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Let F be a field of characteristic zero. We give the following answer to a generalization of a problem of Büchi over $F[t]$: A sequence of 92 or more cubes in $F[t]$, not all constant, with constant third difference equal to 6, consists of cubes of successive elements $x, x+1, \dots$, for some $x \in F[t]$. We use this, in conjunction to the negative answer to Hilbert's tenth problem for $F[t]$, to show that the solvability of systems of degree-one equations, where some of the variables are assumed to be cubes and (or) nonconstant, is an unsolvable problem over $F[t]$.

1. Introduction

Büchi asked the following question, known as the n -squares problem:

Is there a positive integer M such that any sequence of at least M integer squares, with constant second difference 2, is equal to a sequence of squares of successive integers?

He intended to apply a possible positive answer to obtain a result in logic (we discuss this below). The question was made public in [Lipshitz 1990]. P. Vojta [2000] proved that a positive answer to the analogous question for rational numbers is implied by a conjecture of S. Lang, or by a positive answer to a weaker question of E. Bombieri; he also answered in the affirmative the analogue of Büchi's question for meromorphic functions defined on \mathbb{C} or for function fields of curves of characteristic zero. (For rings of functions, one naturally demands that the elements of the sequence be nonconstant.) Further discussion can be found in [Mazur 1994]. The original n -squares problem is still open.

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D. Hensley [1983] proved that the analogue of Büchi's problem in finite fields \mathbb{F}_p , where p is a prime number, has a positive answer with $M = p$. He also gave a nice "heuristic proof" of the conjecture and various lower and upper bounds on the heights of the terms of a Büchi sequence. D. Buell [1987] characterized all the nontrivial integer sequences of length four (we call a sequence of squares of successive numbers *trivial*). R. G. E. Pinch [1993] proved, under a certain condition on the size, that a family of four-term sequences cannot be extended to five-term sequences. J. Browkin and J. Brzeziński [2006] showed that there exist infinitely many nontrivial five- and six-term sequences (originally, Büchi asked the question for five-term sequences), but for certain constants distinct from 2. It is not known to us whether there exist any nontrivial five-term sequences of integers when the constant is 2 as in the original problem.

Vojta's conditional result claims finiteness of the set of eight-term nontrivial sequences of integers. His result actually does not give a value for M , but only implies that there exists an $M \geq 8$ such that Büchi's original problem has a positive answer. We must apologize for citing Vojta's result wrongly in our previous works, where we claimed that his conjectural result was for $M = 8$.

In [Pheidas and Vidaux 2005] we generalized Büchi's question as follows:

Question 1.1. *Let $k > 1$ be an integer. Is there a positive integer M such that any sequence $y = (y_0, \dots, y_{M-1})$ of k -th powers of integers with constant k -th difference equal to $k!$ is necessarily a sequence of k -th powers of successive integers? (That is, one such that $y_n = (x + n)^k$ for a fixed integer x and $n \in \{0, \dots, M-1\}$.)*

Except for Vojta's results mentioned above and those of [Pheidas and Vidaux 2006], the question is open for all k and for any global field in place of the integers. (Recall that in the case of function fields we restrict our attention to sequences of *nonconstant* functions.)

In the present paper we prove a positive answer to the analogue of the Question in the case $k = 3$ and for a polynomial ring $F[t]$ in place of the integers, where F is a field of characteristic zero. We prove:

Theorem 1.2. *Let F be a field of characteristic 0 and t a transcendental element over F . Assume that $x_0, \dots, x_{M-1} \in F[t]$, that at least one of the x_n is non-constant and that M is not less than 92. If the third difference of the sequence $(x_0^3, \dots, x_{M-1}^3)$ is constant and equal to 6, which is to say, if*

$$(1) \quad x_{n+3}^3 - 3x_{n+2}^3 + 3x_{n+1}^3 - x_n^3 = 6 \quad \text{for } n = 0, \dots, M-4,$$

then, for some $x \in F[t]$ and for any $n = 0, \dots, M-1$, we have

$$x_n^3 = (x + n)^3.$$

Here is a consequence of this theorem to mathematical logic:

Theorem 1.3. *Let F be a field of zero characteristic and let t be a variable. Let $L_{3,T}$ be the language $\{0, 1, +, P_3, T\}$. Interpret the unary predicate P_3 as ' $P_3(y)$ if and only if y is a cube (third power) in $F[t]$ ', interpret the unary predicate T as ' $T(x)$ if and only if x is a nonconstant polynomial' and interpret $0, 1$ and $+$ as usual. Let $L_{3,t}$ be the language $\{0, 1, +, P_3, R\}$ where R is a constant-symbol for the function which sends any x to tx (and the remaining symbols are interpreted as above).*

- (a) *Multiplication in $F[t]$ is positive-existentially definable in each of the languages $L_{3,T}$ and $L_{3,t}$.*
- (b) *The positive-existential theory of $F[t]$ in the language $L_{3,T}$ is undecidable.*
- (c) *The positive-existential theory of $F[t]$ in the language $L_{3,t}$ is undecidable.*

This strengthens a result of J. Denef [1978]—an analogue of Hilbert's tenth problem for rings of polynomials in the variable t , in the language $\{+, \cdot, 0, 1, t\}$ (see expositions in [Matiyasevich 1970; Davis 1973; Pheidas and Zahidi 2000; Poonen 2003; Shlapentokh 2000]). It also strengthens the similar result in [Pheidas and Zahidi 1999] referring to the language $\{+, \cdot, 'x \text{ is nonconstant}', 0, 1\}$.

Here is an immediate consequence of [Theorem 1.3](#):

Corollary 1.4 (Undecidability of simultaneous representation by cubic forms). *There is no algorithm (Turing machine) that solves the following problem:*

Let A and B be two matrices with integer entries and with dimensions $m \times n$ and $m \times 1$, respectively. Assume that x_1, \dots, x_n are variables and X is the column matrix of the x_i^3 . Assume that $f_j(Y_1, \dots, Y_n)$ are polynomials of the variables Y_1, \dots, Y_n of degree 1, for $j = 1, \dots, n$. Determine whether the system of equations

$$A \cdot X = B$$

has a solution with $x_1, \dots, x_n \in F[t]$ with the property that for each $j, f_j(x_1^3, \dots, x_n^3) \notin F$.

It would be desirable to be able to prove the similar statement having in place of the conditions $f_j(x_1^3, \dots, x_n^3) \notin F$ conditions only of the form $x_i \notin F$, or, even, 'some of the x_i are nonconstant'. But for the moment we cannot prove any of these. The proofs of [1.3](#) and [1.4](#) (at the end of the paper) show also that the analogous statements (omitting the conditions for nonconstancy) are equivalent over domains such as \mathbb{Z} and \mathbb{Q} . It follows that the analogues of [Corollary 1.4](#) over \mathbb{Z} and over \mathbb{Q} are open problems.

Open problems. It is natural to ask about the truth of the statements of [Theorem 1.2](#) and [1.3](#) for domains other than polynomials. Some examples are:

- (a) The ring of holomorphic and the field of meromorphic functions (on the complex plane or a p -adic plane);
- (b) A polynomial ring $F[t]$ in any characteristic other than 3;
- (c) The ring of algebraic functions of the variable t , integral over $F[t]$ (this would strengthen the result of A. Shlapentokh [Shlapentokh 1992]);
- (d) Fields of rational functions in any characteristic other than 3;
- (e) Fields of algebraic functions in any characteristic other than 3 (this would strengthen, for example, results of K. Eisentraeger and A. Shlapentokh [2007] (see also [Shlapentokh 2002; 2006]) and of K. Zahidi [2000]);
- (f) \mathbb{Z} and \mathbb{Q} (and, in general, global fields).

Outline of the proof. We compute an invariant ν of the sequence which in the end turns out to be an x as in Theorem 1.2. We observe that Equation (1) is equivalent to $x_n^3 = a + nb + (\nu + n)^3$, where a and b are invariants. Differentiating the terms of this equation, combining with the initial one and using an argument involving heights (degrees) we show that a certain invariant of the sequence is equal to 0 (Lemmas 2.7 and 2.9). In this way we obtain a dependence of a on b and ν . Iterating the procedure we obtain b as a function of ν . In consequence the pairs of nontrivial solutions (x_m, x_n) are shown to be on certain curves over F , of genus greater than 0, which is impossible for nonconstant x_n and x_m . We obtain a number of degenerate cases which we have to rule out before we conclude with Theorem 1.2.

Our method can presumably be applied to the analogous problem for $k > 3$ (with k as in Question 1.1) but the number and nature of degenerate cases seems to increase in a way that we have not been able to systematize to this point. Because of the fact that we use derivatives our proof does not transfer to the analogous problem over the integers or the rationals. \square

Remark. Very recently, H. Pasten [2008] proved a strong version of Büchi's problem for squares over polynomial rings. His result gives new evidence that the analogous problem for any (fixed) power could have a positive answer.

2. Büchi's problem for cubes in polynomial rings

From now on we will fix a solution (x_0, \dots, x_{M-1}) of the system (1) and write $u_n = x_n^3$, so

$$(2) \quad u_{n+3} - 3u_{n+2} + 3u_{n+1} - u_n = 6 \quad \text{for } n = 0, \dots, M-4.$$

We call the sequence $u = (u_0, \dots, u_{M-1})$ *trivial* if it is a sequence of cubes of successive elements; that is, if there is $x \in F[t]$ such that $u_n = (x + n)^3$ for all n .

Without loss of generality we can suppose that the field F is algebraically closed. From now on we make the following assumption:

The field F is algebraically closed of characteristic zero, and at least one x_n is not in F .

Lemma 2.1. The system (2) is equivalent to

$$(3) \quad 2u_n = n(n-1)u_2 - 2n(n-2)u_1 + (n-2)(n-1)u_0 + 2(n-2)(n-1)n$$

for $n = 0, \dots, M-1$, and more generally,

$$(4) \quad 2u_n = (k-n)(k-n-1)u_{k+1} - 2(k-n-1)(k-n+1)u_k \\ + (k-n)(k-n+1)u_{k-1} - 2(k-n-1)(k-n)(k-n+1)$$

for any $k = 1, \dots, M-2$.

Proof. A brute-force proof by induction on n is possible, but we will present here a shorter one due to the referee. Since the sequence (w_n) defined by

$$w_n = \frac{1}{2}n(n-1)u_2 - n(n-2)u_1 + \frac{1}{2}(n-2)(n-1)u_0 + (n-2)(n-1)n$$

is a polynomial in n with leading coefficient n^3 , its third difference is the constant sequence (6). Therefore, (w_n) satisfies Equation (2). Since $w_i = u_i$ for $i = 0, 1, 2$, the sequences (w_n) and (u_n) have the same three first terms, hence are equal. This proves that the system (3) holds. The system (4) holds by a similar argument. \square

Lemma 2.2. For any pairwise distinct indices $m, n, q \in \{0, \dots, M-1\}$, the expression

$$(5) \quad v_{m,n,q} = -\frac{1}{3} \left(\frac{(q-n)u_m + (m-q)u_n + (n-m)u_q}{(q-n)(m-q)(n-m)} + m + n + q \right)$$

does not depend on m, n and q .

Proof. Replace u_m, u_n and u_q by the expressions given by (3). \square

For any m, n and q , we will be writing v instead of $v_{m,n,q}$. We will call v the v -invariant of the sequence u . Since

$$3v = \frac{1}{2}(u_2 - 2u_1 + u_0 - 6),$$

the v -invariant of the trivial solution of Büchi's problem (when $x_2 = x_0 + 2$ and $x_1 = x_0 + 1$) is x_0 . To measure how far a solution u of (2) is from being trivial, we will introduce the new variables

$$a = u_0 - v^3 \quad \text{and} \quad b = (u_1 - u_0) - ((v+1)^3 - v^3).$$

We find

$$(6) \quad u_n = a + nb + (v+n)^3$$

(using the expression for $v_{0,n,1}$). Note that (x_n) is the trivial solution if and only if $a = b = 0$.

Definition 2.3. For $x \in F[t] \setminus \{0\}$, we denote by $\deg x$ the degree of x , while $\deg 0 := -\infty$. We denote by e the maximum of the degrees of the u_n for $n = 0, \dots, M - 1$ (hence $e > 0$). If the degree e is divisible by 3 we write $d = \frac{1}{3}e$. In particular, if $u_n = x_n^3$ for each n , then d is the maximum of the degrees of the x_n .

Corollary 2.4. *One of the following is true:*

- (a) *Each u_n has degree e .*
- (b) *There is an index l such that for each $n \neq l$ we have $\deg u_n = e$ and $\deg u_l < e$.*
- (c) *There are indices $l_1 \neq l_2$ such that for each $n \neq l_i, i = 1, 2$, we have $\deg u_n = e$ and $\deg u_{l_i} < e, i = 1, 2$.*

Proof. Assume we are not in cases (a) or (b). Let $l_1 \neq l_2$ such that $\deg u_{l_i} < e$ and let k be an index such that $\deg u_k = e$. By Lemma 2.2 we have

$$3v = v_{k,l_1,l_2} = -\frac{(l_2 - l_1)u_k + (k - l_2)u_{l_1} + (l_1 - k)u_{l_2}}{(l_2 - l_1)(k - l_2)(l_1 - k)} - k - l_1 - l_2$$

hence $\deg v = \deg u_k = e$. So for any index $n \neq l_1, l_2$ we have

$$3v = -\frac{(l_2 - l_1)u_n + (n - l_2)u_{l_1} + (l_1 - n)u_{l_2}}{(l_2 - l_1)(n - l_2)(l_1 - n)} - n - l_1 - l_2,$$

which implies $\deg u_n = \deg v = e$. □

Corollary 2.5. *If m, n, q and r are pairwise distinct indices of the sequence u , then u_m, u_n, u_q and u_r are coprime (the four polynomials do not have any common divisor).*

Proof. We have

$$\begin{aligned} 3v &= 3v_{m,n,q} = -\frac{(q - n)u_m + (m - q)u_n + (n - m)u_q}{(q - n)(m - q)(n - m)} - m - n - q \\ &= 3v_{m,n,r} = -\frac{(r - n)u_m + (m - r)u_n + (n - m)u_r}{(r - n)(m - r)(n - m)} - m - n - r. \end{aligned}$$

Suppose that there is a nonconstant polynomial P dividing u_m, u_n, u_q and u_r . P has a zero in F . Computing the last two quantities of the latter relations at that zero we obtain $m + n + q = m + n + r$, hence $q = r$, which contradicts our hypothesis. □

Definition 2.6. Recalling Corollary 2.4, we let l_1 and l_2 be two indices such that $\deg u_{l_1} \leq e, \deg u_{l_2} \leq e$, and

$$\deg u_n = e \quad \text{for all } n \text{ other than } l_1 \text{ and } l_2.$$

Lemma 2.7. *Let $\{r_1, \dots, r_m\} \subseteq \{0, \dots, M - 1\}$ be a set of m distinct indices. If Q is a nonzero polynomial in $F[t]$ divisible by each x_{r_k} for $k = 1, \dots, m$, then the degree of Q is at least $\frac{1}{3}(m - 2)/d$. In particular, if we choose $M \geq 92$ and $m = M$ then the degree of Q is at least $30d$.*

Proof. Set $R = \{r_1, \dots, r_m\}$. For all $n \in R$, let $P_n \in F[t]$ be such that $Q = x_n P_n$. Since Q is not the zero polynomial, for each $n \in R$, neither x_n nor P_n is the zero polynomial. We write μ for the least common multiple of the elements of the set $\{x_n \mid n \in R\}$. Hence μ divides Q and it is enough to show that the degree of μ is at least $\frac{1}{3}(m - 2)/d$.

We claim that the product $\prod_{n \in R} x_n$ divides μ^3 . Let P be an arbitrary prime of $F[t]$ which divides μ . Write $\text{ord}_P(x)$ for the order of $x \in F[t]$ at P . It suffices to show that

$$\text{ord}_P\left(\prod_{n \in R} x_n\right) \leq 3 \text{ord}_P(\mu).$$

If P does not divide any x_n , the result is obvious. So assume that P divides x_{k_1} for some index k_1 that we choose so that $\text{ord}_P(x_{k_1})$ is maximum:

$$\text{ord}_P(x_{k_1}) = \text{ord}_P(\mu).$$

By [Corollary 2.5](#), P divides either precisely one x_n , or precisely two, or precisely three. Let x_{k_i} , $i = 1, \dots, j$, be the polynomials divisible by P in case j . In order to treat the three cases simultaneously, let x_{k_2} and x_{k_3} be such that P does not divide any x_n with $n \neq k_1, k_2, k_3$. If we choose the indices so that $\text{ord}_P(x_{k_1}) \geq \text{ord}_P(x_{k_2}) \geq \text{ord}_P(x_{k_3})$, we obtain, as required,

$$\text{ord}_P\left(\prod_{n \in R} x_n\right) = \text{ord}_P(x_{k_1}) + \text{ord}_P(x_{k_2}) + \text{ord}_P(x_{k_3}) \leq 3 \text{ord}_P(x_{k_1}) = 3 \text{ord}_P(\mu),$$

It follows from the claim that

$$\sum_{n \in R} \deg x_n \leq 3 \deg \mu,$$

and by [Corollary 2.4](#) we obtain

$$(m - 2)d \leq \sum_{n \in R} \deg x_n,$$

where the -2 corresponds to the indices l_1 and l_2 from [Definition 2.6](#). □

Notation 2.8. We write

$$A = -v''a' + v'a'' + 6v'^3v, \quad B = v''b' - v'b'' - 6v'^3,$$

and if $B \neq 0$

$$q = \frac{A}{B}.$$

Observe that if $Bv' \neq 0$ we can write

$$(7) \quad q = \frac{(a'/v')' + 6vv'}{-(b'/v')' - 6v'}.$$

Lemma 2.9. *Only the following mutually exclusive two cases can occur:*

Case 1: $v' = 0$.

Case 2: $B \neq 0$, $v' \neq 0$ and we have

$$(8) \quad a + bq + (v + q)^3 = 0,$$

$$(9) \quad a' + b'q + 3v'(v + q)^2 = 0.$$

Proof. By differentiating twice the sides of (6) we get

$$(10) \quad u'_n = a' + nb' + 3v'(v + n)^2$$

and

$$(11) \quad u''_n = a'' + nb'' + 6v'^2(v + n) + 3v''(v + n)^2.$$

By plugging into (10) the expression for $3(v + n)^2$ that results from (11) we obtain

$$v''u'_n = v''a' + nv''b' + v'(u''_n - a'' - nb'' - 6v'^2(v + n)),$$

which we can rewrite as

$$(12) \quad nB = A + U_n,$$

where

$$U_n = v''u'_n - v'u''_n.$$

Multiplying (6) by B^3 and (10) by B^2 we get

$$B^3u_n = aB^3 + nbB^3 + (vB + nB)^3,$$

$$B^2u'_n = a'B^2 + nb'B^2 + 3v'(vB + nB)^2;$$

hence, replacing the expression of nB from (12),

$$B^3u_n = aB^3 + (A + U_n)bB^2 + (vB + A + U_n)^3,$$

$$B^2u'_n = a'B^2 + (A + U_n)b'B + 3v'(vB + A + U_n)^2.$$

Separating terms that depend on n from ones that don't, in both equations, we get

$$(13) \quad B^3u_n - U_n(bB^2 + 3(vB + A)^2 + 3(vB + A)U_n + U_n^2) \\ = aB^3 + AbB^2 + (vB + A)^3,$$

$$(14) \quad B^2u'_n - U_n(b'B + 6v'(vB + A) + 3v'U_n) = a'B^2 + Ab'B + 3v'(vB + A)^2.$$

We give names to the right-hand sides of these two equations:

$$\Delta = aB^3 + AbB^2 + (vB + A)^3, \quad \Gamma = a'B^2 + Ab'B + 3v'(vB + A)^2.$$

We now use [Lemma 2.7](#) to prove that $\Delta = \Gamma = 0$. Since $u_n = x_n^3$, its first and second derivatives, u'_n and u''_n , are each a multiple of x_n , hence $U_n = v''u'_n - v'u''_n$ is a multiple of x_n . Therefore, Δ and Γ are both multiples of x_n for each $n \in \{0, \dots, M - 1\}$. Let us compute an upper bound for the degrees of Δ and Γ . Recalling [Definition 2.6](#) we see that the degree of u_n is not more than e , hence that of v is not more than e and we have $\deg a \leq 3e$, $\deg b \leq 2e$, and

$$\deg A \leq 4e - 3, \quad \deg B \leq 3e - 3, \quad \deg U_n \leq 2e - 3 \quad \text{and} \quad \deg(vB + A) \leq 4e - 3.$$

Therefore, computing the degrees of the left-hand sides of [\(13\)](#) and [\(14\)](#), we find

$$\deg \Delta \leq 10e - 9 = 30d - 9 < 30d,$$

$$\deg \Gamma \leq 7e - 7 = 21d - 7 < 30d.$$

We deduce from [Lemma 2.7](#) that we have $\Delta = 0$ and $\Gamma = 0$.

If B is not zero then v' is not zero and we have

$$\frac{\Delta}{B^3} = a + \frac{A}{B}b + \left(v + \frac{A}{B}\right)^3 = 0, \quad \frac{\Gamma}{B^2} = a' + \frac{A}{B}b' + 3v' \left(v + \frac{A}{B}\right)^2 = 0,$$

which proves [\(8\)](#) and [\(9\)](#).

We next assume that $B = 0$ and prove that $v' = 0$. We know from [Equation \(12\)](#) that $A + U_n = 0$ for all n . Since U_n is a multiple of x_n , and $\deg U_n \leq 2e - 3 = 6d - 3$, we deduce from [Lemma 2.7](#) that U_n is zero. From [Corollary 2.4](#), we know that at most two of the u_n may be constant, namely u_{l_1} and u_{l_2} . For all $n \in \{0, \dots, M - 1\}$ distinct from l_1 and l_2 , we may write

$$\frac{U_n}{u_n'^2} = \frac{v''u'_n - v'u''_n}{u_n'^2} = \left(\frac{v'}{u'_n}\right)'$$

and deduce that for those n , the quotient v'/u'_n must be a constant in F , say c_n . So we have $c_n u'_n = v'$ for at least $M - 2$ distinct values of n , so for at least 90 distinct values of n . We conclude by [Lemma 2.7](#): since

$$\deg v' \leq e - 1 = 3d - 1 < \frac{90 - 2}{3}d,$$

we have $v' = 0$. □

We will need the following proposition, whose proof comes from the theory of elliptic curves (see, for example, [[Husemüller 2004](#), Definition (6.2), page 17] or [[Silverman 1986](#), Hurwitz's Theorem, II.5]). The main observation that concerns us here is that a nonsingular cubic curve is of genus 1.

Proposition 2.10. *Let $\mu, \xi \in F$.*

- (a) *The curve with affine equation $Y^3 = \mu X^3 + \xi$ has genus 1 provided that $\mu\xi \neq 0$.*

(b) *The curve with affine equation $Y^2 + \mu Y + \xi = X^3$ has genus 1 provided that $\mu^2 \neq 4\xi$.*

Remark. The general strategy from now on will be the following: we will provide relations among a , b and v that will produce equations that will define curves as in Proposition 2.10, where the coefficients μ and ξ will depend on one or various indices n . These curves will have rational parametrization by polynomials made up of products of various x_n 's; hence they will define curves of genus 0 (for all the indices considered). Proposition 2.10 will then tell us that this can happen for very few values of n (as long as any of x_n or x_0 is nonconstant, and in particular, if n is different from l_1 and l_2). So we will have space to choose the indices such that one of the curves considered is of genus 1, while it admits a rational parametrization, and this will give us a contradiction. The only case that will survive is that in which for all n we have $x_n^3 = (v + n)^3$, which will prove Theorem 1.2.

Lemma 2.11. *Case 1 is impossible, that is, v' can not be zero.*

Proof. We will show first that if v is constant then so is a , and then that v and a cannot be both constant.

Assume that $v' = 0$ and $a' \neq 0$. So we have $a' = u'_0$ from the definition of a , and

$$u'_n = a' + nb', \quad u''_n = a'' + nb''$$

from (6). This leads to $u'_n b'' = a' b'' + nb'' b' = a' b'' + (u''_n - a'') b'$, that is,

$$u'_n b'' - u''_n b' = a' b'' - a'' b'.$$

Since x_n divides u'_n and u''_n and the degree of $u'_n b'' - u''_n b'$ is no more than $3e - 3$, we deduce from Lemma 2.7 that

$$a' b'' - a'' b' = 0.$$

Since $a' \neq 0$, we can write

$$\left(\frac{b'}{a'}\right)' = 0,$$

so $b = ra + s$ for some constants $r, s \in F$. By (6), we have

$$x_n^3 = u_n = a + nb + (v + n)^3 = a + n(ra + s) + (v + n)^3 = (1 + nr)a + ns + (v + n)^3$$

for each n ; hence, recalling the definition of a ,

$$x_n^3 = (1 + nr)x_0^3 + ns + (v + n)^3 - (1 + nr)v^3.$$

Thus, for each n such that x_n is nonconstant (hence for at least 90 distinct values of n), the curve

$$Y^3 = (1 + nr)X^3 + ns + (v + n)^3 - (1 + nr)v^3$$

is a curve over F that admits the parametrization $(X, Y) = (x_0, x_n)$ by nonconstant rational functions, hence is a curve of genus 0. According to [Proposition 2.10](#) this implies that $(1 + nr)(ns + (v + n)^3 - (1 + nr)v^3) = 0$, which cannot happen for more than four values of n . This gives us a contradiction.

Now we prove that v and a cannot be both constant. Recall that

$$x_1^3 = a + b + (v + 1)^3,$$

hence

$$b = x_1^3 - a - (v + 1)^3.$$

Therefore, for each n , we have

$$x_n^3 = a + n(x_1^3 - a - (v + 1)^3) + (v + n)^3 = nx_1^3 + (1 - n)a - n(v + 1)^3 + (v + n)^3.$$

If both v and a are constant, the curve

$$Y^3 = nX^3 + (1 - n)a - n(v + 1)^3 + (v + n)^3$$

is a curve over F that admits the parametrization $(X, Y) = (x_1, x_n)$ by nonconstant rational functions, hence is a curve of genus 0. As in the previous paragraph we conclude that this cannot happen for more than four values of n . □

Lemma 2.12. *In Case 2 of [Lemma 2.9](#) there are two mutually exclusive subcases:*

Case 2.1: *For all n we have $x_n^3 = (v + n)^3$ (that is, the trivial solution).*

Case 2.2: $q' = 0$.

Proof. According to Case 2, we assume that $B \neq 0$ and $v' \neq 0$. Observe that if (x_n) is the trivial solution then $a = b = 0$ and $q = -v$, hence $q' = -v' \neq 0$.

Suppose q' is not zero. By differentiating [\(8\)](#) we get

$$a' + b'q + bq' + 3(v' + q')(v + q)^2 = 0,$$

and subtracting [\(9\)](#), we obtain $bq' + 3q'(v + q)^2 = 0$, that is,

$$(15) \quad b = -3(v + q)^2.$$

Recall that

$$q = \frac{(a'/v')' + 6vv'}{-(b'/v')' - 6v'}.$$

We write $\alpha = a'/v'$ and $\beta = b'/v'$. We obtain

$$q = -\frac{\alpha' + 6vv'}{\beta' + 6v'},$$

hence

$$(16) \quad -\alpha' = q(\beta' + 6v') + 6vv'.$$

On the other hand, dividing by v' in Equation (9) we obtain

$$\alpha + \beta q + 3(v + q)^2 = 0$$

which, by differentiating, gives

$$-\alpha' = \beta'q + \beta q' + 6(v' + q')(v + q),$$

hence

$$-\alpha' = \beta'q + \beta q' + 6(v'v + v'q + q'v + q'q).$$

Substituting the expression for α' from Equation (16) we obtain

$$q(\beta' + 6v') + 6vv' = \beta'q + \beta q' + 6(v'v + v'q + q'v + q'q);$$

hence, simplifying the $q\beta'$, vv' , and qv' ,

$$0 = \beta q' + 6(q'v + q'q),$$

hence

$$\beta = -6(v + q),$$

or again

$$b' = -6v'(v + q).$$

From Equation (15) we obtain

$$b' = -6(v' + q')(v + q),$$

hence $v + q = 0$. Therefore, Equation (15) implies $b = 0$, and Equation (8) implies $a = 0$. By Equation (6), we get

$$u_n = (v + n)^3.$$

This proves the lemma. □

Lemma 2.13. *Case 2.2 of the previous lemma is impossible, that is, $q' \neq 0$.*

Proof. By Equations (6) and (8) we have

$$u_n = (n - q)b + (v + n)^3 - (v + q)^3,$$

therefore

$$u_n = (n - q)b + 3v^2(n - q) + 3v(n^2 - q^2) + n^3 - q^3,$$

so, for all n distinct from q ,

$$\frac{u_n}{n - q} = b + 3v^2 + 3v(n + q) + n^2 + qn + q^2$$

hence

$$\frac{u_n}{n - q} = b + 3v^2 + 3qv + q^2 + n(3v + q) + n^2.$$

If we write

$$w_n = y_n^3 = \frac{u_n}{n - q}, \quad \alpha = b + 3v^2 + 3qv + q^2, \quad \beta = 3v + q,$$

then we have

$$(17) \quad w_n = \alpha + \beta n + n^2,$$

and, differentiating both sides,

$$(18) \quad w'_n = \alpha' + \beta' n.$$

Multiplying (17) by β'^2 and substituting $\beta' n$ from (18) we get

$$\beta'^2 w_n = \beta'^2 \alpha + \beta' \beta (w'_n - \alpha') + (w'_n - \alpha')^2$$

hence

$$(19) \quad \beta'^2 w_n - \beta' \beta w'_n - w_n'^2 + 2\alpha' w'_n = \beta'^2 \alpha - \beta' \beta \alpha' + \alpha'^2.$$

We intend to apply [Lemma 2.7](#).

For the sake of contradiction, in the rest of the proof we assume that q is constant. So, each y_n is a polynomial of the same degree as x_n , and by [Corollary 2.5](#), any four distinct y_n are coprime. Also, we have $\deg \alpha \leq 2e$, $\deg \beta \leq e$ and $\deg w_n \leq e$. Hence, the left-hand side of (19) has degree $\leq 3e - 2 = 9d - 2$. Observe that w_n is a cube and is divisible by x_n^3 . Hence the left-hand side of (19) is divisible by x_n . So we can apply [Lemma 2.7](#) and conclude that

$$(20) \quad \beta'^2 \alpha - \beta' \beta \alpha' + \alpha'^2 = 0.$$

Recall that $v' \neq 0$, so $\beta' \neq 0$. Hence (20) can be written as

$$\left(\frac{\alpha'}{\beta'}\right)^2 - \beta \frac{\alpha'}{\beta'} + \alpha = 0.$$

Therefore, for some $\gamma \in F(t)$, we have

$$(21) \quad \beta^2 - 4\alpha = \gamma^2$$

and

$$(22) \quad \frac{\alpha'}{\beta'} = \frac{1}{2}(\beta + \varepsilon\gamma)$$

for some $\varepsilon \in \{-1, 1\}$.

Substituting the value of α from (21) into (22) we obtain

$$(23) \quad \gamma(\beta' + \varepsilon\gamma') = 0.$$

Thus we have two cases, according to whether $\beta' = -\varepsilon\gamma'$ or $\gamma = 0$.

Case 2.2.1: We assume $\beta' = -\varepsilon\gamma'$. From Equation (22) we obtain

$$\alpha' = c\beta'$$

for some $c \in F$; substituting this expression for α' in (20) we obtain

$$\alpha = c\beta - c^2.$$

Therefore, by (17),

$$y_n^3 = (n + c)\beta + n^2 - c^2.$$

So, for any indices m and n , we have

$$y_m^3 y_n^3 = ((m + c)\beta + m^2 - c^2)((n + c)\beta + n^2 - c^2),$$

hence

$$(24) \quad \lambda_{m,n}^3 y_m^3 y_n^3 = \beta^2 + \mu_{m,n}\beta + \xi_{m,n}$$

where

$$\lambda_{m,n}^3 = \frac{1}{(m + c)(n + c)}, \quad \mu_{m,n} = \frac{(m + c)(n^2 - c^2) + (n + c)(m^2 - c^2)}{(m + c)(n + c)}$$

and

$$\xi_{m,n} = \frac{(m^2 - c^2)(n^2 - c^2)}{(m + c)(n + c)}$$

provided that $(m + c)(n + c) \neq 0$. It is obvious that we can choose $m, n \leq M - 1$ so that $(m + c)(n + c)(\mu_{m,n}^2 - 4\xi_{m,n}) \neq 0$. So, by Proposition 2.10, the curve

$$(25) \quad Y^3 = X^2 + \mu_{m,n}X + \xi_{m,n}$$

is of genus 1. But by Equation (24) the latter is a curve over F that admits the parametrization $(X, Y) = (\beta, \lambda_{m,n}y_m y_n)$ by nonconstant rational functions (recall that $\beta \notin F$), hence is a curve of genus 0, a contradiction that proves that Case 2.2.1 is impossible.

Case 2.2.2: We assume that $\gamma = 0$. From (21) we obtain $4\alpha = \beta^2$, while (17) becomes

$$4y_n^3 = (\beta + 2n)^2.$$

Hence y_n is a square: $y_n = z_n^2$ for some $z_n \in F[t]$. So we have

$$2z_n^3 = \varepsilon_n(\beta + 2n),$$

where $\varepsilon_n = \pm 1$, and we may assume $\varepsilon_n = 1$ for all n by changing z_n by $-z_n$ if necessary. Hence, for each m and n distinct from q, l_1 and l_2 , the curve

$$4X^3 = Y^2 + 2(m + n)Y + 4mn$$

admits the parametrization $(X, Y) = (z_m z_n, \beta)$ by nonconstant rational functions, hence is of genus 0. By [Proposition 2.10](#), we have

$$4(m + n)^2 = 16m^2 n^2.$$

As long as m has been chosen, this can happen for at most two choices of n . So we get a contradiction and conclude that [Case 2.2.2](#) is impossible. \square

Proof of [Theorem 1.2](#). By [Lemmas 2.9, 2.11, 2.12](#) and [2.13](#), the only possible case is [Case 2.1](#) of [Lemma 2.12](#), that is, $x_n^3 = (v + n)^3$ for each n . \square

Proof of [Theorem 1.3](#). (a) By [Theorem 1.2](#), the formula

$$\phi(x, z, w) : \exists y_0 \dots \exists y_{91} \\ x = y_0 \wedge z = y_1 \wedge w = y_2 \wedge \bigwedge_{n=0}^{91} P_3(y_n) \wedge \bigwedge_{n=0}^{88} y_{n+3} - 3y_{n+2} + 3y_{n+1} - y_n = 6$$

is equivalent over $F[t]$ to:

$$\text{Either } x, z, w \text{ are constant polynomials or } x = v^3 \text{ and } z = (v + 1)^3 \text{ and } w = (v + 2)^3 \text{ for some } v \in F[z].$$

Therefore, the formula

$$\psi(v, u) : \exists x \exists z \exists w \psi(x, z, w) \wedge 6v + 6 = (w - z) - (z - x) \wedge z - x = 3u + 3v + 1$$

is equivalent over $F[t]$ to:

$$\text{Either } v, u \in F \text{ or } u = v^2.$$

Both ϕ and ψ are formulas in the intersection of the languages $L_{3,t}$ and $L_{3,T}$.

Let us prove that the formula

$$\psi_1(v, u) : \exists g \exists h \psi(v, u) \wedge \psi(v + t, g) \wedge \psi(v - t, h) \wedge g + h = 2u + 2t^2$$

is satisfied in $F[t]$ if and only if

$$u = v^2.$$

One the one hand, if $u = v^2$ then we can choose $g = (v + t)^2$ and $h = (v - t)^2$. On the other hand, if $\psi_1(v, u)$ is satisfied in $F[t]$, then either $u = v^2$ and we are done, or $u, v \in F$, in which case $v + t, v - t \notin F$, hence $g = (v + t)^2$ and $h = (v - t)^2$, hence $2u + 2t^2 = g + h = 2v^2 + 2t^2$ implies $u = v^2$.

Observe that ψ_1 is equivalent to a positive-existential $L_{3,t}$ -formula. Similarly, the formula

$$\psi_2(v, u) : \exists f \exists g \exists h \exists z$$

$$T(f) \wedge \psi(f, z) \wedge \psi(v, u) \wedge \psi(v + f, g) \wedge \psi(v - f, h) \wedge g + h = 2u + 2z$$

is equivalent to

$$u = v^2.$$

Observe that ψ_2 is equivalent to a positive-existential $L_{3,T}$ -formula.

Therefore squaring over $F[t]$ is positive-existentially definable in each of the languages $L_{3,t}$ and $L_{3,T}$, hence so is multiplication (for details see L. Lipshitz [Lipshitz 1990]).

Statements (b) and (c) follow from (a) and the fact that the positive-existential theory of $F[t]$ in the language $\{0, 1, +, \cdot, T\}$ (resp. $\{0, 1, +, \cdot, t\}$) is undecidable [Pheidas and Zahidi 1999; Denef 1978]. \square

Proof of Corollary 1.4. Any positive-existential $L_{3,T}$ -sentence is equivalent to a disjunction of sentences each of which claims the solvability of a system of linear equations with integer coefficients, together with conditions stating that certain of the variables are cubes plus conditions which state that certain linear polynomials of the variables are nonconstant ($\notin F$). Now observe that for any x we have

$$6x + 6 = (x + 2)^3 - 2(x + 1)^3 + x^3.$$

Hence we can substitute each variable x , which is not assumed to be necessarily a cube, by the expression

$$\frac{1}{6}z_1^3 - \frac{1}{3}z_2^3 + \frac{1}{6}z_3^3 - \frac{1}{6},$$

where the z_j are new variables. Hence any positive-existential $L_{3,T}$ -sentence is equivalent to a disjunction of sentences of form as in the Corollary. Consequently, if the satisfiability problem for such sentences were decidable, so would be the decidability problem for positive-existential sentences of $L_{3,T}$, which would contradict Theorem 1.3. \square

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