$$

Therefore

$$\mathrm{Hess}(h^{-2/(n-2)})(X, X) - \tfrac{1}{n} \Delta(h^{-2/(n-2)}) \sigma^2 = -2^{n/(n-2)} (n-1) \sigma^n + O(\sigma^{2(n-1)}).$$

Also we know

$$h^{2(n-1)/(n-2)} = \big(\tfrac{1}{2}\big)^{2(n-1)/(n-2)} |y|^{-2(n-1)} (1 + O(|y|^{n-2})).$$

Thus we can conclude that (17) is equal to

$$\begin{aligned}
-\tfrac{1}{2}(n-1)(n-2)\sigma^{-1} \int_{\partial B_\sigma} \big(|y|^{-2(n-1)} + O(|y|^{-n})\big) \big(|y|^n + O(|y|^{2(n-1)})\big)\, \sigma^{n-1} d\Sigma_1 \\
= -\tfrac{1}{2}(n-1)(n-2) + O(\sigma^{n-2}).
\end{aligned}$$

Therefore the limit of the right side of (15) is strictly less than 0 when we choose σ to be sufficiently small.

On the other hand, the left side of (15) is

$$\frac{n-2}{2n} c(n)^{-1} \frac{1}{v_i^2(\bar{y})} \int_{B_\sigma} X(K_i v_i^{-\delta_i}) (\lambda_i v_i)^{2n/(n-2)} dy.$$

We write

$$
\begin{aligned}
&\frac{1}{v_i^2(\bar y)}\int_{B_\sigma} X(K_i v_i^{-\delta_i})(\lambda_i v_i)^{2n/(n-2)}dy \\
&= \frac{1}{v_i^2(\bar y)}\int_{B_\sigma} X(K_i)v_i^{p_i+1}\lambda_i^{2n/(n-2)}dy - \frac{\delta_i}{v_i^2(\bar y)}\int_{B_\sigma} K_i\lambda_i^{2n/(n-2)}v_i^{p_i}X(v_i)dy. \qquad (18)
\end{aligned}
$$

The second term of (18) is equal to

$$
\begin{aligned}
&-\frac{\delta_i}{p_i+1}\frac{1}{v_i^2(\bar y)}\int_{B_\sigma} K_i\lambda_i^{2n/(n-2)}X(v_i^{p_i+1})dy \\
&= -\frac{\delta_i}{p_i+1}\frac{1}{v_i^2(\bar y)}\int_{B_\sigma}\Big(\operatorname{div}(K_i\lambda_i^{2n/(n-2)}v_i^{p_i+1}X) - K_i\lambda_i^{2n/(n-2)}v_i^{p_i+1}\operatorname{div}X \\
&\qquad\qquad - \lambda_i^{2n/(n-2)}v_i^{p_i+1}X(K_i) - K_iv_i^{p_i+1}X(\lambda_i^{2n/(n-2)})\Big)dy \\
&= -\frac{\delta_i}{p_i+1}\frac{\sigma}{v_i^2(\bar y)}\int_{\partial B_\sigma} K_i\lambda_i^{2n/(n-2)}v_i^{p_i+1}d\Sigma_\sigma \\
&\quad + \frac{\delta_i}{p_i+1}\frac{1}{v_i^2(\bar y)}\int_{B_\sigma} K_i\lambda_i^{2n/(n-2)}v_i^{p_i+1}\,(n+X(\ln K_i)+2n/(n-2)X(\ln\lambda_i))\,dy.
\end{aligned}
$$

On ∂B_σ, we know $v_i/v_i(\bar y)\to h(\sigma)$ and $v_i\to 0$ uniformly, so

$$
\frac{1}{v_i^2(\bar y)}\int_{\partial B_\sigma} K_i\lambda_i^{2n/(n-2)}v_i^{p_i+1}d\Sigma_\sigma = \int_{\partial B_\sigma} K_i\lambda_i^{2n/(n-2)}\Big(\frac{v_i}{v_i(\bar y)}\Big)^2 v_i^{p_i-1}d\Sigma_\sigma \to 0.
$$

Since $X=r\partial/\partial r$ and $|\partial(\ln K_i)/\partial r|$ and $|\partial(\ln\lambda_i)/\partial r|$ are uniformly bounded, we can choose σ to be small enough (independent of i) to make $n+X(\ln K_i)+2n/(n-2)X(\ln\lambda_i)>0$. Thus when $i\to\infty$, the limit of the second term of (18) is greater than or equal to 0.

Next we will show that the limit of the first term of (18) is 0, or equivalently,

$$
\lim_{i\to\infty} v_i^2(0)\int_{B_\sigma} X(K_i)v_i^{p_i+1}\lambda_i^{2n/(n-2)}dy = 0, \qquad (19)
$$

since $v_i(\bar y)\ge Cv_i(0)^{-1}$ by Proposition 4.1. This then will end the proof because it implies that the limit of the left hand side of (15) is greater than or equal to 0, contradicting the sign of the right hand side.

Note that

$$
\begin{aligned}
X(K_i)(y) &= \Big(\sum\nolimits_j y^j\frac{\partial K_i}{\partial y^j}\Big)(y) \\
&= \Big(\sum\nolimits_j y^j\frac{\partial K_i}{\partial y^j}\Big)(0) + \sum\nolimits_k \frac{\partial}{\partial y^k}\Big(\sum\nolimits_j y^j\frac{\partial K_i}{\partial y^j}\Big)(\varsigma)y^k \quad \text{for some } |\varsigma|\le|y| \\
&= \sum\nolimits_j \frac{\partial K_i}{\partial y^j}(\varsigma)y^j + \sum\nolimits_{j,k}\frac{\partial^2 K_i}{\partial y^k\partial y^j}(\varsigma)\varsigma^j y^k.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&v_i^2(0)\Big|\int_{B_\sigma} X(K_i)v_i^{p_i+1}\lambda_i^{2n/(n-2)}dy\Big| \\
&\le v_i^2(0)\int_{B_\sigma}\sum_j\Big|\frac{\partial K_i}{\partial y^j}(\varsigma)\Big||y|v_i^{p_i+1}\lambda_i^{2n/(n-2)}dy \\
&\qquad\qquad + v_i^2(0)\int_{B_\sigma}\sum_{j,k}\Big|\frac{\partial^2 K_i}{\partial y^j\partial y^k}(\varsigma)\Big||y|^2 v_i^{p_i+1}\lambda_i^{2n/(n-2)}dy \\
&\le Cv_i^2(0)r_i\int_{B_\sigma}|y|v_i^{p_i+1}dy + Cv_i^2(0)r_i^2\int_{B_\sigma}|y|^2v_i^{p_i+1}dy \\
&\le Cv_i^2(0)r_i\cdot v_i(0)^{-2/(n-2)+(n-1)\delta_i/2} + Cv_i^2(0)r_i^2\cdot v_i(0)^{-4/(n-2)+n\delta_i/2} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Lemma 4.2)} \\
&\le Cv_i(0)^{2+2t_i-2/(n-2)} + Cv_i(0)^{2+4t_i-4/(n-2)} \qquad \text{(by Proposition 4.3 and (14))}.
\end{aligned}
$$

By the definition of t_i,

$$
\lim_{i\to\infty} t_i = \lim_{i\to\infty}\Big(1-\frac{(p_i-1)l_i}{2}\Big) = 1-\frac{2}{n-2}\lim_{i\to\infty} l_i < 1-\frac{2}{n-2}\cdot\frac{2n-5}{2} = \frac{3-n}{n-2}.
$$

Thus

$$
\lim_{i\to\infty}\Big(2+2t_i-\frac{2}{n-2}\Big) < 2+2\cdot\frac{3-n}{n-2}-\frac{2}{n-2} = 0 \tag{20}
$$

and

$$
\lim_{i\to\infty}\Big(2+4t_i-\frac{4}{n-2}\Big) < 2+4\cdot\frac{3-n}{n-2}-\frac{4}{n-2} = \frac{4-2n}{n-2} < 0.
$$

Since these are all strict inequalities, we know that

$$
\lim_{i\to\infty}\big(Cv_i(0)^{2+2t_i-2/(n-2)} + Cv_i(0)^{2+4t_i-4/(n-2)}\big) = 0,
$$

and consequently

$$
\lim_{i\to\infty} v_i^2(0)\Big|\int_{B_\sigma} X(K_i)v_i^{p_i+1}\lambda_i^{2n/(n-2)}dy\Big| = 0. \qquad \square
$$

4.5. ***Refined estimates for δ_i and $|\nabla K_i|$.*** Now because $x_0 = \lim_{i\to\infty} x_i$ is a critical point of the function K, which satisfies the flatness condition $(*)$, we have $|\nabla^p K(x_i)| \le C_0|\nabla K(x_i)|^{(n-2-p)/(n-3)}$ when $2 \le p \le n-3$. When $p=2$, this implies, because $g = \lambda^{4/(n-2)}dz^2$, that

$$
\Big|\nabla^2 K\Big(\frac{\partial}{\partial z^{l_1}},\frac{\partial}{\partial z^{l_2}}\Big)(x_i)\Big| = \Big|\frac{\partial^2 K}{\partial z^{l_1}\partial z^{l_2}}(x_i) - \Gamma^l_{l_1l_2}(x_i)\frac{\partial K}{\partial z^l}(x_i)\Big| \le C|\nabla K(x_i)|^{\frac{n-4}{n-3}},
$$

where $l_1, l_2, l = 1, 2, \ldots, n$. Therefore

$$
\Big|\frac{\partial^2 K}{\partial z^{l_1}\partial z^{l_2}}(x_i)\Big| \le C|\nabla K(x_i)| + C|\nabla K(x_i)|^{(n-4)/(n-3)} \le C|\nabla K(x_i)|^{(n-4)/(n-3)},
$$

since $|\nabla K(x_i)| < 1$ for sufficiently large i. That is,

$$|\partial^\alpha K/\partial z^\alpha(x_i)| \le C|\nabla K(x_i)|^{(n-2-|\alpha|)/(n-3)}$$

for $|\alpha| = 2$. Here we have used the notations that $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ with each $\alpha_i \ge 0$, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, and

$$\frac{\partial^\alpha K}{\partial z^\alpha} = \frac{\partial^{\alpha_1}\partial^{\alpha_2}\cdots\partial^{\alpha_n} K}{(\partial z^1)^{\alpha_1}(\partial z^2)^{\alpha_2}\cdots(\partial z^n)^{\alpha_n}}.$$

Generally, when $2 \le p < q \le n-3$, we have

$$|\nabla K(x_i)|^{(n-2-p)/(n-3)} < |\nabla K(x_i)|^{(n-2-q)/(n-3)},$$

so by similar computations we have

$$\left|\frac{\partial^\alpha K}{\partial z^\alpha}(x_i)\right| \le C|\nabla K(x_i)|^{(n-2-|\alpha|)/(n-3)} \quad \text{for } 2 \le |\alpha| \le n-3.$$

Then since $K_i(y) = K(r_i y)$,

$$\left|\frac{\partial^\alpha K_i}{\partial y^\alpha}(0)\right| = r_i^{|\alpha|}\left|\frac{\partial^\alpha K}{\partial z^\alpha}(x_i)\right| \qquad \text{and} \qquad |\nabla K_i(0)| = r_i|\nabla K(x_i)|.$$

Thus

$$\begin{aligned}\left|\frac{\partial^\alpha K_i}{\partial y^\alpha}(0)\right| &\le r_i^{|\alpha|}C|\nabla K(x_i)|^{(n-2-|\alpha|)/(n-3)}\\ &= Cr_i^{(|\alpha|-1)(n-2)/(n-3)}|\nabla K_i(0)|^{(n-2-|\alpha|)/(n-3)}\\ &< Cr_i|\nabla K_i(0)|^{(n-2-|\alpha|)/(n-3)},\end{aligned} \tag{21}$$

where the last step follows from the fact that $(|\alpha|-1)(n-2)/(n-3) > 1$ and $r_i < 1$. With this flatness condition on K_i, we can refine the estimates for δ_i and $|\nabla K_i|$ as follows.

Inequality (9) gives

$$\begin{aligned}\delta_i &\le C\Big(v_i(0)^{2t_i} + \Big|\int_{B_\sigma} \frac{\partial K_i}{\partial r}|y|v_i^{p_i+1}\lambda_i^{2n/(n-2)}dy\Big|\Big)\\ &= C\Big(v_i(0)^{2t_i} + \Big|\int_{B_\sigma} r\frac{\partial K_i}{\partial r}v_i^{p_i+1}\lambda_i^{2n/(n-2)}dy\Big|\Big).\end{aligned}$$

We write $r\partial K_i/\partial r = \sum_j y^j \partial K_i/\partial y^j$. For each $j = 1, \dots, n$,

$$\begin{aligned}\frac{\partial K_i}{\partial y^j}(y) = \frac{\partial K_i}{\partial y^j}(0) + \sum_{|\beta|=1}\frac{\partial^\beta}{\partial y^\beta}\frac{\partial K_i}{\partial y^j}(0)y^\beta + \frac{1}{2!}\sum_{|\beta|=2}\frac{\partial^\beta}{\partial y^\beta}\frac{\partial K_i}{\partial y^j}(0)y^\beta + \cdots\\ + \frac{1}{(n-4)!}\sum_{|\beta|=n-4}\frac{\partial^\beta}{\partial y^\beta}\frac{\partial K_i}{\partial y^j}(0)y^\beta + \frac{1}{(n-3)!}\sum_{|\beta|=n-3}\frac{\partial^\beta}{\partial y^\beta}\frac{\partial K_i}{\partial y^j}(\varsigma)y^\beta,\end{aligned}$$

where $|\varsigma| \le |y|$, and $y^\beta = y_1^{\beta_1} y_2^{\beta_2} \cdots y_n^{\beta_n}$ for $\beta = (\beta_1, \beta_2, \dots, \beta_n)$. Therefore

$$\begin{aligned}
&\int_{B_\sigma} \Big| r \frac{\partial K_i}{\partial r} \Big| v_i^{p_i+1} \lambda_i^{2n/(n-2)} dy \\
&\quad \le C\biggl(\int_{B_\sigma} \Big| \frac{\partial K_i}{\partial y^j}(0) \Big| |y| v_i^{p_i+1} dy + \sum_{|\beta|=1}^{n-4} \int_{B_\sigma} \Big| \frac{\partial^\beta}{\partial y^\beta} \frac{\partial K_i}{\partial y^j}(0) \Big| |y|^{|\beta|+1} v_i^{p_i+1} dy \\
&\qquad\qquad + \sum_{|\beta|=n-3} \int_{B_\sigma} \Big| \frac{\partial^\beta}{\partial y^\beta} \frac{\partial K_i}{\partial y^j}(\varsigma) \Big| |y|^{n-2} v_i^{p_i+1} dy \biggr).
\end{aligned}$$

By Lemma 4.2 and Proposition 4.3, the first term satisfies

$$\int_{B_\sigma} \Big| \frac{\partial K_i}{\partial y^j}(0) \Big| |y| v_i^{p_i+1} dy \le C |\nabla K_i(0)| v_i(0)^{-2/(n-2)},$$

and the last term has

$$\sum_{|\beta|=n-3} \int_{B_\sigma} \Big| \frac{\partial^\beta}{\partial y^\beta} \frac{\partial K_i}{\partial y^j}(\varsigma) \Big| |y|^{n-2} v_i^{p_i+1} dy \le C r_i^{n-2} v_i(0)^{-2}.$$

In addition, (21) gives, for any $1 \le |\beta| \le n-4$,

$$\begin{aligned}
&\int_{B_\sigma} \Big| \frac{\partial^\beta}{\partial y^\beta} \frac{\partial K_i}{\partial y^j}(0) \Big| |y|^{|\beta|+1} v_i^{p_i+1} dy \\
&\qquad \le C r_i \int_{B_\sigma} |\nabla K_i(0)|^{(n-2-(|\beta|+1))/(n-3)} |y|^{|\beta|+1} v_i^{p_i+1} dy \\
&\qquad = C r_i \int_{B_\sigma} |\nabla K_i(0)|^{(n-3-|\beta|)/(n-3)} |y|^{|\beta|} \cdot |y| v_i^{p_i+1} dy \\
&\qquad \le C r_i \int_{B_\sigma} \Bigl(|\nabla K_i(0)|^{\frac{n-3-|\beta|}{n-3} \cdot \frac{n-3}{n-3-|\beta|}} + |y|^{|\beta| \cdot \frac{n-3}{|\beta|}} \Bigr) \cdot |y| v_i^{p_i+1} dy \\
&\qquad\qquad\qquad \text{(by Young's inequality)} \\
&\qquad = C r_i \Bigl(\int_{B_\sigma} |\nabla K_i(0)| \cdot |y| v_i^{p_i+1} dy + \int_{B_\sigma} |y|^{n-2} v_i^{p_i+1} dy \Bigr) \\
&\qquad \le C r_i |\nabla K_i(0)| v_i(0)^{-2/(n-2)} + C r_i v_i(0)^{-2}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\int_{B_\sigma} \Big| r \frac{\partial K_i}{\partial r} \Big| v_i^{p_i+1} \lambda_i^{2n/(n-2)} dy \\
&\qquad \le C |\nabla K_i(0)| v_i(0)^{-2/(n-2)} \\
&\qquad\qquad + \bigl(C r_i |\nabla K_i(0)| v_i(0)^{-2/(n-2)} + C r_i v_i(0)^{-2} \bigr) + C r_i^{n-2} v_i(0)^{-2} \\
&\qquad \le C |\nabla K_i(0)| v_i(0)^{-2/(n-2)} + C r_i v_i(0)^{-2}. \qquad (22)
\end{aligned}$$

Plugging this back into (9) we now have a refined estimate

$$\delta_i \le C\left(v_i(0)^{2t_i} + |\nabla K_i(0)| v_i(0)^{-2/(n-2)} + r_i v_i(0)^{-2}\right). \tag{23}$$

This will enable us to also refine the estimate for $|\nabla K_i(0)|$. Inequality (12) gives $|\int_{B_\sigma} \lambda_i^{2n/(n-2)} v_i^{p_i+1} (\partial K_i/\partial y^1)\, dy| \le C(\delta_i r_i + v_i(0)^{2t_i})$. Again we write

$$\begin{aligned}\frac{\partial K_i}{\partial y^1}(y) = \frac{\partial K_i}{\partial y^1}(0) + \sum_{|\beta|=1} \frac{\partial^\beta}{\partial y^\beta}\frac{\partial K_i}{\partial y^1}(0) y^\beta + \frac{1}{2!}\sum_{|\beta|=2} \frac{\partial^\beta}{\partial y^\beta}\frac{\partial K_i}{\partial y^1}(0) y^\beta + \cdots \\ + \frac{1}{(n-4)!}\sum_{|\beta|=n-4} \frac{\partial^\beta}{\partial y^\beta}\frac{\partial K_i}{\partial y^1}(0) y^\beta + \frac{1}{(n-3)!}\sum_{|\beta|=n-3} \frac{\partial^\beta}{\partial y^\beta}\frac{\partial K_i}{\partial y^1}(\varsigma) y^\beta.\end{aligned}$$

Therefore we have

$$\begin{aligned}&\int_{B_\sigma} \lambda_i^{2n/(n-2)} v_i^{p_i+1} \left|\frac{\partial K_i}{\partial y^1}(0)\right| dy \\ &\qquad \le \left|\int_{B_\sigma} \lambda_i^{2n/(n-2)} v_i^{p_i+1} \frac{\partial K_i}{\partial y^1} dy\right| + C\sum_{|\beta|=1}^{n-4} \int_{B_\sigma} \left|\frac{\partial^\beta}{\partial y^\beta}\frac{\partial K_i}{\partial y^1}(0)\right| |y|^{|\beta|} v_i^{p_i+1} dy \\ &\qquad\qquad + C\sum_{|\beta|=n-3} \int_{B_\sigma} \left|\frac{\partial^\beta}{\partial y^\beta}\frac{\partial K_i}{\partial y^1}(\varsigma)\right| |y|^{n-3} v_i^{p_i+1} dy \\ &\qquad \le C\left(\delta_i r_i + v_i(0)^{2t_i}\right) + C\sum_{|\beta|=1}^{n-4} \int_{B_\sigma} \left|\frac{\partial^\beta}{\partial y^\beta}\frac{\partial K_i}{\partial y^1}(0)\right| |y|^{|\beta|} v_i^{p_i+1} dy \\ &\qquad\qquad + C\sum_{|\beta|=n-3} \int_{B_\sigma} \left|\frac{\partial^\beta}{\partial y^\beta}\frac{\partial K_i}{\partial y^1}(\varsigma)\right| |y|^{n-3} v_i^{p_i+1} dy.\end{aligned}$$

By (8) this implies

$$\begin{aligned}\left|\frac{\partial K_i}{\partial y^1}(0)\right| \le C\left(\delta_i r_i + v_i(0)^{2t_i}\right) + C\sum_{|\beta|=1}^{n-4} \int_{B_\sigma} \left|\frac{\partial^\beta}{\partial y^\beta}\frac{\partial K_i}{\partial y^1}(0)\right| |y|^{|\beta|} v_i^{p_i+1} dy \\ + C\sum_{|\beta|=n-3} \int_{B_\sigma} \left|\frac{\partial^\beta}{\partial y^\beta}\frac{\partial K_i}{\partial y^1}(\varsigma)\right| |y|^{n-3} v_i^{p_i+1} dy.\end{aligned}$$

By Lemma 4.2, Proposition 4.3 and (21), we have, when $1 \le |\beta| \le n-4$,

$$\begin{aligned}&\int_{B_\sigma} \left|\frac{\partial^\beta}{\partial y^\beta}\frac{\partial K_i}{\partial y^1}(0)\right| |y|^{|\beta|} v_i^{p_i+1} dy \\ &\qquad \le C r_i \int_{B_\sigma} |\nabla K_i(0)|^{(n-2-(|\beta|+1))/(n-3)} |y|^{|\beta|} v_i^{p_i+1} dy \\ &\qquad = C r_i \int_{B_\sigma} |\nabla K_i(0)|^{(n-3-|\beta|)/(n-3)} |y|^{|\beta|} v_i^{p_i+1} dy\end{aligned}$$

$$\le Cr_i \int_{B_\sigma} \Big(|\nabla K_i(0)|^{\frac{n-3-|\beta|}{n-3} \cdot \frac{n-3}{n-3-|\beta|}} + |y|^{|\beta| \cdot \frac{n-3}{|\beta|}} \Big) v_i^{p_i+1} dy$$

(by Young's inequality)

$$= Cr_i \Big(\int_{B_\sigma} |\nabla K_i(0)| v_i^{p_i+1} dy + \int_{B_\sigma} |y|^{n-3} v_i^{p_i+1} dy \Big)$$
$$\le Cr_i |\nabla K_i(0)| + Cr_i v_i(0)^{-2(n-3)/(n-2)}.$$

Furthermore,

$$\sum_{|\beta|=n-3} \int_{B_\sigma} \Big| \frac{\partial^\beta}{\partial y^\beta} \frac{\partial K_i}{\partial y^1}(\varsigma) \Big| |y|^{n-3} v_i^{p_i+1} dy \le Cr_i^{n-2} \int_{B_\sigma} |y|^{n-3} v_i^{p_i+1} dy$$
$$\le Cr_i^{n-2} v_i(0)^{-2(n-3)/(n-2)}.$$

Therefore

$$\Big| \frac{\partial K_i}{\partial y^1}(0) \Big| \le C(\delta_i r_i + v_i(0)^{2t_i}) + (Cr_i |\nabla K_i(0)| + Cr_i v_i(0)^{-2(n-3)/(n-2)}) + Cr_i^{n-2} v_i(0)^{-2(n-3)/(n-2)}$$
$$\le C\delta_i r_i + Cv_i(0)^{2t_i} + Cr_i |\nabla K_i(0)| + Cr_i v_i(0)^{-2(n-3)/(n-2)}.$$

The same estimate also holds for $|\partial K_i / \partial y^j(0)|$, where $j = 2, \ldots, n$, so we know

$$|\nabla K_i(0)| \le C\delta_i r_i + Cv_i(0)^{2t_i} + Cr_i |\nabla K_i(0)| + Cr_i v_i(0)^{-2(n-3)/(n-2)}$$
$$\le C(v_i(0)^{2t_i} + |\nabla K_i(0)| v_i(0)^{-2/(n-2)} + r_i v_i(0)^{-2}) r_i + Cv_i(0)^{2t_i} + Cr_i |\nabla K_i(0)| + Cr_i v_i(0)^{-2(n-3)/(n-2)} \qquad \text{(by (23))}.$$

When i is large enough, all the terms involving $|\nabla K_i(0)|$ can be absorbed into the left hand side of this inequality, therefore we get a refined estimate

$$\begin{aligned} |\nabla K_i(0)| &\le Cr_i v_i(0)^{2t_i} + Cr_i^2 v_i(0)^{-2} + Cv_i(0)^{2t_i} + Cr_i v_i(0)^{-2(n-3)/(n-2)} \\ &\le Cr_i^2 v_i(0)^{-2} + Cv_i(0)^{2t_i} + Cr_i v_i(0)^{-2(n-3)/(n-2)}. \end{aligned} \tag{24}$$

Finally, we will prove that (19) holds. As in the proof of Proposition 4.4, this will give the desired contradiction by comparing the signs of both sides of (15), which rules out Case I. We know

$$v_i^2(0) \int_{B_\sigma} |X(K_i)| v_i^{p_i+1} \lambda_i^{2n/(n-2)} dy = v_i^2(0) \int_{B_\sigma} \Big| r \frac{\partial K_i}{\partial r} \Big| v_i^{p_i+1} \lambda_i^{2n/(n-2)} dy$$
$$\le Cv_i^2(0) \big(|\nabla K_i(0)| v_i(0)^{-2/(n-2)} + r_i v_i(0)^{-2} \big) \qquad \text{(by (22))}$$
$$\le Cv_i^2(0) \Big((r_i^2 v_i(0)^{-2} + v_i(0)^{2t_i} + r_i v_i(0)^{-2(n-3)/(n-2)}) v_i(0)^{-2/(n-2)} + r_i v_i(0)^{-2} \Big) \qquad \text{(by (24))}$$
$$= C(r_i^2 v_i(0)^{-2/(n-2)} + v_i(0)^{2+2t_i-2/(n-2)} + 2r_i).$$

By (20) we know $\lim_{i\to\infty}(2+2t_i-2/(n-2))<0$, and therefore

$$\lim_{i\to\infty} v_i(0)^{2+2t_i-2/(n-2)}=0.$$

It follows from this and $\lim_{i\to\infty} r_i^2 v_i(0)^{-2/(n-2)}=\lim_{i\to\infty} r_i=0$ that

$$\lim_{i\to\infty} v_i^2(0)\int_{B_\sigma}|X(K_i)|v_i^{p_i+1}\lambda_i^{2n/(n-2)}dy=0.$$

This completes the proof in Case I.

5. Ruling out Case II

Now we consider Case II, which has been reduced to the following: There is a sequence of functions $\{v_i\}$, each satisfying

$$\Delta_{g^{(i)}}v_i+K(\sigma_i y)v_i^{p_i}=0,$$

where $\sigma_i\to 0$ and $g^{(i)}(y)=g_{\alpha\beta}(\sigma_i y)dy^\alpha dy^\beta$. The sequence $\{v_i\}$ has isolated blow-up point(s) $\{0,\dots\}$.

If 0 is not a simple blow-up point, then we can do another rescaling and repeat the argument in the previous section, with r_i replaced by $r_i\sigma_i$, to get a contradiction. Therefore 0 must be a simple blow-up point for $\{v_i\}$. Then we can still repeat the argument in the previous section, with r_i replaced by σ_i. The only difference is in the expression of $h=\lim_{i\to\infty}v_i(y)/v_i(\bar{y})$. As shown in [Yan 2007, Section 7], because here $|y|^{2/(p_i-1)}\bar{v}_i(|y|)$ doesn't have a second critical point at $|y|=1$, we have a different expression of h: near 0,

$$h(y)=c_1|y|^{2-n}+A+O(|y|),$$

where A is a positive constant. This positive "mass" term $A>0$ guarantees that the limit of the boundary term of the Pohozaev identity (15) is negative, that is,

$$\lim_{i\to\infty}\frac{1}{v_i^2(\bar{y})}\int_{\partial B_\sigma}T_i(X,\nu_i)d\Sigma_i<0.$$

The other parts of the proof remain the same. Therefore Case II can also be ruled out.

Thus we have finished the proof of Theorem 1.4. □

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