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We extend Guillemin's formula for Kähler potentials on toric manifolds to singular quotients of $\mathbb{C}^N$ and $\mathbb{C}P^N$.

1. Introduction

Let G be a torus with Lie algebra $\mathfrak{g}$ and integral lattice $\mathbb{Z}_G \subset \mathfrak{g}$. Let $u_1, \dots, u_N \in \mathbb{Z}_G$ be a set of primitive vectors which span $\mathfrak{g}$ over $\mathbb{R}$. Let $\lambda_1, \dots, \lambda_N \in \mathbb{R}$ and let

$$P = P_{u,\lambda} := \{\eta \in \mathfrak{g}^* \mid \langle \eta, u_j \rangle - \lambda_j \geq 0,\ \ 1 \leq j \leq N\}$$

be the corresponding polyhedral set. We *assume that P has a nonempty interior and that the collection of inequalities defining P is minimal*: if we drop the condition that $\langle \eta, u_j \rangle - \lambda_j \geq 0$ for some index j then the resulting set is strictly bigger than P.

A well-known construction of Delzant, suitably tweaked, produces a symplectic stratified space M_P with an effective Hamiltonian action of the torus G and associated moment map $\phi = \phi_P : M_P \to \mathfrak{g}^*$ such that $\phi(M_P) = P$. We will review the construction below. The space M_P is a symplectic quotient of $\mathbb{C}^N$ by a compact abelian subgroup K of the standard torus $\mathbb{T}^N$. Therefore, by a theorem of Heinzner and Loose [1994] M_P is a complex analytic space. Moreover M_P is a Kähler space; see [Heinzner and Loose 1994, (3.5)] and [Heinzner et al. 1994]. Even though in general the space M_P is singular, the preimages of open faces of P under the moment map ϕ_P are smooth Kähler manifolds. The main results of the paper are formulas for the Kähler forms on these manifolds. In particular we will show that the Kähler form ω on the preimage $\phi_P^{-1}(\mathring{P})$ of the interior $\mathring{P}$ of the polyhedral set P is given by

$$\omega = \sqrt{-1}\, \partial \bar{\partial}\, \phi_P^* \left(\sum_{j=1}^{N} \lambda_j \log(u_j - \lambda_j) + u_j \right), \tag{1-1}$$

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where we think of $u_j \in \mathbb{Z}_G$ as a function on $\mathfrak{g}^*$.

Formula (1-1) was originally proved by Guillemin in the case where M_P is a compact manifold (and thus P is a simple unimodular polytope, also known as a *Delzant polytope*). It was extended to the case of compact orbifolds by Abreu [2001]. Calderbank, David and Gauduchon gave two new proofs of Guillemin's formula (for orbifolds) in [Calderbank et al. 2003]. One of their proofs was simplified further in [Burns and Guillemin 2004].

As we just mentioned, for generic values of λ the polyhedral set P is simple and consequently M_P is at worse an orbifold. But for arbitrary values of λ it may have more serious singularities. Of particular interest is the singular case where P is a cone on a simple polytope. Then there is only one singular point, and the link of the singularity is a Sasakian orbifold. Such orbifolds, especially the ones with Sasaki–Einstein metrics, have attracted some attention in string theory. They play a role in the AdS/CFT correspondence [Martelli and Sparks 2004].

If the polyhedral set P is a polytope, that is, if P is compact, then as a symplectic space M_P may also be obtained as a symplectic quotient of $\mathbb{C}\mathrm{P}^N$. In this case the Fubini–Study form on $\mathbb{C}\mathrm{P}^N$ will induce a Kähler structure on M_P, which is *different* from the one induced by the flat metric on $\mathbb{C}^N$ even in the case where M_P is smooth. We will give a formula for this Kähler structure as well.

The methods of this paper are quite close to that of [Calderbank et al. 2003]. In particular the key Lemma 3.3 is a direct corollary of Proposition 2 in that reference.

2. The "Delzant" construction: toric varieties as Kähler quotients

It will be convenient for us to fix the following notation. As in the introduction, let G be a torus with Lie algebra $\mathfrak{g}$ and integral lattice $\mathbb{Z}_G \subset \mathfrak{g}$. Let $u_1, \dots, u_N \in \mathbb{Z}_G$ be a set of primitive vectors which span $\mathfrak{g}$ over $\mathbb{R}$. Let $\lambda_1, \dots, \lambda_N \in \mathbb{R}$ and let

$$P = P_{u,\lambda} := \{\eta \in \mathfrak{g}^* \mid \langle \eta, u_j\rangle - \lambda_j \geq 0, \quad 1 \leq j \leq N\} \tag{2-1}$$

be the corresponding polyhedral set. As above *we assume that P has the nonempty interior and that the collection of inequalities defining P is minimal.* Let $A : \mathbb{Z}^N \to \mathbb{Z}_G$ be the $\mathbb{Z}$-linear map given by

$$A\,(x_1, \dots, x_N) = \sum x_i u_i\,.$$

That is, A is defined by sending the standard basis vector e_i of $\mathbb{Z}^N$ to u_i. Let A also denote the $\mathbb{R}$-linear extension $\mathbb{R}^N \to \mathfrak{g}$. Let $\mathfrak{k} = \ker A$ and let $B : \mathfrak{k} \to \mathbb{R}^N$ denote the inclusion. The map A induces a surjective map of Lie groups

$$\bar{A} : \mathbb{T}^N = \mathbb{R}^N/\mathbb{Z}^N \to \mathfrak{g}/\mathbb{Z}_G = G.$$

Let $K = \ker \overline{A}$ and let $\overline{B} : K \to \mathbb{T}^N$ denote the corresponding inclusion. The group K is a compact abelian group which need not be connected. It's easy to see that the Lie algebra of K is $\mathfrak{k}$.

We have a short exact sequence of abelian Lie algebras:

$$0 \to \mathfrak{k} \xrightarrow{B} \mathbb{R}^N \xrightarrow{A} \mathfrak{g} \to 0.$$

Let

$$0 \to \mathfrak{g}^* \xrightarrow{A^*} (\mathbb{R}^N)^* \xrightarrow{B^*} \mathfrak{k}^* \to 0$$

be the dual sequence. Note that $\ker B^* = A^*(\mathfrak{g}^*) = \mathfrak{k}^\circ$ where $\mathfrak{k}^\circ$ denotes the annihilator of $\mathfrak{k}$ in $(\mathbb{R}^N)^*$. Let $\{e_i{}^*\}$ denote the dual basis of $(\mathbb{R}^N)^*$ and let $\lambda = \sum \lambda_i e_i{}^*$. We note that

$$(B^*)^{-1}(B^*(-\lambda)) = -\lambda + \mathfrak{k}^\circ = -\lambda + A^*(\mathfrak{g}^*).$$

In particular $(B^*)^{-1}(B^*(-\lambda))$ is the image of the affine embedding

$$\iota_\lambda : \mathfrak{g}^* \hookrightarrow (\mathbb{R}^N)^*, \quad \iota_\lambda(\ell) = -\lambda + A^*(\ell). \tag{2-2}$$

Lemma 2.1. *Let P be the polyhedral set defined by* (2-1) *above.*

(1) *There exists a Kähler space M_P with an effective holomorphic Hamiltonian action of the torus G so that the image of the associated moment map $\phi_P : M_P \to \mathfrak{g}^*$ is P.*

(2) *For every open face $\mathring{F}$, the preimage $\phi_P^{-1}(\mathring{F})$ is the Kähler quotient of a complex torus $(\mathbb{C}^\times)^{N_F}$ by a compact subgroup K_F of the compact torus $\mathbb{T}^{N_F} \subset (\mathbb{C}^\times)^{N_F}$. Here the number N_F and the group K_F depend on the face F.*

(3) *If the set P is bounded, then M_P can also be constructed as a Kähler quotient of $\mathbb{C}\mathrm{P}^N$.*

Proof. For every index i and any $\eta \in \mathfrak{g}^*$

$$\langle \eta, Ae_i \rangle - \lambda_i = \langle A^*\eta, e_i \rangle - \left\langle \sum \lambda_j e_j{}^*, e_i \right\rangle = \langle A^*\eta - \lambda, e_i \rangle = \langle \iota_\lambda(\eta), e_i \rangle.$$

Therefore

$$\iota_\lambda(P) = \left\{ \ell \in (\mathbb{R}^N)^* \;\middle|\; \langle \ell, e_i \rangle \geq 0,\ 1 \leq i \leq N \right\} \cap \iota_\lambda(\mathfrak{g}^*).$$

More generally, if $\mathring{F} \subset P$ is an open face, there is a unique subset $I_F = I \subset \{1, \dots, N\}$ so that

$$\mathring{F} = \bigcap_{j \notin I} \{\eta \in \mathfrak{g}^* \mid \langle \eta, u_j \rangle - \lambda_j > 0\} \cap \bigcap_{j \in I} \{\eta \in \mathfrak{g}^* \mid \langle \eta, u_j \rangle - \lambda_j = 0\}.$$

Therefore

$$\iota_\lambda(\mathring{F}) = \iota_\lambda(\mathfrak{g}^*) \cap \bigcap_{j \notin I} \left\{ \ell \in (\mathbb{R}^N)^* \;\middle|\; \langle \ell, e_j \rangle > 0 \right\} \cap \bigcap_{j \in I} \left\{ \ell \in (\mathbb{R}^N)^* \;\middle|\; \langle \ell, e_j \rangle = 0 \right\}. \tag{2-3}$$

The moment map ϕ for the action of $\mathbb{T}^N$ on $(\mathbb{C}^N, \sqrt{-1}\sum dz_j \wedge d\bar{z}_j)$ is given by

$$\phi(z) = \sum |z_j|^2 {e_j}^*.$$

Hence

$$\phi(\mathbb{C}^N) = \left\{\ell \in (\mathbb{R}^N)^* \;\middle|\; \langle \ell, e_i\rangle \geq 0, \quad 1 \leq i \leq N\right\}.$$

The moment map ϕ_K for the action of K on $\mathbb{C}^N$ is the composition

$$\phi_K = B^* \circ \phi.$$

Let $\nu = B^*(-\lambda)$. We argue that

$$\phi(\phi_K^{-1}(\nu)) = \iota_\lambda(P).$$

Indeed,

$$\begin{aligned}\phi_K^{-1}(\nu) &= \phi^{-1}\big((B^*)^{-1}(\nu)\big) = \phi^{-1}\big((B^*)^{-1}(B^*(-\lambda))\big)\\ &= \phi^{-1}(\iota_\lambda(\mathfrak{g}^*)) = \phi^{-1}(\phi(\mathbb{C}^N) \cap \iota_\lambda(\mathfrak{g}^*)) = \phi^{-1}(\iota_\lambda(P)).\end{aligned}$$

Therefore

$$\phi(\phi_K^{-1}(\nu)) = \iota_\lambda(P).$$

The restriction

$$\phi|_{\phi_K^{-1}(\nu)}$$

descends to a map

$$\bar{\phi} : M_P \equiv \phi_K^{-1}(\nu)/K \to \iota_\lambda(\mathfrak{g}^*).$$

It is not hard to see that the composition ϕ_P of $\bar{\phi}$ with the isomorphism

$$\iota_\lambda(\mathfrak{g}^*) \xrightarrow{\simeq} \mathfrak{g}^*$$

is a moment map for the action of G on the symplectic quotient (symplectic stratified space) M_P. Since the isomorphism $\iota_\lambda(\mathfrak{g}^*) \to \mathfrak{g}^*$ obviously maps $\iota_\lambda(P)$ to P, we conclude that the image of $\phi_P : M_P \to \mathfrak{g}^*$ is exactly P. This proves (1).

To prove (2) we define a bit more notation. For a subset $I \subset \{1, \ldots, N\}$ we define the corresponding coordinate subspace

$$V_I := \left\{z \in \mathbb{C}^N \;\middle|\; j \in I \Rightarrow z_j = 0\right\}.$$

Its "interior" $\mathring{V}_I$ is defined by

$$\mathring{V}_I := \left\{z \in \mathbb{C}^N \;\middle|\; j \in I \Leftrightarrow z_j = 0\right\}.$$

Also, let

$$\mathbb{T}_I^N := \left\{a \in \mathbb{T}^N \;\middle|\; j \notin I \Rightarrow a_j = 1\right\}.$$

The sets V_I, $\mathring{V}_I$ are Kähler submanifolds of $\mathbb{C}^N$ preserved by the action of $\mathbb{T}^N$. Both are fixed by $\mathbb{T}_I^N$, with $\mathring{V}_I$ being precisely the set of points of orbit type $\mathbb{T}_I^N$.

The restriction $\phi_K|_{\mathring{V}_I}$ is a moment map for the action of K on $\mathring{V}_I$. Moreover, for any $\nu \in \mathfrak{k}^*$

$$\phi_K^{-1}(\nu) \cap \mathring{V}_I = (\phi_K|_{\mathring{V}_I})^{-1}(\nu).$$

Hence

$$(\phi_K^{-1}(\nu) \cap \mathring{V}_I)/K = (\phi_K|_{\mathring{V}_I})^{-1}(\nu)/K.$$

While the action of K on $\mathring{V}_I$ need not be free, the action of

$$K_I := K/(K \cap \mathbb{T}^N_I)$$

on $\mathring{V}_I$ *is* free. Therefore, the quotient $(\phi_K^{-1}(\nu) \cap \mathring{V}_I)/K$ may be interpreted as a *regular* Kähler quotient of $\mathring{V}_I$ by the Hamiltonian action of K_I:

$$(\phi_K^{-1}(\nu) \cap \mathring{V}_I)/K = \mathring{V}_I /\!/_{\nu_I} K_I \tag{2-4}$$

for an appropriate value $\nu_I \in \mathfrak{k}_I^*$ of the K_I moment map.

Given a face F, let $I = I_F$ be the corresponding subset of $\{1, \ldots, N\}$. By (2-3),

$$\begin{aligned}\left\{z \in \mathbb{C}^N \;\middle|\; \phi(z) \in \iota_\lambda(\mathring{F})\right\} &= \left\{z \in \mathbb{C}^N \;\middle|\; \phi(z) \in \iota_\lambda(\mathfrak{g}^*),\ \langle \phi(z), e_j\rangle \begin{cases} > 0 & \text{for } j \notin I, \\ = 0 & \text{for } j \in I \end{cases}\right\}\\ &= \phi_K^{-1}(\nu) \cap \mathring{V}_I.\end{aligned}$$

Therefore,

$$\phi_K^{-1}(\nu) \cap \mathring{V}_I = \phi^{-1}(\iota_\lambda(\mathring{F})).$$

It follows from the definition of ϕ_P that

$$(\phi_K^{-1}(\nu) \cap \mathring{V}_I)/K = \phi_P^{-1}(\mathring{F}).$$

By (2-4) we conclude that

$$\phi_P^{-1}(\mathring{F}) = \mathring{V}_I /\!/_{\nu_I} K_I.$$

This proves (2).

If P is compact, then $\iota_\lambda(P) \subset (\mathbb{R}^N)^*$ is bounded. Hence $\iota_\lambda(P)$ is contained in a sufficiently large multiple of the standard simplex. Any such simplex is the image of $\mathbb{C}\mathrm{P}^N$ under the moment map for the standard action of $\mathbb{T}^N$ with the Kähler form on $\mathbb{C}\mathrm{P}^N$ being the appropriate multiple of the standard Fubini–Study form. This proves (3). □

Remark 2.2. It follows from the results of Heinzner and his collaborators (email communication), in particular of Heinzner and Huckleberry [1996], that the action of G on M_P extends to an action the complexified group $G^{\mathbb{C}}$. This action of $G^{\mathbb{C}}$ has a dense open orbit. In other words, M_P is a *toric* Kähler space.

3. Kähler potentials, Legendre transforms and symplectic quotients

As we mentioned in the introduction, the line of argument of this section is quite close to the approach in [Calderbank et al. 2003], and Lemma 3.3 can be easily deduced form Proposition 2 of that reference. We keep our exposition self-contained.

We start by recalling a result of Guillemin [1994, Theorems 4.2, 4.3]:

Lemma 3.1. *Suppose the action of $\mathbb{T}^N$ on $(\mathbb{C}^\times)^N = \mathbb{R}^N \times \sqrt{-1}\mathbb{T}^N$ preserves a Kähler form ω and is Hamiltonian. Then there exists a $\mathbb{T}^N$-invariant function f on $(\mathbb{C}^\times)^N$ such that $\omega = i\partial\bar\partial f$. Additionally*

$$\mathscr{L}_f \circ \pi : (\mathbb{C}^\times)^N \to (\mathbb{R}^N)^*$$

is a moment map for the action of $\mathbb{T}^N$ on $((\mathbb{C}^\times)^N, \omega)$. Here $\pi : \mathbb{R}^N \times \sqrt{-1}\mathbb{T}^N \to \mathbb{R}^N$ is the projection and $\mathscr{L}_f : \mathbb{R}^N \to (\mathbb{R}^N)^*$ is the Legendre transform of f, where we have identified $f \in C^\infty((\mathbb{C}^\times)^N)^{\mathbb{T}^N}$ with a function on $\mathbb{R}^N$.

The same result holds with $(\mathbb{C}^\times)^N$ replaced by $U \times \sqrt{-1}\mathbb{T}^N$ for any contractible open set $U \subset \mathbb{R}^N$.

Lemma 3.2. *Let $f : V \to \mathbb{R}$ be a (strictly) convex function on a finite dimensional vector space V, let $A : W \to V$ be an injective linear map, $x \in V$ be a point and*

$$j : W \to V, \quad j(w) = Aw + x$$

an affine map. Then $f \circ j : W \to \mathbb{R}$ is (strictly) convex and the associated Legendre transform $\mathscr{L}_{f\circ j} : W \to W^*$ is given by

$$\mathscr{L}_{f\circ j} = A^* \circ \mathscr{L}_f \circ j,$$

where $A^* : V^* \to W^*$ is the dual map.

Proof. By the chain rule and the definition of the Legendre transform,

$$\mathscr{L}_{f\circ j}(w) = d(f\circ j)_w = df_{j(w)} \circ dj_w = \mathscr{L}_f(j(w)) \circ A = A^* \circ \mathscr{L}_f \circ j(w)$$

for any $w \in W$. □

Lemma 3.3. *Let $f \in C^\infty(\mathbb{R}^N)$ be a strictly convex function and $\omega = \sqrt{-1}\partial\bar\partial\pi_N^* f$ the corresponding $\mathbb{T}^N$-invariant Kähler form on $(\mathbb{C}^\times)^N = \mathbb{R}^N \times \sqrt{-1}\mathbb{T}^N$ (here $\pi_N : (\mathbb{C}^\times)^N \to \mathbb{R}^N$ is the projection). Let $\phi = \mathscr{L}_f \circ \pi_N : (\mathbb{C}^\times)^N \to (\mathbb{R}^N)^*$ denote the associated moment map.*

Let $K \subset \mathbb{T}^N$ be a closed subgroup and let $G = \mathbb{T}^N/K$. For any $\nu \in \mathfrak{k}^*$ the symplectic quotient

$$(\mathbb{C}^\times)^N /\!/_\nu K$$

is biholomorphic to $U \times \sqrt{-1}G \subset \mathfrak{g} \times \sqrt{-1}G = G^{\mathbb{C}}$ where $U \subset \mathfrak{g}$ is an open contractible set. Hence the reduced Kähler form ω_ν has a potential f_ν.

Moreover, the Legendre–Fenchel dual f_ν^* *of the Kähler potential* f_ν *is given by*

$$f_\nu^* = f^* \circ \iota_\lambda \tag{3-1}$$

where $\iota_\lambda : \mathfrak{g}^* \to (\mathbb{R}^N)^*$ *is the affine embedding* (2-2) *and* $-\lambda$ *is a point in* $(B^*)^{-1}(\nu)$.

Proof. It is no loss of generality to assume that the group K is connected. Then $\mathbb{T}^N \simeq K \times G$. Consequently $\mathbb{R}^N \simeq \mathfrak{k} \times \mathfrak{g}$ and the short exact sequence

$$0 \to \mathfrak{k} \xrightarrow{B} \mathbb{R}^N \xrightarrow{A} \mathfrak{g} \to 0$$

splits. Let

$$\pi_K : \mathbb{R}^N \to \mathfrak{k} \quad \text{and} \quad \iota_{\mathfrak{g}} : \mathfrak{g} \to \mathbb{R}^N$$

denote the maps defined by the splitting. The moment map $\phi_K : (\mathbb{C}^\times)^N \to \mathfrak{k}^*$ for the action of K on $((\mathbb{C}^\times)^N, \omega)$ is the composition

$$\phi_K = B^* \circ \phi = B^* \circ \mathcal{L}_f \circ \pi_N.$$

Let

$$\Delta = (B^*)^{-1}(\nu) \cap \phi((\mathbb{C}^\times)^N) = (B^*)^{-1}(\nu) \cap \mathcal{L}_f(\mathbb{R}^N).$$

Then Δ is the intersection of an affine hyperplane with a convex set, hence is contractible.

Since the action of K on $\phi_K^{-1}(\nu)$ is free, $K^{\mathbb{C}} \cdot \phi_K^{-1}(\nu)$ is an open subset of $(\mathbb{C}^\times)^N$ and $K^{\mathbb{C}}$ acts freely on it. Moreover, for each $x \in \phi_K^{-1}(\nu)$ the orbit $K^{\mathbb{C}} \cdot x$ intersects the level set $\phi_K^{-1}(\nu)$ transversely and

$$K^{\mathbb{C}} \cdot x \cap \phi_K^{-1}(\nu) = K \cdot x$$

(see [Guillemin and Sternberg 1982, pp. 526–527]). It follows that the restriction

$$\pi_K|_{\mathcal{L}_f^{-1}(\Delta)} : \mathcal{L}_f^{-1}(\Delta) \to \mathfrak{k}$$

is 1-1 and a local diffeomorphism. Hence

$$U = \pi_K(\mathcal{L}_f^{-1}(\Delta))$$

is a contractible open set.

On the other hand, the restriction $\omega|_{\phi_K^{-1}(\nu)}$ descends to a Kähler form ω_ν on the symplectic quotient

$$(\mathbb{C}^\times)^N)//_\nu K := \phi_K^{-1}(\nu)/K.$$

Moreover, since ω is $\mathbb{T}^N$ invariant, ω_ν is G-invariant. Note that

$$(\mathbb{C}^\times)^N)//_\nu K \simeq U \times \sqrt{-1}G \subset G^{\mathbb{C}}.$$

By Lemma 3.1 there exists $f_\nu \in C^\infty(U)$ such that

$$\omega_\nu = \sqrt{-1}\partial\bar{\partial} f_\nu.$$

The potential f_ν defines a moment map

$$\phi_G : U \times \sqrt{-1}G \to \mathfrak{g}^*$$

with

$$\phi_G = \mathcal{L}_{f_\nu} \circ \pi_G,$$

where $\pi_G : U \times \sqrt{-1}G \to U$ is the projection. Moreover, by adjusting f_ν [Burns and Guillemin 2004] we may arrange for the diagram

$$\begin{tikzcd} \phi_K^{-1}(\nu) \arrow[r, "\phi"] \arrow[d, "/K"'] & \Delta \subset (\mathbb{R}^N)^* \\ U \times \sqrt{-1}G \arrow[r, "\phi_G"'] & \mathfrak{g}^* \arrow[u, "\iota_\lambda"'] \end{tikzcd} \tag{3-2}$$

to commute. That is, the moment map ϕ_G is defined up to a constant and the potential f_ν is defined up to a pluriharmonic G-invariant function. By adding an appropriate pluriharmonic function to f_ν we can change ϕ_G by any constant we want. Since $\phi = \mathcal{L}_f \circ \pi_N$ and since $\phi_G = \mathcal{L}_{f_\nu} \circ \pi_G$, it follows from (3-2) that the diagram

$$\begin{tikzcd} \mathcal{L}_f^{-1}(\Delta) \arrow[r, "\mathcal{L}_f"] \arrow[d, "\pi_K"'] & \Delta \subset (\mathbb{R}^N)^* \\ U \arrow[r, "\mathcal{L}_{f_\nu}"'] & \mathfrak{g}^* \arrow[u, "\iota_\lambda"'] \end{tikzcd}$$

commutes as well. Since $(\mathcal{L}_{f_\nu})^{-1} = \mathcal{L}_{f_\nu^*}$, where f_ν^* is the Legendre–Fenchel dual of f_ν,

$$\mathcal{L}_{f_\nu^*} = \pi_K \circ (\mathcal{L}_f)^{-1} \circ \iota_\lambda = \pi_K \circ (\mathcal{L}_{f^*}) \circ \iota_\lambda.$$

By Lemma 3.2,

$$\mathcal{L}_{f_\nu^*} = \mathcal{L}_{f^*} \circ \iota_\lambda.$$

Therefore, up to a constant, $f_\nu^* = f^* \circ \iota_\lambda$. □

4. From potentials to dual potentials and back again

We start by making two observations. Let V be a real finite dimensional vector space, V^* its dual, $\mathcal{O} \subset V$ an open set, $\varphi \in C^\infty(\mathcal{O})$ a strictly convex function, $\mathcal{L}_\varphi : \mathcal{O} \to V^*$ the Legendre transform (which we assume to be invertible), $\mathcal{O}^* = \mathcal{L}_\varphi(\mathcal{O})$ and $\varphi^* \in C^\infty(\mathcal{O}^*)$ the Fenchel dual of φ.

Lemma 4.1. *Under these assumptions, $\varphi = (\mathcal{L}_\varphi)^* h$, where $h : \mathcal{O}^* \to \mathbb{R}$ is given by*

$$h(\eta) = \langle \eta, (d\varphi^*)_\eta \rangle - \varphi^*(\eta)$$

where we think of $(d\varphi^*)_\eta \in T_\eta^* \mathcal{O}^*$ as an element of $(V^*)^* = V$.

Proof. By the definition of the Fenchel dual, $\varphi(s)+\varphi^*(\eta)=\langle\eta, s\rangle$ for $\eta=\mathcal{L}_\varphi(s)$. Hence

$$\varphi(s)=\langle\eta, s\rangle-\varphi^*(\eta)=\langle\eta, (\mathcal{L}_\varphi)^{-1}(\eta)\rangle-\varphi^*(\eta)=\langle\eta, \mathcal{L}_{\varphi^*}(\eta)\rangle-\varphi^*(\eta)$$

and the result follows since $\mathcal{L}_{\varphi^*}(\eta)=(d\varphi^*)_\eta$. □

Lemma 4.2. *We keep the above notation. Suppose additionally that the dual potential φ^* has the special form*

$$\varphi^*(\eta)=\sum_{i=1}^{N} f_i(u_i(\eta)-\lambda_i),$$

where $u_1,\dots,u_N$ are vectors in V (thought of as linear functionals $u_i : V^*\to\mathbb{R}$), $\lambda_i\in\mathbb{R}$ are constants and f_i's are functions of one variable. Then

$$h(\eta)=\sum_{i=1}^{N}\bigl(f_i'(u_i(\eta)-\lambda_i)\,u_i(\eta)-f_i(u_i(\eta)-\lambda_i)\bigr). \tag{4-1}$$

Proof. Observe that

$$d(f_i\circ(u_i-\lambda_i))_\eta=f_i'(u_i(\eta)-\lambda_i)\,d(u_i-\lambda_i)_\eta=f_i'(u_i(\eta)-\lambda_i)\,u_i$$

since u_i is linear. Hence

$$\langle\eta, (d\varphi^*)_\eta\rangle=\bigl\langle\eta, \sum f_i'(u_i(\eta)-\lambda_i)u_i\bigr\rangle=\sum f_i'(u_i(\eta)-\lambda_i)\,u_i(\eta)$$

and (4-1) follows from Lemma 4.1. □

Example 4.3. We use the lemma above to argue that for the standard action of $\mathbb{T}^N$ on $(\mathbb{C}^N, \sqrt{-1}\partial\bar\partial\|z\|^2)$, the dual potential φ^* is given by

$$\varphi^*=\sum_{i=1}^{N} e_i\log e_i,$$

where $\{e_1,\dots,e_N\}$ is the standard basis of $\mathbb{R}^N=Lie(\mathbb{T}^N)$.

Indeed, the homogeneous moment map $\Phi:\mathbb{C}^N\to(\mathbb{R}^N)^*$ for the standard action of $\mathbb{T}^N$ is given by

$$\Phi(z)=\sum|z_j|^2 e_j{}^*,$$

where $\{e_j{}^*\}$ is the basis dual to $\{e_j\}$. Hence

$$\|z\|^2=\Phi^*\bigl(\sum e_j\bigr).$$

On the other hand, if $\varphi^*=\sum e_j\log e_j$, then

$$\varphi^*=\sum f\circ e_j$$

where $f(x) = x \log x$. Since $f'(x) = \log x + 1$, (4-1) becomes

$$h = \sum (\log e_j + 1) e_j - \sum e_j \log e_j = \sum e_j.$$

Therefore, $\varphi^* = \sum e_j \log e_j$ is, indeed, the dual potential.

We are now in position to prove (1-1).

Theorem 4.4. *Let G be a torus, $P \subset \mathfrak{g}^*$ the polyhedral set defined by* (2-1), *$M_P = \mathbb{C}^N /\!/_\nu K$ the Kähler G-space with moment map $\phi_P : M_P \to \mathfrak{g}^*$ constructed in Lemma 2.1 (1). Then the Kähler form ω_P on $\mathring{M}_P := \phi_P^{-1}(\mathring{P})$ is given by*

$$\omega_P = \sqrt{-1}\, \partial \bar{\partial}\, \phi_P^* \left(\sum_{j=1}^{N} \lambda_j \log(u_j - \lambda_j) + u_j \right),$$

Proof. By Lemma 2.1, $\mathring{M}_P = (\mathbb{C}^\times)^N /\!/_\nu K$ where $K \subset \mathbb{T}^N$ is a closed subgroup. By Lemma 3.3 the dual potential φ_P^* on $\mathring{P}$ is given by $\varphi_P^* = \varphi^* \circ \iota_\lambda$, where φ^* is the potential on the open orthant in $(\mathbb{R}^N)^*$ dual to the flat metric potential $\varphi(z) = \|z\|^2$ on $(\mathbb{C}^\times)^N$. By Example 4.3 $\varphi^* = \sum e_j \log e_j$. Since $\iota_\lambda^* e_j = u_j - \lambda_j$,

$$\varphi_P^* = \sum (u_j - \lambda_j) \log(u_j - \lambda_j).$$

By Lemmas 4.1 and 4.2, the potential φ_P is given by

$$\varphi_P = \phi_P^* h$$

where

$$h = \sum (\log(u_j - \lambda_j) + 1) u_j - \sum (u_j - \lambda_j) \log(u_j - \lambda_j)$$

(see (4-1)). Therefore

$$\varphi_P = \phi_P^* \left(\sum_{i=1}^{N} (\lambda_j \log(u_j - \lambda_j) + u_j) \right). \qquad \square$$

5. Kähler potentials on the preimages of faces

Once again let $P \subset \mathfrak{g}^*$ be a polyhedral set given by (2-1). Recall that in Section 2 we canonically associated to this set a Kähler quotient M_P of $\mathbb{C}^N$ which carries an effective holomorphic and Hamiltonian action of the torus G with a moment map $\phi_P : M_P \to \mathfrak{g}^*$. Let $F \subset P$ be a face. Its interior $\mathring{F}$ is given by

$$\mathring{F} = \bigcap_{j \notin I} \{\eta \in \mathfrak{g}^* \mid \langle \eta, u_j \rangle - \lambda_j > 0\} \cap \bigcap_{j \in I} \{\eta \in \mathfrak{g}^* \mid \langle \eta, u_j \rangle - \lambda_j = 0\}$$

for some nonempty subset I of $\{1, \dots, N\}$. We have seen in the proof of Lemma 2.1 that the preimage

$$M_{\mathring{F}} := \phi_P^{-1}(\mathring{F})$$

is the Kähler quotient of $\mathring{V}_I$ by a compact abelian group K_I. Therefore there is a potential $\varphi_F^* \in C^\infty(\mathring{F})$ dual to the Kähler potential φ_F on $M_{\mathring{F}}$. The goal of this section is to compute the dual potential φ_F^* "explicitly." Lemmas 4.1 and 4.2 will then give us an analogue of (1-1) for the Kähler metric on $M_{\mathring{F}}$.

The Kähler potential φ_I on $\mathring{V}_I$ for the flat metric induced from $\mathbb{C}^N$ is given by

$$\varphi_I(z) = \sum_{j \notin I} |z_j|^2.$$

The restriction of the moment map $\phi : \mathbb{C}^N \to (\mathbb{R}^N)^*$ to $\mathring{V}_I$ is a moment map for the action of the torus

$$H_I := \mathbb{T}^N / \mathbb{T}^N_I.$$

Note that

$$\phi(\mathring{V}_I) = \left\{ \sum_{i \notin I} a_i {e_i}^* \;\middle|\; a_i > 0 \right\}.$$

This set is an open subset in

$$\operatorname{span}_{i \notin I}\{{e_j}^*\} \simeq \mathfrak{h}_I^*.$$

From now on we identify $\mathfrak{h}_I^*$ with $\operatorname{span}_{i \notin I}\{{e_j}^*\}$. The dual potential $\varphi_I^* \in C^\infty(\phi(\mathring{V}_I))$ is easily seen to be

$$\varphi_I^* = \sum_{j \notin I} e_j \log e_j.$$

The manifold $M_{\mathring{F}}$ is a Hamiltonian G space, but the group G doesn't act effectively. So we cannot yet apply Lemma 3.3 as we would like. Let G_I denote the quotient of G that does act effectively on $M_{\mathring{F}}$. It is isomorphic to the quotient H_I/K_I. The dual of its Lie algebra $\mathfrak{g}_I^*$ is naturally embedded in $\mathfrak{g}^*$:

$$\mathfrak{g}_I^* = \{\eta \in \mathfrak{g}^* \mid \langle \eta, u_i \rangle = 0 \text{ for all } i \in I\}.$$

Note also that the affine span affspan $\mathring{F}$ of $\mathring{F} \subset \mathfrak{g}^*$ is the translation of $\mathfrak{g}_I^*$ by an element $\eta_0 \in \mathring{F}$, as it should be. Let $\gamma_I : \mathfrak{g}_I^* \to \operatorname{affspan} \mathring{F} \subset \mathfrak{g}^*$ denote the affine embedding. Then there exists an affine embedding $\iota_I : \mathfrak{g}_I^* \hookrightarrow \mathfrak{h}_I^*$ so that the diagram

$$\begin{array}{ccc}
\mathfrak{h}_I^* & \longrightarrow & (\mathbb{R}^N)^* \\
{\scriptstyle \iota_I}\big\uparrow & & \big\uparrow{\scriptstyle \iota_\lambda} \\
\mathfrak{g}_I^* & \xrightarrow[\gamma_I]{} & \mathfrak{g}^*
\end{array}$$

commutes. Here the top arrow identifies $\mathfrak{h}_I^*$ with $\operatorname{span}_{i \notin I}\{{e_i}^*\}$. Since γ_I is an embedding, we may think of φ_F^* as living on $\mathring{F} \subset \gamma_I(\mathfrak{g}_I^*)$. Therefore, by Lemma 3.3,

$$\varphi_F^* = (\varphi_I^* \circ \iota_\lambda)|_{\mathring{F}}\,. \tag{5-1}$$

Let

$$v_j = u_j|_{\mathring{F}} \,.$$

These functions are affine, but not necessarily linear. Then

$$(e_j \circ \iota_\lambda)|_{\mathring{F}} = (u_j - \lambda_j)|_{\mathring{F}} = v_j - \lambda_j.$$

Therefore

$$\varphi_F^* = (\varphi_I^* \circ \iota_\lambda)|_{\mathring{F}} = \sum_{j \notin I} (v_j - \lambda_j) \log(v_j - \lambda_j).$$

To get a nicer formula for the potential on $M_{\mathring{F}}$ we now make a simplifying assumption, namely, that $0 \in \mathring{F}$. Then $v_j = u_j|_{\mathfrak{g}_I^*}$ and, in particular, it is *linear* for all j. Hence Lemmas 4.1 and 4.2 apply, and we obtain:

Theorem 5.1. *Under the simplifying assumption above, the Kähler form ω_F on $M_{\mathring{F}}$ is given by*

$$\omega_F = \sqrt{-1}\, \partial\bar\partial\, (\phi_P|_{M_{\mathring{F}}})^* \left(\sum_{j \notin I} \lambda_j \log(v_j - \lambda_j) + v_j \right).$$

Alternatively we may take the isomorphism $\gamma_I : \mathfrak{g}_I^* \to \operatorname{affspan} \mathring{F}$ explicitly into account and think of φ_F^* as living on an open subset of $\mathfrak{g}_I^*$. Then, by Lemma 3.3,

$$\varphi_F^* = \varphi_I^* \circ \iota_\lambda \circ \gamma_I.$$

Since

$$e_i \circ \iota_\lambda \circ \gamma_I = u_i|_{\mathfrak{g}_I^*} + u_i(\eta_0) - \lambda_i,$$

we get

$$\varphi_F^* = \sum_{i \notin I} (u_j|_{\mathfrak{g}_I^*} + u_j(\eta_0) - \lambda_j) \log(u_j|_{\mathfrak{g}_I^*} + u_j(\eta_0) - \lambda_j).$$

We conclude:

Theorem 5.2. *The Kähler form ω_F on $M_{\mathring{F}}$ is given by*

$$\omega_F = \sqrt{-1}\, \partial\bar\partial\, (\phi_P|_{M_{\mathring{F}}})^* \left(\sum_{j \notin I} \big((\lambda_j - u_j(\eta_0)) \log(u_j|_{\mathfrak{g}_I^*} + u_j(\eta_0) - \lambda_j) + u_j|_{\mathfrak{g}_I^*} \big) \right).$$

Variations on the theme. The same technique allows us to prove a variant of (1-1). We keep the notation above. Suppose that the polyhedral set P is compact. That is, suppose that P is actually a polytope. Then

$$\iota_\lambda(P) \subset \{\ell \in (\mathbb{R}^N)^* \mid \langle \ell, e_j \rangle \geq 0 \text{ for all } j\}$$

is bounded. Hence there is $R > 0$ such that $\iota_\lambda(P)$ is contained in a scaled copy Δ_R of the standard simplex

$$\Delta_R = \left\{ \ell \in (\mathbb{R}^N)^* \;\middle|\; \langle \ell, e_j \rangle \geq 0 \text{ for all } j \text{ and } \sum \langle \ell, e_j \rangle \leq R \right\}.$$

Since Δ_1 is the moment map image of $\mathbb{C}\mathrm{P}^N$ under the standard action of $\mathbb{T}^N$, it follows that M_P is also a symplectic quotient of $(\mathbb{C}\mathrm{P}^N, R\omega_{FS})$ by the action of the compact abelian Lie group K defined earlier (ω_{FS} denotes the Fubini–Study form; see Lemma 2.1 (3)). Since

$$\Delta_R = \left\{\ell \in (\mathbb{R}^N)^* \;\middle|\; \langle \ell, e_j\rangle \geq 0,\ 1 \leq j \leq N,\ \left\langle \ell, -\sum e_j\right\rangle + R \geq 0\right\},$$

it follows from (3-1) that the potential f^* dual to the potential for $R\omega_{FS}$ on Δ_R is given by

$$f^* = \sum e_j \log e_j + \bigl(R - \sum e_j\bigr)\log\bigl(R - \sum e_j\bigr).$$

Consequently the potential f^*_ν dual to the potential on the quotient $(\mathbb{C}\mathrm{P}^N/\!/_\nu K, \omega_P)$ is

$$f^*_\nu = \sum (u_j - \lambda_j)\log(u_j - \lambda_j) + \bigl(R - \sum (u_j - \lambda_j)\bigr)\log\bigl(R - \sum (u_j - \lambda_j)\bigr).$$

By Lemma 4.1 the reduced Kähler form ω_P is

$$\omega_P = \sqrt{-1}\partial\bar{\partial}\phi^* h$$

where

$$h(\eta) = \langle \eta, (df_\nu)_\eta\rangle - f_\nu(\eta).$$

A computation similar to the ones in the previous sections gives

$$h = \sum \lambda_j \log(u_j - \lambda_j) - \bigl(R + \sum \lambda_j\bigr)\log\bigl(R - \sum (u_j - \lambda_j)\bigr). \tag{5-2}$$

We have proved the following theorem.

Theorem 5.3. *Let G be a torus, $P \subset \mathfrak{g}^*$ the polyhedral set defined by* (2-1) *which happens to be compact, $M_P = (\mathbb{C}\mathrm{P}^N, R\omega_{FS})/\!/_\nu K$ the Kähler G-space with moment map $\phi_P : M_P \to \mathfrak{g}^*$ constructed in Lemma 2.1 (3). Then the Kähler form ω_P on $\mathring{M}_P := \phi_P^{-1}(\mathring{P})$ is given by*

$$\omega_P = \sqrt{-1}\,\partial\bar{\partial}\,\phi_P^*\Bigl(\sum \lambda_j \log(u_j - \lambda_j) - \bigl(R + \sum \lambda_j\bigr)\log\bigl(R - \sum (u_j - \lambda_j)\bigr)\Bigr).$$

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