

*Pacific
Journal of
Mathematics*

KÄHLER METRICS ON SINGULAR TORIC VARIETIES

DAN BURNS, VICTOR GUILLEMIN AND EUGENE LERMAN

KÄHLER METRICS ON SINGULAR TORIC VARIETIES

DAN BURNS, VICTOR GUILLEMIN AND EUGENE LERMAN

We extend Guillemin's formula for Kähler potentials on toric manifolds to singular quotients of \mathbb{C}^N and $\mathbb{C}P^N$.

1. Introduction

Let G be a torus with Lie algebra \mathfrak{g} and integral lattice $\mathbb{Z}_G \subset \mathfrak{g}$. Let $u_1, \dots, u_N \in \mathbb{Z}_G$ be a set of primitive vectors which span \mathfrak{g} over \mathbb{R} . Let $\lambda_1, \dots, \lambda_N \in \mathbb{R}$ and let

$$P = P_{u,\lambda} := \{\eta \in \mathfrak{g}^* \mid \langle \eta, u_j \rangle - \lambda_j \geq 0, 1 \leq j \leq N\}$$

be the corresponding polyhedral set. We assume that P has a nonempty interior and that the collection of inequalities defining P is minimal: if we drop the condition that $\langle \eta, u_j \rangle - \lambda_j \geq 0$ for some index j then the resulting set is strictly bigger than P .

A well-known construction of Delzant, suitably tweaked, produces a symplectic stratified space M_P with an effective Hamiltonian action of the torus G and associated moment map $\phi = \phi_P : M_P \rightarrow \mathfrak{g}^*$ such that $\phi(M_P) = P$. We will review the construction below. The space M_P is a symplectic quotient of \mathbb{C}^N by a compact abelian subgroup K of the standard torus \mathbb{T}^N . Therefore, by a theorem of Heinzner and Loose [1994] M_P is a complex analytic space. Moreover M_P is a Kähler space; see [Heinzner and Loose 1994, (3.5)] and [Heinzner et al. 1994]. Even though in general the space M_P is singular, the preimages of open faces of P under the moment map ϕ_P are smooth Kähler manifolds. The main results of the paper are formulas for the Kähler forms on these manifolds. In particular we will show that the Kähler form ω on the preimage $\phi_P^{-1}(\overset{\circ}{P})$ of the interior $\overset{\circ}{P}$ of the polyhedral set P is given by

$$(1-1) \quad \omega = \sqrt{-1} \partial \bar{\partial} \phi_P^* \left(\sum_{j=1}^N \lambda_j \log(u_j - \lambda_j) + u_j \right),$$

MSC2000: primary 32J27; secondary 14M25, 53D20.

Keywords: Kähler metric, toric, Kähler potential, Kähler reduction, stratified space.

Supported in part by NSF grants DMS-0104047 and DMS-0514070 (DB), DMS-0104116 and DMS-0408993 (VG), and DMS-0204448(EL).

where we think of $u_j \in \mathbb{Z}_G$ as a function on \mathfrak{g}^* .

Formula (1-1) was originally proved by Guillemin in the case where M_P is a compact manifold (and thus P is a simple unimodular polytope, also known as a *Delzant polytope*). It was extended to the case of compact orbifolds by [Abreu \[2001\]](#). Calderbank, David and Gauduchon gave two new proofs of Guillemin's formula (for orbifolds) in [[Calderbank et al. 2003](#)]. One of their proofs was simplified further in [[Burns and Guillemin 2004](#)].

As we just mentioned, for generic values of λ the polyhedral set P is simple and consequently M_P is at worst an orbifold. But for arbitrary values of λ it may have more serious singularities. Of particular interest is the singular case where P is a cone on a simple polytope. Then there is only one singular point, and the link of the singularity is a Sasakian orbifold. Such orbifolds, especially the ones with Sasaki–Einstein metrics, have attracted some attention in string theory. They play a role in the AdS/CFT correspondence [[Martelli and Sparks 2004](#)].

If the polyhedral set P is a polytope, that is, if P is compact, then as a symplectic space M_P may also be obtained as a symplectic quotient of $\mathbb{C}P^N$. In this case the Fubini–Study form on $\mathbb{C}P^N$ will induce a Kähler structure on M_P , which is *different* from the one induced by the flat metric on \mathbb{C}^N even in the case where M_P is smooth. We will give a formula for this Kähler structure as well.

The methods of this paper are quite close to that of [[Calderbank et al. 2003](#)]. In particular the key [Lemma 3.3](#) is a direct corollary of Proposition 2 in that reference.

2. The “Delzant” construction: toric varieties as Kähler quotients

It will be convenient for us to fix the following notation. As in the introduction, let G be a torus with Lie algebra \mathfrak{g} and integral lattice $\mathbb{Z}_G \subset \mathfrak{g}$. Let $u_1, \dots, u_N \in \mathbb{Z}_G$ be a set of primitive vectors which span \mathfrak{g} over \mathbb{R} . Let $\lambda_1, \dots, \lambda_N \in \mathbb{R}$ and let

$$(2-1) \quad P = P_{u,\lambda} := \{\eta \in \mathfrak{g}^* \mid \langle \eta, u_j \rangle - \lambda_j \geq 0, \quad 1 \leq j \leq N\}$$

be the corresponding polyhedral set. As above we assume that P has the nonempty interior and that the collection of inequalities defining P is minimal. Let $A : \mathbb{Z}^N \rightarrow \mathbb{Z}_G$ be the \mathbb{Z} -linear map given by

$$A(x_1, \dots, x_N) = \sum x_i u_i.$$

That is, A is defined by sending the standard basis vector e_i of \mathbb{Z}^N to u_i . Let A also denote the \mathbb{R} -linear extension $\mathbb{R}^N \rightarrow \mathfrak{g}$. Let $\mathfrak{k} = \ker A$ and let $B : \mathfrak{k} \rightarrow \mathbb{R}^N$ denote the inclusion. The map A induces a surjective map of Lie groups

$$\bar{A} : \mathbb{T}^N = \mathbb{R}^N / \mathbb{Z}^N \rightarrow \mathfrak{g} / \mathbb{Z}_G = G.$$

Let $K = \ker \bar{A}$ and let $\bar{B} : K \rightarrow \mathbb{T}^N$ denote the corresponding inclusion. The group K is a compact abelian group which need not be connected. It's easy to see that the Lie algebra of K is \mathfrak{k} .

We have a short exact sequence of abelian Lie algebras:

$$0 \rightarrow \mathfrak{k} \xrightarrow{B} \mathbb{R}^N \xrightarrow{A} \mathfrak{g} \rightarrow 0.$$

Let

$$0 \rightarrow \mathfrak{g}^* \xrightarrow{A^*} (\mathbb{R}^N)^* \xrightarrow{B^*} \mathfrak{k}^* \rightarrow 0$$

be the dual sequence. Note that $\ker B^* = A^*(\mathfrak{g}^*) = \mathfrak{k}^\circ$ where \mathfrak{k}° denotes the annihilator of \mathfrak{k} in $(\mathbb{R}^N)^*$. Let $\{e_i^*\}$ denote the dual basis of $(\mathbb{R}^N)^*$ and let $\lambda = \sum \lambda_i e_i^*$.

We note that

$$(B^*)^{-1}(B^*(-\lambda)) = -\lambda + \mathfrak{k}^\circ = -\lambda + A^*(\mathfrak{g}^*).$$

In particular $(B^*)^{-1}(B^*(-\lambda))$ is the image of the affine embedding

$$(2-2) \quad \iota_\lambda : \mathfrak{g}^* \hookrightarrow (\mathbb{R}^N)^*, \quad \iota_\lambda(\ell) = -\lambda + A^*(\ell).$$

Lemma 2.1. *Let P be the polyhedral set defined by (2-1) above.*

- (1) *There exists a Kähler space M_P with an effective holomorphic Hamiltonian action of the torus G so that the image of the associated moment map $\phi_P : M_P \rightarrow \mathfrak{g}^*$ is P .*
- (2) *For every open face \mathring{F} , the preimage $\phi_P^{-1}(\mathring{F})$ is the Kähler quotient of a complex torus $(\mathbb{C}^\times)^{N_F}$ by a compact subgroup K_F of the compact torus $\mathbb{T}^{N_F} \subset (\mathbb{C}^\times)^{N_F}$. Here the number N_F and the group K_F depend on the face F .*
- (3) *If the set P is bounded, then M_P can also be constructed as a Kähler quotient of $\mathbb{C}\mathbb{P}^N$.*

Proof. For every index i and any $\eta \in \mathfrak{g}^*$

$$\langle \eta, Ae_i \rangle - \lambda_i = \langle A^*\eta, e_i \rangle - \langle \sum \lambda_j e_j^*, e_i \rangle = \langle A^*\eta - \lambda, e_i \rangle = \langle \iota_\lambda(\eta), e_i \rangle.$$

Therefore

$$\iota_\lambda(P) = \{ \ell \in (\mathbb{R}^N)^* \mid \langle \ell, e_i \rangle \geq 0, 1 \leq i \leq N \} \cap \iota_\lambda(\mathfrak{g}^*).$$

More generally, if $\mathring{F} \subset P$ is an open face, there is a unique subset $I_F = I \subset \{1, \dots, N\}$ so that

$$\mathring{F} = \bigcap_{j \notin I} \{ \eta \in \mathfrak{g}^* \mid \langle \eta, u_j \rangle - \lambda_j > 0 \} \cap \bigcap_{j \in I} \{ \eta \in \mathfrak{g}^* \mid \langle \eta, u_j \rangle - \lambda_j = 0 \}.$$

Therefore

$$(2-3) \quad \iota_\lambda(\mathring{F}) = \iota_\lambda(\mathfrak{g}^*) \cap \bigcap_{j \notin I} \{ \ell \in (\mathbb{R}^N)^* \mid \langle \ell, e_j \rangle > 0 \} \cap \bigcap_{j \in I} \{ \ell \in (\mathbb{R}^N)^* \mid \langle \ell, e_j \rangle = 0 \}.$$

The moment map ϕ for the action of \mathbb{T}^N on $(\mathbb{C}^N, \sqrt{-1} \sum dz_j \wedge d\bar{z}_j)$ is given by

$$\phi(z) = \sum |z_j|^2 e_j^*.$$

Hence

$$\phi(\mathbb{C}^N) = \{\ell \in (\mathbb{R}^N)^* \mid \langle \ell, e_i \rangle \geq 0, \quad 1 \leq i \leq N\}.$$

The moment map ϕ_K for the action of K on \mathbb{C}^N is the composition

$$\phi_K = B^* \circ \phi.$$

Let $\nu = B^*(-\lambda)$. We argue that

$$\phi(\phi_K^{-1}(\nu)) = \iota_\lambda(P).$$

Indeed,

$$\begin{aligned} \phi_K^{-1}(\nu) &= \phi^{-1}((B^*)^{-1}(\nu)) = \phi^{-1}((B^*)^{-1}(B^*(-\lambda))) \\ &= \phi^{-1}(\iota_\lambda(\mathfrak{g}^*)) = \phi^{-1}(\phi(\mathbb{C}^N) \cap \iota_\lambda(\mathfrak{g}^*)) = \phi^{-1}(\iota_\lambda(P)). \end{aligned}$$

Therefore

$$\phi(\phi_K^{-1}(\nu)) = \iota_\lambda(P).$$

The restriction

$$\phi|_{\phi_K^{-1}(\nu)}$$

descends to a map

$$\bar{\phi} : M_P \equiv \phi_K^{-1}(\nu)/K \rightarrow \iota_\lambda(\mathfrak{g}^*).$$

It is not hard to see that the composition ϕ_P of $\bar{\phi}$ with the isomorphism

$$\iota_\lambda(\mathfrak{g}^*) \xrightarrow{\cong} \mathfrak{g}^*$$

is a moment map for the action of G on the symplectic quotient (symplectic stratified space) M_P . Since the isomorphism $\iota_\lambda(\mathfrak{g}^*) \rightarrow \mathfrak{g}^*$ obviously maps $\iota_\lambda(P)$ to P , we conclude that the image of $\phi_P : M_P \rightarrow \mathfrak{g}^*$ is exactly P . This proves (1).

To prove (2) we define a bit more notation. For a subset $I \subset \{1, \dots, N\}$ we define the corresponding coordinate subspace

$$V_I := \{z \in \mathbb{C}^N \mid j \in I \Rightarrow z_j = 0\}.$$

Its ‘‘interior’’ \mathring{V}_I is defined by

$$\mathring{V}_I := \{z \in \mathbb{C}^N \mid j \in I \Leftrightarrow z_j = 0\}.$$

Also, let

$$\mathbb{T}_I^N := \{a \in \mathbb{T}^N \mid j \notin I \Rightarrow a_j = 1\}.$$

The sets V_I, \mathring{V}_I are Kähler submanifolds of \mathbb{C}^N preserved by the action of \mathbb{T}^N . Both are fixed by \mathbb{T}_I^N , with \mathring{V}_I being precisely the set of points of orbit type \mathbb{T}_I^N .

The restriction $\phi_K|_{\mathring{V}_I}$ is a moment map for the action of K on \mathring{V}_I . Moreover, for any $\nu \in \mathfrak{k}^*$

$$\phi_K^{-1}(\nu) \cap \mathring{V}_I = (\phi_K|_{\mathring{V}_I})^{-1}(\nu).$$

Hence

$$(\phi_K^{-1}(\nu) \cap \mathring{V}_I)/K = (\phi_K|_{\mathring{V}_I})^{-1}(\nu)/K.$$

While the action of K on \mathring{V}_I need not be free, the action of

$$K_I := K/(K \cap \mathbb{T}_I^N)$$

on \mathring{V}_I is free. Therefore, the quotient $(\phi_K^{-1}(\nu) \cap \mathring{V}_I)/K$ may be interpreted as a *regular* Kähler quotient of \mathring{V}_I by the Hamiltonian action of K_I :

$$(2-4) \quad (\phi_K^{-1}(\nu) \cap \mathring{V}_I)/K = \mathring{V}_I //_{\nu_I} K_I$$

for an appropriate value $\nu_I \in \mathfrak{k}_I^*$ of the K_I moment map.

Given a face F , let $I = I_F$ be the corresponding subset of $\{1, \dots, N\}$. By (2-3),

$$\begin{aligned} \{z \in \mathbb{C}^N \mid \phi(z) \in \iota_\lambda(\mathring{F})\} &= \left\{ z \in \mathbb{C}^N \mid \phi(z) \in \iota_\lambda(\mathfrak{g}^*), \langle \phi(z), e_j \rangle \begin{cases} > 0 & \text{for } j \notin I, \\ = 0 & \text{for } j \in I \end{cases} \right\} \\ &= \phi_K^{-1}(\nu) \cap \mathring{V}_I. \end{aligned}$$

Therefore,

$$\phi_K^{-1}(\nu) \cap \mathring{V}_I = \phi^{-1}(\iota_\lambda(\mathring{F})).$$

It follows from the definition of ϕ_P that

$$(\phi_K^{-1}(\nu) \cap \mathring{V}_I)/K = \phi_P^{-1}(\mathring{F}).$$

By (2-4) we conclude that

$$\phi_P^{-1}(\mathring{F}) = \mathring{V}_I //_{\nu_I} K_I.$$

This proves (2).

If P is compact, then $\iota_\lambda(P) \subset (\mathbb{R}^N)^*$ is bounded. Hence $\iota_\lambda(P)$ is contained in a sufficiently large multiple of the standard simplex. Any such simplex is the image of $\mathbb{C}\mathbb{P}^N$ under the moment map for the standard action of \mathbb{T}^N with the Kähler form on $\mathbb{C}\mathbb{P}^N$ being the appropriate multiple of the standard Fubini–Study form. This proves (3). \square

Remark 2.2. It follows from the results of Heinzner and his collaborators (email communication), in particular of [Heinzner and Huckleberry \[1996\]](#), that the action of G on M_P extends to an action the complexified group $G^{\mathbb{C}}$. This action of $G^{\mathbb{C}}$ has a dense open orbit. In other words, M_P is a *toric* Kähler space.

3. Kähler potentials, Legendre transforms and symplectic quotients

As we mentioned in the introduction, the line of argument of this section is quite close to the approach in [Calderbank et al. 2003], and Lemma 3.3 can be easily deduced from Proposition 2 of that reference. We keep our exposition self-contained.

We start by recalling a result of Guillemin [1994, Theorems 4.2, 4.3]:

Lemma 3.1. *Suppose the action of \mathbb{T}^N on $(\mathbb{C}^\times)^N = \mathbb{R}^N \times \sqrt{-1}\mathbb{T}^N$ preserves a Kähler form ω and is Hamiltonian. Then there exists a \mathbb{T}^N -invariant function f on $(\mathbb{C}^\times)^N$ such that $\omega = i\partial\bar{\partial}f$. Additionally*

$$\mathcal{L}_f \circ \pi : (\mathbb{C}^\times)^N \rightarrow (\mathbb{R}^N)^*$$

is a moment map for the action of \mathbb{T}^N on $((\mathbb{C}^\times)^N, \omega)$. Here $\pi : \mathbb{R}^N \times \sqrt{-1}\mathbb{T}^N \rightarrow \mathbb{R}^N$ is the projection and $\mathcal{L}_f : \mathbb{R}^N \rightarrow (\mathbb{R}^N)^*$ is the Legendre transform of f , where we have identified $f \in C^\infty((\mathbb{C}^\times)^N)^{\mathbb{T}^N}$ with a function on \mathbb{R}^N .

The same result holds with $(\mathbb{C}^\times)^N$ replaced by $U \times \sqrt{-1}\mathbb{T}^N$ for any contractible open set $U \subset \mathbb{R}^N$.

Lemma 3.2. *Let $f : V \rightarrow \mathbb{R}$ be a (strictly) convex function on a finite dimensional vector space V , let $A : W \rightarrow V$ be an injective linear map, $x \in V$ be a point and*

$$j : W \rightarrow V, \quad j(w) = Aw + x$$

an affine map. Then $f \circ j : W \rightarrow \mathbb{R}$ is (strictly) convex and the associated Legendre transform $\mathcal{L}_{f \circ j} : W \rightarrow W^*$ is given by

$$\mathcal{L}_{f \circ j} = A^* \circ \mathcal{L}_f \circ j,$$

where $A^* : V^* \rightarrow W^*$ is the dual map.

Proof. By the chain rule and the definition of the Legendre transform,

$$\mathcal{L}_{f \circ j}(w) = d(f \circ j)_w = df_{j(w)} \circ dj_w = \mathcal{L}_f(j(w)) \circ A = A^* \circ \mathcal{L}_f \circ j(w)$$

for any $w \in W$. □

Lemma 3.3. *Let $f \in C^\infty(\mathbb{R}^N)$ be a strictly convex function and $\omega = \sqrt{-1}\partial\bar{\partial}\pi_N^*f$ the corresponding \mathbb{T}^N -invariant Kähler form on $(\mathbb{C}^\times)^N = \mathbb{R}^N \times \sqrt{-1}\mathbb{T}^N$ (here $\pi_N : (\mathbb{C}^\times)^N \rightarrow \mathbb{R}^N$ is the projection). Let $\phi = \mathcal{L}_f \circ \pi_N : (\mathbb{C}^\times)^N \rightarrow (\mathbb{R}^N)^*$ denote the associated moment map.*

Let $K \subset \mathbb{T}^N$ be a closed subgroup and let $G = \mathbb{T}^N / K$. For any $\nu \in \mathfrak{k}^*$ the symplectic quotient

$$(\mathbb{C}^\times)^N //_\nu K$$

is biholomorphic to $U \times \sqrt{-1}G \subset \mathfrak{g} \times \sqrt{-1}G = G^{\mathbb{C}}$ where $U \subset \mathfrak{g}$ is an open contractible set. Hence the reduced Kähler form ω_ν has a potential f_ν .

Moreover, the Legendre–Fenchel dual f_v^* of the Kähler potential f_v is given by

$$(3-1) \quad f_v^* = f^* \circ \iota_\lambda$$

where $\iota_\lambda : \mathfrak{g}^* \rightarrow (\mathbb{R}^N)^*$ is the affine embedding (2-2) and $-\lambda$ is a point in $(B^*)^{-1}(v)$.

Proof. It is no loss of generality to assume that the group K is connected. Then $\mathbb{T}^N \simeq K \times G$. Consequently $\mathbb{R}^N \simeq \mathfrak{k} \times \mathfrak{g}$ and the short exact sequence

$$0 \rightarrow \mathfrak{k} \xrightarrow{B} \mathbb{R}^N \xrightarrow{A} \mathfrak{g} \rightarrow 0$$

splits. Let

$$\pi_K : \mathbb{R}^N \rightarrow \mathfrak{k} \quad \text{and} \quad \iota_{\mathfrak{g}} : \mathfrak{g} \rightarrow \mathbb{R}^N$$

denote the maps defined by the splitting. The moment map $\phi_K : (\mathbb{C}^\times)^N \rightarrow \mathfrak{k}^*$ for the action of K on $((\mathbb{C}^\times)^N, \omega)$ is the composition

$$\phi_K = B^* \circ \phi = B^* \circ \mathcal{L}_f \circ \pi_N.$$

Let

$$\Delta = (B^*)^{-1}(v) \cap \phi((\mathbb{C}^\times)^N) = (B^*)^{-1}(v) \cap \mathcal{L}_f(\mathbb{R}^N).$$

Then Δ is the intersection of an affine hyperplane with a convex set, hence is contractible.

Since the action of K on $\phi_K^{-1}(v)$ is free, $K^{\mathbb{C}} \cdot \phi_K^{-1}(v)$ is an open subset of $(\mathbb{C}^\times)^N$ and $K^{\mathbb{C}}$ acts freely on it. Moreover, for each $x \in \phi_K^{-1}(v)$ the orbit $K^{\mathbb{C}} \cdot x$ intersects the level set $\phi_K^{-1}(v)$ transversely and

$$K^{\mathbb{C}} \cdot x \cap \phi_K^{-1}(v) = K \cdot x$$

(see [Guillemin and Sternberg 1982, pp. 526–527]). It follows that the restriction

$$\pi_K|_{\mathcal{L}_f^{-1}(\Delta)} : \mathcal{L}_f^{-1}(\Delta) \rightarrow \mathfrak{k}$$

is 1-1 and a local diffeomorphism. Hence

$$U = \pi_K(\mathcal{L}_f^{-1}(\Delta))$$

is a contractible open set.

On the other hand, the restriction $\omega|_{\phi_K^{-1}(v)}$ descends to a Kähler form ω_v on the symplectic quotient

$$(\mathbb{C}^\times)^N //_v K := \phi_K^{-1}(v) / K.$$

Moreover, since ω is \mathbb{T}^N invariant, ω_v is G -invariant. Note that

$$(\mathbb{C}^\times)^N //_v K \simeq U \times \sqrt{-1}G \subset G^{\mathbb{C}}.$$

By Lemma 3.1 there exists $f_v \in C^\infty(U)$ such that

$$\omega_v = \sqrt{-1} \partial \bar{\partial} f_v.$$

The potential f_ν defines a moment map

$$\phi_G : U \times \sqrt{-1}G \rightarrow \mathfrak{g}^*$$

with

$$\phi_G = \mathcal{L}_{f_\nu} \circ \pi_G,$$

where $\pi_G : U \times \sqrt{-1}G \rightarrow U$ is the projection. Moreover, by adjusting f_ν [Burns and Guillemin 2004] we may arrange for the diagram

$$(3-2) \quad \begin{array}{ccc} \phi_K^{-1}(\nu) & \xrightarrow{\phi} & \Delta \subset (\mathbb{R}^N)^* \\ /K \downarrow & & \uparrow \iota_\lambda \\ U \times \sqrt{-1}G & \xrightarrow{\phi_G} & \mathfrak{g}^* \end{array}$$

to commute. That is, the moment map ϕ_G is defined up to a constant and the potential f_ν is defined up to a pluriharmonic G -invariant function. By adding an appropriate pluriharmonic function to f_ν we can change ϕ_G by any constant we want. Since $\phi = \mathcal{L}_f \circ \pi_N$ and since $\phi_G = \mathcal{L}_{f_\nu} \circ \pi_G$, it follows from (3-2) that the diagram

$$\begin{array}{ccc} \mathcal{L}_f^{-1}(\Delta) & \xrightarrow{\mathcal{L}_f} & \Delta \subset (\mathbb{R}^N)^* \\ \pi_K \downarrow & & \uparrow \iota_\lambda \\ U & \xrightarrow{\mathcal{L}_{f_\nu}} & \mathfrak{g}^* \end{array}$$

commutes as well. Since $(\mathcal{L}_{f_\nu})^{-1} = \mathcal{L}_{f_\nu^*}$, where f_ν^* is the Legendre–Fenchel dual of f_ν ,

$$\mathcal{L}_{f_\nu^*} = \pi_K \circ (\mathcal{L}_f)^{-1} \circ \iota_\lambda = \pi_K \circ (\mathcal{L}_f)^* \circ \iota_\lambda.$$

By Lemma 3.2,

$$\mathcal{L}_{f_\nu^*} = \mathcal{L}_{f^*} \circ \iota_\lambda.$$

Therefore, up to a constant, $f_\nu^* = f^* \circ \iota_\lambda$. □

4. From potentials to dual potentials and back again

We start by making two observations. Let V be a real finite dimensional vector space, V^* its dual, $\mathbb{O} \subset V$ an open set, $\varphi \in C^\infty(\mathbb{O})$ a strictly convex function, $\mathcal{L}_\varphi : \mathbb{O} \rightarrow V^*$ the Legendre transform (which we assume to be invertible), $\mathbb{O}^* = \mathcal{L}_\varphi(\mathbb{O})$ and $\varphi^* \in C^\infty(\mathbb{O}^*)$ the Fenchel dual of φ .

Lemma 4.1. *Under these assumptions, $\varphi = (\mathcal{L}_\varphi)^*h$, where $h : \mathbb{O}^* \rightarrow \mathbb{R}$ is given by*

$$h(\eta) = \langle \eta, (d\varphi^*)_\eta \rangle - \varphi^*(\eta)$$

where we think of $(d\varphi^*)_\eta \in T_\eta^*\mathbb{O}^*$ as an element of $(V^*)^* = V$.

Proof. By the definition of the Fenchel dual, $\varphi(s) + \varphi^*(\eta) = \langle \eta, s \rangle$ for $\eta = \mathcal{L}_\varphi(s)$. Hence

$$\varphi(s) = \langle \eta, s \rangle - \varphi^*(\eta) = \langle \eta, (\mathcal{L}_\varphi)^{-1}(\eta) \rangle - \varphi^*(\eta) = \langle \eta, \mathcal{L}_{\varphi^*}(\eta) \rangle - \varphi^*(\eta)$$

and the result follows since $\mathcal{L}_{\varphi^*}(\eta) = (d\varphi^*)_\eta$. \square

Lemma 4.2. *We keep the above notation. Suppose additionally that the dual potential φ^* has the special form*

$$\varphi^*(\eta) = \sum_{i=1}^N f_i(u_i(\eta) - \lambda_i),$$

where u_1, \dots, u_N are vectors in V (thought of as linear functionals $u_i : V^* \rightarrow \mathbb{R}$), $\lambda_i \in \mathbb{R}$ are constants and f_i 's are functions of one variable. Then

$$(4-1) \quad h(\eta) = \sum_{i=1}^N (f'_i(u_i(\eta) - \lambda_i) u_i(\eta) - f_i(u_i(\eta) - \lambda_i)).$$

Proof. Observe that

$$d(f_i \circ (u_i - \lambda_i))_\eta = f'_i(u_i(\eta) - \lambda_i) d(u_i - \lambda_i)_\eta = f'_i(u_i(\eta) - \lambda_i) u_i$$

since u_i is linear. Hence

$$\langle \eta, (d\varphi^*)_\eta \rangle = \langle \eta, \sum f'_i(u_i(\eta) - \lambda_i) u_i \rangle = \sum f'_i(u_i(\eta) - \lambda_i) u_i(\eta)$$

and (4-1) follows from Lemma 4.1. \square

Example 4.3. We use the lemma above to argue that for the standard action of \mathbb{T}^N on $(\mathbb{C}^N, \sqrt{-1}\partial\bar{\partial}\|z\|^2)$, the dual potential φ^* is given by

$$\varphi^* = \sum_{i=1}^N e_i \log e_i,$$

where $\{e_1, \dots, e_N\}$ is the standard basis of $\mathbb{R}^N = \text{Lie}(\mathbb{T}^N)$.

Indeed, the homogeneous moment map $\Phi : \mathbb{C}^N \rightarrow (\mathbb{R}^N)^*$ for the standard action of \mathbb{T}^N is given by

$$\Phi(z) = \sum |z_j|^2 e_j^*,$$

where $\{e_j^*\}$ is the basis dual to $\{e_j\}$. Hence

$$\|z\|^2 = \Phi^*(\sum e_j).$$

On the other hand, if $\varphi^* = \sum e_j \log e_j$, then

$$\varphi^* = \sum f \circ e_j$$

where $f(x) = x \log x$. Since $f'(x) = \log x + 1$, (4-1) becomes

$$h = \sum (\log e_j + 1) e_j - \sum e_j \log e_j = \sum e_j.$$

Therefore, $\varphi^* = \sum e_j \log e_j$ is, indeed, the dual potential.

We are now in position to prove (1-1).

Theorem 4.4. *Let G be a torus, $P \subset \mathfrak{g}^*$ the polyhedral set defined by (2-1), $M_P = \mathbb{C}^N //_{\nu} K$ the Kähler G -space with moment map $\phi_P : M_P \rightarrow \mathfrak{g}^*$ constructed in Lemma 2.1 (1). Then the Kähler form ω_P on $\mathring{M}_P := \phi_P^{-1}(\mathring{P})$ is given by*

$$\omega_P = \sqrt{-1} \partial \bar{\partial} \phi_P^* \left(\sum_{j=1}^N \lambda_j \log(u_j - \lambda_j) + u_j \right),$$

Proof. By Lemma 2.1, $\mathring{M}_P = (\mathbb{C}^\times)^N //_{\nu} K$ where $K \subset \mathbb{T}^N$ is a closed subgroup. By Lemma 3.3 the dual potential φ_P^* on \mathring{P} is given by $\varphi_P^* = \varphi^* \circ \iota_{\lambda}$, where φ^* is the potential on the open orthant in $(\mathbb{R}^N)^*$ dual to the flat metric potential $\varphi(z) = \|z\|^2$ on $(\mathbb{C}^\times)^N$. By Example 4.3 $\varphi^* = \sum e_j \log e_j$. Since $\iota_{\lambda}^* e_j = u_j - \lambda_j$,

$$\varphi_P^* = \sum (u_j - \lambda_j) \log(u_j - \lambda_j).$$

By Lemmas 4.1 and 4.2, the potential φ_P is given by

$$\varphi_P = \phi_P^* h$$

where

$$h = \sum (\log(u_j - \lambda_j) + 1) u_j - \sum (u_j - \lambda_j) \log(u_j - \lambda_j)$$

(see (4-1)). Therefore

$$\varphi_P = \phi_P^* \left(\sum_{i=1}^N (\lambda_i \log(u_j - \lambda_j) + u_j) \right). \quad \square$$

5. Kähler potentials on the preimages of faces

Once again let $P \subset \mathfrak{g}^*$ be a polyhedral set given by (2-1). Recall that in Section 2 we canonically associated to this set a Kähler quotient M_P of \mathbb{C}^N which carries an effective holomorphic and Hamiltonian action of the torus G with a moment map $\phi_P : M_P \rightarrow \mathfrak{g}^*$. Let $F \subset P$ be a face. Its interior \mathring{F} is given by

$$\mathring{F} = \bigcap_{j \notin I} \{ \eta \in \mathfrak{g}^* \mid \langle \eta, u_j \rangle - \lambda_j > 0 \} \cap \bigcap_{j \in I} \{ \eta \in \mathfrak{g}^* \mid \langle \eta, u_j \rangle - \lambda_j = 0 \}$$

for some nonempty subset I of $\{1, \dots, N\}$. We have seen in the proof of Lemma 2.1 that the preimage

$$M_{\mathring{F}} := \phi_P^{-1}(\mathring{F})$$

is the Kähler quotient of \mathring{V}_I by a compact abelian group K_I . Therefore there is a potential $\varphi_F^* \in C^\infty(\mathring{F})$ dual to the Kähler potential φ_F on $M_{\mathring{F}}$. The goal of this section is to compute the dual potential φ_F^* “explicitly.” Lemmas 4.1 and 4.2 will then give us an analogue of (1-1) for the Kähler metric on $M_{\mathring{F}}$.

The Kähler potential φ_I on \mathring{V}_I for the flat metric induced from \mathbb{C}^N is given by

$$\varphi_I(z) = \sum_{j \notin I} |z_j|^2.$$

The restriction of the moment map $\phi : \mathbb{C}^N \rightarrow (\mathbb{R}^N)^*$ to \mathring{V}_I is a moment map for the action of the torus

$$H_I := \mathbb{T}^N / \mathbb{T}_I^N.$$

Note that

$$\phi(\mathring{V}_I) = \left\{ \sum_{i \notin I} a_i e_i^* \mid a_i > 0 \right\}.$$

This set is an open subset in

$$\text{span}_{i \notin I} \{e_j^*\} \simeq \mathfrak{h}_I^*.$$

From now on we identify \mathfrak{h}_I^* with $\text{span}_{i \notin I} \{e_j^*\}$. The dual potential $\varphi_I^* \in C^\infty(\phi(\mathring{V}_I))$ is easily seen to be

$$\varphi_I^* = \sum_{j \notin I} e_j \log e_j.$$

The manifold $M_{\mathring{F}}$ is a Hamiltonian G space, but the group G doesn't act effectively. So we cannot yet apply Lemma 3.3 as we would like. Let G_I denote the quotient of G that does act effectively on $M_{\mathring{F}}$. It is isomorphic to the quotient H_I/K_I . The dual of its Lie algebra \mathfrak{g}_I^* is naturally embedded in \mathfrak{g}^* :

$$\mathfrak{g}_I^* = \{\eta \in \mathfrak{g}^* \mid \langle \eta, u_i \rangle = 0 \text{ for all } i \in I\}.$$

Note also that the affine span $\text{affspan } \mathring{F}$ of $\mathring{F} \subset \mathfrak{g}^*$ is the translation of \mathfrak{g}_I^* by an element $\eta_0 \in \mathring{F}$, as it should be. Let $\gamma_I : \mathfrak{g}_I^* \rightarrow \text{affspan } \mathring{F} \subset \mathfrak{g}^*$ denote the affine embedding. Then there exists an affine embedding $\iota_I : \mathfrak{g}_I^* \hookrightarrow \mathfrak{h}_I^*$ so that the diagram

$$\begin{array}{ccc} \mathfrak{h}_I^* & \longrightarrow & (\mathbb{R}^N)^* \\ \iota_I \uparrow & & \uparrow \iota_\lambda \\ \mathfrak{g}_I^* & \xrightarrow{\gamma_I} & \mathfrak{g}^* \end{array}$$

commutes. Here the top arrow identifies \mathfrak{h}_I^* with $\text{span}_{i \notin I} \{e_i^*\}$. Since γ_I is an embedding, we may think of φ_F^* as living on $\mathring{F} \subset \gamma_I(\mathfrak{g}_I^*)$. Therefore, by Lemma 3.3,

$$(5-1) \quad \varphi_F^* = (\varphi_I^* \circ \iota_\lambda)|_{\mathring{F}}.$$

Let

$$v_j = u_j|_{\hat{F}}.$$

These functions are affine, but not necessarily linear. Then

$$(e_j \circ \iota_\lambda)|_{\hat{F}} = (u_j - \lambda_j)|_{\hat{F}} = v_j - \lambda_j.$$

Therefore

$$\varphi_F^* = (\varphi_I^* \circ \iota_\lambda)|_{\hat{F}} = \sum_{j \notin I} (v_j - \lambda_j) \log(v_j - \lambda_j).$$

To get a nicer formula for the potential on $M_{\hat{F}}$ we now make a simplifying assumption, namely, that $0 \in \hat{F}$. Then $v_j = u_j|_{\mathfrak{g}_j^*}$ and, in particular, it is *linear* for all j . Hence Lemmas 4.1 and 4.2 apply, and we obtain:

Theorem 5.1. *Under the simplifying assumption above, the Kähler form ω_F on $M_{\hat{F}}$ is given by*

$$\omega_F = \sqrt{-1} \partial \bar{\partial} (\phi_P|_{M_{\hat{F}}})^* \left(\sum_{j \notin I} \lambda_j \log(v_j - \lambda_j) + v_j \right).$$

Alternatively we may take the isomorphism $\gamma_I : \mathfrak{g}_I^* \rightarrow \text{affspan } \hat{F}$ explicitly into account and think of φ_F^* as living on an open subset of \mathfrak{g}_I^* . Then, by Lemma 3.3,

$$\varphi_F^* = \varphi_I^* \circ \iota_\lambda \circ \gamma_I.$$

Since

$$e_i \circ \iota_\lambda \circ \gamma_I = u_i|_{\mathfrak{g}_i^*} + u_i(\eta_0) - \lambda_i,$$

we get

$$\varphi_F^* = \sum_{i \notin I} (u_i|_{\mathfrak{g}_i^*} + u_i(\eta_0) - \lambda_i) \log(u_i|_{\mathfrak{g}_i^*} + u_i(\eta_0) - \lambda_i).$$

We conclude:

Theorem 5.2. *The Kähler form ω_F on $M_{\hat{F}}$ is given by*

$$\omega_F = \sqrt{-1} \partial \bar{\partial} (\phi_P|_{M_{\hat{F}}})^* \left(\sum_{j \notin I} ((\lambda_j - u_j(\eta_0)) \log(u_j|_{\mathfrak{g}_j^*} + u_j(\eta_0) - \lambda_j) + u_j|_{\mathfrak{g}_j^*}) \right).$$

Variations on the theme. The same technique allows us to prove a variant of (1-1). We keep the notation above. Suppose that the polyhedral set P is compact. That is, suppose that P is actually a polytope. Then

$$\iota_\lambda(P) \subset \{\ell \in (\mathbb{R}^N)^* \mid \langle \ell, e_j \rangle \geq 0 \text{ for all } j\}$$

is bounded. Hence there is $R > 0$ such that $\iota_\lambda(P)$ is contained in a scaled copy Δ_R of the standard simplex

$$\Delta_R = \{\ell \in (\mathbb{R}^N)^* \mid \langle \ell, e_j \rangle \geq 0 \text{ for all } j \text{ and } \sum \langle \ell, e_j \rangle \leq R\}.$$

Since Δ_1 is the moment map image of $\mathbb{C}P^N$ under the standard action of \mathbb{T}^N , it follows that M_P is also a symplectic quotient of $(\mathbb{C}P^N, R\omega_{FS})$ by the action of the compact abelian Lie group K defined earlier (ω_{FS} denotes the Fubini–Study form; see [Lemma 2.1 \(3\)](#)). Since

$$\Delta_R = \{ \ell \in (\mathbb{R}^N)^* \mid \langle \ell, e_j \rangle \geq 0, 1 \leq j \leq N, \langle \ell, -\sum e_j \rangle + R \geq 0 \},$$

it follows from (3-1) that the potential f^* dual to the potential for $R\omega_{FS}$ on Δ_R is given by

$$f^* = \sum e_j \log e_j + (R - \sum e_j) \log(R - \sum e_j).$$

Consequently the potential f_v^* dual to the potential on the quotient $(\mathbb{C}P^N //_{\nu} K, \omega_P)$ is

$$f_v^* = \sum (u_j - \lambda_j) \log(u_j - \lambda_j) + (R - \sum (u_j - \lambda_j)) \log(R - \sum (u_j - \lambda_j)).$$

By [Lemma 4.1](#) the reduced Kähler form ω_P is

$$\omega_P = \sqrt{-1} \partial \bar{\partial} \phi^* h$$

where

$$h(\eta) = \langle \eta, (df_v)_{\eta} \rangle - f_v(\eta).$$

A computation similar to the ones in the previous sections gives

$$(5-2) \quad h = \sum \lambda_j \log(u_j - \lambda_j) - (R + \sum \lambda_j) \log(R - \sum (u_j - \lambda_j)).$$

We have proved the following theorem.

Theorem 5.3. *Let G be a torus, $P \subset \mathfrak{g}^*$ the polyhedral set defined by (2-1) which happens to be compact, $M_P = (\mathbb{C}P^N, R\omega_{FS}) //_{\nu} K$ the Kähler G -space with moment map $\phi_P : M_P \rightarrow \mathfrak{g}^*$ constructed in [Lemma 2.1 \(3\)](#). Then the Kähler form ω_P on $\hat{M}_P := \phi_P^{-1}(\hat{P})$ is given by*

$$\omega_P = \sqrt{-1} \partial \bar{\partial} \phi_P^* (\sum \lambda_j \log(u_j - \lambda_j) - (R + \sum \lambda_j) \log(R - \sum (u_j - \lambda_j))).$$

Acknowledgments

We thank Peter Heinzner for patiently answering our queries and the referee for valuable comments.

References

- [Abreu 2001] M. Abreu, “Kähler metrics on toric orbifolds”, *J. Differential Geom.* **58**:1 (2001), 151–187. [MR 2003b:53046](#) [Zbl 1035.53055](#)
- [Burns and Guillemin 2004] D. Burns and V. Guillemin, “Potential functions and actions of tori on Kähler manifolds”, *Comm. Anal. Geom.* **12**:1-2 (2004), 281–303. [MR 2005e:53133](#) [Zbl 1073.53113](#)

- [Calderbank et al. 2003] D. M. J. Calderbank, L. David, and P. Gauduchon, “The Guillemin formula and Kähler metrics on toric symplectic manifolds”, *J. Symplectic Geom.* **1**:4 (2003), 767–784. [MR 2005a:53145](#) [Zbl 02199608](#)
- [Guillemin 1994] V. Guillemin, “Kähler structures on toric varieties”, *J. Differential Geom.* **40**:2 (1994), 285–309. [MR 95h:32029](#) [Zbl 0813.53042](#)
- [Guillemin and Sternberg 1982] V. Guillemin and S. Sternberg, “Geometric quantization and multiplicities of group representations”, *Invent. Math.* **67**:3 (1982), 515–538. [MR 83m:58040](#) [Zbl 0503.58018](#)
- [Heinzner and Huckleberry 1996] P. Heinzner and A. Huckleberry, “Kählerian potentials and convexity properties of the moment map”, *Invent. Math.* **126**:1 (1996), 65–84. [MR 98e:58075](#) [Zbl 0855.58025](#)
- [Heinzner and Loose 1994] P. Heinzner and F. Loose, “Reduction of complex Hamiltonian G -spaces”, *Geom. Funct. Anal.* **4**:3 (1994), 288–297. [MR 95j:58050](#) [Zbl 0816.53018](#)
- [Heinzner et al. 1994] P. Heinzner, A. T. Huckleberry, and F. Loose, “Kählerian extensions of the symplectic reduction”, *J. Reine Angew. Math.* **455** (1994), 123–140. [MR 95k:58061](#) [Zbl 0803.53042](#)
- [Martelli and Sparks 2004] D. Martelli and J. Sparks, “Toric geometry, Sasaki-Einstein manifolds and a new infinite class of AdS/CFT duals”, preprint, 2004. [arXiv hep-th/0411238](#)

Received September 10, 2007. Revised February 29, 2008.

DAN BURNS
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MICHIGAN
ANN ARBOR, MI 48109-1043
UNITED STATES
dburns@umich.edu

VICTOR GUILLEMIN
DEPARTMENT OF MATHEMATICS
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
CAMBRIDGE, MA 02139-4307
UNITED STATES
vwg@math.mit.edu

EUGENE LERMAN
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ILLINOIS
URBANA, IL 61801
UNITED STATES
lerman@math.uiuc.edu