

*Pacific
Journal of
Mathematics*

**HECKE–CLIFFORD ALGEBRAS
AND SPIN HECKE ALGEBRAS, II:
THE RATIONAL DOUBLE AFFINE TYPE**

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This paper introduces the notion of rational spin double affine Hecke algebras (sDaHa) and rational double affine Hecke–Clifford algebras (DaHCa) associated to classical Weyl groups, and establishes the basic properties of these algebras, such as the PBW basis and Dunkl operator representations. We obtain an algebra isomorphism relating the rational DaHCa to the rational sDaHa. We further develop a link between the usual rational Cherednik algebra and the rational sDaHa by introducing a notion of rational covering double affine Hecke algebras.

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1. Introduction

The rational Cherednik algebra (see Etingof and Ginzburg [2002] and also Drinfeld [1986] for a more general deformation construction) is a degenerate version of the double affine Hecke algebra [Cherednik 2005], and it admits a polynomial representation via the Dunkl operators [1989]. A similar degeneration in the case of affine Hecke algebras was introduced and studied earlier in [Drinfeld 1986] in the type A case and by Lusztig in general [1989; 2002]. The rational Cherednik algebra has a rich representation theory and it affords various interesting connections to

MSC2000: 20C08.

Keywords: Cherednik algebras, Hecke algebras, symplectic reflection algebras, Clifford algebras, superalgebras, projective representation theory.

Wang is partially supported by an NSF grant.

integrable systems, noncommutative geometry, and so on. We refer to Etingof [2007] and Rouquier [2005] for reviews and extensive references. The rational Cherednik algebra with one particular parameter being zero, denoted by $\check{\mathfrak{H}}_W$ in this introduction, is known to have a large center [Etingof and Ginzburg 2002]; see Gordon [2003].

In [Wang 2006], the second author introduced (degenerate) spin Hecke algebras of affine and double affine type as well as double affine Hecke–Clifford algebra, associated to I. Schur’s spin symmetric group [1911]. These algebras were shown to be closely related to the affine Hecke–Clifford algebra of Nazarov [1997]. The spin affine Hecke algebras and affine Hecke–Clifford algebras associated to all classical Weyl groups have been recently constructed by the authors [Khongsap and Wang 2008].

In this paper we shall construct three classes of closely related (super) algebras associated to each *classical* finite Weyl group W : the rational double affine Hecke–Clifford algebra (DaHCa) $\check{\mathfrak{H}}_W^c$, the rational spin double affine Hecke algebra (sDaHa) $\check{\mathfrak{H}}_W^-$, and the rational covering double affine Hecke algebra (cDaHa) $\check{\mathfrak{H}}_W^\sim$. We show that the algebras $\check{\mathfrak{H}}_W^c$ and $\check{\mathfrak{H}}_W^-$ are Morita superequivalent (in the terminology of [Wang 2007]) and that $\check{\mathfrak{H}}_W^\sim$ has both the rational Cherednik algebra $\check{\mathfrak{H}}_W$ and the sDaHa $\check{\mathfrak{H}}_W^-$ as its natural quotients. We further establish some basic properties including the PBW basis theorem and Dunkl operator realizations of these algebras.

We expect that these algebras afford very interesting representation theory and connections with noncommutative geometry.

Morris [1976] and Khongsap and Wang [2008] considered a double cover \tilde{W} of the finite Weyl group W associated to a distinguished 2-cocycle

$$1 \rightarrow \mathbb{Z}_2 \longrightarrow \tilde{W} \longrightarrow W \rightarrow 1.$$

Denote $\mathbb{Z}_2 = \{1, z\}$. From now on, let W be one of the classical Weyl groups. In this paper, we define the algebras $\check{\mathfrak{H}}_W^c$, $\check{\mathfrak{H}}_W^-$ and $\check{\mathfrak{H}}_W^\sim$ for every W of type A_{n-1} , D_n , B_n in terms of explicit generators and relations, where the number of parameters in each of these algebras is the number of conjugacy classes of reflections in W . The compatibility among the defining relations for $\check{\mathfrak{H}}_W^c$ (which would imply the PBW basis property and similar compatibility for $\check{\mathfrak{H}}_W^-$ and $\check{\mathfrak{H}}_W^\sim$ when combined with other results) requires lengthy but elementary case-by-case verifications. In a suitable sense, the defining relations are naturally and uniquely dictated by the compatibility of these relations.

As is well known, the rational Cherednik algebra $\check{\mathfrak{H}}_W$ has a triangular decomposition with the group algebra $\mathbb{C}W$ as its middle term. We show that the algebras $\check{\mathfrak{H}}_W^c$, $\check{\mathfrak{H}}_W^-$ and $\check{\mathfrak{H}}_W^\sim$ also afford triangular decompositions which contain $\mathcal{C}_n \rtimes \mathbb{C}W$, $\mathbb{C}W^-$ and $\mathbb{C}\tilde{W}$ respectively as the middle terms, where \mathcal{C}_n denotes the Clifford algebra of the reflection representation of W with a natural W -action. For instance, the

rational DaHCa $\check{\check{H}}_W^c$ and sDaHa $\check{\check{H}}_W^-$ have the triangular decompositions

$$\begin{aligned} \check{\check{H}}_W^c &\cong \mathbb{C}[x_1, \dots, x_n] \otimes (\mathcal{C}_n \rtimes \mathbb{C}W) \otimes \mathbb{C}[y_1, \dots, y_n], \\ \check{\check{H}}_W^- &\cong \mathcal{C}[\xi_1, \dots, \xi_n] \otimes \mathbb{C}W^- \otimes \mathbb{C}[y_1, \dots, y_n], \end{aligned}$$

where $\mathcal{C}[\xi_1, \dots, \xi_n]$ is a noncommutative algebra with $\xi_i \xi_j = -\xi_j \xi_i$ for $i \neq j$. The relations between $\mathbb{C}W^-$ and $\mathcal{C}[\xi_1, \dots, \xi_n]$ involve subtle signs similar to those appearing in the spin affine Hecke algebras defined in [Khongsap and Wang 2008; Wang 2006].

We further show that the algebras $\check{\check{H}}_W^c$ and $\check{\check{H}}_W^-$ have large centers which contain $\mathbb{C}[y_1, \dots, y_n]^W$ and $\mathbb{C}[x_1^2, \dots, x_n^2]^W$ (and $\mathbb{C}[\xi_1^2, \dots, \xi_n^2]^W$ respectively) as subalgebras. In particular, the algebras $\check{\check{H}}_W^c$ and $\check{\check{H}}_W^-$ are module-finite over their centers.

The group algebra $\mathbb{C}W$ and the spin Weyl group algebra $\mathbb{C}W^-$ appear as natural quotients of $\mathbb{C}\tilde{W}$ by the ideals $\langle z \mp 1 \rangle$ respectively. We show that these quotient maps, denoted by Υ_{\pm} , extend to the setup of double affine Hecke algebras. All these statements can be summarized in the following commutative diagram, with the vertical arrows being natural inclusions:

$$\begin{array}{ccccc} \mathbb{C}W & \xleftarrow{\Upsilon_+} & \mathbb{C}\tilde{W} & \xrightarrow{\Upsilon_-} & \mathbb{C}W^- \\ \downarrow & & \downarrow & & \downarrow \\ \check{\check{H}}_W & \xleftarrow{\Upsilon_+} & \check{\check{H}}_W^- & \xrightarrow{\Upsilon_-} & \check{\check{H}}_W^- \end{array}$$

In [Khongsap and Wang 2008], we established a superalgebra isomorphism

$$\Phi : \mathcal{C}_n \rtimes \mathbb{C}W \xrightarrow{\cong} \mathcal{C}_n \otimes \mathbb{C}W^-$$

(which actually holds also for exceptional Weyl groups), generalizing the type A result of Sergeev and Yamaguchi. In this paper, we shall establish a Morita superequivalence between $\check{\check{H}}_W^c$ and $\check{\check{H}}_W^-$, or more explicitly, a superalgebra isomorphism between $\check{\check{H}}_W^c$ and the tensor algebra $\mathcal{C}_n \otimes \check{\check{H}}_W^-$ that extends the isomorphism Φ (see [Wang 2006] for the type A case). This can be summarized conveniently in the following commutative diagram, with the vertical arrows being natural inclusions:

$$\begin{array}{ccc} \mathcal{C}_n \rtimes \mathbb{C}W & \xrightarrow[\Phi]{\cong} & \mathcal{C}_n \otimes \mathbb{C}W^- \\ \downarrow & & \downarrow \\ \check{\check{H}}_W^c & \xrightarrow[\Phi]{\cong} & \mathcal{C}_n \otimes \check{\check{H}}_W^- \end{array}$$

As our constructions in a way rely on a choice of orthonormal basis of \mathfrak{h} , they do not seem to be easily extendable to the exceptional Weyl groups. Also, in contrast to the setup of rational Cherednik algebras in [Etingof and Ginzburg 2002],

our Hecke algebras do not seem to afford an extra parameter in a natural way to trivialize their center.

There has been another attempt (see [Chmutova 2005]) to generalize the rational Cherednik algebras and more generally symplectic reflection algebras by adding a twist with a 2-cocycle of a finite group. But the approach there does not produce intrinsically interesting new algebras with nontrivial 2-cocycles of the Weyl groups as does our approach.

The paper is organized as follows. In Section 2, we recall some facts about the distinguished double covers of the Weyl groups. For more detailed treatment, consult [Khongsap and Wang 2008]. We introduce in Section 3 the rational DaHCa $\check{\mathfrak{H}}_W^c$ and establish its PBW basis property. Section 4 obtains the Dunkl operator representations of $\check{\mathfrak{H}}_W^c$. Sections 5 and 6 are the counterparts for the sDaHa $\check{\mathfrak{H}}_W^-$ of Section 3 and 4, respectively. In addition, Section 5 establishes the superalgebra isomorphism Φ relating the sDaHa and DaHCa. Finally, Section 7 introduces the rational cDaHa $\check{\mathfrak{H}}_W^\sim$, which provides a link between the sDaHa $\check{\mathfrak{H}}_W^-$ and the usual DaHa $\check{\mathfrak{H}}_W$. Finally, in the appendix (Section 8), we present the proofs of several lemmas in Section 3 and 4.

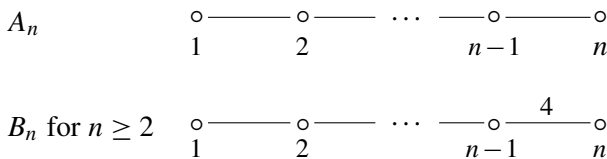
2. The spin Weyl groups

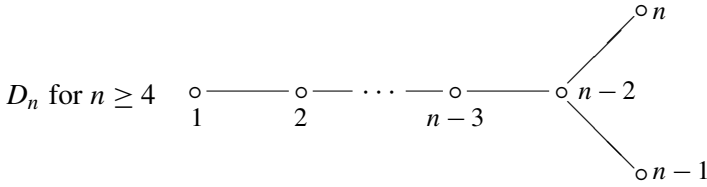
In this section, we recall from [Khongsap and Wang 2008] some preliminary setups which lead to Theorem 2.1 below, but here we will restrict ourselves to classical Weyl groups only, as this is all we need subsequently.

2.1. A double covering of Weyl groups. Let W be an (irreducible) finite Weyl group of classical type (that is, of type A, B, D) with the presentation

$$(2-1) \quad \langle s_1, \dots, s_n \mid (s_i s_j)^{m_{ij}} = 1, m_{ii} = 1, m_{ij} = m_{ji} \in \mathbb{Z}_{\geq 2} \text{ for } i \neq j \rangle.$$

The integers m_{ij} are specified by the Coxeter–Dynkin diagrams whose vertices correspond to the generators of W below. By convention, we only mark the edge connecting i and j with $m_{ij} \geq 4$. We have $m_{ij} = 3$ for $i \neq j$ connected by an unmarked edge, and $m_{ij} = 2$ if i and j are not connected by an edge.





We shall be concerned about a distinguished double covering \tilde{W} of W :

$$1 \rightarrow \mathbb{Z}_2 \longrightarrow \tilde{W} \longrightarrow W \rightarrow 1.$$

We let $\mathbb{Z}_2 = \{1, z\}$ and denote by \tilde{t}_i a fixed preimage of the generators s_i of W for each i . The group \tilde{W} is generated by $z, \tilde{t}_1, \dots, \tilde{t}_n$ with relations (besides the obvious relation that z is central of order 2) listed in Table 1, which corresponds to setting the α_i for all i in Karpilovsky [1987, Table 7.1] to be z .

The quotient algebra $\mathbb{C}W^- := \mathbb{C}\tilde{W}/\langle z+1 \rangle$ of $\mathbb{C}\tilde{W}$ by the ideal generated by $z+1$ will be called the *spin Weyl group algebra* associated to W . Denote by $t_i \in \mathbb{C}W^-$ the image of \tilde{t}_i . The spin Weyl group algebra $\mathbb{C}W^-$ has a natural superalgebra (that is, \mathbb{Z}_2 -graded algebra) structure by letting each t_i be odd. The algebra $\mathbb{C}W^-$ is generated by t_1, \dots, t_n with the labeling as in the Coxeter–Dynkin diagrams and the explicit relations summarized in Table 2.

By definition, the quotient by the ideal $\langle z-1 \rangle$ of the group algebra $\mathbb{C}\tilde{W}$ is isomorphic to $\mathbb{C}W$.

2.2. A superalgebra isomorphism. Denote by $\mathfrak{h} = \mathbb{C}^n$ the natural representation (respectively the reflection representation) of the Weyl group W of type A_{n-1} (respectively of type B_n and D_n). Note that \mathfrak{h} carries a W -invariant nondegenerate

W	Defining Relations for \tilde{W}
A_n	$\tilde{t}_i^2 = 1$ for $1 \leq i \leq n$, $\tilde{t}_i \tilde{t}_{i+1} \tilde{t}_i = \tilde{t}_{i+1} \tilde{t}_i \tilde{t}_{i+1}$ for $1 \leq i \leq n-1$, $\tilde{t}_i \tilde{t}_j = z \tilde{t}_j \tilde{t}_i$ if $m_{ij} = 2$.
B_n ($n \geq 2$)	$\tilde{t}_i^2 = 1$ for $1 \leq i \leq n$, $\tilde{t}_i \tilde{t}_{i+1} \tilde{t}_i = \tilde{t}_{i+1} \tilde{t}_i \tilde{t}_{i+1}$ for $1 \leq i \leq n-2$, $\tilde{t}_i \tilde{t}_j = z \tilde{t}_j \tilde{t}_i$ for $1 \leq i < j \leq n-1$, $m_{ij} = 2$, $\tilde{t}_i \tilde{t}_n = z \tilde{t}_n \tilde{t}_i$ for $1 \leq i \leq n-2$, $(\tilde{t}_{n-1} \tilde{t}_n)^2 = z(\tilde{t}_n \tilde{t}_{n-1})^2$.
D_n ($n \geq 4$)	$\tilde{t}_i^2 = 1$ for $1 \leq i \leq n$, $\tilde{t}_i \tilde{t}_j \tilde{t}_i = \tilde{t}_j \tilde{t}_i \tilde{t}_j$ if $m_{ij} = 3$, $\tilde{t}_i \tilde{t}_j = z \tilde{t}_j \tilde{t}_i$ for $1 \leq i < j \leq n$, $m_{ij} = 2$.

Table 1. The defining relations of \tilde{W} .

Type of W	Defining Relations for $\mathbb{C}W^-$
A_n	$t_i^2 = 1, \quad t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1},$ $(t_i t_j)^2 = -1$ if $ i - j > 1.$
B_n	t_1, \dots, t_{n-1} satisfy these relations for $\mathbb{C}W_{A_{n-1}}^-:$ $t_n^2 = 1, \quad (t_i t_n)^2 = -1$ if $i \neq n - 1, n,$ $(t_{n-1} t_n)^4 = -1.$
D_n	t_1, \dots, t_{n-1} satisfy these relations for $\mathbb{C}W_{A_{n-1}}^-:$ $t_n^2 = 1, \quad (t_i t_n)^2 = -1$ if $i \neq n - 2, n,$ $t_{n-2} t_n t_{n-2} = t_n t_{n-2} t_n$

Table 2. The defining relations of $\mathbb{C}W^-$.

bilinear form (\cdot, \cdot) , which gives rise to an identification $\mathfrak{h}^* \cong \mathfrak{h}$ and also a bilinear form on \mathfrak{h}^* which will be again denoted by (\cdot, \cdot) .

Denote by \mathcal{C}_n the Clifford algebra associated to $(\mathfrak{h}, (\cdot, \cdot))$. We shall denote by $\{c_i\}$ the generators in \mathcal{C}_n corresponding to a standard orthonormal basis $\{e_i\}$ of \mathbb{C}^n , and denote by $\{\beta_i\}$ the elements of \mathcal{C}_n corresponding to the simple roots $\{\alpha_i\}$, normalized with $\beta_i^2 = 1$. More explicitly, \mathcal{C}_n is generated by c_1, \dots, c_n subject to the relations

$$(2-2) \quad c_i^2 = 1 \quad \text{and} \quad c_i c_j = -c_j c_i \quad \text{for } i \neq j.$$

For type A_{n-1} , we have $\beta_i = (c_i - c_{i+1})/\sqrt{2}$ for $1 \leq i \leq n - 1$. For type B_n , we have an additional $\beta_n = c_n$, and $\beta_n = (c_{n-1} + c_n)/\sqrt{2}$ for type D_n .

The action of W on \mathfrak{h} and \mathfrak{h}^* preserves the bilinear form (\cdot, \cdot) and thus it acts by automorphisms of the algebra \mathcal{C}_n . This gives rise to a semidirect product $\mathcal{C}_n \rtimes \mathbb{C}W$. Moreover, the algebra $\mathcal{C}_n \rtimes \mathbb{C}W$ naturally inherits the superalgebra structure by letting elements in W be even and each c_i be odd.

Given two superalgebras \mathcal{A} and \mathcal{B} , we view the tensor product of superalgebras $\mathcal{A} \otimes \mathcal{B}$ as a superalgebra with multiplication defined by

$$(a \otimes b)(a' \otimes b') = (-1)^{|b||a'|} (aa' \otimes bb') \quad \text{for } a, a' \in \mathcal{A} \text{ and } b, b' \in \mathcal{B},$$

where $|b|$ denotes the \mathbb{Z}_2 -degree of b , and so on. Also, we shall use shorthand notation ab for $(a \otimes b) \in \mathcal{A} \otimes \mathcal{B}$ with $a = a \otimes 1$ and $b = 1 \otimes b$.

Theorem 2.1 [Khongsap and Wang 2008]. *We have an isomorphism*

$$\Phi : \mathcal{C}_n \rtimes \mathbb{C}W \xrightarrow{\cong} \mathcal{C}_n \otimes \mathbb{C}W^-$$

of superalgebras extending the identity map on \mathcal{C}_n and sends $s_i \mapsto -\sqrt{-1} \beta_i t_i$. The inverse map Ψ is the extension of the identity map on \mathcal{C}_n that sends $t_i \mapsto \sqrt{-1} \beta_i s_i$.

(In the terminology of [Wang 2007], the superalgebras $\mathcal{C}_n \rtimes \mathbb{C}W$ and $\mathbb{C}W^-$ are Morita superequivalent.)

Remark 2.2. Theorem 2.1 was formulated and proved in [Khongsap and Wang 2008] for every finite Weyl group including the exceptional types, and the type A case was treated by Sergeev and Yamaguchi.

3. Rational double affine Hecke–Clifford algebras (DaHCa)

In this section, we introduce the rational double affine Hecke–Clifford algebras associated to the Weyl group W of type A , D and B , and then establish the PBW property. The type A case was treated in [Wang 2006].

The definition of the algebras $\check{\mathfrak{H}}_W^c$.

The algebra $\check{\mathfrak{H}}_W^c$ of type A_{n-1} . We will make the identifications

$$\mathbb{C}[\mathfrak{h}^*] \cong \mathbb{C}[x_1, \dots, x_n] \quad \text{and} \quad \mathbb{C}[\mathfrak{h}] \cong \mathbb{C}[y_1, \dots, y_n],$$

where the x_i and y_i correspond to the standard orthonormal basis $\{e_i\}$ for \mathfrak{h}^* and its dual basis $\{e_i^*\}$ for \mathfrak{h} .

The following algebra $\check{\mathfrak{H}}_{A_{n-1}}^c$ was introduced in [Wang 2006] under the notation \mathcal{A}_u . We recall it for convenience and usage in the subsequent subsections. For x, y in an algebra A , we denote as usual that $[x, y] = xy - yx \in A$.

Definition 3.1. Let $u \in \mathbb{C}$ and $W = W_{A_{n-1}} \cong S_n$. The rational double affine Hecke–Clifford algebra (DaHCa) of type A_{n-1} , denoted by $\check{\mathfrak{H}}_W^c$ or $\check{\mathfrak{H}}_{A_{n-1}}^c$, is the algebra generated by x_i, y_i and c_i for $1 \leq i \leq n$ and W , subject to the relation (2-2) among the c_i and the following relations (where we identify $\mathfrak{h}^* = \mathbb{C}x_1 + \dots + \mathbb{C}x_n$ and $\mathfrak{h} = \mathbb{C}y_1 + \dots + \mathbb{C}y_n$):

$$(3-1) \quad \left\{ \begin{array}{ll} x_i x_j = x_j x_i, \quad y_i y_j = y_j y_i, \quad y_i c_j = c_j y_i & \text{for all } i, j, \\ x_i c_i = -c_i x_i, \quad x_i c_j = c_j x_i & \text{for } i \neq j, \\ wxw^{-1} = w(x) & \text{for all } x \in \mathfrak{h}^* \text{ and } w \in W, \\ wyw^{-1} = w(y) & \text{for all } y \in \mathfrak{h} \text{ and } w \in W, \\ wcw^{-1} = w(c) & \text{for all } c \in \mathcal{C}_n \text{ and } w \in W, \end{array} \right.$$

and

$$(3-2a) \quad [y_j, x_i] = u(1 + c_j c_i) s_{ji} \quad \text{for } i \neq j,$$

$$(3-2b) \quad [y_i, x_i] = -u \sum_{k \neq i} (1 + c_k c_i) s_{ki}.$$

Alternatively, we may view u as a formal variable and $\check{\mathfrak{H}}_W^c$ as a $\mathbb{C}(u)$ -algebra. Similar remarks apply to all DaHCa, sDaHa, and cDaHa introduced in this paper.

The algebra $\check{\check{\mathfrak{H}}}_W^c$ of type D_n . Let $W = W_{D_n}$. Regarding elements in W as even signed permutations of $1, 2, \dots, n$ as usual, we identify the generators $s_i \in W$ for $1 \leq i \leq n-1$ with transposition $(i, i+1)$, and $s_n \in W$ with the transposition of $(n-1, n)$ coupled with the sign changes at $n-1$ and n . For $1 \leq i \neq j \leq n$, we denote by $s_{ij} \equiv (i, j) \in W$ the transposition of i and j , and by $\bar{s}_{ij} \equiv (\bar{i}, \bar{j}) \in W$ the transposition of i and j coupled with the sign changes at i, j . By convention,

$$\bar{s}_{n-1, n} \equiv \overline{(n-1, n)} = s_n \quad \text{and} \quad \bar{s}_{ij} \equiv \overline{(i, j)} = s_{jn}s_{i, n-1}s_n s_{i, n-1}s_{jn}.$$

Definition 3.2. Let $u \in \mathbb{C}$ and $W = W_{D_n}$. The rational double affine Hecke–Clifford algebra of type D_n , denoted by $\check{\check{\mathfrak{H}}}_W^c$ or $\check{\check{\mathfrak{H}}}_{D_n}^c$, is the algebra generated by x_i, y_i and c_i for $1 \leq i \leq n$ and W , subject to the relation (2-2) among the c_i , (3-1) with the current W , and the relations

$$(3-3a) \quad [y_j, x_i] = u((1 + c_j c_i)s_{ij} - (1 - c_j c_i)\bar{s}_{ij}) \quad \text{for } i \neq j,$$

$$(3-3b) \quad [y_i, x_i] = -u \sum_{k \neq i} ((1 + c_k c_i)s_{ki} + (1 - c_k c_i)\bar{s}_{ki}).$$

The algebra $\check{\check{\mathfrak{H}}}_W^c$ of type B_n . Let $W = W_{B_n}$. We identify W as usual with the signed permutations on $1, \dots, n$. Regarding W_{D_n} as a subgroup of W , we have $s_{ij}, \bar{s}_{ij} \in W$ for $1 \leq i \neq j \leq n$. Further denote by $\tau_i \equiv (\bar{i}) \in W$ the sign change at i for $1 \leq i \leq n$. By definition, we have $\tau_n \equiv \overline{(n)} = s_n$ and $\tau_i \equiv (\bar{i}) = s_{in}s_n s_{in}$.

Definition 3.3. Let $u, v \in \mathbb{C}$, and $W = W_{B_n}$. The rational double affine Hecke–Clifford algebra of type B_n , denoted by $\check{\check{\mathfrak{H}}}_W^c$ or $\check{\check{\mathfrak{H}}}_{B_n}^c$, is the algebra generated by x_i, y_i and c_i for $1 \leq i \leq n$ and W , subject to the relations (2-2) for the c_i , (3-1) with the current W , and the relations

$$(3-4a) \quad [y_j, x_i] = u((1 + c_j c_i)s_{ij} - (1 - c_j c_i)\bar{s}_{ij}) \quad \text{for } i \neq j,$$

$$(3-4b) \quad [y_i, x_i] = -u \sum_{k \neq i} ((1 + c_k c_i)s_{ki} + (1 - c_k c_i)\bar{s}_{ki}) - \sqrt{2}v\tau_i.$$

When it is necessary to indicate the dependence of the algebra $\check{\check{\mathfrak{H}}}_W^c$ on u and v , we will write $\check{\check{\mathfrak{H}}}_W^c(u, v)$ for $\check{\check{\mathfrak{H}}}_W^c$.

Remark 3.4. The factor $\sqrt{2}$ in (3-4b) is inserted to make the definition of $\check{\check{\mathfrak{H}}}_{B_n}^c$ compatible with the notion of sDaHa $\check{\check{\mathfrak{H}}}_{B_n}^-$ below under a Morita superequivalence Φ . See Theorem 5.5.

The PBW basis for $\check{\check{\mathfrak{H}}}_W^c$. For any classical Weyl group W , the algebra $\check{\check{\mathfrak{H}}}_W^c$ is a superalgebra by letting elements of W and x_i and y_i for all i be even, and letting each c_i be odd.

Theorem 3.5. Let W be $W_{A_{n-1}}, W_{D_n}$ or W_{B_n} . The multiplication of the subalgebras $\mathbb{C}[\mathfrak{h}^*], \mathbb{C}[\mathfrak{h}], \mathcal{C}_n$, and $\mathbb{C}W$ induces a vector space isomorphism

$$\mathbb{C}[\mathfrak{h}^*] \otimes \mathcal{C}_n \otimes \mathbb{C}W \otimes \mathbb{C}[\mathfrak{h}] \xrightarrow{\cong} \check{\check{\mathfrak{H}}}_W^c.$$

Equivalently, the elements $\{x^\alpha c^\epsilon w y^\gamma \mid \alpha, \gamma \in \mathbb{Z}_+^n, \epsilon \in \mathbb{Z}_2^n, w \in W\}$ form a linear basis for $\check{\check{\mathfrak{H}}}_W^c$ (the PBW basis).

Proof. Recall that W acts diagonally on $V = \mathfrak{h}^* \oplus \mathfrak{h}$. The proof is similar to the proof of [Etingof and Ginzburg 2002, Theorem 1.3] with one crucial modification, as first observed in [Wang 2006].

Clearly $K := \mathcal{C}_n \rtimes \mathbb{C}W$ is a semisimple algebra. Observe that $E := V \otimes_{\mathbb{C}} K$ is a natural K -bimodule (even though V is not) with the right K -module structure on E given by right multiplication and the left K -module structure on E by letting

$$\begin{aligned} w.(v \otimes a) &= v^w \otimes wa, \\ c_i.(x_j \otimes a) &= (-1)^{\delta_{ij}} x_j \otimes (c_i a), \\ c_i.(y_j \otimes a) &= y_j \otimes (c_i a), \end{aligned}$$

where $v \in V, w \in W$ and $a \in K$.

The rest of the proof can proceed in the same way as in the proof of [Etingof and Ginzburg 2002, Theorem 1.3]. It boils down to the verifications in Lemmas 3.7, 3.8 and 3.9 below of the conjugation invariance (by c_i and W) of the defining relations (3-2), (3-3), or (3-4) for type A, D , or B , respectively, and of the Jacobi identities among the generators x_i and y_i . \square

Remark 3.6. Note that $\mathcal{C}_n \rtimes \mathbb{C}W$ is actually a subalgebra of $\check{\check{\mathfrak{H}}}_W^c$ and the tensor product in the above theorem indicates that $\check{\check{\mathfrak{H}}}_W^c$ has the structure of an algebra with triangular decomposition:

$$\check{\check{\mathfrak{H}}}_W^c \cong \mathbb{C}[\mathfrak{h}^*] \otimes (\mathcal{C}_n \rtimes \mathbb{C}W) \otimes \mathbb{C}[\mathfrak{h}].$$

We will prove Lemmas 3.7, 3.8 and 3.9 in the appendix.

Lemma 3.7. *Let $W = W_{A_{n-1}}, W_{D_n}$, or W_{B_n} . Then the relations (3-2), (3-3), or (3-4) are invariant under the conjugation by c_i for $1 \leq i \leq n$.*

Lemma 3.8. *The relations (3-2), (3-3), or (3-4) are invariant under the conjugation by elements in $W_{A_{n-1}}, W_{D_n}$, or W_{B_n} , respectively.*

Lemma 3.9. *Let $W = W_{A_{n-1}}, W_{D_n}$, or W_{B_n} . Then the Jacobi identity holds for any triple among x_i and y_i in $\check{\check{\mathfrak{H}}}_W^c$ for $1 \leq i \leq n$.*

4. The Dunkl operators for DaHCA

The Dunkl representations. The algebra $\check{\check{\mathfrak{H}}}_W^c$ is a superalgebra by letting elements of W and x_i and y_i for all i be even, and letting each c_i be odd. Recall that $\check{\check{\mathfrak{H}}}_W^c$ admits the triangular decomposition $\check{\check{\mathfrak{H}}}_W^c \cong \mathbb{C}[\mathfrak{h}^*] \otimes K \otimes \mathbb{C}[\mathfrak{h}]$, where we have written $K = \mathcal{C}_n \rtimes \mathbb{C}W$. In contrast to the usual DaHa $\check{\mathfrak{H}}_W$, the DaHCA $\check{\check{\mathfrak{H}}}_W^c$ has no automorphism that switches the subalgebras $\mathbb{C}[\mathfrak{h}]$ and $\mathbb{C}[\mathfrak{h}^*]$. Denote by \mathfrak{H}_x and \mathfrak{H}_y

the subalgebras of $\ddot{\mathfrak{H}}_W^c$ generated by K and x_1, \dots, x_n , and generated by K and y_1, \dots, y_n , respectively.

A K -module M can be extended to either an \mathfrak{H}_x -module or an \mathfrak{H}_y -module by demanding the action of the x_i and the y_i to be trivial, respectively. We define the induced $\ddot{\mathfrak{H}}_W^c$ -modules

$$M_x := \ddot{\mathfrak{H}}_W^c \otimes_{\mathfrak{H}_x} M \quad \text{and} \quad M_y := \ddot{\mathfrak{H}}_W^c \otimes_{\mathfrak{H}_y} M.$$

Below we will always use the vector space identifications

$$M_x = \mathbb{C}[y_1, \dots, y_n] \otimes M \quad \text{and} \quad M_y = \mathbb{C}[x_1, \dots, x_n] \otimes M.$$

The action of $\ddot{\mathfrak{H}}_W^c$ on M_x (respectively M_y) is transferred to $\mathbb{C}[y_1, \dots, y_n] \otimes M$ (respectively $\mathbb{C}[x_1, \dots, x_n] \otimes M$) as follows. On $\mathbb{C}[y_1, \dots, y_n] \otimes M$, K acts diagonally. More explicitly, K acts on $\mathbb{C}[x_1, \dots, x_n] \otimes M$ by

$$w.(x_j \otimes m) = x_j^w \otimes wm \quad \text{and} \quad c_i.(x_j \otimes m) = (-1)^{\delta_{ij}} x_j \otimes c_i m,$$

where $c_i \in \mathcal{C}_n$ and $w \in W$. Moreover, y_i acts by left multiplication in the first tensor factor, and the action of x_i is given by the so-called Dunkl operators (which are generalizations of those in [Dunkl 1989]). Similarly, on $\mathbb{C}[x_1, \dots, x_n] \otimes M$, x_i acts by left multiplication, and y_i acts by another version of Dunkl operators. In the remainder of this section we shall describe these Dunkl operators explicitly.

Remark 4.1. A canonical choice for a K -module is \mathcal{C}_n , whose K -module structure is defined by letting \mathcal{C}_n act by left multiplication and letting W act as usual; see Section 2.2.

The Dunkl operators for $\ddot{\mathfrak{H}}_{A_{n-1}}^c$. We first prepare a few lemmas. We shall denote the action of $\sigma \in W$ on $\mathbb{C}[\mathfrak{h}]$ and $\mathbb{C}[\mathfrak{h}^*]$ by $f \mapsto f^\sigma$.

Lemma 4.2. *Let $W = W_{A_{n-1}}$. Then the following hold in $\ddot{\mathfrak{H}}_W^c$ for $l \in \mathbb{Z}_+$ and $i \neq j$:*

$$\begin{aligned} [y_i, x_j^l] &= u \left(\frac{x_j^l - x_i^l}{x_j - x_i} + \frac{x_j^l - (-x_i)^l}{x_j + x_i} c_i c_j \right) s_{ij}, \\ [y_i, x_i^l] &= -u \sum_{k \neq i} \left(\frac{x_i^l - x_k^l}{x_i - x_k} + \frac{x_i^l - (-x_k)^l}{x_i + x_k} c_k c_i \right) s_{ki}. \end{aligned}$$

It is understood here and in similar ratios of operators below that $h/g = (1/g) \cdot h$.

Proof. This lemma is a type A counterpart of Lemma 4.8 for type B below. A proof can be simply obtained by modifying the proof of Lemma 4.8 with the removal of those terms involving \bar{s}_{ij} , \bar{s}_{ki} and τ_i . We skip the details. \square

Lemma 4.3. *Let $W = W_{A_{n-1}}$ and $f \in \mathbb{C}[x_1, \dots, x_n]$. Then the following identity holds in $\check{\check{S}}_W^\xi$:*

$$[y_i, f] = -u \sum_{k \neq i} \left(\frac{f - f^{s_{ki}}}{x_i - x_k} + \frac{f c_k c_i - c_k c_i f^{s_{ki}}}{x_i + x_k} \right) s_{ki}.$$

Proof. It suffices to check the formula for every monomial f of the form $x_1^{l_1} \cdots x_n^{l_n}$, which follows by [Lemma 4.2](#) and an induction on a based on the identity

$$[y_i, x_1^{l_1} \cdots x_a^{l_a} x_{a+1}^{l_{a+1}}] = [y_i, x_1^{l_1} \cdots x_a^{l_a}] x_{a+1}^{l_{a+1}} + x_1^{l_1} \cdots x_a^{l_a} [y_i, x_{a+1}^{l_{a+1}}]. \quad \square$$

Now we are ready to compute the Dunkl operator for the y_i .

Theorem 4.4. *Let $W = W_{A_{n-1}}$, and let M be a K -module. The action of y_i on the $\check{\check{S}}_W^\xi$ -module $\mathbb{C}[x_1, \dots, x_n] \otimes M$ is realized as a Dunkl operator as follows. For any polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$ and $m \in M$, we have*

$$y_i \circ (f \otimes m) = -u \sum_{k \neq i} \left(\frac{f - f^{s_{ki}}}{x_i - x_k} + \frac{f c_k c_i - c_k c_i f^{s_{ki}}}{x_i + x_k} \right) \otimes s_{ki} m.$$

Proof. We calculate that $y_i \circ (f \otimes m) = [y_i, f] \otimes m + f \otimes y_i m = [y_i, f] \otimes m$. Now the result follows from [Lemma 4.3](#). \square

Lemma 4.5. *Let $W = W_{A_{n-1}}$. Then the following holds in $\check{\check{S}}_W^\xi$ for $l \in \mathbb{Z}_+$ and $i \neq j$:*

$$[y_j^l, x_i] = u \frac{y_j^l - y_i^l}{y_j - y_i} (1 + c_j c_i) s_{ij} \quad \text{and} \quad [y_i^l, x_i] = -u \sum_{k \neq i} \frac{y_i^l - y_k^l}{y_i - y_k} (1 + c_k c_i) s_{ki}.$$

Proof. This lemma is a type A counterpart of [Lemma 4.11](#) for type B below, with the removal of those terms involving \bar{s}_{ij} , \bar{s}_{ki} and τ_i . We omit the details. \square

Lemma 4.6. *Let $W = W_{A_{n-1}}$, and let $f \in \mathbb{C}[y_1, \dots, y_n]$. Then the following identity holds in $\check{\check{S}}_W^\xi$:*

$$[f, x_i] = -u \sum_{k \neq i} \frac{f - f^{s_{ki}}}{y_i - y_k} (1 + c_k c_i) s_{ki}.$$

Proof. It suffices to check the formula for every monomial f , which can be done as for the formula in [Lemma 4.3](#), now using [Lemma 4.5](#) instead of [Lemma 4.2](#). \square

Now we are ready to compute the Dunkl operator for the x_i .

Theorem 4.7. *Let $W = W_{A_{n-1}}$, and let M be a K -module. The action of x_i on $\mathbb{C}[y_1, \dots, y_n] \otimes M$ is realized as a Dunkl operator as follows. For any polynomial $f \in \mathbb{C}[y_1, \dots, y_n]$ and $m \in M$, we have*

$$x_i \circ (f \otimes m) = u \sum_{k \neq i} \frac{f - f^{s_{ki}}}{y_i - y_k} \otimes (1 + c_k c_i) s_{ki} m.$$

Proof. We observe that

$$x_i \circ (f \otimes m) = [x_i, f] \otimes m + f \otimes x_i m = [x_i, f] \otimes m.$$

Now the result follows by [Lemma 4.6](#). \square

The Dunkl operators for $\check{\mathfrak{H}}_{B_n}^c$. We first prepare a few lemmas. The proofs of [Lemmas 4.8, 4.9, and 4.11](#) appear in the appendix.

Lemma 4.8. *Let $W = W_{B_n}$. Then the following holds in $\check{\mathfrak{H}}_W^c$ for $l \in \mathbb{Z}_+$ and $i \neq j$:*

$$\begin{aligned} [y_i, x_j^l] &= u \left(\frac{x_j^l - x_i^l}{x_j - x_i} + \frac{x_j^l - (-x_i)^l}{x_j + x_i} c_i c_j \right) s_{ij} - u \left(\frac{x_j^l - (-x_i)^l}{x_j + x_i} - \frac{x_j^l - x_i^l}{x_j - x_i} c_i c_j \right) \bar{s}_{ij}, \\ [y_i, x_i^l] &= -u \sum_{k \neq i} \left(\frac{x_i^l - x_k^l}{x_i - x_k} + \frac{x_i^l - (-x_k)^l}{x_i + x_k} c_k c_i \right) s_{ki} \\ &\quad - u \sum_{k \neq i} \left(\frac{x_i^l - (-x_k)^l}{x_i + x_k} - \frac{x_i^l - x_k^l}{x_i - x_k} c_k c_i \right) \bar{s}_{ki} - \sqrt{2} v \frac{x_i^l - (-x_i)^l}{2x_i} \tau_i. \end{aligned}$$

Lemma 4.9. *Let $W = W_{B_n}$, and let $f \in \mathbb{C}[x_1, \dots, x_n]$. Then the following holds in $\check{\mathfrak{H}}_W^c$:*

$$\begin{aligned} [y_i, f] &= -u \sum_{k \neq i} \left(\frac{f - f^{s_{ki}}}{x_i - x_k} + \frac{f - f^{\bar{s}_{ki}}}{x_i + x_k} c_k c_i \right) s_{ki} \\ &\quad - u \sum_{k \neq i} \left(\frac{f - f^{\bar{s}_{ki}}}{x_i + x_k} - \frac{f - f^{s_{ki}}}{x_i - x_k} c_k c_i \right) \bar{s}_{ki} - \sqrt{2} v \frac{f - f^{\tau_i}}{2x_i} \tau_i. \end{aligned}$$

Now we are ready to compute the Dunkl operator for the y_i .

Theorem 4.10. *Let $W = W_{B_n}$, and let M be a K -module. The action of y_i on $\mathbb{C}[x_1, \dots, x_n] \otimes M$ is realized as follows. For any polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$ and $m \in M$, we have*

$$\begin{aligned} y_i \circ (f \otimes m) &= -u \sum_{k \neq i} \left(\frac{f - f^{s_{ki}}}{x_i - x_k} + \frac{f - f^{\bar{s}_{ki}}}{x_i + x_k} c_k c_i \right) \otimes s_{ki} m \\ &\quad - u \sum_{k \neq i} \left(\frac{f - f^{\bar{s}_{ki}}}{x_i + x_k} - \frac{f - f^{s_{ki}}}{x_i - x_k} c_k c_i \right) \otimes \bar{s}_{ki} m - \sqrt{2} v \frac{f - f^{\tau_i}}{2x_i} \otimes \tau_i m. \end{aligned}$$

Proof. We observe that

$$y_i \circ (f \otimes m) = [y_i, f] \otimes m + f \otimes y_i m = [y_i, f] \otimes m.$$

Now the result follows from [Lemma 4.9](#). \square

Lemma 4.11. *Let $W = W_{B_n}$. Then the following hold in $\check{\check{\mathfrak{H}}}_W^c$ for $l \in \mathbb{Z}_+$ and $i \neq j$:*

$$\begin{aligned} [y_j^l, x_i] &= u \left(\frac{y_j^l - y_i^l}{y_j - y_i} (1 + c_j c_i) s_{ij} - \frac{y_j^l - (-y_i)^l}{y_j + y_i} (1 - c_j c_i) \bar{s}_{ij} \right). \\ [y_i^l, x_i] &= -u \sum_{k \neq i} \frac{y_i^l - y_k^l}{y_i - y_k} (1 + c_k c_i) s_{ki} - u \sum_{k \neq i} \frac{y_i^l - (-y_k)^l}{y_i + y_k} (1 - c_k c_i) \bar{s}_{ki} \\ &\quad - \sqrt{2} v \frac{y_i^l - (-y_i)^l}{2y_i} \tau_i. \end{aligned}$$

In a similar fashion, we can derive the next lemma from [Lemma 4.11](#).

Lemma 4.12. *Let $W = W_{B_n}$, and let $f \in \mathbb{C}[y_1, \dots, y_n]$. Then the following identity holds in $\check{\check{\mathfrak{H}}}_W^c$:*

$$\begin{aligned} [f, x_i] &= -u \sum_{k \neq i} \frac{f - f^{s_{ki}}}{y_i - y_k} (1 + c_k c_i) s_{ki} - u \sum_{k \neq i} \frac{f - f^{\bar{s}_{ki}}}{y_i + y_k} (1 - c_k c_i) \bar{s}_{ki} \\ &\quad - \sqrt{2} v \frac{f - f^{\tau_i}}{2y_i} \tau_i. \end{aligned}$$

Now we are ready to compute the Dunkl operator for the x_i .

Theorem 4.13. *Let $W = W_{B_n}$. The action of x_i on $\mathbb{C}[y_1, \dots, y_n] \otimes M$ is realized as follows. For any polynomial $f \in \mathbb{C}[y_1, \dots, y_n]$ and $m \in M$, we have*

$$\begin{aligned} x_i \circ (f \otimes m) &= u \sum_{k \neq i} \frac{f - f^{s_{ki}}}{y_i - y_k} \otimes (1 + c_k c_i) s_{ki} m + u \sum_{k \neq i} \frac{f - f^{\bar{s}_{ki}}}{y_i + y_k} \otimes (1 - c_k c_i) \bar{s}_{ki} m \\ &\quad + \sqrt{2} v \frac{f - f^{\tau_i}}{2y_i} \otimes \tau_i m. \end{aligned}$$

Proof. We observe that $x_i \circ (f \otimes m) = [x_i, f] \otimes m + f \otimes x_i m = [x_i, f] \otimes m$. Now the result follows from [Lemma 4.12](#). \square

The Dunkl operators for $\check{\check{\mathfrak{H}}}_{D_n}^c$. Below, the actions of the x_i and the y_i are realized as Dunkl operators. Due to the similarity of the bracket relations $[\cdot, \cdot]$ in $\check{\check{\mathfrak{H}}}_{D_n}^c$ and $\check{\check{\mathfrak{H}}}_{B_n}^c$ (for instance, compare the type D relation [\(3-3b\)](#) with the type B relation [\(3-4b\)](#)), the formulas below for type D_n are obtained from their type B_n counterparts in the previous subsection by dropping the terms involving the parameter v . The proofs are the same as for the type B , and thus will be skipped.

Lemma 4.14. *Let $W = W_{D_n}$, and $f \in \mathbb{C}[x_1, \dots, x_n]$. Then the following holds in $\check{\check{\mathfrak{H}}}_W^c$:*

$$[y_i, f] = -u \sum_{k \neq i} \left(\frac{f - f^{s_{ki}}}{x_i - x_k} + \frac{f - f^{\bar{s}_{ki}}}{x_i + x_k} c_k c_i \right) s_{ki} - u \sum_{k \neq i} \left(\frac{f - f^{\bar{s}_{ki}}}{x_i + x_k} - \frac{f - f^{s_{ki}}}{x_i - x_k} c_k c_i \right) \bar{s}_{ki}.$$

Lemma 4.15. *Let $W = W_{D_n}$, and $f \in \mathbb{C}[y_1, \dots, y_n]$. Then the following identity holds in $\check{\check{\mathfrak{H}}}_W^{\mathfrak{c}}$:*

$$[f, x_i] = -u \sum_{k \neq i} \frac{f - f^{s_{ki}}}{y_i - y_k} (1 + c_k c_i) s_{ki} - u \sum_{k \neq i} \frac{f - f^{\bar{s}_{ki}}}{y_i + y_k} (1 - c_k c_i) \bar{s}_{ki}.$$

Theorem 4.16. *Let $W = W_{D_n}$, and let M be a K -module. The action of y_i on $\mathbb{C}[x_1, \dots, x_n] \otimes M$ is realized as a Dunkl operator as follows. For any polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$ and $m \in M$, we have*

$$\begin{aligned} y_i \circ (f \otimes m) &= -u \sum_{k \neq i} \left(\frac{f - f^{s_{ki}}}{x_i - x_k} + \frac{f - f^{\bar{s}_{ki}}}{x_i + x_k} c_k c_i \right) \otimes s_{ki} m \\ &\quad - u \sum_{k \neq i} \left(\frac{f - f^{\bar{s}_{ki}}}{x_i + x_k} - \frac{f - f^{s_{ki}}}{x_i - x_k} c_k c_i \right) \otimes \bar{s}_{ki} m. \end{aligned}$$

Theorem 4.17. *Let $W = W_{D_n}$, and let M be a K -module. The action of x_i on $\mathbb{C}[y_1, \dots, y_n] \otimes M$ is realized as follows. For any $f \in \mathbb{C}[y_1, \dots, y_n]$ and $m \in M$,*

$$x_i \circ (f \otimes m) = u \sum_{k \neq i} \frac{f - f^{s_{ki}}}{y_i - y_k} \otimes (1 + c_k c_i) s_{ki} m + u \sum_{k \neq i} \frac{f - f^{\bar{s}_{ki}}}{y_i + y_k} \otimes (1 - c_k c_i) \bar{s}_{ki} m.$$

The even center for $\check{\check{\mathfrak{H}}}_W^{\mathfrak{c}}$. Recall that the even center $\mathfrak{Z}(A)$ of a superalgebra A consists of the even central elements of A . It turns out that the algebra $\check{\check{\mathfrak{H}}}_W^{\mathfrak{c}}$ has a large center.

Proposition 4.18. *Let W be $W_{A_{n-1}}$, W_{D_n} or W_{B_n} . The even center $\mathfrak{Z}(\check{\check{\mathfrak{H}}}_W^{\mathfrak{c}})$ contains $\mathbb{C}[y_1, \dots, y_n]^W$ and $\mathbb{C}[x_1^2, \dots, x_n^2]^W$ as subalgebras. In particular, $\check{\check{\mathfrak{H}}}_W^{\mathfrak{c}}$ is module-finite over its even center.*

Proof. Let $f \in \mathbb{C}[y_1, \dots, y_n]^W$. Then by the definition of $\check{\check{\mathfrak{H}}}_W^{\mathfrak{c}}$, f commutes with \mathcal{C}_n , W , and y_i for all $1 \leq i \leq n$. Since $f = f^w$ for all $w \in W$, it follows by Lemmas 4.6, 4.12 or 4.15 (for type A , D or B , respectively) that $[f, x_i] = 0$ for each i . Hence f commutes with \mathcal{C}_n , W , and $\mathbb{C}[x_1, \dots, x_n]$. Therefore f is in the even center $\mathfrak{Z}(\check{\check{\mathfrak{H}}}_W^{\mathfrak{c}})$.

Suppose now that $f \in \mathbb{C}[x_1^2, \dots, x_n^2]^W$. Then by the definition of $\check{\check{\mathfrak{H}}}_W^{\mathfrak{c}}$, f commutes with \mathcal{C}_n , W , and x_i for all $1 \leq i \leq n$. By Lemma 4.3, 4.9 or 4.14 (for type A , D or B respectively), we have $[y_i, f] = 0$ for each i . Therefore f is in the even center.

The module-finiteness over the even center now follows from the PBW property of $\check{\check{\mathfrak{H}}}_W^{\mathfrak{c}}$ (see Theorem 3.5). \square

5. Rational spin double affine Hecke algebras (sDaHa)

In this section, we introduce the rational spin double affine Hecke algebras associated to the Weyl group W of type A_{n-1} , D_n and B_n , and then establish their PBW property.

Elements in $\mathbb{C}W^-$ of order 2. Recall that the spin group algebra $\mathbb{C}W^-$ has a presentation with generator t_i given in Section 2. Introduce the notations

$$t_{i\uparrow j} = \begin{cases} t_i t_{i+1} \cdots t_j & \text{if } i \leq j, \\ 1 & \text{otherwise,} \end{cases} \quad t_{i\downarrow j} = \begin{cases} t_i t_{i-1} \cdots t_j & \text{if } i \geq j, \\ 1 & \text{otherwise.} \end{cases}$$

Define the following odd elements in $\mathbb{C}W^-$ of order 2, which are analogs of reflections in W , for $1 \leq i < j \leq n$:

$$\begin{aligned} t_{ij} &\equiv [i, j] = (-1)^{j-i-1} t_{j-1} \cdots t_{i+1} t_i t_{i+1} \cdots t_{j-1} \\ t_{ji} &\equiv [j, i] = -[i, j] \\ \bar{t}_{ij} &\equiv \overline{[i, j]} = \begin{cases} (-1)^{j-i-1} t_{j\uparrow n-1} t_{i\uparrow n-2} t_n t_{n-2\downarrow i} t_{n-1\downarrow j} & \text{for type } D_n, \\ (-1)^{j-i} t_{j\uparrow n-1} t_{i\uparrow n-2} t_n t_{n-1} t_n t_{n-2\downarrow i} t_{n-1\downarrow j} & \text{for type } B_n \end{cases} \\ \bar{t}_{ji} &\equiv \overline{[j, i]} = \overline{[i, j]} \\ \bar{t}_i &\equiv \overline{[i]} = (-1)^{n-i} t_i \cdots t_{n-1} t_n t_{n-1} \cdots t_i \quad \text{for } 1 \leq i \leq n. \end{aligned}$$

Note the natural inclusions of algebras $\mathbb{C}W_{A_{n-1}}^- \leq \mathbb{C}W_{D_n}^- \leq \mathbb{C}W_{B_n}^-$. In particular, t_1, \dots, t_{n-1} and $t_n t_{n-1} t_n$ generate a subalgebra of $\mathbb{C}W_{B_n}^-$ that is isomorphic to $\mathbb{C}W_{D_n}^-$ (where $-t_n t_{n-1} t_n$ corresponds to the n -th generator for $\mathbb{C}W_{D_n}^-$). Hence, the notations $[i, j]$ and $\overline{[i, j]}$ here are consistent with such a subalgebra structure. Although we will not use it in this paper, we can show for $i < j$ that $\overline{[i, j]} = [j, n][i, n-1]t_n[i, n-1][j, n]$.

The algebra $\check{\mathfrak{H}}_{\bar{W}}^-$ of type A_{n-1} . The following algebra, $\check{\mathfrak{H}}_{A_{n-1}}^-$, was introduced in [Wang 2006]. We recall the definition here for convenience in the subsequent subsections.

Definition 5.1. Let $u \in \mathbb{C}$, and let $W = W_{A_{n-1}}$. The rational spin double affine Hecke algebra of type A_{n-1} , denoted by $\check{\mathfrak{H}}_{\bar{W}}^-$ or $\check{\mathfrak{H}}_{A_{n-1}}^-$, is the algebra generated by ξ_i and y_i for $1 \leq i \leq n$ and $\mathbb{C}W^-$, subject to the relations

$$(5-1) \quad \begin{cases} y_i y_j = y_j y_i, & \xi_i \xi_j = -\xi_j \xi_i & \text{for } i \neq j, \\ t_i y_i = y_{i+1} t_i, & t_i \xi_i = -\xi_{i+1} t_i \\ t_i y_j = y_j t_i, & t_i \xi_j = -\xi_j t_i & \text{for } j \neq i, i+1, \end{cases}$$

$$\begin{aligned} [y_j, \xi_i] &= -u[i, j] \quad \text{for } i \neq j, \\ [y_i, \xi_i] &= u \sum_{k \neq i} [i, k]. \end{aligned}$$

The algebra $\check{\check{\mathfrak{H}}}_W^-$ of type D_n .

Definition 5.2. Let $u \in \mathbb{C}$, and let $W = W_{D_n}$. The rational spin double affine Hecke algebra of type D_n , denoted by $\check{\check{\mathfrak{H}}}_W^-$ or $\check{\check{\mathfrak{H}}}_{D_n}^-$, is the algebra generated by ξ_i, y_i for $1 \leq i \leq n$ and $\mathbb{C}W^-$, subject to (5-1) and the additional relations

$$\begin{aligned} t_n y_n &= -y_{n-1} t_n, & t_n \xi_n &= -\xi_{n-1} t_n, \\ t_n y_j &= y_j t_n, & t_n \xi_j &= -\xi_j t_n, \quad \text{for } j \neq n-1, n, \\ [y_j, \xi_i] &= -u[i, j] + u[\overline{i}, j] & \text{for } i \neq j, \\ [y_i, \xi_i] &= u \sum_{k \neq i} ([i, k] + [\overline{i}, k]). \end{aligned}$$

The algebra $\check{\check{\mathfrak{H}}}_W^-$ of type B_n .

Definition 5.3. Let $u, v \in \mathbb{C}$, and let $W = W_{B_n}$. The rational spin double affine Hecke algebra of type B_n , denoted by $\check{\check{\mathfrak{H}}}_W^-$ or $\check{\check{\mathfrak{H}}}_{B_n}^-$, is the algebra generated by ξ_i and y_i for $1 \leq i \leq n$ and $\mathbb{C}W_{B_n}^-$, subject to (5-1) and the additional relations

$$\begin{aligned} t_n y_n &= -y_n t_n, & t_n \xi_n &= -\xi_n t_n \\ t_n y_j &= y_j t_n, & t_n \xi_j &= -\xi_j t_n \quad \text{for } j \neq n, \\ [y_j, \xi_i] &= -u[i, j] + u[\overline{i}, j] & \text{for } i \neq j, \\ [y_i, \xi_i] &= u \sum_{k \neq i} ([i, k] + [\overline{i}, k]) + v[\overline{i}]. \end{aligned}$$

If necessary, we write $\check{\check{\mathfrak{H}}}_W^-(u, v)$ for $\check{\check{\mathfrak{H}}}_W^-$ to indicate the dependence on u and v .

Isomorphism of superalgebras. The algebra $\check{\check{\mathfrak{H}}}_W^-$ contains several distinguished subalgebras: the skew-polynomial algebra $\mathbb{C}[\xi_1, \dots, \xi_n]$, the spin Weyl group algebra $\mathbb{C}W^-$, and the polynomial algebra $\mathbb{C}[y_1, \dots, y_n]$. The algebra $\check{\check{\mathfrak{H}}}_W^-$ has a superalgebra structure with y_i even and ξ_i and t_i odd for all i .

Lemma 5.4. Let W be one of the Weyl groups $W_{A_{n-1}}, W_{D_n}$ or W_{B_n} . The map $\Phi : \mathbb{C}_n \rtimes \mathbb{C}W \rightarrow \mathbb{C}_n \otimes \mathbb{C}W^-$ (which, by [Theorem 2.1](#), is an isomorphism) sends

$$(5-2) \quad (c_k - c_i) s_{ik} \mapsto -\sqrt{-2} [k, i],$$

$$(5-3) \quad (c_k + c_i) \bar{s}_{ik} \mapsto -\sqrt{-2} [\overline{k}, \overline{i}],$$

$$(5-4) \quad c_i \tau_i \mapsto -\sqrt{-1} [\overline{i}]$$

for $i \neq k$, whenever it is applicable.

Proof. We may assume that $i > k$ without loss of generality.

We prove (5-2) by induction on i . First, (5-2) holds for $i = k+1$ by [Theorem 2.1](#). Assuming that (5-2) holds for i , that is, $\Phi((c_k - c_i) s_{ik}) = -\sqrt{-2} [k, i]$, we have

by [Theorem 2.1](#) and the definition of $[k, i]$ that

$$\begin{aligned}\Phi((c_k - c_{i+1})s_{i+1,k}) &= \Phi(s_i(c_k - c_i)s_{ik}s_i) \\ &= (-\sqrt{-1}\beta_i t_i)(-\sqrt{-2}[k, i])(-\sqrt{-1}\beta_i t_i) \\ &= \sqrt{-2}t_i[k, i]t_i = -\sqrt{-2}[k, i+1].\end{aligned}$$

We now prove (5-4) by a similar downward induction on i , whose initial case $i = n$ is taken care of by [Theorem 2.1](#). Assume that (5-4) holds for $i+1 \leq n$, that is, $\Phi(c_{i+1}\tau_{i+1}) = -\sqrt{-1}[i+1]$. Then, by [Theorem 2.1](#) and the definition of $[i]$,

$$\begin{aligned}\Phi(c_i\tau_i) &= \Phi(s_i c_{i+1} \tau_{i+1} s_i) \\ &= (-\sqrt{-1}\beta_i t_i)(-\sqrt{-1}[i+1])(-\sqrt{-1}\beta_i t_i) \\ &= \sqrt{-1}t_i[i+1]t_i = -\sqrt{-1}[i].\end{aligned}$$

Next, we prove (5-3) by downward induction, first on k and then on i , for $W = W_{D_n}$. The initial case $i = n$ and $k = n-1$ holds by [Theorem 2.1](#). Then, it follows by the induction assumption that $\Phi((c_{k+1} + c_n)\bar{s}_{n,k+1}) = -\sqrt{-2}[k+1, n]$; it follows by [Theorem 2.1](#) and the definition of $[k, n]$ that

$$\begin{aligned}\Phi((c_k + c_n)\bar{s}_{nk}) &= \Phi(s_k(c_{k+1} + c_n)\bar{s}_{n,k+1}s_k) \\ &= (-\sqrt{-1}\beta_k t_k) \cdot (-\sqrt{-2}[k+1, n]) \cdot (-\sqrt{-1}\beta_k t_k) \\ &= \sqrt{-2}t_k[k+1, n]t_k = -\sqrt{-2}[k, n].\end{aligned}$$

This in turn becomes the initial step when $i = n$ for proving (5-3) by downward induction on i (with fixed $k < n$). By the induction assumption, (5-3) holds for $i > k+1$. Then

$$\begin{aligned}\Phi((c_k + c_{i-1})\bar{s}_{i-1,k}) &= \Phi(s_{i-1}(c_k + c_i)\bar{s}_{ik}s_{i-1}) \\ &= (-\sqrt{-1}\beta_{i-1}t_{i-1})(-\sqrt{-2}[k, i])(-\sqrt{-1}\beta_{i-1}t_{i-1}) \\ &= \sqrt{-2}t_{i-1}[k, i]t_{i-1} = -\sqrt{-2}[k, i-1].\end{aligned}$$

This completes the proof of (5-3) for type D .

The formula (5-3) for $W = W_{B_n}$ is similarly proved by double downward inductions on k and then on i . The only difference from the type D case is that for type B we have to check the initial case when $k = n-1$ and $i = n$, which uses (5-2) and (5-4):

$$\begin{aligned}\Phi((c_{n-1} + c_n)\bar{s}_{n-1,n}) &= \Phi(\tau_n(c_{k+1} - c_n)s_{n-1,n}\tau_n) \\ &= (-\sqrt{-1}c_n t_n) \cdot (-\sqrt{-2}t_{n-1}) \cdot (-\sqrt{-1}c_n t_n) \\ &= \sqrt{-2}t_n t_{n-1} t_n = -\sqrt{-2}[n-1, n].\end{aligned}$$

Thus the lemma is proved. □

Recall the isomorphism $\Phi : \mathcal{C}_n \rtimes \mathbb{C}W \rightarrow \mathcal{C}_n \otimes \mathbb{C}W^-$ of superalgebras and its inverse Ψ given in [Theorem 2.1](#).

Theorem 5.5. *Let W be one of the Weyl groups $W_{A_{n-1}}$, W_{D_n} or W_{B_n} .*

(1) *There exists an isomorphism*

$$\Phi : \check{\mathfrak{H}}_W^c \longrightarrow \mathcal{C}_n \otimes \check{\mathfrak{H}}_W^-$$

of superalgebras that extends $\Phi : \mathcal{C}_n \rtimes \mathbb{C}W \rightarrow \mathcal{C}_n \otimes \mathbb{C}W^-$ and sends

$$y_i \mapsto y_i, \quad x_i \mapsto \sqrt{-2}c_i\xi_i, \quad s_i \mapsto -\sqrt{-1}\beta_i t_i, \quad c_i \mapsto c_i \quad \text{for all } i.$$

(2) *The inverse*

$$\Psi : \mathcal{C}_n \otimes \check{\mathfrak{H}}_W^- \longrightarrow \check{\mathfrak{H}}_W^c$$

extends $\Psi : \mathcal{C}_n \otimes \mathbb{C}W^- \rightarrow \mathcal{C}_n \rtimes \mathbb{C}W$ and sends

$$y_i \mapsto y_i, \quad \xi_i \mapsto \frac{c_i x_i}{\sqrt{-2}}, \quad t_i \mapsto \sqrt{-1}\beta_i s_i, \quad c_i \mapsto c_i \quad \text{for all } i.$$

In the terminology of [[Wang 2007](#)], $\check{\mathfrak{H}}_W^c$ and $\check{\mathfrak{H}}_W^-$ are Morita superequivalent by [Theorem 5.5](#).

Proof. Recall that Φ extends the isomorphism $\mathcal{C}_n \rtimes \mathbb{C}W \xrightarrow{\cong} \mathcal{C}_n \otimes \mathbb{C}W^-$. Among all the relations (3-1)–(3-4) for $\check{\mathfrak{H}}_W^c$, it is easy to check that those in (3-1) are preserved by Φ . So it remains to check that Φ preserves the relations in (3-2), (3-3), (3-4) for $W = W_{A_{n-1}}$, W_{D_n} , and W_{B_n} , respectively.

We shall verify in detail that Φ preserves (3-4) with $W = W_{B_n}$. Indeed, by [Lemma 5.4](#), we have for $i \neq j$ that

$$\begin{aligned} \Phi(\text{left side of (3-4a)}) &= \sqrt{-2}[y_j, c_i \xi_i] \\ &= \sqrt{-2}c_i(-u[i, j] + u[\overline{i}, j]) \\ &= \Phi(u((1 + c_j c_i)s_{ji} - (1 - c_j c_i)\bar{s}_{ij})) \\ &= \Phi(\text{right side of (3-4a)}). \end{aligned}$$

Also, by [Lemma 5.4](#), we have

$$\begin{aligned} \Phi(\text{left side of (3-4b)}) &= \sqrt{-2}[y_i, c_i \xi_i] \\ &= \sqrt{-2}uc_i \sum_{k \neq i} ([i, k] + [\overline{i}, k]) + \sqrt{-2}vc_i[\overline{i}] \\ &= \Phi(-u \sum_{k \neq i} ((1 + c_k c_i)s_{ki} + (1 - c_k c_i)\bar{s}_{ki}) - \sqrt{2}v\tau_i) \\ &= \Phi(\text{right side of (3-4b)}). \end{aligned}$$

By dropping the terms involving v in the above equations, we verify that the relations (3-3) with $W = W_{D_n}$ are preserved by Φ . By further dropping the terms involving $[\overline{ij}]$, \bar{s}_{ij} and so on, we can also verify (3-2) with $W = W_{A_{n-1}}$.

So, the homomorphism Φ is well defined. Similarly, one shows that Ψ is a well-defined algebra homomorphism. For example, the relation $t_n \xi_n = -\xi_{n-1} t_n$ in $\check{\check{\mathfrak{H}}}_W^-$ for $W = W_{D_n}$ is preserved by Ψ , since

$$\begin{aligned} \Psi(t_n \xi_n) &= \frac{\sqrt{-1}}{\sqrt{2}}(c_{n-1} + c_n) s_n \frac{1}{\sqrt{-2}} c_n x_n \\ &= \frac{1}{2}(c_{n-1} + c_n) c_{n-1} x_{n-1} s_n \\ &= \frac{1}{2} c_{n-1} x_{n-1} (-c_{n-1} - c_n) s_n = -\Psi(\xi_{n-1} t_n). \end{aligned}$$

On the other hand, the relation $t_n \xi_n = -\xi_n t_n$ in $\check{\check{\mathfrak{H}}}_W^-$ for $W = W_{B_n}$ is preserved by Ψ , since

$$\Psi(t_n \xi_n) = \sqrt{-1} c_n s_n \frac{1}{\sqrt{-2}} c_n x_n = \frac{1}{\sqrt{2}} x_n s_n = -\frac{1}{\sqrt{2}} c_n x_n c_n s_n = -\Psi(\xi_n t_n).$$

Since Φ and Ψ are inverses on generators, they are (inverse) algebra isomorphisms. \square

The PBW property for $\check{\check{\mathfrak{H}}}_W^-$. We have the following PBW type property for the algebra $\check{\check{\mathfrak{H}}}_W^-$.

Theorem 5.6. *Let W be one of the Weyl groups $W_{A_{n-1}}$, W_{D_n} or W_{B_n} . The multiplication of the subalgebras induces an isomorphism of vector spaces*

$$\mathbb{C}[\xi_1, \dots, \xi_n] \otimes \mathbb{C}W^- \otimes \mathbb{C}[y_1, \dots, y_n] \rightarrow \check{\check{\mathfrak{H}}}_W^-.$$

Equivalently, the set $\{\xi^\alpha \sigma y^\gamma\}$ forms a basis for $\check{\check{\mathfrak{H}}}_W^-$, where σ runs over a basis for $\mathbb{C}W^-$, and $\alpha, \gamma \in \mathbb{Z}_+^n$.

Proof. It follows from the defining relations for $\check{\check{\mathfrak{H}}}_W^-$ that $\check{\check{\mathfrak{H}}}_W^-$ is spanned by the elements $\xi^\alpha \sigma y^\gamma$, where σ runs over a basis for $\mathbb{C}W^-$, and $\alpha, \gamma \in \mathbb{Z}_+^n$. By the isomorphism $\Psi : \mathbb{C}_n \otimes \check{\check{\mathfrak{H}}}_W^- \rightarrow \check{\check{\mathfrak{H}}}_W^c$ in [Theorem 5.5](#), we see that the images $\Psi(\xi^\alpha \sigma y^\gamma)$ are linearly independent in $\check{\check{\mathfrak{H}}}_W^c$ by the PBW property for $\check{\check{\mathfrak{H}}}_W^c$ (see [Theorem 3.5](#)). So the elements $\xi^\alpha \sigma y^\gamma$ are linearly independent in $\check{\check{\mathfrak{H}}}_W^-$.

Therefore, the set $\{\xi^\alpha \sigma y^\gamma\}$ forms a basis for $\check{\check{\mathfrak{H}}}_W^-$. \square

The tensor product in the above theorem gives a triangular decomposition of the algebra $\check{\check{\mathfrak{H}}}_W^-$.

6. The Dunkl operators for sDaHa

Denote by \mathfrak{h}_ξ the subalgebra of $\check{\check{\mathfrak{H}}}_W^-$ generated by the ξ_i for $1 \leq i \leq n$ and $\mathbb{C}W^-$. For a $\mathbb{C}W^-$ -module V , it can be extended to a \mathfrak{h}_ξ -modules by letting the actions of the ξ_i on V be trivial. We define the induced $\check{\check{\mathfrak{H}}}_W^-$ -module

$$V_\xi := \check{\check{\mathfrak{H}}}_W^- \otimes_{\mathfrak{h}_\xi} V \cong C[y_1, \dots, y_n] \otimes V.$$

We will always identify $V_\xi = \mathbb{C}[y_1, \dots, y_n] \otimes V$. On $\mathbb{C}[y_1, \dots, y_n] \otimes V$, the element $t_i \in \mathbb{C}W^-$ acts as $s_i \otimes t_i$, the element y_i acts by left multiplication, and ξ_i acts as a Dunkl operator, which we will describe in this section.

Under [Lemma 5.4](#) and the superalgebra isomorphism $\Phi : \check{\mathfrak{H}}_W^c \rightarrow \mathcal{C}_n \otimes \check{\mathfrak{H}}_W^-$ in [Theorem 5.5](#), the results in this section are fairly straightforward counterparts of those in [Section 4](#), and we omit the proofs.

The Dunkl operator for $\check{\mathfrak{H}}_{A_{n-1}}^-$. The next lemma is the counterpart of [Lemma 4.6](#).

Lemma 6.1. *Let $W = W_{A_{n-1}}$, and $f \in \mathbb{C}[y_1, \dots, y_n]$. Then the following identity holds in $\check{\mathfrak{H}}_W^-$:*

$$[f, \xi_i] = -u \sum_{k \neq i} \frac{f - f^{s_{ki}}}{y_i - y_k} [k, i].$$

The following is the counterpart of [Theorem 4.7](#).

Proposition 6.2. *Let $W = W_{A_{n-1}}$, and let V be a $\mathbb{C}W^-$ -module. The action of ξ_i on $\mathbb{C}[y_1, \dots, y_n] \otimes V$ is realized as a Dunkl operator as follows. For any polynomial $f \in \mathbb{C}[y_1, \dots, y_n]$ and $v \in V$, we have*

$$\xi_i \circ (f \otimes v) = u \sum_{k \neq i} \frac{f - f^{s_{ki}}}{y_i - y_k} \otimes [k, i]v.$$

The Dunkl operator for $\check{\mathfrak{H}}_{B_n}^-$. This lemma is the counterpart of [Lemma 4.11](#):

Lemma 6.3. *Let $W = W_{B_n}$ and $l \in \mathbb{Z}_+$. Then*

$$[y_j^l, \xi_i] = u \frac{y_j^l - y_i^l}{y_j - y_i} [j, i] + u \frac{y_j^l - (-y_i)^l}{y_j + y_i} [\overline{j}, i].$$

$$[y_i^l, \xi_i] = -u \sum_{k \neq i} \frac{y_i^l - y_k^l}{y_i - y_k} [k, i] + u \sum_{k \neq i} \frac{y_i^l - (-y_k)^l}{y_i + y_k} [\overline{k}, i] + v \frac{y_i^l - (-y_i)^l}{2y_i} [\overline{i}].$$

This lemma is the counterpart of [Lemma 4.12](#):

Lemma 6.4. *Let $W = W_{B_n}$. The following identity holds in $\check{\mathfrak{H}}_W^-$:*

$$[f, \xi] = -u \sum_{k \neq i} \frac{f - f^{s_{ki}}}{y_i - y_k} [k, i] + u \sum_{k \neq i} \frac{f - f^{\bar{s}_{ki}}}{y_i + y_k} [\overline{k}, i] + v \frac{f - f^{\tau_i}}{2y_i} [\overline{i}].$$

This proposition is the counterpart of [Theorem 4.13](#):

Proposition 6.5. *Let $W = W_{B_n}$, and let V be a $\mathbb{C}W^-$ -module. The action of ξ_i on $\mathbb{C}[y_1, \dots, y_n] \otimes V$ is realized as a Dunkl operator as follows. For any polynomial $f \in \mathbb{C}[y_1, \dots, y_n]$ and $m \in V$, we have*

$$\xi_i \circ (f \otimes m) = u \sum_{k \neq i} \frac{f - f^{s_{ki}}}{y_i - y_k} \otimes [k, i]m - u \sum_{k \neq i} \frac{f - f^{\bar{s}_{ki}}}{y_i + y_k} \otimes [\overline{k}, i]m - v \frac{f - f^{\tau_i}}{2y_i} \otimes [\overline{i}]m.$$

The Dunkl operator for $\ddot{\mathfrak{H}}_{D_n}^-$.

Proposition 6.6. *Let $W = W_{D_n}$, and let V be a $\mathbb{C}W^-$ -module. The action of ξ_i on $\mathbb{C}[y_1, \dots, y_n] \otimes V$ is realized as a Dunkl operator as follows. For any polynomial $f \in \mathbb{C}[y_1, \dots, y_n]$ and $v \in V$,*

$$\xi_i \circ (f \otimes v) = u \sum_{k \neq i} \frac{f - f^{\delta_{ki}}}{y_i - y_k} \otimes [k, i]v - u \sum_{k \neq i} \frac{f - f^{\bar{\delta}_{ki}}}{y_i + y_k} \otimes \overline{[k, i]}v.$$

The even center for $\ddot{\mathfrak{H}}_{\bar{W}}^-$.

Proposition 6.7. *Let W be one of the Weyl groups $W_{A_{n-1}}$, W_{D_n} or W_{B_n} . The even center for $\ddot{\mathfrak{H}}_{\bar{W}}^-$ contains $\mathbb{C}[y_1, \dots, y_n]^W$ and $\mathbb{C}[\xi_1^2, \dots, \xi_n^2]^W$. In particular, $\ddot{\mathfrak{H}}_{\bar{W}}^-$ is module-finite over its even center.*

Proof.

By the isomorphism $\Phi : \ddot{\mathfrak{H}}_W^c \rightarrow \mathcal{C}_n \otimes \ddot{\mathfrak{H}}_{\bar{W}}^-$ (see [Theorem 5.5](#)) and [Proposition 4.18](#), we have

$$\begin{aligned} \mathbb{C}[y_1, \dots, y_n]^W &\subseteq \Phi(Z(\ddot{\mathfrak{H}}_W^c)) = Z(\mathcal{C}_n \otimes \ddot{\mathfrak{H}}_{\bar{W}}^-), \\ \mathbb{C}[\xi_1^2, \dots, \xi_n^2]^W &\subseteq \Phi(Z(\ddot{\mathfrak{H}}_W^c)) = Z(\mathcal{C}_n \otimes \ddot{\mathfrak{H}}_{\bar{W}}^-). \end{aligned}$$

The first statement follows by noting that $\mathbb{C}[y_1, \dots, y_n]^W$ and $\mathbb{C}[\xi_1^2, \dots, \xi_n^2]^W$ actually lie in $\ddot{\mathfrak{H}}_{\bar{W}}^-$. The second statement now follows from the PBW property of $\ddot{\mathfrak{H}}_{\bar{W}}^-$ (see [Theorem 5.6](#)). □

7. Rational covering double affine Hecke algebras (cDaHa)

This section introduces the rational covering double affine Hecke algebras (cDaHa) $\ddot{\mathfrak{H}}_{\bar{W}}^-$ associated to classical Weyl groups W . It has as its natural quotients the usual rational DaHa $\check{\mathfrak{H}}_W$ [[Etingof and Ginzburg 2002](#)] (which will be recalled below) and the rational sDaHa $\check{\mathfrak{H}}_{\bar{W}}^-$ introduced in [Section 5](#).

“Reflections” in \widetilde{W} . Recall from [Section 2.1](#) the distinguished double cover \widetilde{W} of a Weyl group W with generators \tilde{t}_i .

Introduce the notations

$$\tilde{t}_{i \uparrow j} = \begin{cases} \tilde{t}_i \tilde{t}_{i+1} \cdots \tilde{t}_j & \text{if } i \leq j, \\ 1 & \text{otherwise,} \end{cases} \quad \tilde{t}_{i \downarrow j} = \begin{cases} \tilde{t}_i \tilde{t}_{i-1} \cdots \tilde{t}_j & \text{if } i \geq j, \\ 1 & \text{otherwise.} \end{cases}$$

Define the following elements in \widetilde{W} , which are distinguished preimages of reflections in W under the canonical map $\widetilde{W} \rightarrow W$ for $1 \leq i < j \leq n$:

$$\begin{aligned} \{i, j\} &= z^{j-i-1} \tilde{t}_{j-1} \dots \tilde{t}_{i+1} \tilde{t}_i \tilde{t}_{i+1} \dots \tilde{t}_{j-1}, \\ \{j, i\} &= z\{i, j\}, \\ \overline{\{i, j\}} &= \begin{cases} z^{j-i-1} \tilde{t}_{j \uparrow n-1} \tilde{t}_{i \uparrow n-2} \tilde{t}_n \tilde{t}_{n-2} \downarrow_i \tilde{t}_{n-1} \downarrow_j & \text{for type } D_n, \\ z^{j-i} \tilde{t}_{j \uparrow n-1} \tilde{t}_{i \uparrow n-2} \tilde{t}_n \tilde{t}_{n-1} \tilde{t}_n \tilde{t}_{n-2} \downarrow_i \tilde{t}_{n-1} \downarrow_j & \text{for type } B_n, \end{cases} \\ \overline{\{j, i\}} &= \overline{\{i, j\}}, \\ \{i\} &= z^{n-i} \tilde{t}_i \dots \tilde{t}_{n-1} \tilde{t}_n \tilde{t}_{n-1} \dots \tilde{t}_i \quad \text{for } 1 \leq i \leq n. \end{aligned}$$

We have $\{i, j\} \in \widetilde{W}_{A_{n-1}}$ and $\overline{\{i, j\}} \in \widetilde{W}_{D_n}$ for $1 \leq i < j \leq n$, and $\{i\} \in \widetilde{W}_{B_n}$ for $1 \leq i \leq n$. We have a sequence of subgroups $\widetilde{W}_{A_{n-1}} \leq \widetilde{W}_{D_n} \leq \widetilde{W}_{B_n}$. The next lemma is straightforward from the definitions, and it helps to explain our choices of notations (recall $s_{ij} = (i, j)$ and $\bar{s}_{ij} = \overline{(i, j)}$).

Lemma 7.1. *Let W be $\widetilde{W}_{A_{n-1}}$, \widetilde{W}_{D_n} , or \widetilde{W}_{B_n} . For $i \neq j$, the canonical quotient map $\Upsilon_+ : \mathbb{C}\widetilde{W} \rightarrow \mathbb{C}W$ sends*

$$\{i, j\} \mapsto (i, j), \quad \overline{\{i, j\}} \mapsto \overline{(i, j)}, \quad \{i\} \mapsto \tau_i,$$

and, again for $i \neq j$, the canonical quotient map $\Upsilon_- : \mathbb{C}\widetilde{W} \rightarrow \mathbb{C}W^-$ sends

$$\{i, j\} \mapsto [i, j], \quad \overline{\{i, j\}} \mapsto \overline{[i, j]}, \quad \{i\} \mapsto \overline{[i]}$$

whenever it makes sense for the given W .

The rational Cherednik algebras. Recall that $\mathfrak{h} = \mathbb{C}^n$, and we have identified $\mathbb{C}[\mathfrak{h}] = \mathbb{C}[x_1, \dots, x_n]$ and $\mathbb{C}[\mathfrak{h}^*] = \mathbb{C}[y_1, \dots, y_n]$. Below we shall recall, in a more concrete form, the definition from [Etingof and Ginzburg 2002] of rational double affine Hecke algebras (also called rational Cherednik algebras) associated to the classical Weyl groups.

Let $t, u \in \mathbb{C}$. Let W be one of the Weyl groups $W_{A_{n-1}}$, W_{D_n} , or W_{B_n} respectively. The rational Cherednik algebra $\check{\mathfrak{H}}_W$ is the algebra generated by x_i and y_i for $1 \leq i \leq n$ and W , subject to the common relations

$$\begin{aligned} x_i x_j &= x_j x_i, & y_i y_j &= y_j y_i & \text{for all } i, j, \\ \sigma x &= x^\sigma \sigma, & \sigma y &= y^\sigma \sigma & \text{for } \sigma \in W, x \in \mathfrak{h}^*, y \in \mathfrak{h} \end{aligned}$$

and the additional relations

$$\left. \begin{aligned} [y_j, x_i] &= u s_{ij} & \text{for } i \neq j, \\ [y_i, x_i] &= t \cdot 1 - u \sum_{k \neq i} s_{ki}, \end{aligned} \right\} \quad \text{for type } A,$$

$$\left. \begin{aligned} [y_j, x_i] &= u(s_{ij} - \bar{s}_{ij}) \quad \text{for } i \neq j, \\ [y_i, x_i] &= t \cdot 1 - u \sum_{k \neq i} (s_{ki} + \bar{s}_{ki}), \end{aligned} \right\} \quad \text{for type } D,$$

$$\left. \begin{aligned} [y_j, x_i] &= u(s_{ij} - \bar{s}_{ij}) \quad \text{for } i \neq j, \\ [y_i, x_i] &= t \cdot 1 - u \sum_{k \neq i} (s_{ki} + \bar{s}_{ki}) - v\tau_i, \end{aligned} \right\} \quad \text{for type } B.$$

The algebra $\check{\mathfrak{H}}_W$ has the following well-known PBW property: the multiplication of the subalgebras induces a vector space isomorphism

$$\mathbb{C}[\mathfrak{h}^*] \otimes \mathbb{C}W \otimes \mathbb{C}[\mathfrak{h}] \xrightarrow{\cong} \check{\mathfrak{H}}_W.$$

Equivalently, the set $\{x^\alpha w y^\gamma \mid \alpha, \gamma \in \mathbb{Z}_+^n, w \in W\}$ forms a PBW basis for $\check{\mathfrak{H}}_W$.

The rational covering double affine Hecke algebra $\check{\mathfrak{H}}_W^\sim$. Recall that the group \tilde{W} from Section 2 has the defining relations given in Table 1, and \tilde{W} contains a central element z of order 2.

Definition 7.2. Let $W = W_{A_{n-1}}$, and let $t, u \in \mathbb{C}$. The rational covering double affine Hecke algebra of type A_{n-1} , denoted by $\check{\mathfrak{H}}_W^\sim$ or $\check{\mathfrak{H}}_{A_{n-1}}^\sim$, is the algebra generated by \tilde{x}_i and \tilde{y}_i for $1 \leq i \leq n$ and $z, \tilde{t}_1, \dots, \tilde{t}_{n-1}$ subject to the relations for \tilde{W} , and the following relations: z is central and

$$(7-1) \quad \begin{cases} \tilde{x}_i \tilde{x}_j = z \tilde{x}_j \tilde{x}_i, & \tilde{y}_i \tilde{y}_j = \tilde{y}_j \tilde{y}_i \quad \text{for } i \neq j, \\ \tilde{t}_i \tilde{x}_j = z \tilde{x}_j \tilde{t}_i, & \tilde{t}_i \tilde{y}_j = \tilde{y}_j \tilde{t}_i \quad \text{for } j \neq i, i+1, \\ \tilde{t}_i \tilde{x}_{i+1} = z \tilde{x}_i \tilde{t}_i, & \tilde{t}_i \tilde{y}_{i+1} = \tilde{y}_i \tilde{t}_i, \end{cases}$$

$$[\tilde{y}_j, \tilde{x}_i] = uz\{i, j\} \quad \text{for } j \neq i,$$

$$[\tilde{y}_i, \tilde{x}_i] = -uz \sum_{k \neq i} \{i, k\}.$$

Definition 7.3. Let $W = W_{D_n}$, and let $u \in \mathbb{C}$. The rational covering double affine Hecke algebra of type D_n , denoted by $\check{\mathfrak{H}}_W^\sim$ or $\check{\mathfrak{H}}_{D_n}^\sim$, is the algebra generated by \tilde{x}_i and \tilde{y}_i for $1 \leq i \leq n$ and $z, \tilde{t}_1, \dots, \tilde{t}_n$, subject to the relations for \tilde{W} , relations (7-1), and the additional relations that z is central and

$$\begin{aligned} \tilde{t}_n \tilde{x}_j &= z \tilde{x}_j \tilde{t}_n, & \tilde{t}_n \tilde{y}_j &= \tilde{y}_j \tilde{t}_n & \text{for } i \neq n-1, n, \\ \tilde{t}_n \tilde{x}_n &= -\tilde{x}_{n-1} \tilde{t}_n, & \tilde{t}_n \tilde{y}_n &= -\tilde{y}_{n-1} \tilde{t}_n, \\ [\tilde{y}_j, \tilde{x}_i] &= uz(\{i, j\} - \overline{\{i, j\}}) \quad \text{for } j \neq i, \\ [\tilde{y}_i, \tilde{x}_i] &= -uz \sum_{k \neq i} (\{i, k\} + \overline{\{i, k\}}). \end{aligned}$$

Definition 7.4. Let $W = W_{B_n}$, and let $u, v \in \mathbb{C}$. The rational covering double affine Hecke algebra of type B_n , denoted by $\check{\mathfrak{H}}_W^\sim$ or $\check{\mathfrak{H}}_{B_n}^\sim$, is the algebra generated by \tilde{x}_i

and \tilde{y}_i for $1 \leq i \leq n$ and $z, \tilde{t}_1, \dots, \tilde{t}_n$, subject to the relations for \tilde{W} , relations (7-1), and the additional relations that z is central and

$$\begin{aligned} \tilde{t}_n \tilde{x}_i &= z \tilde{x}_i \tilde{t}_n, & \tilde{t}_n \tilde{y}_i &= \tilde{y}_i \tilde{t}_n & \text{for } i \neq n, \\ \tilde{t}_n \tilde{x}_n &= -\tilde{x}_n \tilde{t}_n, & \tilde{t}_n \tilde{y}_n &= -\tilde{y}_n \tilde{t}_n, \\ [\tilde{y}_j, \tilde{x}_i] &= uz(\{i, j\} - \overline{\{i, j\}}) & \text{for } j \neq i, \\ [\tilde{y}_i, \tilde{x}_i] &= -uz \sum_{k \neq i} (\{i, k\} + \overline{\{i, k\}}) - vz\{i\}. \end{aligned}$$

PBW basis for $\ddot{\mathfrak{H}}_{\tilde{W}}$. The next result uses the notion of rational cDaHa to provide a link between the rational Cherednik algebra $\ddot{\mathfrak{H}}_{\tilde{W}}^{t=0}$ with the specialization $t = 0$ and the rational sDaHa .

Proposition 7.5. *Let $W = W_{A_{n-1}}, W_{D_n}$, or W_{B_n} . Then the quotient of the rational cDaHa $\ddot{\mathfrak{H}}_{\tilde{W}}$ by the ideal $\langle z - 1 \rangle$ (respectively, by the ideal $\langle z + 1 \rangle$) is isomorphic to the rational Cherednik algebra $\ddot{\mathfrak{H}}_W^{t=0}$ (respectively, the rational sDaHa $\ddot{\mathfrak{H}}_{\overline{W}}$).*

Proof. We will merely construct the isomorphisms of superalgebras explicitly, while noting that the verification follows directly from the definitions of the various algebras involved.

The canonical isomorphism map $\Upsilon_+ : \mathbb{C}\tilde{W}/\langle z - 1 \rangle \rightarrow \mathbb{C}W$ (see Lemma 7.1) can be extended to the isomorphism of superalgebras

$$\Upsilon_+ : \ddot{\mathfrak{H}}_{\tilde{W}}/\langle z - 1 \rangle \rightarrow \ddot{\mathfrak{H}}_W^{t=0}, \quad \tilde{t}_i \mapsto s_i, \quad \tilde{x}_i \mapsto x_i, \quad \tilde{y}_i \mapsto y_i.$$

Also, the canonical isomorphism map $\Upsilon_- : \mathbb{C}\tilde{W}/\langle z + 1 \rangle \rightarrow \mathbb{C}W^-$ (see Lemma 7.1) can be extended to the isomorphism of superalgebras $\Upsilon_+ : \ddot{\mathfrak{H}}_{\tilde{W}}/\langle z + 1 \rangle \rightarrow \ddot{\mathfrak{H}}_{\overline{W}}$ by sending $\tilde{t}_i \mapsto t_i$, $\tilde{x}_i \mapsto \xi_i$ and $\tilde{y}_i \mapsto y_i$. \square

The next theorem follows from Proposition 7.5, the PBW basis theorem (Theorem 5.6) for $\ddot{\mathfrak{H}}_{\overline{W}}$, and the PBW property for $\ddot{\mathfrak{H}}_W$ (see [Etingof and Ginzburg 2002]), by the same type of argument for [Wang 2007, Proposition 3.10] or [Khongsap and Wang 2008, Theorem 5.5].

Theorem 7.6. *Let $W = W_{A_{n-1}}, W_{D_n}$, or W_{B_n} . Then the elements $\tilde{x}^\alpha \tilde{w} \tilde{y}^\gamma$, where $\alpha, \gamma \in \mathbb{Z}_+^n$ and $\tilde{w} \in \tilde{W}$, form a basis for $\ddot{\mathfrak{H}}_{\tilde{W}}$.*

8. Appendix: Proofs of several lemmas

Proof of Lemma 3.7. We will show that the relations (3-4) are invariant under the conjugation by elements c_l for $1 \leq l \leq n$. The verifications for the invariants of other relations under the conjugation by c_l are similar and will be omitted.

Consider the relation (3-4a) first. Clearly, (3-4a) is invariant under the conjugation by c_l for $l \neq i, j$. Moreover, we calculate that

$$\begin{aligned} c_i(\text{right side of (3-4a)})c_i &= u((c_i c_j - 1)s_{ji} - (-c_i c_j - 1)\bar{s}_{ij}) \\ &= -[y_j, x_i] = c_i(\text{left side of (3-4a)})c_i, \\ c_j(\text{right side of (3-4a)})c_j &= u((c_j c_i + 1)s_{ji} - (-c_j c_i + 1)\bar{s}_{ij}) \\ &= [y_j, x_i] = c_j(\text{left side of (3-4a)})c_j. \end{aligned}$$

Thus (3-4a) is invariant under conjugation by any c_l .

Next, we will show that the relation (3-4b) is invariant under the conjugation by each c_l . Indeed, we have

$$\begin{aligned} c_i(\text{right side of (3-4b)})c_i &= -\sqrt{2}v c_i \tau_i c_i - u \sum_{k \neq i} c_i((1 + c_k c_i)s_{ki} + (1 - c_k c_i)\bar{s}_{ki})c_i \\ &= \sqrt{2}v \tau_i - u \sum_{k \neq i} ((c_i c_k - 1)s_{ki} + (-c_i c_k - 1)\bar{s}_{ki}) \\ &= \sqrt{2}v \tau_i + u \sum_{k \neq i} ((1 + c_k c_i)s_{ki} + (1 - c_k c_i)\bar{s}_{ki}) \\ &= -[y_i, x_i] = c_i(\text{left side of (3-4b)})c_i. \end{aligned}$$

For $j \neq i$, we have

$$\begin{aligned} c_j(\text{right side of (3-4b)})c_j &= -\sqrt{2}v \tau_i - u c_j((1 + c_j c_i)s_{ji} + (1 - c_j c_i)\bar{s}_{ji})c_j \\ &\quad - u \sum_{k \neq i, j} c_j((1 + c_k c_i)s_{ki} + (1 - c_k c_i)\bar{s}_{ki})c_j \\ &= -\sqrt{2}v \tau_i - u((c_j c_i + 1)s_{ji} + (-c_j c_i + 1)\bar{s}_{ji})c_j \\ &\quad - u \sum_{k \neq i, j} ((1 + c_k c_i)s_{ki} + (1 - c_k c_i)\bar{s}_{ki}) \\ &= c_j(\text{left side of (3-4b)})c_j. \quad \square \end{aligned}$$

Proof of Lemma 3.8. We will show below that the relations (3-4) are invariant under the conjugation by elements in W_{B_n} . The proof can be readily modified to yield the Weyl group invariance of the relations (3-2) and (3-3) in type A and D cases respectively, and we leave the details to the interested reader.

Case (i): We check the invariance of (3-4a) under W_{B_n} . Consider first the conjugation invariance by the transposition s_{lk} . If $\{l, k\} \cap \{i, j\} = \emptyset$, then

$$\begin{aligned} s_{lk}(\text{right side of (3-4a)})s_{lk} &= u((1 + c_j c_i)s_{ji} - (1 - c_j c_i)\bar{s}_{ij}) \\ &= [y_j, x_i] = s_{lk}(\text{left side of (3-4a)})s_{lk}. \end{aligned}$$

If $\{l, k\} \cap \{i, j\} = \{j\}$, then we may assume $l = j$ and we have

$$\begin{aligned} s_{jk}(\text{right side of (3-4a)})s_{jk} &= u((1 + c_k c_i)s_{ik} - (1 - c_k c_i)\bar{s}_{ik}) \\ &= [y_k, x_i] = s_{jk}(\text{left side of (3-4a)})s_{jk}. \end{aligned}$$

We leave an entirely analogous computation when $\{l, k\} \cap \{i, j\} = \{i\}$ to the reader.

Now, if $\{l, k\} = \{i, j\}$, then

$$\begin{aligned} s_{ji}(\text{right side of (3-4a)})s_{ji} &= u((1 + c_i c_j)s_{ij} - (1 - c_i c_j)\bar{s}_{ij}) \\ &= [y_i, x_j] = s_{ji}(\text{left side of (3-4a)})s_{ji}. \end{aligned}$$

So (3-4a) is invariant under the conjugation by each transposition s_{lk} .

It remains to show that (3-4a) is invariant under the conjugation by the simple reflection $s_n = \tau_n$. Observe that (3-4a) is clearly invariant under conjugation by s_n for $n \neq j, i$. Moreover, if $j = n$, then

$$\begin{aligned} s_n(\text{right side of (3-4a)})s_n &= u((1 - c_j c_i)\bar{s}_{ji} - (1 + c_j c_i)s_{ij}) \\ &= -[y_j, x_i] = s_n(\text{left side of (3-4a)})s_n. \end{aligned}$$

If $i = n$, then

$$\begin{aligned} s_n(\text{right side of (3-4a)})s_n &= u((1 - c_j c_i)\bar{s}_{ji} - (1 + c_j c_i)s_{ij}) \\ &= -[y_j, x_i] = s_n(\text{left side of (3-4a)})s_n. \end{aligned}$$

This completes Case (i).

Case (ii): We check the invariance of (3-4b) under W_{B_n} . Consider first the conjugation invariance by a transposition s_{jl} . If $\{j, l\} \cap \{i\} = \emptyset$, then

$$\begin{aligned} s_{jl}(\text{right side of (3-4b)})s_{jl} &= -us_{jl}((1 + c_j c_i)s_{ji} + (1 - c_j c_i)\bar{s}_{ji})s_{jl} \\ &\quad - us_{jl}((1 + c_l c_i)s_{li} + (1 - c_l c_i)\bar{s}_{li})s_{jl} \\ &\quad - u \sum_{k \neq i, j, l} s_{jl}((1 + c_k c_i)s_{ki} + (1 - c_k c_i)\bar{s}_{ki})s_{jl} - \sqrt{2}v\tau_i \\ &= [y_i, x_i] = s_{jl}(\text{left side of (3-4b)})s_{jl}. \end{aligned}$$

If $\{j, l\} \cap \{i\} = \{i\}$, we may assume that $j = i$, and then

$$\begin{aligned} s_{il}(\text{right side of (3-4b)})s_{il} &= -us_{il}((1 + c_l c_i)s_{ji} + (1 - c_l c_i)\bar{s}_{li})s_{il} \\ &\quad - u \sum_{k \neq i, l} s_{il}((1 + c_k c_l)s_{kl} + (1 - c_k c_l)\bar{s}_{kl})s_{il} - \sqrt{2}v\tau_l \\ &= [y_l, x_l] = s_{il}(\text{left side of (3-4b)})s_{il}. \end{aligned}$$

It remains to show that (3-4b) is invariant under the conjugation by the simple reflection $s_n \equiv \tau_n \in W_{B_n}$. If $i \neq n$, we have

$$\begin{aligned}
& s_n(\text{right side of (3-4b)})s_n \\
&= -\sqrt{2}v\tau_i - us_n((1 + c_n c_i)s_{ni} + (1 - c_n c_i)\bar{s}_{ni})s_n \\
&\quad - u \sum_{k \neq i, n} s_n((1 + c_k c_i)s_{ki} + (1 - c_k c_i)\bar{s}_{ki})s_n \\
&= -\sqrt{2}v\tau_i - u((1 - c_n c_i)\bar{s}_{ni} + (1 + c_n c_i)s_{ni}) \\
&\quad - u \sum_{k \neq i, n} ((1 + c_k c_i)s_{ki} + (1 - c_k c_i)\bar{s}_{ki}) \\
&= [y_i, x_i] = s_n(\text{left side of (3-4b)})s_n.
\end{aligned}$$

If $i = n$, then

$$\begin{aligned}
& s_n(\text{right side of (3-4b)})s_n \\
&= -\sqrt{2}v\tau_n - u \sum_{k \neq n} ((1 - c_k c_n)\bar{s}_{kn} + (1 + c_k c_n)s_{kn}) \\
&= [y_n, x_n] = s_n(\text{left side of (3-4b)})s_n.
\end{aligned}$$

This completes the proof of (ii). Hence the lemma is proved. \square

Proof of Lemma 3.9. We will establish the Jacobi identity for $W = W_{B_n}$. The proof can be easily modified for the cases of type A and D , and we leave the details to the reader.

The Jacobi identity holds trivially among triple the x_i or triple the y_i .

Now, we consider the triple with two y 's and one x . The case with two identical y_i is trivial. So we first consider x_i, y_j , and y_l where i, j, l are all distinct. The Jacobi identity holds in this case since

$$\begin{aligned}
& [x_i, [y_j, y_l]] + [y_l, [x_i, y_j]] + [y_j, [y_l, x_i]] \\
&= 0 + [y_l, -u((1 + c_j c_i)s_{ji} - (1 - c_j c_i)\bar{s}_{ij})] \\
&\quad + [y_j, u((1 + c_l c_i)s_{li} - (1 - c_l c_i)\bar{s}_{il})] = 0.
\end{aligned}$$

Now for $i \neq j$, we have

$$\begin{aligned}
& [x_i, [y_i, y_j]] + [y_j, [x_i, y_i]] + [y_i, [y_j, x_i]] \\
&= 0 + [y_j, u \sum_{k \neq i} ((1 + c_k c_i)s_{ki} + (1 - c_k c_i)\bar{s}_{ki}) + \sqrt{2}v\tau_i] \\
&\quad + [y_i, u((1 + c_j c_i)s_{ji} - (1 - c_j c_i)\bar{s}_{ij})] \\
&= [y_j, u \sum_{k \neq i, j} ((1 + c_k c_i)s_{ki} + (1 - c_k c_i)\bar{s}_{ki})] \\
&\quad + [y_j, u((1 + c_j c_i)s_{ji} + (1 - c_j c_i)\bar{s}_{ij})] \\
&\quad + [y_i, u((1 + c_j c_i)s_{ji} - (1 - c_j c_i)\bar{s}_{ij})] \\
&= 0 + u((1 + c_j c_i)y_j s_{ji} + (1 - c_j c_i)y_j \bar{s}_{ij})
\end{aligned}$$

$$\begin{aligned}
& -u((1+c_jc_i)s_{ji}y_j + (1-c_jc_i)\bar{s}_{ji}y_j) \\
& +u((1+c_jc_i)y_i s_{ji} - (1-c_jc_i)y_i \bar{s}_{ij}) \\
& -u((1+c_jc_i)s_{ji}y_i - (1-c_jc_i)\bar{s}_{ij}y_i) = 0.
\end{aligned}$$

Now we consider the Jacobi identity with one y and two x 's. The case with all distinct indices can be easily verified as above. Moreover, for $i \neq j$, we have

$$\begin{aligned}
& [x_i, [y_i, x_j]] + [x_j, [x_i, y_i]] + [y_i, [x_j, x_i]] \\
& = [x_i, u((1+c_i c_j)s_{ij} - (1-c_i c_j)\bar{s}_{ij})] \\
& \quad + [x_j, u \sum_{k \neq i} ((1+c_k c_i)s_{ki} + (1-c_k c_i)\bar{s}_{ki}) + \sqrt{2}v\tau_i] + 0 \\
& = [x_i, u((1+c_i c_j)s_{ij} - (1-c_i c_j)\bar{s}_{ij})] \\
& \quad + [x_j, u((1+c_j c_i)s_{ij} + (1-c_j c_i)\bar{s}_{ij})] \\
& = u((1-c_i c_j)x_i s_{ij} - (1+c_i c_j)x_i \bar{s}_{ij}) - u((1+c_i c_j)s_{ij}x_i - (1-c_i c_j)\bar{s}_{ij}x_i) \\
& \quad + u((1-c_j c_i)x_j s_{ij} + (1+c_j c_i)x_j \bar{s}_{ij}) \\
& \quad - u((1+c_j c_i)s_{ij}x_j + (1-c_j c_i)\bar{s}_{ij}x_j) = 0.
\end{aligned}$$

This completes the verification of the Jacobi identity for any triples. \square

Proof of Lemma 4.8. We will proceed by induction. For $l = 1$, then the equations hold by (3-4). Now assume that the statement is true for l . Then, for $i \neq j$

$$\begin{aligned}
[y_i, x_j^{l+1}] &= [y_i, x_j^l]x_j + x_j^l[y_i, x_j] \\
&= u\left(\frac{x_j^l - x_i^l}{x_j - x_i} + \frac{x_j^l - (-x_i)^l}{x_j + x_i}c_i c_j\right)s_{ij}x_j - u\left(\frac{x_j^l - (-x_i)^l}{x_j + x_i} - \frac{x_j^l - x_i^l}{x_j - x_i}c_i c_j\right)\bar{s}_{ij}x_j \\
& \quad + x_j^l u((1+c_i c_j)s_{ij} - (1-c_i c_j)\bar{s}_{ij}) \\
&= u\left(\frac{x_j^{l+1} - x_i^{l+1}}{x_j - x_i} + \frac{x_j^{l+1} - (-x_i)^{l+1}}{x_j + x_i}c_i c_j\right)s_{ij} \\
& \quad - u\left(\frac{x_j^{l+1} - (-x_i)^{l+1}}{x_j + x_i} - \frac{x_j^{l+1} - x_i^{l+1}}{x_j - x_i}c_i c_j\right)\bar{s}_{ij},
\end{aligned}$$

and

$$\begin{aligned}
[y_i, x_i^{l+1}] &= [y_i, x_i^l]x_i + x_i^l[y_i, x_i] \\
&= -u \sum_{k \neq i} \left(\frac{x_i^l - x_k^l}{x_i - x_k} + \frac{x_i^l - (-x_k)^l}{x_i + x_k}c_k c_i\right)s_{ki}x_i \\
& \quad - u \sum_{k \neq i} \left(\frac{x_i^l - (-x_k)^l}{x_i + x_k} - \frac{x_i^l - x_k^l}{x_i - x_k}c_k c_i\right)\bar{s}_{ki}x_i - \sqrt{2}v \frac{x_i^l \tau_i - \tau_i x_i^l}{2x_i}x_i
\end{aligned}$$

$$\begin{aligned}
 & -u x_i^l \sum_{k \neq i} ((1 + c_k c_i) s_{ki} + (1 - c_k c_i) \bar{s}_{ki}) - \sqrt{2} v x_i^l \tau_i \\
 = & -u \sum_{k \neq i} \left(\frac{x_i^{l+1} - x_k^{l+1}}{x_i - x_k} + \frac{x_i^{l+1} - (-x_k)^{l+1}}{x_i + x_k} c_k c_i \right) s_{ki} \\
 & - u \sum_{k \neq i} \left(\frac{x_i^{l+1} - (-x_k)^{l+1}}{x_i + x_k} - \frac{x_i^{l+1} - x_k^{l+1}}{x_i - x_k} c_k c_i \right) \bar{s}_{ki} \\
 & - \sqrt{2} v \frac{x_i^{l+1} \tau_i - \tau_i x_i^{l+1}}{2x_i}. \quad \square
 \end{aligned}$$

Proof of Lemma 4.9. It suffices to check the formula for every monomial f . First, we consider the monomial $g = \prod_{j \neq i} x_j^{a_j}$. By induction and Lemma 4.8, we can show that the formula holds for the monomial of the form $g = \prod_{j \neq i} x_j^{a_j}$ (the detail of the induction step does not differ much from the following calculation). Now consider the monomial $f = x_i^l g$. By Lemma 4.8, we have

$$\begin{aligned}
 [y_i, f] &= [y_i, x_i^l] g + x_i^l [y_i, g] \\
 &= -u \sum_{k \neq i} \left(\frac{x_i^l - x_k^l}{x_i - x_k} + \frac{x_i^l - (-x_k)^l}{x_i + x_k} c_k c_i \right) s_{ki} g \\
 &\quad - u \sum_{k \neq i} \left(\frac{x_i^l - (-x_k)^l}{x_i + x_k} - \frac{x_i^l - x_k^l}{x_i - x_k} c_k c_i \right) \bar{s}_{ki} g - \sqrt{2} v \frac{x_i^l - (-x_i)^l}{2x_i} \tau_i g \\
 &\quad - u \sum_{k \neq i} x_i^l \left(\frac{g - g^{s_{ki}}}{x_i - x_k} + \frac{g - g^{\bar{s}_{ki}}}{x_i + x_k} c_k c_i \right) s_{ki} - u \sum_{k \neq i} x_i^l \left(\frac{g - g^{\bar{s}_{ki}}}{x_i + x_k} - \frac{g - g^{s_{ki}}}{x_i - x_k} c_k c_i \right) \bar{s}_{ki} \\
 &= -u \sum_{k \neq i} \left(\frac{f - f^{s_{ki}}}{x_i - x_k} + \frac{f - f^{\bar{s}_{ki}}}{x_i + x_k} c_k c_i \right) s_{ki} - u \sum_{k \neq i} \left(\frac{f - f^{\bar{s}_{ki}}}{x_i + x_k} - \frac{f - f^{s_{ki}}}{x_i - x_k} c_k c_i \right) \bar{s}_{ki} \\
 &\quad - \sqrt{2} v \frac{f - f^{\tau_i}}{2x_i} \tau_i. \quad \square
 \end{aligned}$$

Proof of Lemma 4.11. We will proceed by induction. For $l = 1$, the equations hold by (3-4). Now assume that the statement is true for l . Then, for $i \neq j$,

$$\begin{aligned}
 [y_j^{l+1}, x_i] &= y_j [y_j^l, x_i] + [y_j, x_i] y_j^l \\
 &= u y_j \left(\frac{y_j^l - y_i^l}{y_j - y_i} (1 + c_j c_i) s_{ij} - \frac{y_j^l - (-y_i)^l}{y_j + y_i} (1 - c_j c_i) \bar{s}_{ij} \right) \\
 &\quad + u ((1 + c_j c_i) s_{ji} - (1 - c_j c_i) \bar{s}_{ij}) y_j^l \\
 &= u \left(\frac{y_j^{l+1} - y_i^{l+1}}{y_j - y_i} (1 + c_j c_i) s_{ij} - \frac{y_j^{l+1} - (-y_i)^{l+1}}{y_j + y_i} (1 - c_j c_i) \bar{s}_{ij} \right).
 \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
 [y_i^{l+1}, x_j] &= y_i [y_i^l, x_j] + [y_i, x_j] y_i^l \\
 &= -u \sum_{k \neq i} y_i \frac{y_i^l - y_k^l}{y_i - y_k} (1 + c_k c_i) s_{ki} - u \sum_{k \neq i} y_i \frac{y_i^l - (-y_k)^l}{y_i + y_k} (1 - c_k c_i) \bar{s}_{ki} \\
 &\quad - \sqrt{2} v y_i \frac{y_i^l - (-y_i)^l}{2y_i} \tau_i \\
 &\quad - u \sum_{k \neq i} ((1 + c_k c_i) s_{ki} + (1 - c_k c_i) \bar{s}_{ki}) y_i^l - \sqrt{2} v \tau_i y_i^l \\
 &= -u \sum_{k \neq i} \frac{y_i^{l+1} - y_k^{l+1}}{y_i - y_k} (1 + c_k c_i) s_{ki} - u \sum_{k \neq i} \frac{y_i^{l+1} - (-y_k)^{l+1}}{y_i + y_k} (1 - c_k c_i) \bar{s}_{ki} \\
 &\quad - \sqrt{2} v \frac{y_i^{l+1} - (-y_i)^{l+1}}{2y_i} \tau_i. \quad \square
 \end{aligned}$$

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Received February 14, 2008. Revised August 6, 2008.

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