

# *Pacific Journal of Mathematics*

**TWISTED CONJUGACY CLASSES IN R. THOMPSON'S  
GROUP  $F$**

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**In this article, we prove that any automorphism of R. Thompson's group  $F$  has infinitely many twisted conjugacy classes. The result follows from the work of Brin, together with standard facts about R. Thompson's group  $F$ , and elementary properties of the Reidemeister numbers.**

## 1. Introduction

Work in Nielsen–Reidemeister fixed point theory [Jiang 1983; Fel'shtyn 2000], in Selberg theory [Shokranian 1992; Arthur and Clozel 1989], and in algebraic geometry [Deligne 1977] has led to a program to obtain a twisted analogue of the celebrated Burnside–Frobenius theorem. If one assumes  $\phi : G \rightarrow G$  is an automorphism of a group  $G$ , then the relation

$$x \sim gx\phi(g^{-1})$$

partitions  $G$  into equivalence classes, called the *Reidemeister classes* of  $\phi$  or *twisted conjugacy classes* of  $\phi$ . The program mentioned above consists of establishing the coincidence of the Reidemeister number  $R(\phi)$  of the automorphism  $\phi$  (the number of its Reidemeister classes) and the number of fixed points of an induced homeomorphism for an appropriate dual object.

In this paper, we show that  $R(\phi) = \infty$  for any automorphism  $\phi$  of R. Thompson's group  $F$ .

Following the naming convention suggested in [Taback and Wong 2006b], we say that if  $G$  is a group for which  $R(\phi) = \infty$  for every automorphism  $\phi$  of  $G$ , then  $G$  has property  $R_\infty$ .

The long-term project of discovering which groups have property  $R_\infty$  began in [Fel'shtyn and Hill 1994]. Currently, we know that the following groups belong to this class:

- (1) nonelementary Gromov hyperbolic groups [Fel'shtyn 2001; Levitt and Lustig 2000];

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MSC2000: 20E45, 37C25, 55M20.

Keywords: Reidemeister number, twisted conjugacy class, R. Thompson's group  $F$ ,  $R_\infty$  property.

## (2) Baumslag–Solitar groups

$$BS(m, n) = \langle a, b \mid ba^m b^{-1} = a^n \rangle$$

except for  $BS(1, 1)$  [Fel'shtyn and Gonçalves 2008];

- (3) generalized Baumslag–Solitar groups, that is, finitely generated groups which act on a tree with all edge and vertex stabilizers infinite cyclic [Levitt 2005];
- (4) lamplighter groups  $\mathbb{Z}_n \wr \mathbb{Z}$  if and only if  $2 \mid n$  or  $3 \mid n$  [Gonçalves and Wong 2006];
- (5) the solvable generalization  $\Gamma$  of  $BS(1, n)$  given by the short exact sequence

$$1 \rightarrow \mathbb{Z} \left[ \begin{array}{c} 1 \\ n \end{array} \right] \rightarrow \Gamma \rightarrow \mathbb{Z}^k \rightarrow 1$$

as well as any group quasiisometric to  $\Gamma$  [Taback and Wong 2006b], and groups which are quasiisometric to  $BS(1, n)$  [Taback and Wong 2006a] (it is interesting to note that having the  $\mathbb{R}_\infty$  property in general is not a quasiisometry invariant of a group);

- (6) saturated weakly branch groups, including the Grigorchuk and Gupta–Sidki groups [Fel'shtyn et al. 2008].

Our work relies heavily on [Brin 1996], where the automorphisms of R. Thompson's groups  $F$  and  $T$  are classified. By that work, it is also reasonable to investigate whether or not  $T$  has property  $R_\infty$ . We do not answer the question for  $T$  here, as  $T$  is simple, and thus requires an entirely different approach than what we use for  $F$ . However, we note that generalizations of the approach we use for  $F$  should help to answer the question below, for specific  $n$  (this extension is complicated by the existence of “exotic” automorphisms of  $F_n$  for various  $n$ ; see [Brin and Guzmán 1998]).

**Question 1.1.** Do the generalized Thompson's groups  $F_n$  have the  $R_\infty$  property?

## 2. Relevant facts about R. Thompson's group $F$

Following the definition in [Brin 1996],  $F$  consists of a restricted class of homeomorphisms of the real line under the operation of composition. A homeomorphism  $\alpha : \mathbb{R} \rightarrow \mathbb{R}$  is in  $F$  if and only if  $\alpha$

- (1) is piecewise-linear (admitting finitely many breaks in slope),
- (2) is orientation-preserving,
- (3) has all slopes of affine portions of its graph in the set  $\{2^k \mid k \in \mathbb{Z}\}$ ,
- (4) has all breaks in slope occurring over the dyadic rationals  $\mathbb{Z}[\frac{1}{2}]$ ,
- (5) maps  $\mathbb{Z}[\frac{1}{2}]$  into  $\mathbb{Z}[\frac{1}{2}]$ , and

(6) has first and last affine components equal to pure translations by (potentially distinct) integers  $\alpha_l$  and  $\alpha_r$ .

We rely heavily on a deep result in [Brin 1996]. In order to state his result in full, we must first isolate some relevant automorphisms.

Define the set of *eventually  $T$ -like* piecewise linear self-homeomorphisms of  $\mathbb{R}$  as follows.

A homeomorphism  $\alpha : \mathbb{R} \rightarrow \mathbb{R}$  is *eventually  $T$ -like* if and only if  $\alpha$  satisfies conditions (2)–(5) as in the definition of an element of  $F$  above, as well as conditions (1') and (6') as below.

- (1')  $\alpha$  is piecewise-linear with potentially infinitely many breaks in slope, where only a finite collection of breaks in slope can occur over any compact subset of the domain, and
- (6')  $\alpha$  has minimal domain value  $R_\alpha \geq 0$  and maximal domain value  $L_\alpha \leq 0$  so that  $\alpha(x + 1) = \alpha(x) + 1$  for all  $x \geq R_\alpha$  and  $\alpha(x + 1) = \alpha(x) + 1$  for all  $x \leq L_\alpha - 1$ .

It is easy to check that an inner automorphism of  $\text{Homeo}(\mathbb{R})$  defined by conjugating elements of  $\text{Homeo}(\mathbb{R})$  by any specific eventually  $T$ -like element restricts to an automorphism of  $F$ . We will call such an automorphism of  $F$  an *eventually  $T$ -like conjugation* of  $F$ .

Now define  $\text{Rev} : F \rightarrow F$  to be the automorphism of  $F$  produced by conjugating an element of  $F$  by the real homeomorphism  $x \mapsto -x$  (again, the reader can check that this does induce an automorphism of  $F$ ).

The following is a restatement of [Brin 1996, Theorem 1] using our language.

**Theorem 2.1** (Brin). *The automorphism group of  $F$  is generated by  $\text{Rev}$  and the eventually  $T$ -like conjugations of  $F$ .*

In our context, the abelianization of  $F$  is given by the map  $Ab : F \rightarrow \mathbb{Z}^2$ , which is defined by the rule

$$Ab(f) = (f_l, f_r)$$

(see [Cannon et al. 1996, Theorem 4.1]) where here,  $f_l$  is the translational part of  $f$  near  $-\infty$ , and  $f_r$  is the translational part of  $f$  near  $\infty$ .

Below, given any  $k \in F$ , let us denote by  $k_l$  and  $k_r$  the translational parts of  $k$  (near  $-\infty$  and  $\infty$  respectively).

### 3. Observations and arguments

Our argument consists of noticing relevant facts about the automorphisms mentioned by Brin, and then chasing implications through the relevant algebraic machinery.

All individual steps from this point on are both simple and elementary, so any statement without proof is to be taken as an exercise for the reader.

**Remark 3.1.** Let  $f \in F$ , and let  $g : \mathbb{R} \rightarrow \mathbb{R}$  be eventually  $T$ -like. If  $k = f^g$ , then  $f_l = k_l$  and  $f_r = k_r$ .

Let  $Ab_* : \text{Aut}(F) \rightarrow \text{Aut}(\mathbb{Z} \times \mathbb{Z})$  be the map induced by the abelianization map of  $F$ . [Theorem 2.1](#) and [Remark 3.1](#) together imply the straightforward points (all matrices throughout the paper are assuming the standard basis on  $\mathbb{Z}^2$ ):

**Corollary 3.2.** (1) *The matrix of the element  $Ab_*(\text{Rev})$  is*

$$M = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

(2) *Let  $g : \mathbb{R} \rightarrow \mathbb{R}$  be eventually  $T$ -like and  $\varphi_g$  the automorphism of  $F$  which is conjugation by  $g$ . The matrix of  $Ab_*(\varphi_g)$  is the identity matrix.*

(3)  *$Ab_*(\text{Aut}(F))$  is isomorphic to  $\mathbb{Z}_2$  and corresponds to the group generated by*

$$M = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

We can now state the main result.

**Theorem 3.3.** *For any automorphism  $\phi$  of  $R$ . Thompson's group  $F$ , the Reidemeister number  $R(\phi)$  is infinite.*

*Proof.* Suppose  $\varphi : F \rightarrow F$  is an automorphism of  $F$ . It is immediate that the Reidemeister classes of  $\varphi$  project surjectively onto the Reidemeister classes of  $Ab_*(\varphi)$ .

The matrix  $M$  of  $Ab_*(\varphi)$  is either the identity or

$$M = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

In the first case, each point in  $\mathbb{Z}^2$  represents its own Reidemeister class. In the second case, if we use

$$x = (a, b), \quad g = (c, d), \quad \text{and} \quad \phi = M$$

in the formula in the first paragraph of the introduction, we see that

$$(a, b) \sim (c, d) + (a, b) + M(-c, -d) = (a + c + d, b + c + d).$$

As  $c + d$  can be any integer, we see that  $(a, b) \sim (a + t, b + t)$  for any integer  $t$ . In particular, the set  $\{(0, t) | t \in \mathbb{Z}\}$  represents an infinite set of representatives of distinct Reidemeister classes for  $Ab_*(\varphi)$ .  $\square$

## Acknowledgments

Bleak thanks Professor Brin for helpful conversations and discussion of his work [1996], upon which this note strongly relies. Fel'shtyn thanks V. Guba and M. Sapir for stimulating discussions and comments.

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Received March 22, 2008. Revised June 25, 2008.

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